



Existence and uniqueness results for a semilinear fuzzy fractional elliptic equation

Aziz El Ghazouani^a, Amale Talhaoui^a, M'hamed Elomari^a, Said Melliani^a

^aLaboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, P.O. Box 523, Beni Mellal, 23000, Morocco

Abstract. The purpose of this study is to look at a family of starting value problem for semilinear fuzzy fractional elliptic equation with fractional Caputo derivatives. Firstly, we are going to extend the definition of laplacian operator under generalized H-differentiability in the Fuzzy systems. Secondly, the fuzzy integral equation are founded. Then, the existence and uniqueness of a fuzzy solution are established utilizing the Banach fixed point assessment method under Lipschitz conditions. Finally, we conclude our work by a conclusion.

1. Introduction

It is common knowledge that fuzzy mathematics elegantly simulates unpredictable processes [1, 2] as well as investigated in discourse analysis, psychology, information science, choice, and other relevant industrial and applied scientific domains; This is due to its incredible versatility and usefulness (see [2]). Because there is still the potential of uncertainty in real life, fuzzy ambiguity must be considered in way to properly adapt theory to practice [3]. One of the fundamental properties of fuzzy sets is the use of membership functions over realistic facts to mitigate information loss [4].

In recent years, fraction differential operators, a type of absolute operator, have provided a larger degree of flexibility [5–8]. Caputo, as each of us aware, invented the notion of Caputo fractional derivative in 1967. A fewer-known thurt is that the notion of fractional derivative was established 20 years before Caputo by the Russian mathematician Gerasimov. As a result, it's also known as the Gerasimov-Caputo derivative [9]. Moreover, fractional order differential equations combine and properly characterize difficulties [4] and collect all function information in a weighted version [10]. Consequently, fractional-order differential equations are frequently employed in modelling viscous-elastic, chaotic, non - linear physiological functions, as well as other real world processes, particularly in explaining Memories and heredity features, and help to advance vital fields such as biochemistry and physics [11]. That is, real-world situations may be completely explained theoretically using fractional PDEs, and they might aid us in reaching more precise information [12].

2020 *Mathematics Subject Classification.* Primary 47S40; Secondary 34K36, 34A07

Keywords. Existence and uniqueness, Semilinear fuzzy fractional elliptic equation, Banach fixed point theorem, Lipschitz conditions, Fractional Caputo derivatives.

Received: 05 March 2023; Revised: 27 April 2023; Accepted: 03 May 2023

Communicated by Maria Alessandra Ragusa

Research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Email addresses: aziz.elghazouani@usms.ac.ma (Aziz El Ghazouani), m.l.talhaoui@gmail.com (Amale Talhaoui), m.elomari@usms.ma (M'hamed Elomari), s.melliani@usms.ma (Said Melliani)

Iqbal and Niazi investigated a type of equations of Caputo’s fuzzy fractional evolution in 2021 and came to several noteworthy discoveries, including the controllability and the existence and uniqueness of mild solutions (see [13, 14]). Around ten years previously, Arshad and Lupulescu [16], and Agarwal et al. [15] used a variety of techniques to demonstrate the existence and uniqueness of solutions to fuzzy FODE.

Inspired by the above studies, We explore the existence and uniqueness of fuzzy solutions to the underlying semi-linear fuzzy fractional elliptic problem in this research.

$${}^C_{gH}D_t^q u \oplus \Delta u = F(y, t, u(y, t)), \quad (y, t) \in \Omega \times J = (0, T) =: \Delta_T, \tag{1}$$

with the following conditions:

$$\begin{cases} u(y, t) = \tilde{0}, & (y, t) \in \partial\Omega \times J, \\ u(y, 0) = f(y), & y \in \Omega \\ u_i(y, 0) = g(y), & y \in \Omega \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}$ is a bounded space with a smooth boundary $\partial\Omega$, and $T > 0$ is a predetermined number. In 1, $q \in (1, 2)$ is the fractional order and ${}^C_{gH}D_t^q$ signifies the fuzzy Caputo fractional derivative with regard to t .

The manuscript is laid out as pursuing: Following this introduction, We have offered some options related to fuzzy sets theory in section 2. we expand the concept of laplacian operator under extended H-differentiability in Section 3. In Section 4, we introduce the fuzzy nonlinear integral equation satisfied by the solution of the dilemma (1)–(2). We provide our key conclusions, the existence and uniqueness of a fuzzy solution for the semilinear fuzzy differential equation using the Banach fixed point theorem, in Section 5.

2. Preliminaries

In this section we recall fundamental tools of fuzzy theory that will be used through this research.

Let E^n represents the set of all fuzzy numbers Φ in \mathbb{R}^n , in particular, E^1 reflects the set of all fuzzy numbers Φ over \mathbb{R} .

Definition 1. [17] $\Phi : \mathbb{R} \rightarrow [0, 1]$ A fuzzy membership function is alluded to as a fuzzy number if and only if the very next cases are met:

- (1) Φ is normal. This indicates that there’s a ξ such as $\Phi(\xi) = 1$.
- (2) Φ is fuzzy convex.
- (3) Φ is upper semi continuous.
- (4) $\text{Supp}(\Phi) = \{\rho \in \mathbb{R} \mid \Phi(\rho) > 0\}$ is a compact set as a support set.

If Φ is a fuzzy number on \mathbb{R} , therefore, the α -cut of Φ is $[\Phi]^\alpha = \{s \in \mathbb{R} \mid \Phi(s) \geq \alpha\}$, for $\alpha \in (0, 1]$.

Since $[\Phi]^\alpha$ is a compact set of all $\alpha \in [0, 1]$, then we can represent $[\Phi]^\alpha$ by $[\Phi_l(\alpha), \Phi_r(\alpha)]$.

Definition 2. [17] Assume that Υ and Ψ are two level-wise fuzzy sets. The generalized Hukuhara difference $\Upsilon \ominus_g \Psi$ is established as described in the following:

$$\Upsilon \ominus_g \Psi = \omega \Leftrightarrow \begin{cases} \Upsilon = \Psi + \omega & (i) \\ \text{or } \Psi = \Upsilon + (-1)\omega, & (ii) \end{cases} \tag{3}$$

For the α -levels,

$$(\Upsilon \ominus_g \Psi)^\alpha = [\min\{\Upsilon_l(\alpha) - \Psi_l(\alpha), \Upsilon_r(\alpha) - \Psi_r(\alpha)\}, \max\{\Upsilon_l(\alpha) - \Psi_l(\alpha), \Upsilon_r(\alpha) - \Psi_r(\alpha)\}]$$

Lemma 1. (See [18].) If $u, v \in E^n$, thus for $\alpha \in (0, 1]$,

$$\begin{aligned} [u + v]^\alpha &= [u_l^\alpha + v_l^\alpha, u_r^\alpha + v_r^\alpha], \\ [u \cdot v]^\alpha &= [\min\{u_i^\alpha v_j^\alpha\}, \max\{u_i^\alpha v_j^\alpha\}], \quad i, j = l, r, \\ [u - v]^\alpha &= [u_l^\alpha - v_r^\alpha, u_r^\alpha - v_l^\alpha]. \end{aligned}$$

Let ι signify an element in \mathbb{R}^n and \mathcal{S} symbolize a non-empty subsoace of \mathbb{R}^n . The distance $d(\iota, \mathcal{S})$ separating ι and \mathcal{S} is given as

$$d(\iota, \mathcal{S}) = \inf\{\|\iota - \vartheta\| : \vartheta \in \mathcal{S}\}$$

Next consider \mathcal{S} and \mathcal{X} to be non-empty subspace of \mathbb{R}^n . The Hausdorff split of \mathcal{X} and \mathcal{S} is stated as

$$d_H^*(\mathcal{X}, \mathcal{S}) = \sup\{d(v, \mathcal{S}) : v \in \mathcal{X}\}$$

In practice,

$$d_H^*(\mathcal{S}, \mathcal{X}) \neq d_H^*(\mathcal{X}, \mathcal{S}).$$

The Hausdorff length of nonempty subsets of \mathcal{S} and \mathcal{X} of \mathbb{R}^n is given as

$$d_H(\mathcal{S}, \mathcal{X}) = \max\{d_H^*(\mathcal{S}, \mathcal{X}), d_H^*(\mathcal{X}, \mathcal{S})\}. \tag{4}$$

It is now symmetrical in \mathcal{S} and \mathcal{X} . Additionally,

- (1) $d_H(\mathcal{S}, \mathcal{X}) \geq 0$ with $d_H(\mathcal{S}, \mathcal{X}) = 0$ iff $\bar{\mathcal{S}} = \bar{\mathcal{X}}$,
- (2) $d_H(\mathcal{S}, \mathcal{B}) = d_H(\mathcal{X}, \mathcal{S})$
- (3) $d_H(\mathcal{S}, \mathcal{X}) \leq d_H(\mathcal{S}, \mathcal{Y}) + d_H(\mathcal{Y}, \mathcal{X})$

for all non-empty subsets of \mathbb{R}^n 's \mathcal{S}, \mathcal{X} , and \mathcal{Y} . The Hausdorff measure (4) is a kind of metric.

The supremum measure d_∞ on E^n is characterized as:

$$d_\infty(\phi, \varphi) = \sup\{d_H([\phi]^\alpha, [\varphi]^\alpha) : \alpha \in (0, 1]\},$$

for all $\phi, \varphi \in E^n$, and is clearly distance on E^n .

The supremum measure H_1 on $C(J, E^n)$ is characterized as follows:

$$H_1(\psi, \Theta) = \sup\{d_\infty(\psi(s), \Theta(s)) : s \in J\},$$

for each $\psi, \Theta \in C(J, E^n)$.

Definition 3. [19] Let $f \in L^{E^1}(J)$. The fuzzy Riemann Liouville integral of f with order $0 < q$ is stated as:

$$I_{RL}^q f(s) = \frac{1}{\Gamma(q)} \odot \int_0^s (s-t)^{q-1} \odot f(t) ds, \tag{5}$$

where Γ is the Gamma function defined as

$$\Gamma(t) = \int_0^\infty s^{t-1} e^{-s} ds$$

Definition 4. [20] Let $f \in L^{E^1}(J)$. The definition of Caputo gH derivative of $f(t)$ is as follow

$${}^C_{gH}D^q f(s) = \begin{cases} \frac{1}{\Gamma(n-q)} \odot \int_0^s (s-\tau)^{n-q-1} \odot f_{gH}^{(n)}(\tau) d\tau, & n-1 < q < n \\ \left(\frac{d}{ds}\right)^{n-1} f(s), & q = n-1 \end{cases} \tag{6}$$

Lemma 2. Let $\Phi \in L^{E^1}(J)$ and $\forall q \in (n - 1, n)$ we gain

$$(i) \quad {}^C D^q I^q \Phi(s) = \Phi(s)$$

$$(ii) \quad I^q {}^C D^q \Phi(s) = \Phi(s) \ominus_{gH} \Phi(0) \ominus_{gH} (s) \odot \Phi'(0) \ominus_{gH} \cdots \ominus_{gH} \frac{(s)^{(n-1)}}{(n-1)!} \odot \Phi^{(n-1)}(0).$$

Definition 5. [21] Let $g : [0, \infty) \rightarrow Y \subset \mathbb{R}_{\mathcal{F}}$ be continuous such that $e^{-s\tau} \odot g(\tau)$ is integrable. The fuzzy Laplace transform of g , indicated by $L[g(\tau)]$, is therefore computed as

$$L[g(\tau)] := G(s) = \int_0^\infty e^{-s\tau} \odot g(\tau) d\tau, s > 0.$$

Proposition 1. If Φ is a fuzzy piecewise continuous function on $[0, \infty]$ with exponential levels a , so

$$L((\Phi \star \Psi)(x)) = L(\Phi(x)) \odot L(\Psi(x))$$

where Ψ is a piecewise continuous real function on $[0, \infty)$.

Proof.

$$\begin{aligned} L(\Phi(x)) \odot L(\Psi(x)) &= \left(\int_0^\infty e^{-s\tau} \odot \Phi(\tau) d\tau \right) \odot \left(\int_0^\infty e^{-s\sigma} \odot \Psi(\sigma) d\sigma \right) \\ &= \int_0^\infty \left(\int_0^\infty e^{-s(\tau+\sigma)} \odot \Phi(\tau) d\tau \right) \odot \Psi(\sigma) d\sigma \end{aligned}$$

Holding τ constant in the inside integral, We gain by replacing $x = \tau + \sigma$ and $d\sigma = dx$

$$\begin{aligned} L(\Phi(x)) \odot L(\Psi(x)) &= \int_0^\infty \left(\int_\sigma^\infty e^{-sx} \odot \Phi(\tau) \odot \Psi(x - \tau) dx \right) d\tau \\ &= \int_0^\infty \int_\sigma^\infty e^{-sx} \odot \Phi(\tau) \odot \Psi(x - \tau) dx d\tau \\ &= \int_0^\infty e^{-s\sigma} \odot \left(\int_0^x \Phi(x - \sigma) \odot \Psi(\sigma) d\tau \right) d\sigma \\ &= L((\Phi \star \Psi)(x)) \end{aligned}$$

□

Proposition 2. For every $\alpha > 0$, we get the subsequent outcome

$$\int_0^t E_{\alpha,1}(As^\alpha) ds = tE_{\alpha,2}(At^\alpha)$$

Proof.

$$\begin{aligned} \int_0^t E_{\alpha,1}(As^\alpha) ds &= \int_0^t \sum_{n=0}^\infty \frac{s^{n\alpha}}{\Gamma(n\alpha + 1)} A^n ds \\ &= \sum_{n=0}^\infty \frac{\int_0^t s^{n\alpha} ds}{\Gamma(n\alpha + 1)} A^n \\ &= \sum_{n=0}^\infty \frac{t^{n\alpha+1}}{(n\alpha + 1)\Gamma(n\alpha + 1)} A^n \\ &= \sum_{n=0}^\infty \frac{t^{n\alpha+1}}{\Gamma(n\alpha + 2)} A^n \\ &= tE_{\alpha,2}(At^\alpha) \end{aligned}$$

□

Lemma 3. For all $\alpha \in [1, 2]$ and $s > 0$,

1. $s^{\alpha-1} (s^\alpha - A)^{-1} = \mathcal{L} (E_{\alpha,1} (At^\alpha)) (s)$,
2. $s^{\alpha-2} (s^\alpha - A)^{-1} = \mathcal{L} (tE_{\alpha,2} (At^\alpha)) (s)$,
3. $(s^\alpha - A)^{-1} = \frac{1}{\Gamma(\alpha-1)} \mathcal{L} \left(\int_0^t (t-s)^{\alpha-2} E_{\alpha,1} (As^\alpha) ds \right)$.

Proof. 1. For $s > 0$,

$$\begin{aligned} \mathcal{L} (E_{\alpha,1} (At^\alpha)) (s) &= \mathcal{L} \left(\sum_{n=0}^{+\infty} \frac{t^{\alpha n} A^n}{\Gamma(\alpha n + 1)} \right) \\ &= \sum_{n=0}^{+\infty} \mathcal{L} (t^{\alpha n}) \frac{A^n}{\Gamma(\alpha n + 1)} \\ &= \sum_{n=0}^{+\infty} \frac{1}{s^{\alpha n + 1}} A^n \\ &= s^{\alpha-1} (s^\alpha - A)^{-1} \end{aligned}$$

2. For $s > 0, s^{\alpha-1} (s^\alpha - A)^{-1} = \mathcal{L} (E_{\alpha,1} (At^\alpha)) (s)$, then

$$\begin{aligned} s^{\alpha-2} (s^\alpha - A)^{-1} &= s^{-1} s^{\alpha-1} (s^\alpha - A)^{-1} \\ &= \mathcal{L}(1)(s) \mathcal{L} (E_{\alpha,1} (At^\alpha)) (s) \\ &= \mathcal{L} (1 * E_{\alpha,1} (At^\alpha)) (s) \\ &= \mathcal{L} \left(\int_0^t E_{\alpha,1} (At^\alpha) \right) (s) \\ &= \mathcal{L} (tE_{\alpha,2} (t^\alpha A)) (s) \end{aligned}$$

3. From (1), we get

$$\begin{aligned} (s^\alpha - A)^{-1} &= s^{1-\alpha} \mathcal{L} (E_{\alpha,1} (At^\alpha)) (s) \\ &= \mathcal{L} \left(\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} \right) \mathcal{L} (E_{\alpha,1} (At^\alpha)) (s) \\ &= \mathcal{L} \left(\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} * E_{\alpha,1} (At^\alpha) \right) (s) \\ &= \mathcal{L} \left(\int_0^t \frac{(t-\delta)^{\alpha-2}}{\Gamma(\alpha-1)} E_{\alpha,1} (A\delta^\alpha) d\delta \right) (s), \end{aligned}$$

As a consequence, the intended goal. \square

Lemma 4. [21]

(1) Allow $\varphi, \phi : [0, \infty) \rightarrow Y \subset \mathbb{R}_{\mathcal{F}}$ to be continuous functions, $c_1, c_2 \in \mathbb{R}^+$. Therefore

$$\mathcal{L} [c_1 \odot \varphi(s) + c_2 \odot \phi(s)] = c_1 \odot \mathcal{L}[\varphi(s)] + c_2 \odot \mathcal{L}[\phi(s)]$$

(2) Allow $\varphi : [0, \infty) \rightarrow Y \subset \mathbb{R}_{\mathcal{F}}$ to be a continuous function. Thus

$$\mathcal{L} [e^{as} \odot \varphi(t)] = \varphi(t - c), t - c > 0.$$

Definition 6. [22] Assume \mathcal{G} to be a vector set in \mathbb{R} . A fuzzy inner product on \mathcal{G} is a mapping $\langle \cdot, \cdot \rangle : \mathcal{G} \times \mathcal{G} \rightarrow E^1$ such as $\forall \Theta, \Upsilon, \Xi \in \mathcal{G}$ and $\lambda \in \mathbb{R}$, we obtain:

- 1) $\langle \Theta + \Upsilon, \Xi \rangle = \langle \Theta, \Xi \rangle \oplus \langle \Upsilon, \Xi \rangle$,
- 2) $\langle \lambda \Theta, \Upsilon \rangle = \tilde{\lambda} \langle \Theta, \Upsilon \rangle$,

- 3) $\langle \Theta, \Upsilon \rangle = \langle \Upsilon, \Theta \rangle,$
- 4) $\langle \Theta, \Theta \rangle \geq \bar{0},$
- 5) $\inf_{\alpha \in (0,1]} \langle \Theta, \Theta \rangle_{\alpha}^{-} > 0$ if $\Theta \neq 0,$
- 6) $\langle \Theta, \Theta \rangle = \bar{0}$ iff $\Theta = 0.$

A fuzzy inner product set is a vector space \mathcal{G} that admire a fuzzy inner product.

3. Fuzzy Laplacian operator

In this section, we are going to extend the definition of laplacian operator under generalized H-differentiability in the fuzzy theory.

Definition 7. [23] Take $\Phi : J \rightarrow \mathbb{R}_{\mathcal{F}}$ and fix $s_0 \in J$. We say Φ is (i)-differentiable at s_0 , if there's an element $\Phi'(s_0) \in \mathbb{R}_{\mathcal{F}}$ such as $\forall h > 0$ enough close to 0 , exist $\Phi(s_0 + h) \ominus \Phi(s_0), \Phi(s_0) \ominus \Phi(s_0 - h)$ and the limits

$$\lim_{h \rightarrow 0^+} \frac{\Phi(s_0 + h) \ominus \Phi(s_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\Phi(s_0) \ominus \Phi(s_0 - h)}{h} = \Phi'(s_0).$$

In this case we denote $\Phi'(s_0)$ by $\partial_1^1 \Phi(s_0)$. And Φ is (ii)-differentiable if for all $h > 0$ enough close to 0 , exist $\Phi(s_0 + h) \ominus \Phi(s_0), \Phi(s_0) \ominus \Phi(s_0 - h)$ and the limits

$$\lim_{h \rightarrow 0^+} \frac{\Phi(s_0) \ominus \Phi(s_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\Phi(s_0 - h) \ominus \Phi(s_0)}{-h} = \Phi'(s_0).$$

This derivative is indicated in this instance by $\partial_2^1 \Phi(s_0)$.

Here we remember certain notions and proofs for the initial order derivative [23] and 2nd order derivatives [24] dependent on the selection of derivative kind in every stage of differentiating.

Theorem 1. Allow $\Phi : J \rightarrow \mathbb{R}_F$ be fuzzy function, where $[\Phi(s)]^\beta = [\Phi_l(s, \beta), \Phi_r(s, \beta)]$ for any $\beta \in [0, 1]$. Then

- (1) If Φ is (i)-differentiable thus $\Phi_l(s, \beta)$ and $\Phi_r(s, \beta)$ are differentiable functions and $[\partial_{(i)}^1 \Phi(s)]^\beta = [\Phi'_l(s, \beta), \Phi'_r(s, \beta)]$.
- (2) If Φ is (ii)-differentiable thus $\Phi_l(s, \beta)$ and $\Phi_r(s, \beta)$ are differentiable functions and $[\partial_{(ii)}^1 \Phi(s)]^\beta = [\Phi'_r(s, \beta), \Phi'_l(s, \beta)]$.

Proof. See [23]. \square

Definition 8. Allow $\Phi : J \rightarrow \mathbb{R}_{\mathcal{F}}$ and $n, m = (i), (ii)$. We declare Φ is (n, m) -differentiable at $s_0 \in J$, if $\partial_n^1 \Phi$ occur in the near of s_0 as a fuzzy function and it's (m) -differentiable at s_0 . The 2nd derivatives of Φ is noted by $\partial_{n,m}^2 \Phi(t_0)$ for $n, m = (i), (ii)$.

According to Definition 7 we have:

Theorem 2. Let $\partial_{(i)}^1 \Phi : J \rightarrow \mathbb{R}_{\mathcal{F}}$ or $\partial_{(ii)}^1 \Phi : J \rightarrow \mathbb{R}_{\mathcal{F}}$ be fuzzy functions, with $[\Phi(s)]^\beta = [\Phi_l(s, \beta), \Phi_r(s, \beta)]$. Then

- (1) if $\partial_{(i)}^1 \Phi$ is (i)-differentiable, then $\Phi'_l(s, \beta)$ and $\Phi'_r(s, \beta)$ are differentiable functions and $[\partial_{(i),(i)}^2 \Phi(s)]^\beta = [\Phi''_l(s, \beta), \Phi''_r(s, \beta)]$.
- (2) if $\partial_{(i)}^1 \Phi$ is (ii)-differentiable, then $\Phi'_l(s, \beta)$ and $\Phi'_r(s, \beta)$ are differentiable functions and $[\partial_{(i),(ii)}^2 \Phi(s)]^\beta = [\Phi''_r(s, \beta), \Phi''_l(s, \beta)]$.
- (3) if $\partial_{(ii)}^1 \Phi$ is (i)-differentiable, then $\Phi'_l(s, \beta)$ and $\Phi'_r(s, \beta)$ are differentiable functions and $[\partial_{(ii),(i)}^2 \Phi(s)]^\beta = [\Phi''_r(s, \beta), \Phi''_l(s, \beta)]$.
- (4) if $\partial_{(ii)}^1 \Phi$ is (ii)-differentiable, then $\Phi'_l(s, \beta)$ and $\Phi'_r(s, \beta)$ are differentiable functions and $[\partial_{(ii),(ii)}^2 \Phi(s)]^\beta = [\Phi''_l(s, \beta), \Phi''_r(s, \beta)]$.

Proof. See the Proof of the Theorem 3.9 in [24]. \square

Definition 9. The fuzzy Laplace operator of Φ is the summation of all the fuzzy 2nd partial derivatives in Cartesian coordinate system t_j :

$$\Delta\Phi = \sum_{j=1}^n \frac{\partial^2\Phi}{\partial t_j^2}$$

The fuzzy Laplace operator, as a 2nd fuzzy differential operator, transfers C^k fuzzy functions to C^{k-2} fuzzy functions with $k \geq 2$. It is a linear operator $\Delta : C^k(\mathbf{R}^n) \rightarrow C^{k-2}(\mathbf{R}^n)$, or more broadly, an operator $\Delta : C^k(\Omega) \rightarrow C^{k-2}(\Omega)$ for every open set $\Omega \subseteq \mathbf{R}^n$.

Theorem 3. Let Φ and Φ' be differentiable fuzzy value functions, and if α -cut representation of f is denoted by $[\Phi]^\alpha = [f\Phi_l^\alpha, \Phi_r^\alpha]$, then the fuzzy Laplacian operator denoted Δ defined as

- (1) if $\partial_{(i)}^1\Phi$ is (i)-differentiable or if $\partial_{(ii)}^1\Phi$ is (ii)-differentiable, thus $\Phi'_l(\cdot, \alpha)$ and $\Phi'_r(\cdot, \alpha)$ are differentiable functions and $[\Delta\Phi]^\alpha = [\Delta\Phi_l^\alpha, \Delta\Phi_r^\alpha]$.
- (2) if $\partial_{(i)}^1\Phi$ is (ii)-differentiable or if $\partial_{(ii)}^1\Phi$ is (i)-differentiable, thus $\Phi'_l(\cdot, \alpha)$ and $\Phi'_r(\cdot, \alpha)$ are differentiable functions and $[\Delta\Phi]^\alpha = [\Delta\Phi_r^\alpha, \Delta\Phi_l^\alpha]$.

where Δ is the usual laplacian operator.

Proof. We just offer the specifics for scenario (1) because the other situations are comparable.

If $h > 0$ and $\alpha \in [0, 1]$, we obtain

$$\left[\partial_1^{(1)}\Phi(t_j + h) \ominus \partial_1^{(1)}\Phi(t_j) \right]^\alpha = \left[\Phi'_l(t_j + h, \alpha) - \Phi'_l(t_j, \alpha), \Phi'_r(t_j + h, \alpha) - \Phi'_r(t_j, \alpha) \right],$$

and then multiply by $1/h$, we get

$$\frac{\left[\partial_1^{(1)}\Phi(t_j + h) \ominus \partial_1^{(1)}\Phi(t_j) \right]^\alpha}{h} = \left[\frac{\Phi'_l(t_j + h, \alpha) - \Phi'_l(t_j, \alpha)}{h}, \frac{\Phi'_r(t_j + h, \alpha) - \Phi'_r(t_j, \alpha)}{h} \right].$$

Similarly, we obtain

$$\frac{\left[\partial_1^{(1)}\Phi(t_j) \ominus \partial_1^{(1)}\Phi(t_j - h) \right]^\alpha}{h} = \left[\frac{\Phi'_l(t_j, \alpha) - \Phi'_l(t_j - h, \alpha)}{h}, \frac{\Phi'_r(t_j, \alpha) - \Phi'_r(t_j - h, \alpha)}{h} \right].$$

Getting to the limits, we gain

$$\left[\partial_{1,1}^{(2)}\Phi(t_j) \right]^\alpha = \left[\partial^{(2)}\Phi_l(t_j, \alpha), \partial^{(2)}\Phi_r(t_j, \alpha) \right].$$

by applying the sum, we get

$$\left[\sum_{j=1}^n \partial_{1,1}^{(2)}\Phi(t_j) \right]^\alpha = \left[\sum_{j=1}^n \partial^{(2)}\Phi_l(t_j, \alpha), \sum_{j=1}^n \partial^{(2)}\Phi_r(t_j, \alpha) \right].$$

therefore,

$$[\Delta\Phi]^\alpha = [\Delta\Phi_l^\alpha, \Delta\Phi_r^\alpha].$$

This concludes the theorem’s demonstration. \square

4. The fuzzy integral equation

Consider the following fuzzy eigenvalue problem for the fuzzy Laplacian on a bounded domain Ω .

$$\begin{cases} \ominus \Delta \phi_j(y) = \lambda_j \odot \phi_j(y), & y \in \Omega \\ \phi_j(y) = \tilde{0}, & y \in \partial\Omega, \end{cases}$$

where $\tilde{0}$ is a fuzzy number. Then, the above equation is expanded in accordance with its left and right functions as follows:

$$\begin{aligned} (-\Delta \phi_{j,l}(y), -\Delta \phi_{j,r}(y)) &= \lambda_j \odot (\phi_{j,l}(y), \phi_{j,r}(y)) \\ (\phi_{j,l}(y), \phi_{j,r}(y)) &= (0, 0), \end{aligned}$$

Now, we look at these equations according to the two following cases. The equation with lower functions is

$$\begin{cases} -\Delta \phi_{j,l}(y, \alpha) = \lambda_j \phi_{j,l}(y, \alpha), & y \in \Omega \\ \phi_{j,l}(y) = 0, & y \in \partial\Omega, \end{cases} \tag{7}$$

and with upper functions is

$$\begin{cases} -\Delta \phi_{j,r}(y, \alpha) = \lambda_j \phi_{j,r}(y, \alpha), & y \in \Omega \\ \phi_{j,r}(y) = 0, & y \in \partial\Omega, \end{cases} \tag{8}$$

The boundary value problems (7) and (8) is the Dirichlet problems for the Helmholtz system, and thus λ_j is classified as a Dirichlet eigen-value for Ω .

By using the theorem of compact self-adjoint operator spectral one can demonstrate that the eigen-spaces are size limitations and that the Dirichlet eigen-values λ_j are real, positive, and unbounded. As a result, they may be ordered in ascending order :

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_j \leq \dots \text{ and } \lambda_j \rightarrow \infty \text{ as } j \rightarrow \infty.$$

Consider the related eigen-functions $\phi_j \in H_0^1(\Omega)$.

Next, assume that the difficulty (1) has a solution u in the kind

$$u(y, t) = \sum_{j=1}^{\infty} u_j(t) \odot \phi_j(y).$$

Thus, $u_j(t)$ solves the pursuing fuzzy fractional ordinary differential equation with conditions:

$$\begin{cases} {}^C_{gH} D_t^q u_j \ominus \lambda_j \odot u_j(t) = \langle F(y, t, u(y, t)), \phi_j \rangle, & t \in J, \\ u_j(0) = \langle f, \phi_j \rangle, \\ \frac{du_j}{dt}(0) = \langle g, \phi_j \rangle, \end{cases} \tag{9}$$

where $\langle \cdot, \cdot \rangle$ indicate the fuzzy inner product in $L^2(\Omega)$. By using the lemmas (2), (3) and the proposition (1), we gain the solution of (9) as below:

(i) If u is Caputo (i) – gH differentiable, then

$$\begin{aligned} u_j(t) &= E_{q,1}(\lambda_j \odot t^q) \odot \langle f, \phi_j \rangle \oplus t \odot E_{q,2}(\lambda_j \odot t^q) \odot \langle g, \phi_j \rangle \\ &\oplus \int_0^t \int_s^t \frac{(t-\tau)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}(\lambda_j \odot (\tau-s)^q) \odot \langle F(\cdot, s, u(\cdot, s)), \phi_j \rangle d\tau ds \end{aligned} \tag{10}$$

(ii) If u is Caputo (ii) – gH differentiable, therefore

$$\begin{aligned}
 u_j(t) = & E_{q,1}(\lambda_j \odot t^q) \odot \langle f, \phi_j \rangle \oplus t \odot E_{q,2}(\lambda_j \odot t^q) \odot \langle g, \phi_j \rangle \\
 & \ominus (-1) \odot \int_0^t \int_s^t \frac{(t-\tau)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}(\lambda_j \odot (\tau-s)^q) \odot \langle F(\cdot, s, u(\cdot, s)), \phi_j \rangle d\tau ds
 \end{aligned} \tag{11}$$

therefore, the problem (1) solution is as follow

(i) If u is Caputo (i) – gH differentiable, then

$$\begin{aligned}
 u(y, t) = & \sum_{j=1}^{\infty} \left[E_{q,1}(\lambda_j \odot t^q) \odot \langle f, \phi_j \rangle \oplus t \odot E_{q,2}(\lambda_j \odot t^q) \odot \langle g, \phi_j \rangle \right] \odot \phi_j(y) \\
 & \oplus \sum_{j=1}^{\infty} \left[\int_0^t \int_s^t \frac{(t-\tau)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}(\lambda_j \odot (\tau-s)^q) \odot \langle F(y, s, u(y, s)), \phi_j \rangle d\tau ds \right] \odot \phi_j(y).
 \end{aligned} \tag{12}$$

(ii) If u is Caputo (ii) – gH differentiable, then

$$\begin{aligned}
 u(y, t) = & \sum_{j=1}^{\infty} \left[E_{q,1}(\lambda_j \odot t^q) \odot \langle f, \phi_j \rangle \oplus t \odot E_{q,2}(\lambda_j \odot t^q) \odot \langle g, \phi_j \rangle \right] \odot \phi_j(y) \\
 & \ominus (-1) \odot \sum_{j=1}^{\infty} \left[\int_0^t \int_s^t \frac{(t-\tau)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}(\lambda_j \odot (\tau-s)^q) \odot \langle F(y, s, u(y, s)), \phi_j \rangle d\tau ds \right] \odot \phi_j(y).
 \end{aligned} \tag{13}$$

5. Existence and uniqueness results

Consider the points that follow.

(H1) The inhomogeneous term $F : \Delta_T \times C(\Delta_T, L^2(\Delta_T)) \rightarrow C(\Delta_T, L^2(\Delta_T))$ is a continuous function that meets the globally Lipschitz criterion

$$d_H([f(y, s, \xi(y, s))]^\alpha, [f(y, s, \zeta(y, s))]^\alpha) \leq K d_H([\xi(y, s)]^\alpha, [\zeta(y, s)]^\alpha),$$

for all $\xi(y, s), \zeta(y, s) \in C(\Delta_T, L^2(\Delta_T))$, and a constant $K > 0$.

(H2) $\mathcal{Q}(t)$ is a fuzzy set that is appropriate for $u \in C(\Delta_T; L^2(\Delta_T))$, the equation

$$\mathcal{Q}(t-s)F(u)(y, s) := \sum_{j=1}^{\infty} \int_s^t \frac{(t-\tau)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}(\lambda_j \odot (\tau-s)^q) \odot \langle F(u)(y, s), \phi_j \rangle \odot \phi_j(y) d\tau$$

such as

$$[\mathcal{Q}(t)]^\alpha = [\mathcal{Q}_l^\alpha(t), \mathcal{Q}_r^\alpha(t)],$$

and $\mathcal{Q}_i^\alpha(t) (i = l, r)$ is continuous. That is, there's a constant $M > 0$ such as $|\mathcal{Q}_i^\alpha(t)| \leq M \quad \forall t \in J$

Theorem 4. *Assume that assumptions (H1)-(H2) are correct. Thus, for any $f, g \in L^2(\Omega)$, the fuzzy initial value problem (1), (2) has a unique solution $u \in C(\Delta_T; L^2(\Delta_T))$.*

Proof. Denote

$$\mathcal{S}(t)f := \sum_{j=1}^{\infty} E_{\alpha,1}(\lambda_j \odot t^q) \odot \langle f, \phi_j \rangle \odot \phi_j, \quad \mathcal{P}(t)g := \sum_{j=1}^{\infty} t \odot E_{\alpha,2}(\lambda_j \odot t^q) \odot \langle g, \phi_j \rangle \odot \phi_j,$$

The solutions is then defined as meeting the equation as

$$u(y, t) = \mathcal{S}(t)f(y) \oplus \mathcal{P}(t)g(y) \oplus \int_0^t \mathcal{Q}(t-s)F(u)(y, s)ds \tag{14}$$

where

$$F(u)(y, s) := F(y, s, u(y, s)),$$

$$\mathcal{Q}(t-s)F(u)(y, s) := \sum_{j=1}^{\infty} \int_s^t \frac{(t-\tau)^{q-2}}{\Gamma(q-1)} \odot E_{q,1}(\lambda_j \odot (\tau-s)^q) \odot \langle F(u)(y, s), \phi_j \rangle \odot \phi_j(y) d\tau.$$

For each $\xi(t) \in C(\Delta_T, L^2(\Delta_T)), t, y \in \Delta_T$ define

$$(\Phi\xi)(y, t) = \mathcal{S}(t)f(y) \oplus \mathcal{P}(t)g(y) \oplus \int_0^t \mathcal{Q}(t-s)F(u)(y, s)ds.$$

Thus, $(\Phi\xi)(y, t) : \Delta_T \rightarrow C(\Delta_T, L^2(\Delta_T))$ is continuous in regard to t , and $\Phi : C(\Delta_T; L^2(\Delta_T)) \rightarrow C(\Delta_T; L^2(\Delta_T))$. It is obvious that fixed points of Φ are solutions to the initial value problem (1),(2). For $\xi(t), \zeta(t) \in C(\Delta_T; L^2(\Delta_T))$, we have

$$\begin{aligned} & d_H([\Phi\xi](y, t)^\alpha, [\Phi\zeta](y, t)^\alpha) \\ &= d_H\left(\left[\mathcal{S}(t)f(y) \oplus \mathcal{P}(t)g(y) \oplus \int_0^t \mathcal{Q}(t-s)F(\xi)(y, s)ds\right]^\alpha, \right. \\ & \quad \left. \left[\mathcal{S}(t)f(y) \oplus \mathcal{P}(t)g(y) \oplus \int_0^t \mathcal{Q}(t-s)F(\zeta)(y, s)ds\right]^\alpha\right) \\ &\leq d_H\left([\mathcal{S}(t)f(y)]^\alpha \oplus [\mathcal{P}(t)g(y)]^\alpha \oplus \left[\int_0^t \mathcal{Q}(t-s)F(\xi)(y, s)ds\right]^\alpha, \right. \\ & \quad \left. [\mathcal{S}(t)f(y)]^\alpha \oplus [\mathcal{P}(t)g(y)]^\alpha \oplus \left[\int_0^t \mathcal{Q}(t-s)F(\zeta)(y, s)ds\right]^\alpha\right) \\ &\leq d_H\left(\left[\int_0^t \mathcal{Q}(t-s)F(\xi)(y, s)ds\right]^\alpha, \left[\int_0^t \mathcal{Q}(t-s)F(\zeta)(y, s)ds\right]^\alpha\right) \\ &\leq \int_0^t d_H\left([\mathcal{Q}_l^\alpha(t-s)F_l^\alpha(\xi)(y, s), \mathcal{Q}_r^\alpha(t-s)F_r^\alpha(\xi)(y, s)], [\mathcal{Q}_l^\alpha(t-s)F_l^\alpha(\zeta)(y, s), \mathcal{Q}_r^\alpha(t-s)F_r^\alpha(\zeta)(y, s)]\right) ds \\ &\leq \int_0^t \max(|\mathcal{Q}_l^\alpha(t-s)[F_l^\alpha(\xi)(y, s) - F_l^\alpha(\zeta)(y, s)]|, |\mathcal{Q}_r^\alpha(t-s)[F_r^\alpha(\xi)(y, s) - F_r^\alpha(\zeta)(y, s)]|) ds \\ &\leq M \int_0^t \max(|[F_l^\alpha(\xi)(y, s) - F_l^\alpha(\zeta)(y, s)]|, |[F_r^\alpha(\xi)(y, s) - F_r^\alpha(\zeta)(y, s)]|) ds \\ &\leq M \int_0^t \max([F_l^\alpha(\xi)(y, s), F_r^\alpha(\xi)(y, s)], [F_l^\alpha(\zeta)(y, s), F_r^\alpha(\zeta)(y, s)]) ds \\ &\leq M \int_0^t d_H([F(\xi)(y, s)]^\alpha, [F(\zeta)(y, s)]^\alpha) ds \\ &\leq MK \int_0^t d_H([\xi(y, s)]^\alpha, [\zeta(y, s)]^\alpha) ds \end{aligned}$$

Thus,

$$\begin{aligned} d_{\infty}((\Phi\xi)(y, t), (\Phi\zeta)(y, t)) &= \sup_{\alpha \in (0,1]} d_H([\xi(y, t)]^{\alpha}, [\zeta(y, t)]^{\alpha}) \\ &\leq MK \int_0^t \sup_{\alpha \in (0,1]} d_H([\xi(y, s)]^{\alpha}, [\zeta(y, s)]^{\alpha}) ds \\ &= MK \int_0^t d_{\infty}(\xi(y, s), \zeta(y, s)) ds. \end{aligned}$$

Consequently,

$$\begin{aligned} H_1(\Phi\xi, \Phi\zeta) &= \sup_{t \in J} d_{\infty}((\Phi\xi)(y, t), (\Phi\zeta)(y, t)) \\ &\leq MK \sup_{t \in J} \int_0^t d_{\infty}(\xi(y, s), \zeta(y, s)) ds \\ &\leq MKTH_1(\xi(y, s), \zeta(y, s)). \end{aligned}$$

Pick T so that $T < \frac{1}{MK}$. Hence, Φ is a contraction mapp. According to the Banach fixed point theorem, the semilinear fuzzy fractional elliptic equation has a unique fixed point $u \in C(\Delta_T; L^2(\Delta_T))$. \square

6. Conclusion

The goal of this study is to look at a family of starting value issues for semilinear fuzzy fractional elliptic equations with fractional Caputo derivatives. Before that, In the fuzzy theory, we will broaden the definition of the laplacian operator under extended H-differentiability. The fuzzy integral equation is first established, and then the existence and uniqueness of a fuzzy solution are established by utilising the Banach fixed point assessment technique under Lipschitz conditions.

References

- [1] L. A. Zadeh. Fuzzy sets. *Inf. Control* 1965, 8, 338–353.
- [2] R. Alikhani, F. Bahrami, S. Parvizi. Differential calculus of fuzzy multi-variable functions and its applications to fuzzy partial differential equations. *Fuzzy Set. Syst.* 2019, 375, 100–120.
- [3] Y.T. Wu, H.Y. Lan, C.J. Liu On implicit coupled systems of fuzzy fractional delay differential equations with triangular fuzzy functions. *AIMS Math.* 2021, 6, 3741–3760.
- [4] M. Senol, S. Atpinar, Z. Zararsiz, S. Salahshour, Ahmadian, A. Approximate solution of time-fractional fuzzy partial differential equations. *Comput. Appl. Math.* 2019, 38, 18.
- [5] K. Shah, A. R. Seadawy, M. Arfan. Evaluation of one dimensional fuzzy fractional partial differential equations. *Alex. Eng. J.* 2020, 59, 3347–3353.
- [6] Abdelouahab B., Oussaeif T.E., Ouannas A., Saad K.M., Jahanshahi H., Diar A., Aljuaid A.M., Aly A.A., A nonlinear fractional problem with a second kind integral condition for time-fractional partial differential equation, *Journal of Function Spaces*, vol.2022, art.n. , (2022).
- [7] De Gaetano A., Jleli M., Ragusa M.A., Samet B., Nonexistence results for nonlinear fractional differential inequalities involving weighted, *Discrete and Continuous Dynamical Systems-Series S*, doi : 10.3934/dcdss.2022185.
- [8] Zhang F, Lan H.Y., Xu H.Y., Generalized Hukuhara weak solutions for a class of coupled systems of fuzzy fractional order partial differential equations without Lipschitz conditions, *Mathematics*, 10 (21), (2022).
- [9] V. Kiryakova . FCAA related news, events and books (Ed. Note, FCAA–Volume 20–2–2017). *Fract. Calc. Appl. Anal.* 2017, 20, 293–306.
- [10] A. M. Saeed. Improved rotated finite difference method for solving fractional elliptic partial differential equations. *American Sci. Res. J. Eng. Tech. Sci.* 2016, 26, 261–270.
- [11] B. Zheng. A new fractional Jacobi elliptic equation method for solving fractional partial differential equations. *Adv. Differ. Equ.* 2014, 2014, 228.
- [12] M. Al-Smadi, O. A. Arqub, S. Hadid. An attractive analytical technique for coupled system of fractional partial differential equations in shallow water waves with conformable derivative. *Commun. Theor. Phys.* 2020, 72, 85001–85018.
- [13] A.U.K. Niazi, N. Iqbal, R. Shah, F. Wannalookkhee. Controllability for fuzzy fractional evolution equations in credibility space. *Fractal Fract.* 2021, 5, 112.
- [14] N. Iqbal, A.U.K. Niazi, R. Shafqat, S. Zaland. Existence and uniqueness of mild solution for fractional-order controlled fuzzy evolution equation. *J. Funct. Spaces* 2021, 2021, 5795065.

- [15] R.P. Agarwal, S. Arshad, D. O'Regan, V. Lupulescu. Fuzzy fractional integral equations under compactness type condition. *Fract. Calc. Appl. Anal.* 2012, 15, 572–590.
- [16] S. Arshad, V. Lupulescu. On the fractional differential equations with uncertainty. *Nonlinear Anal.* 2011, 74, 3685–3693.
- [17] A. Tofigh. Fuzzy fractional differential operators and equations: fuzzy fractional differential equations. Vol. 397. Springer Nature, 2020.
- [18] S. Seikkala, On the fuzzy initial value problem, *Fuzzy Sets and Systems* 24, 319-330, (1987).
- [19] A. Armand, Z. Gouyandeh, Fuzzy fractional integro-differential equations under generalized Caputo differentiability. *Annals of Fuzzy Mathematics and Informatics* Volume 10, No. 5, (November 2015), pp. 789-798.
- [20] T. Allahviranloo. "Fuzzy fractional differential operators and equations." *Studies in fuzziness and soft computing* 397 (2021).
- [21] T. Allahviranloo, M. B. Ahmadi. Fuzzy laplace transforms. *Soft Computing*, 14(3), 235-243, (2010).
- [22] A. Hasankhani, A. Nazari, M. Saheli. Some properties of fuzzy Hilbert spaces and norm of operators, (2010).
- [23] Y. Chalco-Cano, H. Román-Flores, On new solutions of fuzzy differential equations, *Chaos, Solitons and Fractals* 38 (2008) 112–119.
- [24] A. Khastan, F. Bahrami, K. Ivaz, New Results on Multiple Solutions for Nth-order Fuzzy Differential Equations under Generalized Differentiability, preprint.