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Generalized solutions for time ψ-fractional heat equation

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Abstract. This paper focuses on the time fractional heat problem with the use of a new fractional derivative. Using Banach's fixed point theorem and Laplace transforms, we give and prove the integral solution of the problem. In Colombeau's algebra, The existence and uniqueness of the solution are demonstrated using the Gronwall lemma.

1. Introduction

The study on Convolution-type derivatives has evolved a focus area of research for the reason that some dynamical models could be better precisely described with fractional derivatives compared to those that have integer-order derivatives [4].

In recent years many researchers have focused on the study of phenomena whose modeling is given by differential equations with a singularity [3, 5, 8], to do this, it is necessary to define the multiplication of two distributions in a manner that is consistent with the standard multiplication, the thing that led us automatically to do the study in Colombeau's algebra. This algebra which is commutative, associative, differential in which we can imbed the space of distributions so that the product of the infinitely differentiable functions and the regular derivative are respected [6, 11].

In the last twenty years or more, mathematicians have become increasingly interested in Heat equations, particularly because of their applicability to optics. In fact, certain Heat equations result from reduced forms or limits of Zakharov's system. In particular, the Hartree-Fock theory and quantum field theory both involve Heat equations. Heat equations, which combine the characteristics of parabolic and hyperbolic equations [8], seem to be a tricky subject from a mathematical perspective. In fact, it possesses essentially reversible behavior, conservation laws, and certain dispersive characteristics similar to those of the Kelin-Gordon equation, but it propagates at an unlimited speed. The time-reversibility, on the other hand, inhibits the generation of an analytic semigroup by Heat equations, despite the fact that they share a smoothing effect with parabolic issues. for more details, one can see the papers [1, 2, 5, 12, 16]. On the other hand, the first definition of the fractional derivative was introduced at the end of the nineteenth century by Liouville and Riemann, but the concept of non-integer derivative and integral, as a generalization of the traditional integer order differential and integral calculus was mentioned already in 1695

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by Leibniz and L'Hospital. Recently, fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics and economics. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic models. Actually, fractional differential equations are considered as an alternative model to integer differential equations.

In this paper we characterize a new method for solving the fractional Heat problem with initial data and potential are singular (singular distribution) as we can see in the following

$$\begin{cases} {}_tD^c_\psi x(y,t) - \Delta x(y,t) + v(y)x(y,t) = 0, \quad t \in [0,T] \\ x(y,0) = a_0(y) \end{cases} \tag{1.1}$$

Where a_0 is singular generalized functions and ${}_tD^c_\psi$ is $\psi-$Caputo derivative of order $\alpha, \alpha \in]n-1;n] \quad n \in \mathbb{N}$.

The study is structured as follows: in section 2 we mention some concepts of Colombeau's algebra, in section 3 we will give and demonstrate the existence of $\psi-$Caputo derivative in Colombeau algebra $\mathcal{G}$, in section 4 we gave and demonstrate the integral solution of the issue, in section 5, we demonstrated the existence and uniqueness of the solution in colombeau algebra.

2. Preliminaries

In this section we will introduce basic notations and definitions from Colombeau theory (see also [3, 9]).

Definition 2.1. *$\mathcal{A}_0(\mathbb{R}^n)$ is a set of functions ϕ in $C_0^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(t)dt = 1$. For $q \in \mathbb{N}, \mathcal{A}_q(\mathbb{R}^n) = \left\{\phi \in \mathcal{A}_0 : \int_{\mathbb{R}^n} t^i\phi(t)dt = 0, 0 < |i| \le q\right\}$, where $t^i = t_1^{i_1} \cdots t_n^{i_n}$.*

In [9] sets

$$\overline{\mathcal{A}}_q(\mathbb{R}^n) = \left\{\Phi(x_1,..,x_n) = \Phi(x_1)..\Phi(x_n) : \phi(x_i) \in \mathcal{A}_q(\mathbb{R})\right\},$$

are used because of applications to initial value problems. We shall follow the Colombeau original definition.

Obviously, if $\phi \in \mathcal{A}_q, q \in \mathbb{N}_0$, then for every $\varepsilon > 0, \phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi\left(\frac{x}{\varepsilon}\right), x \in \mathbb{R}^n$, belongs to $\mathcal{A}_q$. If $\phi \in \mathcal{A}_0$, then its support number $d(\phi)$ is defined by

$$d(\phi) = \sup\{|x| : \phi(x) \neq 0\}.$$

$\mathcal{E}(\Omega)$ represents the set of

$$R : \mathcal{A}_0 \times \Omega \to \mathbb{C}, (\Phi, x) \mapsto R(\Phi, x),$$

which are in $C^\infty(\Omega)$ for every fixed ϕ. In the other words elements of $\mathcal{E}$ are functions $R : \mathcal{A}_0 \to C^\infty$. Note that for any $f \in C^\infty$, the mapping

$$(\phi, x) \mapsto f(x), (\phi, x) \in \mathcal{A}_0 \times \Omega,$$

defines an element in $\mathcal{E}(\Omega)$ which does not depend on ϕ. Conversely, if an element F in $\mathcal{E}(\Omega)$ does not depend on $\Phi \in \mathcal{A}_0$, we have:

$$F(\Phi, x) = F(\Psi, x), \quad x \in \Omega, \text{ for every } \Phi, \Psi \in \mathcal{A}_0,$$

then it defines a function $f \in C^\infty(\Omega)$,

$$f(x) = F(\Phi, x), x \in \Omega, \text{ for every } \phi \in \mathcal{A}_0.$$

In this sense, we identify $C^\infty(\Omega)$ with the corresponding subspace of $\mathcal{E}(\Omega)$.

Definition 2.2. *A component $R \in \mathcal{E}(\Omega)$ is moderate if $\forall L \subset\subset \Omega$, $\alpha \in \mathbb{N}$, $\exists N \in \mathbb{N}$ such that for every $\Phi \in \mathcal{A}_N$, $\exists \eta > 0$ and $C > 0$ such that:*

$$\|\partial^{\alpha} R\left(\Phi_{\epsilon}, x\right)\| \leq C\epsilon^{-N} \qquad x \in L, 0 < \epsilon < \eta.$$

The ensemble of all mild components is expressed as $\mathcal{E}_M(\Omega)$.

Definition 2.3. *An element $R \in \mathcal{E}_0(\mathbb{C})$ is moderate if $\exists N \in \mathbb{N}_0$ such that for every $\phi \in \mathcal{A}_N$, $\exists \eta > 0$, $C > 0$ such that:*

$$\|R\left(\phi_{\varepsilon}\right)\| < C\varepsilon^{-N}, 0 < \varepsilon < \eta.$$

The ensemble of mild components is expressed by $\mathcal{E}_{0M}(\mathbb{C})$ (resp. $\mathcal{E}_{0M}(\mathbb{R})$).

Definition 2.4. *A component $R \in \mathcal{E}_M(\Omega)$ is named null if for every $L \subset\subset \Omega$ and every $\alpha \in \mathbb{N}_0^n$, $\exists N \in \mathbb{N}_0$ and $\left\{a_q\right\} \in \Gamma$ such that for every $q \geq N$ and every $\phi \in \mathcal{A}_q$, $\exists \eta > 0$ and $C > 0$ such that:*

$$\left\|\partial^{\alpha} R\left(\phi_{\varepsilon}, x\right)\right\| \leq C\varepsilon^{a_q - N} \qquad x \in L, 0 < \varepsilon < \eta.$$

The ensemble of null components is expressed by $\mathcal{N}(\Omega)$.

Definition 2.5. *The spaces of generalized functions $\mathcal{G}(\Omega)$ expressed by*

$$\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$$

The following description describes what the term "association" means in colombeau algebra.

Definition 2.6. *[3] Let $f, g \in \mathcal{G}(\mathbb{R})$.*
We said that f,g are associated if $\forall$ $h(\varphi_{\epsilon}, x)$ and $m(\varphi_{\epsilon}, x)$ and arbitrary $\xi \in \mathcal{D}(\mathbb{R})$ there is a $n \in \mathbb{N}$ such that $\forall \varphi(x) \in \mathcal{A}_n(\mathbb{R})$, we have:

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \|h(\varphi_{\epsilon}, x) - m(\varphi_{\epsilon}, x)\| \xi(x) dx = 0$$

and we denoted by $f \approx g$.

3. $\psi-$Fractional Derivative in colombeau algebra $\mathcal{G}$

Let $(f_{\epsilon}(t))_{\epsilon}$ be a representative of a Colombeau generalized function $f(t) \in \mathcal{G}(\mathbb{R}^+)$ and let $\psi \in C^n(\mathbb{R}^+)$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in \mathbb{R}^+$.

The ψ-Caputo fractional derivative of $(f_{\epsilon}(t))_{\epsilon}$, is defined by

$$D_{\psi}^{c} f_{\epsilon}(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (\psi(t) - \psi(s))^{n-\alpha-1} f_{\epsilon}^{[n]}(s)\psi'(s) ds, & \alpha \in]n-1, n[, \\ f_{\epsilon}^{(n)}(t) = f_{\epsilon}^{[n]}(t) = (\frac{1}{\psi'(t)} \frac{d}{dt})^n f_{\epsilon}(t), & \alpha = n, \end{cases} \tag{3.1}$$

$n \in \mathbb{N}, \epsilon \in (0, 1)$.

Lemma 3.1. *Let $(f_{\epsilon}(t))_{\epsilon}$ be a representative of $f(t) \in \mathcal{G}(\mathbb{R}^+)$. Then, for every $\alpha > 0$, $\sup_{t\in[0,T]} \mid D_{\psi}^{c} f_{\epsilon}(t) \mid$ has a moderate bound.*

Proof. Fix $\epsilon \in (0, 1)$.
Let $\alpha \in]n-1, n]$,

Then,
$\sup_{t\in[0,T]} \mid D_{\psi}^{c} f_{\epsilon}(t) \mid \leq \frac{1}{\Gamma(n-\alpha)} \sup_{t\in[0,T]} \int_0^t \mid (\psi(t) - \psi(s))^{n-\alpha-1} f_{\epsilon}^{[n]}(s)\psi'(s) \mid ds$

$$= \frac{1}{\Gamma(n-\alpha)} \sup_{s\in[0,T]} \mid f_{\epsilon}^{[n]}(s) \mid \sup_{t\in[0,T]} \mid \frac{(\psi(t) - \psi(0))^{n-\alpha}}{n-\alpha} \mid$$

$$\leq \frac{1}{\Gamma(n-\alpha)} \frac{R^{n-\alpha}}{n-\alpha} \sup_{s\in[0,T]} \mid f_\epsilon^{[n]}(s) \mid .$$

With $R = \psi(T) - \psi(0)$.
Since $f(t) \in \mathcal{G}([0, +\infty))$, as a result $\sup_{s\in[0,T]} \mid f_\epsilon^{[n]}(s) \mid$ has a moderate bound.

Thus, $\exists M \in \mathbb{N}$, such that

$$\sup_{t\in[0,T]} \mid D_\psi^c f_\epsilon(t) \mid = O\left(\varepsilon^{-M}\right), \quad \varepsilon \to 0$$

Then, $\sup_{t\in[0,T]} \mid D_\psi^c f_\epsilon(t) \mid$ has a moderate bound, $\forall \alpha > 0$.
□

Lemma 3.2. *Let $(f_{1\epsilon}(t))_\epsilon$, $(f_{2\epsilon}(t))_\epsilon$ be two distinct representatives of $f(t) \in \mathcal{G}(\mathbb{R}^+)$. Then, for every $\alpha > 0$, $\sup_{t\in[0,T]} \mid D_\psi^c f_{1\epsilon}(t) - D_\psi^c f_{2\epsilon}(t) \mid$ is negligible.*

Proof. Fix $\epsilon \in (0, 1)$.
Let $\alpha \in]n-1, n]$,
Then,

$$\sup_{t\in[0,T]} \mid D_\psi^c f_{1\epsilon}(t) - D_\psi^c f_{2\epsilon}(t) \mid \leq \frac{1}{\Gamma(n-\alpha)} \frac{R^{n-\alpha}}{n-\alpha} \sup_{s\in[0,T]} \mid f_{1\epsilon}^{[n]}(s) - f_{2\epsilon}^{[n]}(s) \mid .$$

With $R = \psi(T) - \psi(0)$.
Since $(f_{1\epsilon}(t))_\epsilon$ and $(f_{2\epsilon}(t))_\epsilon$ represent the same Colombeau generalized function $f(t)$, so $\sup_{s\in[0,T]} \mid f_{1\epsilon}^{[n]}(s) - f_{2\epsilon}^{[n]}(s) \mid$ is negligible, then for all p$\in \mathbb{N}$

$$\sup_{t\in[0,T]} \mid D_\psi^c f_{1\epsilon}(t) - D_\psi^c f_{2\epsilon}(t) \mid = O\left(\varepsilon^{-p}\right), \quad \varepsilon \to 0$$

Therefore, $\sup_{t\in[0,T]} \mid D_\psi^c f_{1\epsilon}(t) - D_\psi^c f_{2\epsilon}(t) \mid$ is negligible. □

We may now initiate the ψ-Caputo fractional derivative of a Colombeau generalized function on $\mathbb{R}^+$ after establishing the first two lemmas.

Definition 3.1. *Let $f(t) \in \mathcal{G}(\mathbb{R}^+)$ be a Colombeau function on $\mathbb{R}^+$.*
The $\psi-$Caputo fractional derivative of $f(t)$, utilizing the notation $D_\psi^c f(t) = \left[\left(D_\psi^c f_\epsilon(t)\right)_\epsilon\right]$, $\alpha > 0$, is the element of $\mathcal{G}(\mathbb{R}^+)$ satisfying (3.1).

Remark 3.1. *For $\alpha \in]n-1, n]$.*
The first derivative of $(\mathrm{d}/\mathrm{d}t)D_\psi^c f_\epsilon(t)$ is $(\mathrm{d}/\mathrm{d}t)D_\psi^c f_\epsilon(t) =$
$(1/\Gamma(1-\alpha))\left[\int_0^t \left(\frac{\psi\prime(s)}{(\psi(t)-\psi(s))^{\alpha+1-n}} f_\varepsilon^{[n+1]}(s)\right)\mathrm{d}s + \frac{\psi\prime(0)}{(\psi(t)-\psi(0))^{\alpha+1-n}} f_\epsilon^{[m]}(0)\right]$ *and it is not defined in zero, unless $f_\epsilon^{[m]}(0) = 0$.*

Theorem 3.1. *Let $f(t) \in \mathcal{G}$ be a Colombeau function. The $\psi-$Caputo fractional derivative $D_\psi^\alpha f(t)$ is a Colombeau generalized function, if $f_\varepsilon^{[n]}(0) = f_\epsilon^{[n+1]}(0) = f_\epsilon^{[n+2]}(0) = \cdots = 0$.*

Proof. Let $\alpha \in]n-1, n]$.
In Lemma 1, we proved that $\sup_{t\in[0,T]} \mid D_\psi^\alpha f_\epsilon(t) \mid$ has a moderate limit for indefinite Colombeau generalized function. To get a moderate limit for the initial derivative $(\mathrm{d}/\mathrm{d}\ t)D_\psi^\alpha f_\epsilon(t)$ we utilize the expression acquired in Remark 1 and for $f_\epsilon^{[n]}(0) = 0$, we obtain

$$(\mathrm{d}/\mathrm{d}t)D_\psi^\alpha f_\epsilon(t) = (1/\Gamma(1-\alpha)) \int_0^t \left(\frac{\psi\prime(S)}{(\psi(t)-\psi(s))^{\alpha+1-n}} f_\varepsilon^{[n+1]}(s)\right)\mathrm{d}s$$

.
Now, in the same way as in Lemma 1 we acquires a moderate limit for $\sup_{t\in[0,T]} \mid (\mathrm{d}/\mathrm{d}t)D_\psi^\alpha f_\epsilon(t) \mid$.
Using the conditions, higher-order derivatives can be estimated similarly. $f_\epsilon^{[n]}(0) = f_\epsilon^{[n+1]}(0) = f_\epsilon^{[n+2]}$

$(0) = \cdots = 0$.
Finally, if $f_\epsilon^{[n]}(0) = 0$, therefore, it follows that for each $\alpha > 0$, all derivatives of $D_\psi^\alpha f_\epsilon(t)$ have moderate representations. □

Definition 3.2. *Let $(f_\epsilon(t))_\epsilon$ be a representative of $f(t) \in \mathcal{G}(\mathbb{R}^+)$.*
The regularized ψ−Caputo fractional derivative of $(f_\epsilon(t))_\epsilon$, is given by

$$\tilde{D}_\psi^c f_\epsilon(t) = \begin{cases} \left(D_\psi^c f_\epsilon(t) * \varphi_\varepsilon\right)(t), & \alpha \in]n-1, n] \\ f_\epsilon^{(n)}(t) = f_\epsilon^{[n]}(t) = (\frac{1}{\psi'(t)}\frac{\mathrm{d}}{dt})^n f_\epsilon(t), & \alpha = n, \end{cases} \tag{3.2}$$

$n \in \mathbb{N}, \epsilon \in (0,1)$.

where $D_\psi^c f_\epsilon(t)$ is provided by (3.1).

The convolution in (3.2) is $\left(D_\psi^c f_\epsilon(t) * \varphi_\epsilon\right)(t) = \int_0^\infty D_\psi^c f_\epsilon(t)\varphi_\epsilon(t-s)\mathrm{d}s$.

4. The integral solution of Heat equation

Definition 4.1. *[15] Define the Mittag-Leffler function by:*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(k\alpha+\beta)}.$$

Definition 4.2. *[25] Describe the Laplace transform of a function g by*

$$\mathcal{L}(g(x))(s) = \int_0^{+\infty} e^{-sx} g(x) dx.$$

Proposition 4.1. *[25] Let f and g two functions, we have*

$$\mathcal{L}\left((f * g)(x)\right)(s) = \mathcal{L}\left(f(x)\right)(s)\mathcal{L}\left(g(x)\right)(s).$$

Definition 4.3. *[15]*

1. *The Gamma function is given by*

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt, \forall x > 0$$

2. *The $\mathbb{B}$ function is described by*

$$\forall x, y > 0, \quad \mathbb{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

Proposition 4.2. *[15]*

1. $\forall x, y \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, $\mathbb{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.
2. *For all $x > 0$, $\Gamma(x+1) = x\Gamma(x)$.*

Definition 4.4. *[23] The Wright type function is represented by*

$$\begin{aligned} \phi_\alpha(x) &= \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\Gamma(-\alpha n + 1 - \alpha)} \\ &= \sum_{n=0}^{\infty} \frac{(-x)^n \Gamma(\alpha(n+1)) \sin(\pi(n+1)\alpha)}{n!} \end{aligned}$$

for $\alpha \in (0,1)$ and $x \in \mathbb{C}$.

Proposition 4.3. *[23] The Wright function ϕ_α is a complete function with the following characteristics:*

(i) $\int_0^\infty \phi_\alpha(\theta)\theta^r d\theta = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}$ for $r > -1$;
(ii) $\phi_\alpha(\theta) \geq 0$ for $\theta \geq 0$ and $\int_0^\infty \phi_\alpha(\theta)d\theta = 1$
(iii) $\int_0^\infty \phi_\alpha(\theta)e^{-z\theta}d\theta = E_\alpha(-z), \quad z \in \mathbb{C}$;
(iv) $\alpha\int_0^\infty \theta\phi_\alpha(\theta)e^{-z\theta}d\theta = E_{\alpha,\alpha}(-z), \quad z \in \mathbb{C}$.

Definition 4.5. *[23] We proceed with the observed one-sided steady probability density in*

$$\rho_\alpha(\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\theta^{-\alpha k-1}\frac{\Gamma(\alpha k+1)}{k!}\sin(k\pi\alpha), \quad \theta \in (0,\infty)$$

And we have,

$$\int_0^\infty e^{-\lambda\theta}\rho_\alpha(\theta)d\theta = e^{-\lambda^\alpha}, \text{ where } \alpha \in (0,1). \tag{4.1}$$

Lemma 4.1. *Let $f : C(J,X) \to C(J,X)$ be continuous.*

The issue (1.1) is equal to the mild equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(\psi(t)-\psi(s))^{1-\alpha}}\psi'(s)Ax(s)ds, \quad t \in J, \tag{4.2}$$

With:
$x : D(A) \longrightarrow D(A)$ offered that the integral in 4.2 exists, and $A = v - \Delta$.

We will need the following lemma.

Lemma 4.2. *For all $\alpha \in]n-1,n] \quad n \in \mathbb{N}$ and $s > 0$, and let $\phi \in C^n(\mathbb{R}^+)$ be an increasing function with $\phi'(t) \neq 0 \quad$ for all $t \in \mathbb{R}^+$. We have,*

1) $s^{\alpha-1}(s^\alpha - A)^{-1} = \mathcal{L}\Big(\int_0^\infty \rho_\alpha(\theta)T\left(\frac{(\phi(t)-\phi(0))^\alpha}{\theta^\alpha}\right)d\theta\Big)(s)$,

2) $(s^\alpha - A)^{-1}X(s) = \mathcal{L}\Big(\Big(\int_0^\tau\int_0^\infty \alpha\rho_\alpha(\theta)\frac{(\phi(\tau)-\phi(s))^{\alpha-1}}{\theta^\alpha}T\left(\frac{(\phi(\tau)-\phi(s))^\alpha}{\theta^\alpha}\right)$
$x(s)\phi'(s)d\theta ds\Big)\Big)(s)$.

With,
$X(s) = \int_0^\infty e^{-\lambda(\phi(s)-\phi(0))}x(s)\phi'(s)ds$

Proof. 1) For $s > 0$,
$s^{\alpha-1}(s^\alpha - A)^{-1} = s^{\alpha-1}\int_0^\infty e^{-s^\alpha}T(\tau)d\tau = \alpha\int_0^\infty (s\hat{t})^{\alpha-1}e^{-(s\hat{t})^\alpha}T\left(\hat{t}^\alpha\right)dt$
Where $\{T\}_{t\geq 0}$ is $C_0-semigroup$ defined by
$Ax = \lim_{t\longrightarrow 0^+}\frac{T(t)x-x}{x}$ *and* $(\lambda^\alpha I - A)^{-1}x = \int_0^\infty exp(-\lambda^\alpha t)T(t)xdt$
Putting $\hat{t} = \phi(t) - \phi(0)$, *we have*

$= \alpha\int_0^\infty s^{\alpha-1}(\phi(t)-\phi(0))^{\alpha-1}e^{-\left(s(\phi(t)-\phi(0)\right)^\alpha} \times T\left((\phi(t)-\phi(0))^\alpha\right)\psi'(t)dt$

$= \int_0^\infty -\frac{1}{s}\frac{d}{dt}\left(e^{-(s(\phi(t)-\phi(0)))^\alpha}\right)T\left((\phi(t)-\phi(0))^\alpha\right)dt.$

Using (4), we get

$= \int_0^\infty\int_0^\infty \theta\rho_\alpha(\theta)e^{-s(\phi(t)-\phi(0))^\theta}T\left((\phi(t)-\phi(0))^\alpha\right)\psi'(t)d\theta dt$

$= \int_0^\infty e^{-s(\phi(t)-\phi(0))} \left(\int_0^\infty \rho_\alpha(\theta) T\left(\frac{(\phi(t)-\phi(0))^\alpha}{\theta^\alpha}\right) d\theta\right) \psi'(t)dt$

$= \mathcal{L}\Big(\int_0^\infty \rho_\alpha(\theta) T\left(\frac{(\phi(t)-\phi(0))^\alpha}{\theta^\alpha}\right) d\theta\Big)(s)$

2) For $s > 0$,

$$
\begin{aligned}
(s^\alpha - A)^{-1} X(s) &= \int_0^\infty e^{-s^\alpha \tau} T(\tau) X(s) d\tau \\
&= \alpha \int_0^\infty \hat{\tau}^{\alpha-1} e^{-(s\tau))^\alpha} T\left(\hat{\tau}^\alpha\right) X(s) d\tau \\
& \textit{Where } \{T\}_{t\geq 0} \textit{ is } C_0-\textit{semigroup defined by} \\
& Ax = \lim_{t\longrightarrow 0^+} \frac{T(t)x - x}{x} \quad \textit{and} \\
& (\lambda^\alpha I - A)^{-1} x = \int_0^\infty exp(-\lambda^\alpha t) T(t) x dt \\
& \textit{Putting } \hat{t} = \phi(t) - \phi(0), \textit{ we have} \\
&= \int_0^\infty \alpha(\phi(\tau) - \phi(0))^{\alpha-1} e^{-(s(\phi(\tau)-\phi(0)))^\alpha} \\
& \times T\left((\phi(\tau) - \phi(0))^\alpha\right) \phi'(\tau) X(s) d\tau \\
&= \int_0^\infty \int_0^\infty \alpha(\phi(\tau) - \phi(0))^{\alpha-1} e^{-(s(\phi(\tau)-\phi(0)))^\alpha} \\
& T\left((\phi(\tau) - \phi(0))^\alpha\right) \times e^{-(\lambda(\phi(r)-\phi(0)))} x(r)\psi'(r)\phi'(\tau) dr d\tau,
\end{aligned}
$$

Using (4), we get

$= \int_0^\infty \int_0^\infty \int_0^\infty \alpha(\phi(\tau) - \phi(0))^{\alpha-1} \rho_\alpha(\theta) e^{-s(\phi(\tau)-\phi(0))\theta'} T\left((\phi(\tau) - \phi(0))^\alpha\right)$
$\times e^{-s(\phi(r)-\phi(0))} x(r)\phi'(r)\phi'(\tau) d\theta dr d\tau$

$= \int_0^\infty \int_0^\infty \int_0^\infty \alpha e^{-s(\phi(\tau)+\phi(r)-2\phi(0))} \frac{(\phi(\tau)-\phi(0))^{\alpha-1}}{\theta^\alpha} \rho_\alpha(\theta)$
$\times T\left(\frac{(\phi(\tau)-\phi(0))^\alpha}{\theta^\alpha}\right) x(r)\phi'(r)\phi'(\tau) d\theta dr d\tau$

$= \int_0^\infty \int_t^\infty \int_0^\infty \alpha e^{-s(\phi(\tau)-\phi(0))} \rho_\alpha(\theta) \frac{(\phi(t)-\phi(0))^{\alpha-1}}{\theta^\alpha} T\left(\frac{(\phi(t)-\phi(0))^\alpha}{\theta^\alpha}\right)$
$x\left(\phi^{-1}(\phi(\tau) - \phi(t) + \phi(0))\right)$
$\phi'(\tau)\phi'(t) d\theta d\tau dt$

$= \int_0^\infty \int_0^\tau \int_0^\infty \alpha e^{-s(\phi(\tau)-\phi(0))} \rho_\alpha(\theta) \frac{(\phi(t)-\phi(0))^{\alpha-1}}{\theta^\alpha} T\left(\frac{(\phi(t)-\phi(0))^\alpha}{\theta^\alpha}\right)$
$x\left(\phi^{-1}(\phi(\tau) - \phi(t) + \phi(0))\right) \phi'(\tau)$
$\phi'(t) d\theta dt d\tau$

$= \int_0^\infty e^{-s(\phi(\tau)-\phi(0))} \Big(\int_0^\tau \int_0^\infty \alpha \rho_\alpha(\theta) \frac{(\phi(\tau)-\phi(r))^{\alpha-1}}{\theta^\alpha} T\left(\frac{(\phi(\tau)-\phi(r))^\alpha}{\theta^\alpha}\right)$
$x(r)\phi'(r) d\theta dr\Big) \times \phi'(\tau) d\tau.$

$= \mathcal{L}\Big(\Big(\int_0^\tau \int_0^\infty \alpha \rho_\alpha(\theta) \frac{(\phi(\tau)-\phi(r))^{\alpha-1}}{\theta^\alpha} T\left(\frac{(\phi(\tau)-\phi(r))^\alpha}{\theta^\alpha}\right)$
$x(r)\phi'(r) d\theta dr\Big)\Big)(s)$

□

Proposition 4.4. *If*

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \psi'(s) Ax(s) ds,$$

holds, then we have

$$x(t) = E(t)x_0 + \alpha \int_0^t E(t)(\psi(t) - \psi(s))^{\alpha-1} x(s)\psi'(s)ds.$$

With,
$E(t) = \int_0^\infty \phi_a(\theta) T\left((\psi(t) - \psi(0))^\alpha \theta\right) d\theta$

Proof. Since $x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(\psi(t)-\psi(s))^{1-\alpha}} \psi'(s)Ax(s)ds$, using the Laplace transform, we obtain

$$\begin{aligned}
\mathcal{L}\Big(x(t)\Big)(s) &= \mathcal{L}\Big(x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(\psi(t) - \psi(\tau))^{1-\alpha}} \psi'(\tau)Ax(\tau)d\tau\Big)(s) \\
&= \mathcal{L}(x_0)(s) + \mathcal{L}\Big(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(\psi(t) - \psi(\tau))^{1-\alpha}} \psi'(\tau)Ax(\tau)d\tau\Big)(s) \\
&= \frac{x_0}{s} + \mathcal{L}\Big(\frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(\psi(t) - \psi(\tau))^{1-\alpha}} \psi'(\tau)Ax(\tau)d\tau\Big)(s) \\
&= \frac{x_0}{s} + \frac{1}{s^\alpha} A\mathcal{L}\left(x(t)\right)(s) \\
&= \frac{x_0}{s} + \frac{1}{s^\alpha} A \int_0^\infty e^{-\lambda(\psi(s)-\psi(0))} x(s)\psi'(s)ds
\end{aligned}$$

We can deduce
$\mathcal{L}\Big(x(t)\Big)(s) = s^{\alpha-1}\left(s^\alpha - A\right)^{-1} x_0 + \left(s^\alpha - A\right)^{-1} X(s).$
Now, use the lemma 4.2, then
$\mathcal{L}\Big(x(t)\Big)(s) = \mathcal{L}\Big(\int_0^\infty \rho_\alpha(\theta) T\left(\frac{(\psi(t)-\psi(0))^\alpha}{\theta^\alpha}\right) d\theta\Big)(s)x_0 +$
$\mathcal{L}\Big(\Big(\int_0^\tau \int_0^\infty \alpha\rho_\alpha(\theta) \frac{(\psi(\tau)-\psi(s))^{\alpha-1}}{\theta^\alpha} T\left(\frac{(\psi(\tau)-\psi(s))^\alpha}{\theta^\alpha}\right)$
$x(s)\psi'(s)d\theta ds\Big)\Big)(s)$

We can now invert the Laplace transform to obtain the result

$\forall x \in X$, characterize operators $S_\psi^\alpha(t,s)$ and $T_\psi^\alpha(t,s)$ by

$$S_\psi^\alpha(t,s)x = \int_0^\infty \phi_\alpha(\theta) T\left((\psi(t) - \psi(s))^\alpha \theta\right) x d\theta$$

And

$$T_\psi^\alpha(t,s)x = \alpha \int_0^\infty \theta\phi_a(\theta) T\left((\psi(t) - \psi(s))^\alpha \theta\right) x d\theta$$

for $0 \le s \le t \le T$. □

Lemma 4.3. *S_ψ^α and T_ψ^α provide the following characteristics :*

(i) The operators $S_\psi^\alpha(t,s)$ and $T_\psi^\alpha(t,s)$ are strongly continuous for all $t \ge s \ge 0$, that is, for every $x \in X$ and $0 \le s \le t_1 < t_2 \le T$ we have

$$\| S_\psi^\alpha(t_2,s)x - S_\psi^\alpha(t_1,s)x\| \to 0 \text{ and } \| T_\psi^\alpha(t_2,s)x - T_\psi^\alpha(t_1,s)x\| \to 0$$

as $t_1 \to t_2$.

(ii) For any fixed $t \ge s \ge 0$, $S_\psi^\alpha(t,s)$ and $T_\psi^\alpha(t,s)$ are bounded linear operators with

$$\| S_\psi^\alpha(t,s)(x)\| \le M\|x\| \text{ and } \left\|T_\psi^\alpha(t,s)(x)\right\| \le \frac{\alpha M}{\Gamma(1+\alpha)}\|x\| = \frac{M}{\Gamma(\alpha)}\|x\|$$

for all $x \in X$.

Proof. Similar demonstration existe in [27] □

5. Existence and Uniqueness of the Solution in colombeau algebra $\mathcal{G}$

In this section consider the following fractional Heat problem:

$$\begin{cases} {}_tD^c_{\psi}x(y,t) = Ax(y,t), \quad t \in [0,T] \\ x(y,0) = a_0(y) \end{cases}$$

with $a_0(y) \in D'(\mathbb{R}^n)$ *and* $A = \Delta - v$.

Now we will transform the problem in the Colombeau algebra.

$$\begin{cases} {}_tD^c_{\psi}x_\varepsilon(y,t) = A_\varepsilon x_\varepsilon(y,t) \quad y \in \mathbb{R}^n, \quad t \geq 0 \\ \quad x_\varepsilon(y,0) = a_{0,\varepsilon}(y) \end{cases} \tag{5.1}$$

with $a_{0,\varepsilon}(y)$ is the regularizetion of $a_0(y)$, and $A = [(A_\varepsilon)] = [(\Delta - v_\epsilon)]$ is the infinitesimal generator of $\{T_\epsilon(t)\}$ $C_0 -$ *semigroup.*

Theorem 5.1. *If the generalized operators S^α_ψ and T^α_ψ verify the Lemma (4.3). Then the problem (5.1) has a unique solution in $\mathcal{G}(\mathbb{R}^n \times \mathbb{R}^+)$.*

Proof. **Existence**

The integral solution of the problem (5.1) is given through the previous section:

$$\begin{aligned} x_\varepsilon(t) = & \int_0^\infty \phi_{\varepsilon,\alpha}(\theta) T\left((\psi_\varepsilon(t) - \psi_\varepsilon(0))^\alpha \theta\right) x_{\varepsilon,0} d\theta \\ & + \alpha \int_0^t \int_0^\infty \phi_{\varepsilon,\alpha}(\theta)(\psi_\varepsilon(t) - \psi_\varepsilon(s))^{\alpha-1} T\left((\psi_\varepsilon(t) - \psi_\varepsilon(0))^\alpha \theta\right) \\ & x_\epsilon(s)\psi'_\varepsilon(s) d\theta ds. \\ & = S^\alpha_\psi(t,s)x_{\varepsilon,0} + \int_0^t (\psi_\varepsilon(t) - \psi_\varepsilon(s))^{\alpha-1} T^\psi_\alpha(t,s)x_\epsilon(s)\psi'_\varepsilon ds \end{aligned}$$

Which implies that:

$\|x_\varepsilon(t,.)\| \leq \left\|S^\alpha_{\psi,\varepsilon}(t,0)x_{\varepsilon,0}\right\| + \int_0^t \left\|(\psi_\varepsilon(t) - \psi_\varepsilon(s))^{\alpha-1} T^\alpha_{\psi,\varepsilon}(t,s)x_\epsilon(s)\psi'_\varepsilon(s)\right\| ds$

$\leq M\left\|x_{\varepsilon,0}\right\| + \int_0^t (\psi_\varepsilon(t) - \psi_\varepsilon(s))^{\alpha-1} \left\|T^\alpha_{\psi,\varepsilon}(t,s)x_\epsilon(s)\right\| \psi'_\varepsilon(s) ds$

Then

$$\|x_\varepsilon(t,.)\| \leq M\left\|x_{\varepsilon,0}\right\| + \frac{M}{\Gamma(\alpha)} \int_0^t (\psi_\varepsilon(t) - \psi_\varepsilon(s))^{\alpha-1} \|x_\varepsilon(s,.)\| \psi'_\varepsilon(s) ds$$

By the Granwall's inequality

$$\|x_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq M\left\|a_{\varepsilon,0}\right\| \times \exp\left(\frac{M}{\Gamma(\alpha+1)}(\psi_\varepsilon(t) - \psi_\varepsilon(0))^\alpha\right).$$

Since $\psi_\varepsilon \in G(\mathbb{R}^+)$, $a_{0,\epsilon} \in \mathcal{G}(\mathbb{R}^n)$ there exist $K \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} \|x_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} = O\left(\varepsilon^{-K}\right), \quad \varepsilon \to 0$$

So

$$x_\epsilon \in \mathcal{G}(\mathbb{R}^+ \times \mathbb{R}^n)$$

Uniqueness

Let's say there are two solutions $x_{1,\epsilon}(t,.)$, $x_{2,\epsilon}(t,.)$ to the problem (5.1), consequently :

$$\begin{cases} D^c_{\psi} x_{1,\epsilon}(y,t) + A_\epsilon x_{1,\epsilon}(y,t) - D^c_{\psi} x_{2,\epsilon}(y,t) - A_\epsilon x_{2,\epsilon}(y,t) = 0 \\ y \in \mathbb{R}^n, \quad t \geq 0 \\ x_{1,\epsilon}(y,0) - x_{2,\epsilon}(y,0) = N_{0,\epsilon}(y) \end{cases}$$

Then:

$$\begin{cases} D^c_{\psi} \left(x_{1,\epsilon}(y,t) - x_{2,\epsilon}(y,t)\right) + A_\epsilon \left(x_{1,\epsilon}(y,t) - x_{2,\epsilon}(y,t)\right) = 0 \\ y \in \mathbb{R}^n, \quad t \geq 0 \\ x_{1,\epsilon}(y,0) - x_{2,\epsilon}(y,0) = N_{0,\epsilon}(y) \end{cases} \tag{5.2}$$

With $(N_{0,\varepsilon})_\varepsilon \in \mathcal{N}(\mathbb{R}^+)$.
The integral solution of the problem (5.2) is:

$$x_\varepsilon(t) = \int_0^\infty \phi_{\varepsilon,\alpha}(\theta) T\left((\psi_\varepsilon(t) - \psi_\varepsilon(0))^\alpha \theta\right) N_{0,\epsilon}(y) d\theta$$
$$+\alpha \int_0^t \int_0^\infty \phi_{\varepsilon,\alpha}(\theta)(\psi_\varepsilon(t) - \psi_\varepsilon(s))^{\alpha-1} T\left((\psi_\varepsilon(t) - \psi_\varepsilon(0))^\alpha \theta\right)$$
$$\times \left(x_{1,\varepsilon}(s) - x_{2,\varepsilon}(s)\right) \psi'_\varepsilon(s) d\theta ds.$$
$$= S^\alpha_\psi(t,s) N_{0,\epsilon}(y) + \int_0^t (\psi_\varepsilon(t) - \psi_\varepsilon(s))^{\alpha-1} T^\alpha_\psi(t,s) \times \left(x_{1,\varepsilon}(s) - x_{2,\varepsilon}(s)\right) \psi'_\varepsilon ds.$$

Then

$$\left\|x_{1,\varepsilon}(t,.) - x_{2,\varepsilon}(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq \left\|S^\alpha_{\psi,\varepsilon}(t,0) N_{0,\varepsilon}(.)\right\|$$
$$+ \int_0^t \left\|(\psi_\varepsilon(t) - \psi_\varepsilon(s))^{\alpha-1} T^\alpha_{\psi,\varepsilon}(t,s) \left(x_{1,\varepsilon}(s) - x_{2,\varepsilon}(s)\right) \psi'_\varepsilon(s)\right\| ds$$
$$\leq M \left\|N_{0,\varepsilon}(.)\right\| + \int_0^t (\psi_\varepsilon(t) - \psi_\varepsilon(s))^{\alpha-1} \left\|T^\alpha_{\psi,\varepsilon}(t,s) \left(x_{1,\varepsilon}(s) - x_{2,\varepsilon}(s)\right)\right\|$$
$$\times \psi'_\varepsilon(s) ds$$
$$\leq M \left\|N_{0,\varepsilon}(.)\right\| + \frac{M}{\Gamma(\alpha)} \int_0^t (\psi_\varepsilon(t) - \psi_\varepsilon(s))^{\alpha-1} \left\|\left(x_{1,\varepsilon}(s) - x_{2,\varepsilon}(s)\right)\right\| \times \psi'_\varepsilon(s) ds$$

Using the Granwall's inequalit

$$\left\|x_{1,\varepsilon}(t,.) - x_{2,\varepsilon}(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq M \left\|N_{0,\varepsilon}(.)\right\| \times \exp\left(\frac{M}{\Gamma(\alpha+1)} (\psi_\varepsilon(T) - \psi_\varepsilon(0))^\alpha\right)$$

Since
$\psi_\varepsilon \in G(\mathbb{R}^+)$, $\quad (N_{0,\varepsilon})_\varepsilon \in \mathcal{N}(\mathbb{R}^+)$, then for every $q \in \mathbb{N}$ such that:

$$\sup_{t \in [0,T]} \left\|x_{1,\varepsilon}(t,.) - x_{2,\varepsilon}(t,.)\right\|_{L^\infty} = O\left(\varepsilon^q\right) \quad \varepsilon \to 0$$

So,

$$x_{1,\epsilon} \approx x_{2,\epsilon}$$

□

6. Conclusions

In this work, first we give and demonstrated the existence of $\psi-$fractional caputo derivative in colombeau algebra and secondly we have solved the frational Heat equation and then proved the existence and uniqueness of this solution in colombeau algebra.

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