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A fixed point result on a DCMLS and applications to matrix equations and random integral equations

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Abstract. In this work, we present a fixed point result in the context of a double controlled metric-like space involving its control functions. A matrix equation has been solved via a fixed point technique. We also give some concrete and numerical examples. Finally, we ensure the existence of a solution of a random integral equation of Fredholm type.

1. Introduction

The fixed point iteration techniques use the notion of a fixed point in a iterated manner in order to compute a solution of certain equations. It is known that the Banach contraction principle (BCP) [1] is the useful and essential tool in solving several concrete applications. Namely, it is arising in the existence of solutions of integral, differential and difference equations. Variant generalizations and extensions of the BCP appear in literature, see [2–4]. This principle is an essential result in the fixed point theory. Also, there are several generalizations of metric spaces, like b-metric spaces [5, 6] and extended b-metric spaces [7]. For related fixed point results, see [8–19]. In a b-metric setting, a coefficient $s \geq 1$ appears in the right part of the triangular inequality. Among the generalizations of a $b-$metric space, there is the double controlled metric space X presented first by Abdeljawad et al. [20], where the triangular inequality is extended via two control functions $\varpi, \epsilon : X \times X \longrightarrow [1, \infty)$.

Definition 1.1. *Consider a nonempty set X. A function $d : X \times X \longrightarrow [0, \infty)$ is called a double controlled metric (DCM in abbreviation) with controlled functions $\varpi, \epsilon : X \times X \longrightarrow [1, \infty)$ if the following conditions hold for all $x, \tau, \varsigma \in X$,*
(μ_1): $d(x, \tau) = 0 \Longleftrightarrow x = \tau$;
(μ_2): $d(x, \tau) = d(\tau, x)$;
(μ_3): $d(x, \tau) \leq \varpi(x, \varsigma)d(x, \varsigma) + \epsilon(\varsigma, \tau)d(\varsigma, \tau)$.
The pair (X, d) is called a double controlled metric space (DCMS).

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In 2020, the double controlled metric-like is initiated by Mlaiki [21] where within this new setting, the distance between two elements in not necessarily equal to zero.

Definition 1.2. *[21] Given a nonempty set X. The function $d : X \times X \longrightarrow [0, \infty)$ is termed as a double controlled metric-like (DCML, in short) with controlled functions $\varpi, \epsilon : X \times X \longrightarrow [1, \infty)$ if the following conditions hold for all $x, \tau, \varsigma \in X$,*
(d_1): $d(x, \tau) = 0 \Longrightarrow x = \tau$;
(d_2): $d(x, \tau) = d(\tau, x)$;
(d_3): $d(x, \tau) \leq \varpi(x, \varsigma)d(x, \varsigma) + \epsilon(\varsigma, \tau)d(\varsigma, \tau)$.
The pair (X, d) is said to be a double controlled metric-like space (DCMLS).

To illustrate Definition 1.2, we give two examples.

Example 1.3. *Let $X = [0, \infty)$. Define $d : X \times X \longrightarrow [0, \infty)$ by*

$$d(x,\tau) = \begin{cases} 0 & \textit{if} \quad x = \tau \neq 0 \\ \frac{1}{2} & \textit{if} \quad x = 0 \quad \textit{and} \quad \tau = 0 \\ \frac{1}{x} & \textit{if} \quad x \geq 1 \quad \textit{and} \quad \tau \in [0,1) \\ \frac{1}{\tau} & \textit{if} \quad \tau \geq 1 \quad \textit{and} \quad x \in [0,1) \\ 1 & \textit{otherwise.} \end{cases}$$

Consider the following functions $\varpi, \epsilon : X \times X \longrightarrow [1, \infty)$ given as

$$\varpi(x,\tau) = \begin{cases} x & \textit{if } x, \tau \geq 1 \\ 1 & \textit{otherwise} \end{cases} \qquad \textit{and} \quad \epsilon(x,\tau) = \begin{cases} 1 & \textit{if} \quad x, \tau < 1 \\ \max(x,\tau) & \textit{if not}. \end{cases}$$

Here, (X, d) is a DCMLS.

Example 1.4. *Let $X = [0, \infty]$. Given $d : X \times X \longrightarrow [0, \infty)$ as*

$$d(x,\tau) = \begin{cases} 0 & \textit{if} \quad x = \tau \\ \frac{x}{x+1} & \textit{if} \quad x \neq 0 \quad \textit{and} \quad \tau = 0 \\ \frac{\tau}{\tau+1} & \textit{if} \quad \tau \neq 0 \quad \textit{and} \quad x = 0 \\ x + \tau & \textit{if} \quad 0 \neq x \neq \tau \neq 0. \end{cases}$$

Take $\varpi, \epsilon : X \times X \longrightarrow [1, \infty)$ as $\varpi(x, \tau) = \epsilon(x, \tau) = 2x + 2\tau + 2$. Then (X, d) is a DCMLS.

On the other hand, matrix equations have an essential role in variant problems of applied mathematics and engineering. Several matrix equations arise in control theory [22] and stability analysis [23]. There is an extensive work dealing with the existence of solutions of nonlinear matrix equations. Among the used methods to ensure such an existence, the fixed point theory plays an essential tool in this regard. For more details, see [24–26]. In this manuscript, we first prove a classical fixed point result in a DCMLS. We also present an application to solve a nonlinear matrix equation of the form

$$X = Q + \sum_{j=1}^{m} A_j^* F(X) A_j.$$

Some nontrivial and numerical examples concerning the iterative algorithm related to this matrix equation have been provided. At the end, a random integral equation of Fredholm type is investigated.

2. Main results

Our main classical fixed point result is given as follows:

Theorem 2.1. *Let (X, d) be a complete DCMLS and T be a self-mapping on X satisfying the following conditions:*

(i) : There exists $k \in [0; 1)$ such that $d(Tx; Ty) \leq k\varpi(x, y)\epsilon(x, y)d(x, y)$, for all $x, y \in X$;

(ii) : For $x_0 \in X$, we have $\sup_{m\geq 1} \lim_{i\to\infty} \varpi(x_{i+1}, x_{i+2})\epsilon(x_i, x_{i+1})\epsilon(x_{i+1}, x_m) < \frac{1}{k}$, where $x_j = T^j x_0$ for all $j \in \mathbb{N}$;

(iii) : $\lim_{n\to\infty} \varpi(x_n, x)$ and $\lim_{n\to\infty} \epsilon(x_n, x)$ exist and are finite for any $x \in X$.

Then T has a fixed point x^*. Moreover, assume that for every two fixed points x and y of T in X,

$$\limsup_{n\to\infty} \varpi(T^n x, T^n y)\epsilon(T^n x, T^n y) < \frac{1}{k}, \tag{2.1}$$

so such a fixed point is unique.

Proof. Let $x_n = T^n x_0 = Tx_{n-1}$ for all $n \geq 1$. By using (i) n-times, we have

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\leq k\varpi(x_{n-1}, x_n)\epsilon(x_{n-1}, x_n)d(x_{n-1}, x_n) \\ &\leq k^n \prod_{i=1}^{n} \varpi(x_{i-1}, x_i)\epsilon(x_{i-1}, x_i)d(x_0, x_1). \end{aligned}$$

Let $b_n = k^n \prod_{i=1}^{n} \varpi(x_{i-1}, x_i)\epsilon(x_{i-1}, x_i)d(x_0, x_1)$, so we have

$$\frac{b_{n+1}}{b_n} = \varpi(x_n, x_{n+1})\epsilon(x_n, x_{n+1}).k < 1.$$

Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.2}$$

For $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq \varpi(x_n, x_{n+1})d(x_n, x_{n+1}) + \epsilon(x_{n+1}, x_m)d(x_{n+1}, x_m) \\ &\leq \varpi(x_n, x_{n+1})d(x_n, x_{n+1}) + \epsilon(x_{n+1}, x_m)[\varpi(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) + \\ &\epsilon(x_{n+2}, x_m)d(x_{n+2}, x_m)] \\ &= \varpi(x_n, x_{n+1})d(x_n, x_{n+1}) + \epsilon(x_{n+1}, x_m)\varpi(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\ &+ \epsilon(x_{n+1}, x_m)\epsilon(x_{n+2}, x_m)d(x_{n+2}, x_m) \\ &. \\ &. \\ &. \\ &\leq \varpi(x_n, x_{n+1})d(x_n, x_{n+1}) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \epsilon(x_j, x_m) \right) \varpi(x_i, x_{i+1})d(x_i, x_{i+1}) \\ &+ \left(\prod_{j=n+1}^{m-1} \epsilon(x_j, x_m) \right) d(x_{m-1}, x_m). \end{aligned}$$

By using the fact that $\varpi(x,y) \geq 1$ and $\epsilon(x,y) \geq 1$ for all $x,y \in X$, we deduce

$$\begin{aligned} d(x_n,x_m) &\leq \varpi(x_n,x_{n+1})d(x_n,x_{n+1}) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\epsilon(x_j,x_m)\right)\varpi(x_i,x_{i+1})d(x_i,x_{i+1}) \\ &\leq \sum_{i=n}^{m-1}\left(\prod_{j=n+1}^{i}\epsilon(x_j,x_m)\right)\varpi(x_i,x_{i+1})d(x_i,x_{i+1}) \\ &\leq \sum_{i=n}^{m-1}\left(\prod_{j=n+1}^{i}\epsilon(x_j,x_m)\right)\varpi(x_i,x_{i+1})k^i\left(\prod_{j=1}^{i}\varpi(x_{j-1},x_j)\epsilon(x_{j-1},x_j)\right)d(x_0,x_1). \end{aligned}$$

Choose

$$a_i = \left(\prod_{j=n+1}^{i}\epsilon(x_j,x_m)\right)\varpi(x_i,x_{i+1})k^i\left(\prod_{j=1}^{i}\varpi(x_{j-1},x_j)\epsilon(x_{j-1},x_j)\right)d(x_0,x_1),$$

and $\Omega_p = \sum_{i=1}^{p} a_i$. Then we have

$$d(x_n,x_m) \leq \Omega_{m-1} - \Omega_{n-1}. \tag{2.3}$$

Since $\frac{a_{i+1}}{a_i} = \varpi(x_{i+1},x_{i+2})\epsilon(x_i,x_{i+1})\epsilon(x_{i+1},x_m).k$, by condition (*ii*) one gets $\frac{a_{i+1}}{a_i} < 1$, so we conclude that the real sequence $\{\Omega_p\}$ converges, then it is a Cauchy sequence in $\mathbb{R}$. By (2.3), we conclude that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\{x_n\}$ converges to some $x^* \in X$. Next, we show that $Tx^* = x^*$. The triangle inequality of DCMLS implies that

$$d(x^*,x_{n+1}) \leq \varpi(x^*,x_n)d(x^*,x_n) + \epsilon(x_n,x_{n+1})d(x_n,x_{n+1}).$$

Using the condition (*iii*) and (2.2), we deduce that $d(x^*,x_{n+1}) \longrightarrow_{n\to\infty} 0$. By triangular inequality and the condition (*i*), we get

$$\begin{aligned} d(Tx^*,x^*) &\leq \varpi(Tx^*,x_{n+1})d(Tx^*,x_{n+1}) + \epsilon(x_{n+1},x^*)d(x_{n+1},x^*) \\ &= \varpi(Tx^*,x_{n+1})d(Tx^*,Tx_n) + \epsilon(x_{n+1},x^*)d(x_{n+1},x^*) \\ &\leq \varpi(Tx^*,x_{n+1})k\varpi(x^*,x_n)\epsilon(x^*,x_n)d(x^*,x_n) + \epsilon(x_{n+1},x^*)d(x_{n+1},x^*). \end{aligned}$$

Letting $n \to \infty$ implies that $d(Tx^*,x^*) = 0$. That is, x^* is a fixed point of T. To prove the uniqueness of the fixed point, let u and v be two distinct fixed points of T. We have

$$\begin{aligned} d(u,v) = d(Tu,Tv) &\leq k\varpi(u,v)\epsilon(u,v)d(u,v) \\ &= k\varpi(T^n u,T^n v)\epsilon(T^n u,T^n v)d(u,v). \end{aligned}$$

Letting $n \to \infty$, one gets using (2.1), $0 < d(u,v) < d(u,v)$, which is a contradiction. Hence, the fixed point of T is unique. □

Example 2.2. *Let (X,d) be the complete DCMLS given as in Example 1.3. Define $T : X \longrightarrow X$ by $Tx = \frac{x}{\alpha+\alpha x}$ where $\alpha > 13$. We will show that all conditions in Theorem 2.1 hold. First, for $x \in X$, we have*

$$T^n x = \frac{x}{\alpha^n + \left(\sum_{k=1}^{n}\alpha^k\right)x}.$$

It is obvious that $x_n = T^n x \longrightarrow 0$ as $n \longrightarrow \infty$, and so for each $x \in X$,

$$\lim_{n\to\infty}\varpi(x_n,x) = \lim_{n\to\infty}\epsilon(x_n,x) = 2 + 2x < \infty.$$

For all $x,\tau \in X$, we will show that $d(Tx,T\tau) \leq \frac{1}{k}\varpi(x,\tau)\epsilon(x,\tau)d(x,\tau)$. For this, we consider the following cases:
Case 1: *$x = \tau$. we have $0 = d(Tx,T\tau) \leq \frac{1}{k}\varpi(x,\tau)\epsilon(x,\tau)d(x,\tau)$.*

Case 2: *($x \neq 0$ and $\tau = 0$) or ($\tau \neq 0$ and $x = 0$). Without loss of generality, we may assume that $x \neq 0$ and $\tau = 0$. In this case, we have*

$$\begin{aligned}
d(Tx, T0) = d(\frac{x}{\alpha + \alpha x}, 0) &= \frac{\frac{x}{\alpha+\alpha x}}{\frac{x}{\alpha+\alpha x} + 1} \\
&= \frac{x}{\alpha + (\alpha + 1)x} \\
&\leq \frac{x}{\alpha} \\
&\leq \frac{1}{\alpha}(2x + 2)^2 \left(\frac{x}{x + 1}\right) \\
&= \frac{1}{\alpha}\varpi(x, 0)\epsilon(x, 0)d(x, \tau).
\end{aligned}$$

Case 3: $0 \neq x \neq \tau \neq 0$. *Here, we have* $0 \neq \frac{x}{\alpha+\alpha x} \neq \frac{\tau}{\alpha+\alpha\tau} \neq 0$. *One writes*

$$\begin{aligned}
d(Tx, T\tau) &= d(\frac{x}{\alpha + \alpha x}, \frac{\tau}{\alpha + \alpha\tau}) \\
&= \frac{x}{\alpha + \alpha x} + \frac{\tau}{\alpha + \alpha\tau} \\
&\leq \frac{x}{\alpha} + \frac{\tau}{\alpha} \\
&\leq \frac{1}{\alpha}(2x + 2\tau + 2)^2 (x + \tau) \\
&= \frac{1}{\alpha}\varpi(x, \tau)\epsilon(x, \tau)d(x, \tau).
\end{aligned}$$

Now, let $x \in X$, then

$$\begin{aligned}
\sup_{m\geq 1}\lim_{i\to\infty}\varpi(x_{i+1}, x_{i+2})\epsilon(x_i, x_{i+1})\epsilon(x_{i+1}, x_m) &= \sup_{m\geq 1}\lim_{i\to\infty}(2 + 2x_{i+1} + 2x_{i+2}) \\
&\quad (2 + 2x_i + 2x_{i+1})(2 + 2x_{i+1} + 2x_m) \\
&= \sup_m 4(2 + 2x_m) \\
&\leq 4\left(2 + 2\frac{x}{\alpha + \alpha x}\right) \\
&\leq 4(2 + 1) \\
&\leq 13.
\end{aligned}$$

For $x, \tau \in X$, we have

$$\begin{aligned}
\lim_{n\to\infty}\sup\varpi(T^n x, T^n\tau)\epsilon(T^n x, T^n\tau) &= \lim_{n\to\infty}\sup(2 + 2x_n + 2\tau_n)(2 + 2x_n + 2\tau_n) \\
&\leq 4.
\end{aligned}$$

Thus, T fulfills all the axioms in Theorem 2.1 *with $k = \frac{1}{13}$. Thus, T admits a unique fixed point, which is* 0.

3. An application to matrix equations

In the sequel, we need the following notations:
$M_n(\mathbb{C})$ denotes the set of all $n \times n$ complex matrices;
$H(n)$ denotes the set of all $n \times n$ Hermitian matrices;
$P(n)$ denotes the set of all $n \times n$ positive definite Hermitian matrices;

$H^+(n)$ denotes the set of all $n \times n$ positive semi definite Hermitian matrices.
We will use the trace norm $\|E\|_t = \sum_{j=1}^{n} s_j(E)$, where $s_j(E), j = 1, \cdots, n$ are the singular values of $E \in H(n)$, which are the roots of the eigenvalues of E^*E. In this direction, let $d : H(n) \times H(n) \longrightarrow [0, +\infty)$ be defined by

$$d(U, \Upsilon) = (\|U - \Upsilon\|_t)^p$$

for all $U, \Upsilon \in H(n)$. Then $(H(n), d)$ is a complete DCMLS, where

$$\varpi(U, \Upsilon) = \epsilon(U, \Upsilon) = \left(2 + \frac{2}{1 + tr(U^2\Upsilon^2)}\right)^{\frac{p-1}{2}}.$$

Moreover, for $D, E \in H(n)$, consider

$$D \preceq E \Longleftrightarrow E - D \in H^+(n)$$

and

$$D \prec E \Longleftrightarrow E - D \in P(n).$$

Now, consider the matrix equation:

$$U = \Lambda + \sum_{j=1}^{q} \Omega_j^* F(U) \Omega_j. \tag{3.4}$$

Here, Λ is positive definite and the Ω_j are arbitrary $n \times n$ matrices. Suppose that F is a continuous order-preserving mapping (with respect to $\preceq$) from $P(n)$ into $P(n)$. Define the self-mapping G on $H(n)$ by

$$G(U) = \Lambda + \sum_{j=1}^{q} \Omega_j^* F(U) \Omega_j.$$

Note that G is well-defined. Note that a fixed point of G is also a solution of (3.4). The following lemma is useful in the next.

Lemma 3.1. *Let $D, E \in H^+(n)$. Then $0 \leq tr(DE) \leq \|D\|\, tr(E)$, where $\|D\| = \sqrt{\lambda^+(D^*D)}$ is the spectral norm for D. Here, $\lambda^+(D^*D)$ represents the largest eigenvalue of D^*D and D^* designs the conjugate transpose of D.*

Proof. It is known that $tr(DE) \geq 0$. Also, because $D \preceq \|D\|\, I_n$, one writes $0 \leq tr((\|D\|\, I_n - D)E) = tr(\|D\|\, E - DE) = \|D\|\, tr(E) - tr(DE)$. The proof is ended. □

Lemma 3.2. *If $D \in H(n)$ and $D \prec I_n$, then $\|D\| \leq 1$.*

Theorem 3.3. *Assume that there are positive reals M and $p \geq 1$ such that:*

(i) For all $U, \Upsilon \in H(n)$, we have

$$|tr(F(\Upsilon) - F(U))|^p \leq k \frac{\left(2 + \frac{2}{1+tr(U^2\Upsilon^2)}\right)^{p-1}}{M^p} |tr(\Upsilon - U)|^p$$

, for some $k \in \left[0, \frac{1}{\left(2+\frac{2}{1+tr(\Lambda^4)}\right)^{\frac{3(p-1)}{2}}}\right)$;

(ii) $\sum_{j=1}^{q} \Omega_j \Omega_j^* \prec M I_n$ and $0 \prec \sum_{j=1}^{q} \Omega_j^* F(\Lambda) \Omega_j$;

(iii) There is $U_0 \in P(n)$ so that $\Lambda + \sum_{j=1}^{q} \Omega_j^* F(U_0) \Omega_j \preceq U_0$.

Then the matrix equation (3.4) has a unique solution. Moreover, the sequence $\{U_n\}$ defined by the iteration $U_n = \Lambda + \sum_{j=1}^{q} \Omega_j^* F(U_{n-1}) \Omega_j$ converges to the solution of the matrix equation (3.4).

Proof. For all $U, \Upsilon \in H(n)$, we have

$$\begin{aligned}
d(G(U), G(\Upsilon)) &= (|tr(G(U) - G(\Upsilon))|)^p \\
&= \left(\left| tr(\sum_{j=1}^{q} \Omega_j^* (F(U) - F(\Upsilon))\Omega_j) \right| \right)^p \\
&= \left(\left| \sum_{j=1}^{q} tr\left(\Omega_j^* (F(U) - F(\Upsilon))\Omega_j \right) \right| \right)^p \\
&= \left(\left| \sum_{j=1}^{q} tr\left(\Omega_j \Omega_j^* (F(U) - F(\Upsilon)) \right) \right| \right)^p \\
&= \left(\left| tr(\left(\sum_{j=1}^{q} \Omega_j \Omega_j^* \right) (F(U) - F(\Upsilon)) \right| \right)^p \\
&\leq \left\| \sum_{j=1}^{q} \Omega_j \Omega_j^* \right\|^p (\|F(U) - F(\Upsilon)\|_t)^p \\
&\leq k \frac{\left\| \sum_{j=1}^{q} \Omega_j \Omega_j^* \right\|^p}{M^p} (2 + \frac{2}{1 + tr(U^2 \Upsilon^2)})^{p-1} (|tr(\Upsilon - U)|)^p \\
&= k \left(\frac{\left\| \sum_{j=1}^{q} \Omega_j \Omega_j^* \right\|}{q} \right)^p (2 + \frac{2}{1 + tr(U^2 \Upsilon^2)})^{p-1} d(U, \Upsilon) \\
&= k\varpi(U, \Upsilon)\epsilon(U, \Upsilon)d(U, \Upsilon),
\end{aligned}$$

where $\varpi, \epsilon : (U, \Upsilon) \longmapsto \left(2 + \frac{2}{1+tr(U^2\Upsilon^2)} \right)^{\frac{p-1}{2}}$. That is, the first assumption in Theorem 2.1 holds. Recall that G maps $P(n)$ into the set $\{U \in H(n)/U \succeq \Lambda\}$. So, the solution should be there and so it belongs to $P(n)$.
Under the assumption that $G(U_0) \preceq U_0$ and with the assumption that F is order-preserving, one asserts that G is order-preserving. Then for any $X \in [\Lambda, U_0]$ we have

$$\Lambda \preceq G(\Lambda) \preceq G(U) \preceq G(U_0) \preceq U_0.$$

Thus, $[\Lambda, U_0]$ is mapped into itself, that is, $\{G^j(\Lambda)\}_{j\geq 0}$ is an increasing sequence (with respect to $\preceq$) and $\{G^j(U_0)\}_{j\geq 0}$ is a decreasing sequence.
Moreover, for any j and ι, we have $G^j(\Lambda) \preceq G^\iota(U_0)$. Also, the sequence $\{G^j(U_0)\}_{j\geq 0}$ is decreasing and bounded below, thus both sequences converge to some limit, say U_∞. We find that $\Lambda \preceq U_\infty \preceq U_0$.

The function $(U, \Upsilon) \longmapsto \left(2 + \frac{2}{1+tr(U^2\Upsilon^2)}\right)^{\frac{p-1}{2}}$ is continuous. Then

$$\sup_{m\geq 1}\lim_{i\to\infty} \varpi(U_{i+1}, U_{i+2})\epsilon(U_i, U_{i+1})\epsilon(U_{i+1}, U_m) = (2 + \frac{2}{1+tr(U_\infty^4)})^{p-1}\left(2 + \frac{2}{1+tr(U_\infty^2\Lambda^2)})\right)^{\frac{p-1}{2}}$$
$$\leq \left(2 + \frac{2}{1+tr(\Lambda^4)})\right)^{\frac{3(p-1)}{2}} < \frac{1}{k}.$$

Hence, the condition (*ii*) in Theorem 2.1 is checked.

Now, let U and Υ be two fixed points of G, then $U, \Upsilon \in [\Lambda, U_0]$. Also,

$$\limsup_{n\to\infty} \varpi(G^n(U), G^n(\Upsilon))\epsilon(G^n(U), G^n(\Upsilon)) = \left(2 + \frac{2}{1+tr(U^2\Upsilon^2)}\right)^{p-1}$$
$$\leq \left(2 + \frac{2}{1+tr(\Lambda^4})\right)^{p-1} < \frac{1}{k}.$$

Thus, all conditions of Theorem 2.1 are fulfilled. Therefore, the matrix equation (3.4) has a unique solution in $[\Lambda, U_0]$. Moreover, the sequence $\{U_n\}$ defined by the iteration $U_n = \Lambda + \sum_{j=1}^{q} \Omega_j^* F(U_{n-1})\Omega_j$ converges (in the sense of DCMLS) to the solution of the matrix equation (3.4). $\square$

In this part, we illustrate Theorem 3.3 by some numerical examples and some illustrated curves.

Example 3.4. *Consider*

$$\Lambda = \begin{pmatrix} 7 & 3 & 0 & 0 \\ 3 & 7 & 3 & 0 \\ 0 & 3 & 7 & 3 \\ 0 & 0 & 3 & 7 \end{pmatrix}, \quad \Omega_1 = 10^{-4} \times \begin{pmatrix} -588 & 64 & 0 & -2069 \\ -6 & -21 & 0 & 0 \\ 201 & 0 & 0 & 1201 \\ 1212 & 3131 & 3 & 424 \end{pmatrix}$$

$$\Omega_2 = 10^{-4} \times \begin{pmatrix} 0 & 121 & 666 & 1301 \\ 341 & 141 & 1201 & 2004 \\ 11 & 0 & 0 & 11 \\ 114 & 541 & 1111 & 511 \end{pmatrix}, \quad \Omega_3 = 10^{-4} \times \begin{pmatrix} 14 & 21 & -421 & -1158 \\ 0 & 1471 & -451 & 112 \\ 125 & 1214 & 1142 & 2999 \\ 1254 & -1010 & 1241 & 0 \end{pmatrix}.$$

Define $F : H(n) \longrightarrow H(n)$ *by* $F(U) = U$. *The equation* (3.4) *becomes*

$$U = \Lambda + \Omega_1^* U \Omega_1 + \Omega_2^* U \Omega_2 + \Omega_3^* U \Omega_3. \tag{3.5}$$

After calculations, we get

$$\sum_{j=1}^{3} \Omega_j \Omega_j^* = \begin{pmatrix} 0.0830 & 0.0352 & -0.0652 & -0.0045 \\ 0.0352 & 0.0352 & 0.0163 & 0.0036 \\ -0.0652 & 0.0163 & 0.1327 & 0.0111 \\ -0.0045 & 0.0036 & 0.0111 & 0.1739 \end{pmatrix}. \tag{3.6}$$

Note that the eigenvalues of matrix $\sum_{j=1}^{3} \Omega_j \Omega_j^*$ *are* $\lambda_1 = 0.0153$, $\lambda_2 = 0.1017$, $\lambda_3 = 0.1647$ *and* $\lambda_4 = 0.1876$, *and therefore we have* $\sum_{j=1}^{3} \Omega_j \Omega_j^* \leq \frac{1}{2} I_3$.

Thus, all assertions in Theorem 3.3 hold with $M = \frac{1}{2}$, $k = 0.3$ *and* $p = 2$. *Take the iterative algorithm*

$$\begin{cases} U_0 = \alpha\Lambda \quad (\alpha > 1) \\ U_n = \Lambda + \Omega_1^* U_{n-1} \Omega_1 + \Omega_2^* U_{n-1} \Omega_2 + \Omega_3^* U_{n-1} \Omega_3, \quad n \geq 1. \end{cases} \tag{3.7}$$

By considering $\alpha = 2$, after 6 successive iterations, we obtain the following positive definite solution (the limit of the convergent sequence $\{U_n\}$)

$$U_6 = \begin{pmatrix} 7.3365 & 3.3336 & 0.2600 & 0.4450 \\ 3.3336 & 8.2911 & 3.0693 & 0.5684 \\ 0.2600 & 3.0693 & 7.6810 & 3.8390 \\ 0.4450 & 0.5684 & 3.8390 & 8.9171 \end{pmatrix}.$$

While, by considering $\alpha = 3$, after 7 successive iterations, we obtain the following positive definite solution (the limit of the convergent sequence $\{U_n\}$)

$$U_7 = U_6 = \begin{pmatrix} 7.3365 & 3.3336 & 0.2600 & 0.4450 \\ 3.3336 & 8.2911 & 3.0693 & 0.5684 \\ 0.2600 & 3.0693 & 7.6810 & 3.8390 \\ 0.4450 & 0.5684 & 3.8390 & 8.9171 \end{pmatrix}.$$

The graphical view of convergence for $\alpha = 2$ and $\alpha = 3$ is shown in Figure 1.

This figure illustrates the convergence curve of the iterative method (3.7). Note that the curves are perfect lines for $\alpha = 2$ (resp. $\alpha = 3$), i.e., the algorithm (3.7) converges to the theoretical solution of (3.5) after 6 iterations (resp. 7 iterations). Following Figure 1, remark that the speeds of convergence for the case $\alpha = 2$, or $\alpha = 3$ are slightly the

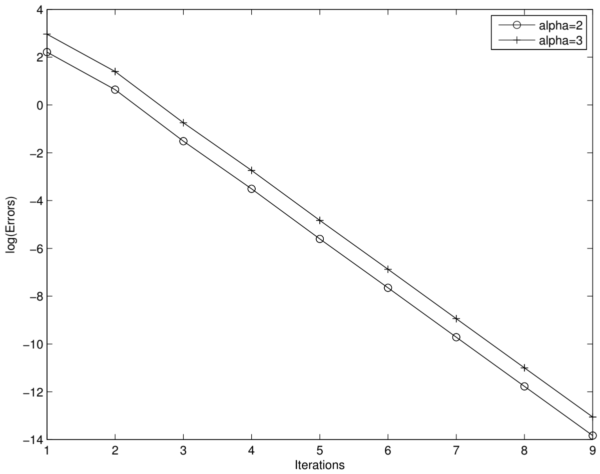

Figure 1: The Convergence curve.

same. This explains the above numbers of iterations corresponding to obtention of convergence ($N = 6$ for $\alpha = 2$ et $N = 7$ for $\alpha = 3$).

Example 3.5. *Take $\Lambda = I_3$ and $U_0 = \alpha I_3$ with $\alpha > 2$. Choose*

$$\Omega_1 = \begin{pmatrix} 0.3 & 0.01 & 0.01 \\ 0 & 0.28 & -0.02 \\ 0.02 & 0.03 & 0.34 \end{pmatrix} \qquad \Omega_2 = \begin{pmatrix} -0.34 & 0 & 0 \\ 0 & -0.34 & 0 \\ 0.01 & 0.01 & -0.32 \end{pmatrix}.$$

Define $F : H(n) \longrightarrow H(n)$ by $F(U) = U$. The equation (3.4) becomes

$$U = \Lambda + \Omega_1^* U \Omega_1 + \Omega_2^* U \Omega_2. \tag{3.8}$$

By a calculation, one gets

$$\sum_{j=1}^{2} \Omega_j \Omega_j^* = \begin{pmatrix} 0.2058 & 0.0026 & 0.0063 \\ 0.0026 & 0.1944 & -0.0018 \\ 0.0063 & -0.0018 & 0.2195 \end{pmatrix}. \tag{3.9}$$

Note that the eigenvalues of the matrix $\sum_{J=1}^{2} \Omega_J \Omega_J^*$ *are* $\lambda_1 = 0,221976$, $\lambda_2 = 0,204297$ *and* $\lambda_3 = 0,193427$. *Thus, one asserts that*

$$\sum_{J=1}^{2} \Omega_J \Omega_J^* \preceq 023 I_3.$$

Let $p = 2$. *The number* $\frac{1}{\left(2+\frac{2}{1+tr(\Lambda^4)}\right)^{\frac{3(p-1)}{2}}}$ *is equal to* $\frac{1}{(2+\frac{2}{1+3})^{\frac{3}{2}}} = 0,2529$. *It remains to check the last condition. Since* $U_0 = \alpha.I_3$, *the eigenvalues of the matrix* $\Lambda + \sum_{J=1}^{2} \Omega_J^* F(U_0)\Omega_J = I_3 + \alpha \sum_{J=1}^{2} \Omega_J^* \Omega_J$ *are* $\delta_1 = 1 + \alpha\lambda_1 = 1 + 0.221976\alpha < \alpha$, $\delta_2 = 1 + \alpha\lambda_2 = 1 + 0.204297\alpha < \alpha$ *and* $\delta_3 = 1 + \alpha\lambda_3 = 1 + 0.193427\alpha < \alpha$. *Thus,*

$$\Lambda + \sum_{J=1}^{2} \Omega_J^* F(U_0)\Omega_J \preceq U_0.$$

All assertions in Theorem 3.3 hold with $M = 0.23$ *and* $k = 0.25$. *We will consider the iterative algorithm*

$$\begin{cases} U_0 = \alpha\Lambda \quad (\alpha > 2) \\ U_n = \Lambda + \Omega_1^* U_{n-1}\Omega_1 + \Omega_2^* U_{n-1}\Omega_2. \end{cases} \tag{3.10}$$

By considering $\alpha = 3$, *after* $N = 6$ *successive iterations, we obtain the following positive definite solution (the limit of the convergent sequence* $\{U_n\}$*)*

$$U_6 = \begin{pmatrix} 1.2597 & 0.0017 & 0.0106 \\ 0.0017 & 1.2423 & 0.0025 \\ 0.0106 & 0.0025 & 1.2796 \end{pmatrix}.$$

Example 3.6. *Take*

$$\Lambda = \begin{pmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{pmatrix}, \quad \Omega_1 = \begin{pmatrix} 0.01 & 0.005 & 0.012 & 0.023 \\ 0.005 & 0.02 & 0.011 & 0.019 \\ 0.012 & 0.011 & 0.031 & 0.013 \\ 0.023 & 0.019 & 0.013 & 0.044 \end{pmatrix},$$

$$\Omega_2 = \begin{pmatrix} 0.015 & 0.01 & 0.027 & 0.04 \\ 0.01 & 0.03 & 0.01 & 0.016 \\ 0.027 & 0.01 & 0.025 & 0 \\ 0.04 & 0.016 & 0 & 0.017 \end{pmatrix}, \quad \Omega_3 = \begin{pmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.04 & 0 & 0 \\ 0 & 0 & 0.07 & 0 \\ 0 & 0 & 0 & 0.08 \end{pmatrix}.$$

Choose $F(U) = U$. *The equation* (3.4) *becomes*

$$U = \Lambda + \Omega_1^* U\Omega_1 + \Omega_2^* U\Omega_2 + \Omega_3^* U\Omega_3. \tag{3.11}$$

We have

$$\sum_{J=1}^{3} \Omega_J \Omega_J^* = \begin{pmatrix} 0.003552 & 0.002079 & 0.002026 & 0.002933 \\ 0.002079 & 0.003863 & 0.001688 & 0.002626 \\ 0.002026 & 0.001688 & 0.007749 & 0.0027 \\ 0.002933 & 0.002626 & 0.0027 & 0.01154 \end{pmatrix}. \tag{3.12}$$

The eigenvalues of the matrix $\sum_{J=1}^{3} \Omega_J \Omega_J^*$ *are* $\lambda_1 = 0,0152804$, $\lambda_2 = 0,00641319$, $\lambda_3 = 0,00343146$ *and* $\lambda_4 = 0,00157896$. *Consequently,* $\sum_{J=1}^{3} \Omega_J \Omega_J^* \preceq 0.2 I_4$. *Let* $p = 2$. *Also, the number* $\frac{1}{\left(2+\frac{2}{1+tr(\Lambda^4)}\right)^{\frac{3(p-1)}{2}}}$ *is equal to* 0.35346956.

Moreover, the eigenvalues of the matrix $\Lambda + \sum_{j=1}^{3} \Omega_j^* F(U_0)\Omega_j = \Lambda + 10\sum_{j=1}^{3} \Omega_j^*\Omega_j$ *are* $\delta_1 = 8.36216 < 10$, $\delta_2 = 6.2811 < 10$, $\delta_3 = 3.81817 < 10$ *and* $\delta_4 = 1.80561 < 10$. *Thus, by choosing* $U_0 = 10I_3$, *one writes*

$$\Lambda + \sum_{j=1}^{3} \Omega_j^* F(U_0)\Omega_j \preceq U_0.$$

Hence, all assertions in Theorem 3.3 hold with $M = 0.2$ *and* $k = 0.3$. *We will consider the iterative algorithm*

$$\begin{cases} U_0 = 10I_4 \\ U_n = \Lambda + \Omega_1^* U_{n-1}\Omega_1 + \Omega_2^* U_{n-1}\Omega_2 + \Omega_3^* U_{n-1}\Omega_3. \end{cases} \tag{3.13}$$

After $N = 3$ *successive iterations, we obtain the following positive definite solution (the limit of the convergent sequence* $\{U_n\}$*)*

$$U_3 = \begin{pmatrix} 10.0361 & 0.0211 & 0.0206 & 0.0298 \\ 0.0211 & 10.0390 & 0.0172 & 0.0267 \\ 0.0206 & 0.0172 & 10.0782 & 0.0275 \\ 0.0298 & 0.0267 & 0.0275 & 10.1168 \end{pmatrix}.$$

Remark 3.7. *The graphical view of convergence for Example 3.5 and Example 3.6 is shown in Figure 2. Note that*

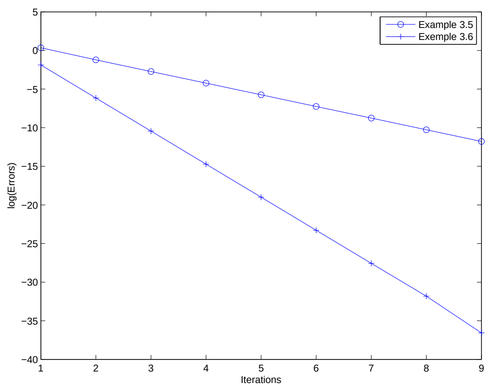

Figure 2: The Convergence curve.

we get the convergence faster in Example 3.6. To get convergence, we just need 3 iterations, while in Example 3.5 we require 6 iterations. That is, the convergence in Example 3.6 is speeder than as in Example 3.5. We see this fact in Figure 2. Indeed, the slope of the curve of convergence for Example 3.6 equal to 4.3093, that it is greater than as in Example 3.5, which is equal to 1.5125.

4. An application to a random Fredholm type integral equation

Probabilistic fundamental analysis is an essential concept of mathematics, which is applied to resolve diverse problems. An equation needing a mathematical tool to model its phenomena is characterised as a random equation. Fixed point techniques for stochastic functions have been investigated in 1950 by the Prague School of Probability. It appears due to the importance of fixed point results in probabilistic functional analysis and probabilistic models along with variant applications. Related issues to measurability of probabilistic, solutions, and statistical concepts of random solutions have arisen because of the initiation of randomness. Observe that random fixed point results are considered as stochastic generalizations of classical fixed point results which are described as deterministic results. For more related results, see [27–30].

Now, we apply Theorem 3.3 to guarantee the existence of a unique solution to the following nonlinear random Fredholm integral equation of the second kind of the form:

$$U(\varpi,t)=\int_0^1 G(\varpi,t,s,U(\varpi,t))ds \tag{4.14}$$

where

(i) $\varpi\in\Omega$ is a supporting set of (Ω,β,μ) a probability measure space;
(ii) $U(\varpi,t)$ is a valued random variables for each $t\in[0,1]$;
(iii) $G(\varpi,t,s,U(\varpi,t))$ is real stochastic kernel for $t,s\in[0,1]$ and measurable in t on $[0,1]$.

Let $X=C([0,1],\mathbb{R})$ be the set of all continuous functions defined on $[0,1]$. Consider

$$\|U(\varpi)\|_\infty =: \|U(\varpi,.)\|_\infty=\sup_{t\in[0,1]}|U(\varpi,t)|, \quad \varpi\in\Omega.$$

Now, define

$$d(U(\varpi),V(\varpi))=\|U(\varpi)-V(\varpi)\|_\infty \quad \text{for all } U(\varpi),V(\varpi)\in X.$$

Note that (X,d) is a complete DCMLS with controlled functions

$$\vartheta(U(\varpi),V(\varpi))=1 \quad \text{and} \quad \epsilon(U(\varpi),V(\varpi))=e^{\|U(\varpi).V(\varpi)\|_\infty} \quad \text{for all } U(\varpi),V(\varpi)\in X.$$

Theorem 4.1. *Assume that for all $U(\varpi),V(\varpi)\in X$ and $\varpi\in\Omega$:*

(h1) $|G(\varpi,t,s,U(\varpi,t))-G(\varpi,t,s,V(\varpi,t))|\le k.e^{\sup_{t\in[0,1]}|U(\varpi,t)||V(\varpi,t)|}|U(\varpi,t)-V(\varpi,t)|$ *for some* $k\in\left[0,\frac{1}{e^{2.(\sup_{t,s\in[0,1]}|G(\varpi,t,s,U(\varpi,t))|)^2}}\right)$;

(h2) $G(\varpi,t,s,\int_0^1 G(\varpi,t,r,U(\varpi,t))dr))<G(\varpi,t,s,U(\varpi,t))$ *for all* $t,s\in[0,1]$.

Then the above Fredholm integral equation (4.14) *has a unique solution.*

Proof. Define the random self operator on X given as $TU:\Omega\times[0,1]\longrightarrow\mathbb{R}$, where for all $(\varpi,t)\in\Omega\times[0,1]$,

$$TU(\varpi,t)=\int_0^1 G(\varpi,t,s,U(\varpi,t))ds.$$

Now, for $\varpi\in\Omega$ and $t\in[0,1]$, we have

$$\begin{aligned}|TU(\varpi,t)-TV(\varpi,t)| &= \left|\int_0^1 (G(\varpi,t,s,U(\varpi,t))-G(\varpi,t,s,V(\varpi,t)))\,ds\right|\\ &\le \int_0^1 |G(\varpi,t,s,U(\varpi,t))-G(\varpi,t,s,V(\varpi,t))|\,ds\\ &\le k.e^{\sup_{t\in[0,1]}|U(\varpi,t)||V(\varpi,t)|}\int_0^1 |U(\varpi,t)-V(\varpi,t)|\,ds\\ &\le k.e^{\sup_{t\in[0,1]}|U(\varpi,t)||V(\varpi,t)|}\sup_{s\in[0,1]}|U(\varpi,t)-V(\varpi,t)|\,ds\\ &= k.\vartheta(U(\varpi),V(\varpi))\epsilon(U(\varpi),V(\varpi))d(U(\varpi),V(\varpi)).\end{aligned}$$

Thus,

$$d(TU(\varpi), TV(\varpi)) \leq k.\vartheta(U(\varpi), V(\varpi))\epsilon(U(\varpi), V(\varpi))d(U(\varpi), V(\varpi)).$$

For all $n \in \mathbb{N}^*$ and $U(\varpi) \in X$, we have

$$\begin{aligned}
T^n U(\varpi, t) &= T(T^{n-1}U)(\varpi, t)\\
&= \int_0^1 G(\varpi, t, s, T^{n-1}U(\varpi, t))ds\\
&= \int_0^1 G(\varpi, t, s, T(T^{n-2}U)(\varpi, t))ds\\
&= \int_0^1 G(\varpi, t, s, \int_0^1 G(\varpi, t, r, T^{n-2}U(\varpi, t))dr)ds\\
&< \int_0^1 G(\varpi, t, s, T^{n-2}U(\varpi, s))ds\\
&= T^{n-1}U(\varpi, t).
\end{aligned}$$

Thus, for all $t \in [0,1]$, we find that $\{U_n\}_n = \{T^n U(\varpi, t)\}_n$ is a bounded below increasing sequence, and so it is convergent, say to l. Since $\{T^n\}_n$ is monotone, it by the Dini theorem it follows that for all $\varpi \in \Omega$, $\sup_{t\in[0,1]} |T^n U(\varpi, t)|$ converges to some $\sigma(\varpi) \leq \sup_{t,s\in[0,1]} |G(\varpi, t, s, U(\varpi, t))|$. Observe that

$$\begin{aligned}
\lim_{n,m\to\infty} \epsilon(T^n U(\varpi), T^m U(\varpi)) &= \lim_{n,m\to\infty} e^{\sup_{t\in[0,1]}|T^n U(\varpi,t)||T^m U(\varpi,t)|}\\
&\to e^{\sigma(\varpi)^2}\\
&\leq e^{(\sup_{t,s\in[0,1]}|G(\varpi,t,s,U(\varpi,t))|)^2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sup_{m\geq 1} \lim_{i\to\infty} \varpi(U_{i+1}, U_{i+2})\epsilon(U_i, U_{i+1})\epsilon(U_{i+1}, U_m) &\leq \sup_{m\geq 1} e^{(\sigma(\varpi))^2} e^{\sigma(\varpi)\sup_{t\in[0,1]}|U_m(\varpi,t)|}\\
&= e^{(\sigma(\varpi))^2} e^{\sigma(\varpi)\sup_{t\in[0,1]}|U_0(\varpi,t)|}\\
&\leq e^{2.(\sup_{t,s\in[0,1]}|G(\varpi,t,s,U(\varpi,t))|)^2}\\
&\leq \frac{1}{k}.
\end{aligned}$$

All the axioms of Theorem 2.1 hold, and hence there is a unique solution of the equation (4.14).
□

5. Conclusion

In this work, we established a fixed point result involving the control functions in the right-hand side of the contraction mapping. We also gave an application on matrix equations and some investigations on numerical parts have been considered and studied. Namely, some convergence results for a class of matrix equations have been derived. Working on stochastic fixed point results for generalized contractive mappings in the framework of variant generalized metric spaces is an interesting subject. The last part of our paper goes with this direction by giving a solution of a random integral equation. It would be better to study random fixed point theory in DCMLKs via sevral contraction mappings. We keep it for next papers.

Data Availability

No data is associated with this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors Contributions

All authors contributed equally and significantly in writing this article.

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