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Strong convergence theorems and a projection method using a balanced mapping in Hadamard spaces

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Abstract. In this paper, we prove a strong convergence theorem generated iterative of Halpern type using a balanced mapping of a countable family of nonexpansive mappings. Further, we propose a projection method with balanced mappings.

1. Introduction

Let H be a Hilbert space, C a nonempty subset of H, and T a nonexpansive mapping of C into itself. The problem of finding a fixed point of T is one of the most important problems in nonlinear analysis. In 2008, Takahashi et al. proposed a strong convergence theorem, which is called *the shrinking projection method*.

Theorem 1.1 (Takahashi et al. [10]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, T a nonexpansive mapping such that $\mathcal{F}(T) \neq \emptyset$, and $\{\alpha_n \mid n \in \mathbb{N}\} \subset [0, a] \subset [0, 1[$. For a point $x \in H$ chosen arbitrarily, generate a sequence $\{x_n\}$ and a sequence $\{C_n\}$ of sets by $x_1 \in C$, $C_0 = C$ and*

$$y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n;$$
$$C_n = \{z \in C \mid \|z - y_n\| \leq \|z - x_n\|\} \cap C_{n-1};$$
$$x_{n+1} = P_{C_n} x$$

for each $n \in \mathbb{N}$, where P_K is the metric projection of C onto a nonempty closed convex subset K of C. Then, $\{x_n\}$ converges strongly to $P_{\mathcal{F}(T)}x \in C$.

On the other hand, as another type of strongly convergent sequence to a fixed point, Kimura et al. proposed the following projection method in a Hilbert space in 2011. It is called *the combining projection method*.

Theorem 1.2 (Kimura et al. [8]). *Let C a nonempty closed convex subset C of a Hilbert space and T_j a nonexpansive mapping of C into itself for $j \in \{1, 2, \ldots, N\}$ such that $\bigcap_{j=1}^{N} \mathcal{F}(T_j) \neq \emptyset$. Put $I_N = \{1, 2, \ldots, N\}$. Let $\{\alpha_n \mid n \in \mathbb{N}\} \subset$*

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$[0,1]$, $\left\{\beta_n^j \mid j \in I_N, n \in \mathbb{N}\right\} \subset [0,1]$ *such that* $\sum_{j\in I_N} \beta_n^j = 1$ *for* $n \in \mathbb{N}$, $\{\gamma_{n,k} \mid n,k \in \mathbb{N}, k \le n\}$ *such that* $\sum_{k=1}^n \gamma_{n,k} = 1$ *for* $n \in \mathbb{N}$, *and* $\{\delta_n \mid n \in \mathbb{N}\} \subset [0,1]$. *Define a sequence* $\{x_n\}$*by* $u, x_1 \in C$ *and*

$$
\begin{aligned}
y_n^j &= \alpha_n x_n + (1-\alpha_n) T_j x_n \text{ for } j \in I_N;\\
C_n^j &= \left\{ z \in C \;\middle|\; \|z - y_n^j\| \le \|z - x_n\| \right\} \text{ for } j \in I_N;\\
v_{n,k}^j &= P_{C_k^j} x_n \text{ for } k \in \{1,2,\dots,n\} \text{ and } j \in I_N;\\
w_{n,k} &= \sum_{j\in I_N} \beta_k^j v_{n,k}^j \text{ for } k \in \{1,2,\dots,n\};\\
x_{n+1} &= \delta_n u + (1-\delta_n) \sum_{k=1}^n \gamma_{n,k} w_{n,k}
\end{aligned}
$$

for each $n \in \mathbb{N}$, *where* P_K *is the metric projection of* H *onto a nonempty closed convex subset* K *of* H. *Suppose the following conditions hold:*

(a) $\liminf_{n\to\infty} \alpha_n < 1$;
(b) $\beta_n^j > 0$ *for all* $j \in I_N$;
(c) $\lim_{n\to\infty} \gamma_{n,k} > 0$ *for all* $k \in \mathbb{N}$ *and* $\sum_{n=1}^\infty \sum_{k=1}^n |\gamma_{n+1,k} - \gamma_{n,k}| < \infty$;
(d) $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^\infty \delta_n = \infty$ *and* $\sum_{n=1}^\infty |\delta_{n+1} - \delta_n| < \infty$.

Then, $\{x_n\}$ *converges strongly to* $P_{\bigcap_{j=1}^N \mathcal{F}(T_j)} u$.

We can prove this theorem by using the following result for a countable family of nonexpansive mapping in a Banach space.

Theorem 1.3 (Aoyama et al. [1])**.** *Let* E *be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable,* C *a nonempty closed convex subset of* E, $\{\alpha_n \mid n \in \mathbb{N}\} \subset [0,1]$, $\{\beta_n^k \mid k,n \in \mathbb{N}, k \le n\} \subset [0,1]$ *such that* $\sum_{k=1}^n \beta_n^k = 1$ *for* $n \in \mathbb{N}$, *and* S_k *a nonexpansive mapping of* C *into itself for* $k \in \mathbb{N}$ *such that* $\bigcap_{k=1}^\infty \mathcal{F}(S_k) \ne \emptyset$. *Define* $\{x_n\}$ *by* $x_1, u \in C$ *and*

$$
x_{n+1} = \alpha_n u + (1-\alpha_n) \sum_{k=1}^n \beta_n^k S_k
$$

for each $n \in \mathbb{N}$. *Suppose the following conditions hold:*

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ *and* $\sum_{n=1}^\infty \sum_{k=1}^n |\alpha_{n+1} - \alpha_n| < \infty$;
(b) $\lim_{n\to\infty} \beta_n^k > 0$ *for* $k \in \mathbb{N}$ *and* $\sum_{n=1}^\infty \sum_{k=1}^n \left|\beta_{n+1}^k - \beta_n^k\right| < \infty$.

Then, $\{x_n\}$ *converges strongly to* Qu, *where* Q *is sunny nonexpansive retraction of* E *onto* $\bigcap_{k=1}^\infty \mathcal{F}(S_k)$.

Huang and Kimura generalized Theorem 1.2 to the setting of Hadamard space [6]. In this result, they repeatedly use a usual convex combination between two points to construct the convex combination among three or more points. There is another approach to take a convex combination among such points; a notion of balanced mapping.

In this paper, we propose a convergence theorem generated by a Halpern type iterative sequence using a balanced mapping of a countable family of nonexpansive mappings. We apply this result to a new method using a balanced mapping of nonexpansive mappings in a Hadamard space, which is similar to [6]. It is different from the method proposed in [7]. In Section 2, we introduce a Hadamard space and a balanced mapping of nonexpansive mappings. In Section 3, we prove a convergence theorem generated by a Halpern iterative sequence using a balanced mapping of a countable family of nonexpansive mappings in a Hadamard space. In Section 4, we propose a projection method using a nonexpansive mapping and prove a convergence theorem.

2. Preliminaries

Let (X, d) be a metric space, and T a mapping of X into itself. The set of all fixed points of T is denoted by $\mathcal{F}(T)$. Let $\{x_n\}$ be a bounded sequence of X. An element $x_0 \in X$ is said to be an *asymptotic center of* $\{x_n\} \subset X$ if the following equality holds:

$$\limsup_{n\to\infty} d(x_n, x_0) = \inf_{x\in X} \limsup_{n\to\infty} d(x_n, x).$$

A sequence $\{x_n\} \subset X$ is said to be Δ*-convergent to* $x_0 \in X$ if x_0 is a unique asymptotic center of all subsequences of $\{x_n\}$. It is denoted by $x_n \xrightarrow{\Delta} x_0$. We say a mapping T is *nonexpansive* if for $x, y \in X$, it follows that $d(Tx, Ty) \le d(x, y)$. If a mapping T is nonexpansive and $\mathcal{F}(T) \neq \emptyset$, it is closed convex. Further, a mapping T is called Δ*-demiclised* if for every $\{x_n\} \subset X$ satisfying $x_n \xrightarrow{\Delta} x_0 \in X$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, it follows that $x_0 \in \mathcal{F}(T)$. We know that if a mapping T is nonexpansive, it is Δ-demiclosed.

Let $x, y \in X$ and γ_{xy} a mapping of $[0, d(x, y)]$ into X. A mapping γ_{xy} is said to be *a geodesic with endpoints x and y* if $\gamma_{xy}(0) = x$, $\gamma_{xy}(d(x, y)) = y$ and $d(\gamma_{xy}(s), \gamma_{xy}(t)) = |s - t|$ for all $s, t \in [0, d(x, y)]$. X is called *a unique geodesic space* if for all $x, y \in X$, there exists a unique geodesic with endpoints x and y. The image of the geodesic with endpoints x and y is denoted by $\operatorname{Im}\gamma_{xy}$. For $x, y \in X$ and $t \in [0, 1]$, there exists $z \in \operatorname{Im}\gamma_{xy}$ such that $d(x, z) = (1 - t)d(x, y)$ and $d(y, z) = td(x, y)$, which is denoted by $z = tx \oplus (1 - t)y$.

Let X be a unique geodesic space and $x, y, z \in X$. Then, *a geodesic triangle of vertices x, y, z* is defined by $\operatorname{Im}\gamma_{xy} \cup \operatorname{Im}\gamma_{yz} \cup \operatorname{Im}\gamma_{zx}$, which is denoted by $\triangle(x, y, z)$. For $x, y, z \in X$, *a comparison triangle* to $\triangle(x, y, z) \subset X$ of vertices $\bar{x}, \bar{y}, \bar{z} \in \mathbb{E}^2$ is defined by $\operatorname{Im}\gamma_{\bar{x}\bar{y}} \cup \operatorname{Im}\gamma_{\bar{y}\bar{z}} \cup \operatorname{Im}\gamma_{\bar{z}\bar{x}}$ with $d(x, y) = d_{\mathbb{E}^2}(\bar{x}, \bar{y})$, $d(y, z) = d_{\mathbb{E}^2}(\bar{y}, \bar{z})$ and $d(z, x) = d_{\mathbb{E}^2}(\bar{z}, \bar{x})$, which is denoted by $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$. A point $\bar{p} \in \operatorname{Im}\gamma_{\bar{x}\bar{y}}$ is called *a comparison point of $p \in \operatorname{Im}\gamma_{xy}$* if $d(x, p) = d_{\mathbb{E}^2}(\bar{x}, \bar{p})$. A unique geodesic space X is called a CAT(0) space if for all $x, y, z \in X$, $p, q \in \triangle(x, y, z)$ and their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$, it follows that $d(p, q) \le d_{\mathbb{E}^2}(\bar{p}, \bar{q})$. A complete CAT(0) space is called *a Hadamard space*. In a CAT(0) space, the following lemmas hold:

Lemma 2.1 (Bačák [2]). *Let X be a* CAT(0) *space, $x, y, z \in X$ and $t \in [0, 1]$. Then the following holds:*

$$d(tx \oplus (1 - t)y, z)^2 \le td(x, z)^2 + (1 - t)d(y, z)^2 - t(1 - t)d(x, y)^2.$$

Lemma 2.2 (He et al. [5]). *Let X be a Hadamard space and $\{x_n\}$ a bounded sequence of X such that $x_n \xrightarrow{\Delta} x \in X$. Then $d(u, x) \le \liminf_{n\to\infty} d(u, x_n)$ for $u \in X$.*

Let X be a Hadamard space and put $I_N = \{1, 2, \ldots, N\}$. Let T_k a nonexpansive mapping of X into itself for $k \in I_N$ and $\left\{\alpha^k \mid k \in I_N\right\} \subset [0, 1]$ with $\sum_{k\in I_N} \alpha^k = 1$. Then *a balanced mapping U of T_k* is defined by

$$Ux = \operatorname*{Argmin}_{y\in X} \sum_{k\in I_N} \alpha^k d(T_k x, y)^2$$

for all $x \in X$; see [4].

Theorem 2.3 (Hasegawa and Kimura [4]). *Let X be a Hadamard space. Put $I_N = \{1, 2, \ldots, N\}$. Let T_k a nonexpansive mapping for all $k \in I_N$ such that $\bigcap_{k\in I_N} \mathcal{F}(T_k)$ is nonempty and $\{\alpha^k : k \in I_N\} \subset [0, 1]$ such that $\sum_{k\in I_N} \alpha^k = 1$. Define $U \colon X \to X$ by*

$$Ux = \operatorname*{Argmin}_{y\in X} \sum_{k\in I_N} \alpha^k d(T_k x, y)^2$$

for all $x \in X$. Then the following hold:

(a) *U is single-valued and nonexpansive;*
(b) *$\mathcal{F}(U) = \bigcap_{k\in I_N} \mathcal{F}(T_k)$;*

(c) *the inequality*

$$\sum_{k=1}^{N} \alpha^k d(T_k x, Ux)^2 \le \sum_{k=1}^{N} \alpha^k d(T_k x, y)^2 - d(Ux, y)^2$$

holds for $x, y \in X$.

The following lemma is important to prove a convergence theorem generated by a Halpern's iterative method using a balanced mapping of a countable family of nonexpansive mappings:

Lemma 2.4 (Aoyama et al. [1]). *Let* $\{s_n\}$ *be a sequence of nonnegative real numbers,* $\{\alpha_n\}$ *a sequence of* $[0,1]$ *with* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{u_n\}$ *a sequence of nonnegative real numbers with* $\sum_{n=1}^{\infty} u_n < \infty$ *and* $\{t_n\}$ *a real numbers with* $\limsup_{n\to\infty} t_n \le 0$. *Suppose that* $s_{n+1} \le (1-\alpha_n)s_n + \alpha_n t_n + u_n$ *for all* $n \in \mathbb{N}$. *Then* $\lim_{n\to\infty} s_n = 0$.

3. A convergence theorem with balanced mappings

In this section, we generate a Halpern type iterative sequence using a balanced mapping of a countable family of nonexpansive mappings and prove the convergence theorem. We first show the properties of a balanced mapping of a countable family of nonexpansive mappings:

Lemma 3.1. *Let* X *be a Hadamard space,* T_k *a nonexpansive mapping of* X *into itself for* $k \in \mathbb{N}$ *with* $\bigcap_{k=1}^{\infty} \mathcal{F}(T_k) \ne \emptyset$, $\{\alpha^k \mid k = 1, 2, \dots, n\} \subset [0,1]$ *and* $\{\beta^k \mid k = 1, 2, \dots, n+1\} \subset [0,1]$ *such that* $\sum_{k=1}^{n} \alpha^k = \sum_{k=1}^{n+1} \beta^k = 1$. *Put*

$$Ux = \operatorname*{Argmin}_{y\in X} \sum_{k=1}^{n} \alpha^k d(T_k x, y)^2 \text{ and } Vx = \operatorname*{Argmin}_{y\in X} \sum_{k=1}^{n+1} \beta^k d(T_k x, y)^2$$

for all $x \in X$. *Then the inequality*

$$d(Ux, Vx) \le 4d(x, z) \sum_{k=1}^{n} \left|\beta^k - \alpha^k\right|$$

holds for all $x \in X$ *and* $z \in \bigcap_{k=1}^{\infty} \mathcal{F}(T_k)$.

Proof. Let $x \in X$. If $Ux = Vx$, we get the result obviously. Suppose $Ux \ne Vx$. Let $t \in]0,1[$. By Lemma 2.1, we get

$$\begin{aligned}
\sum_{k=1}^{n} \alpha^k d(T_k x, Ux)^2 &\le \sum_{k=1}^{n} \alpha^k d(T_k x, tUx \oplus (1-t)Vx)^2 \\
&\le \sum_{k=1}^{n} \alpha^k (td(T_k x, Ux)^2 + (1-t)d(T_k x, Vx)^2 - t(1-t)d(Ux, Vx)^2) \\
&= t\sum_{k=1}^{n} \alpha^k d(T_k x, Ux)^2 + (1-t)\sum_{k=1}^{n} \alpha_n^k d(T_k x, Vx)^2 - t(1-t)d(Ux, Vx)^2
\end{aligned}$$

and hence

$$t(1-t)d(Ux, Vx)^2 \le (1-t)\left(\sum_{k=1}^{n} \alpha^k d(T_k x, Vx)^2 - \sum_{k=1}^{n} \alpha^k d(T_k x, Ux)^2\right).$$

Dividing $1-t > 0$ and letting $t \to 1$, we get

$$d(Ux, Vx)^2 \le \sum_{k=1}^{n} \alpha^k d(T_k x, Vx)^2 - \sum_{k=1}^{n} \alpha_n^k d(T_k x, Ux)^2 = \sum_{k=1}^{n} \alpha^k (d(T_k x, Vx)^2 - d(T_k x, Ux)^2). \tag{1}$$

Similarly, we get

$$d(Vx,Ux)^2 \le \sum_{k=1}^{n+1} \beta^k (d(T_k x, Ux)^2 - d(T_k x, Vx)^2).$$

Then we have

$$\begin{aligned} d(Vx,Ux)^2 &= \sum_{k=1}^{n} \beta^k (d(T_k x, Ux)^2 - d(T_k x, Vx)^2) + \beta^{n+1}(d(T_{n+1}x, Ux)^2 - d(T_{n+1}x, Vx)^2) \\ &= \sum_{k=1}^{n} \beta^k (d(T_k x, Ux)^2 - d(T_k x, Vx)^2) + \left(1 - \sum_{k=1}^{n} \beta^k\right)\left(d(T_{n+1}x, U_n x)^2 - d(T_{n+1}x, U_{n+1}x)^2\right) \\ &= \sum_{k=1}^{n} \beta^k (d(T_k x, Ux)^2 - d(T_k x, Vx)^2) + \sum_{k=1}^{n} (\alpha^k - \beta^k)\left(d(T_{n+1}x, Ux)^2 - d(T_{n+1}x, Vx)^2\right) \\ &\le \sum_{k=1}^{n} \beta^k (d(T_k x, Ux)^2 - d(T_k x, Vx)^2) + \sum_{k=1}^{n} \left|\beta^k - \alpha^k\right| \left|d(T_{n+1}x, Ux)^2 - d(T_{n+1}x, Vx)^2\right| \end{aligned}$$

and hence

$$d(Ux,Vx)^2 \le \sum_{k=1}^{n} \beta^k (d(T_k x, Ux)^2 - d(T_k x, Vx)^2) + \sum_{k=1}^{n} \left|\beta^k - \alpha^k\right| \left|d(T_{n+1}x, Ux)^2 - d(T_{n+1}x, Vx)^2\right|. \tag{2}$$

Adding (1) and (2), we get

$$\begin{aligned} 2d(Ux,Vx)^2 &\le \sum_{k=1}^{n} \alpha^k (d(T_k x, Vx)^2 - d(T_k x, Ux)^2) + \sum_{k=1}^{n} \beta^k (d(T_k x, Ux)^2 - d(T_k x, Vx)^2) \\ &\quad + \sum_{k=1}^{n} \left|\beta^k - \alpha^k\right| \left|d(T_{n+1}x, Ux)^2 - d(T_{n+1}x, Vx)^2\right| \\ &\le \sum_{k=1}^{n} \left|\beta^k - \alpha^k\right| \left(\left|d(T_k x, Ux)^2 - d(T_k x, Vx)^2\right|\right) + \sum_{k=1}^{n} \left|\beta^k - \alpha^k\right| \left(\left|d(T_{n+1}x, Ux)^2 - d(T_{n+1}x, Vx)^2\right|\right) \\ &\le \sum_{k=1}^{n} \left|\beta^k - \alpha^k\right| (d(T_k x, Ux) + d(T_k x, Vx)) d(Ux, Vx) \\ &\quad + \sum_{k=1}^{n} \left|\beta^k - \alpha^k\right| (d(T_{n+1}x, Ux) + d(T_{n+1}x, Vx)) d(Ux, Vx). \end{aligned}$$

Dividing $2d(Ux,Vx) > 0$, we get

$$d(Ux,Vx) \le \frac{1}{2}\sum_{k=1}^{n} \left|\beta^k - \alpha^k\right| ((d(T_k x, Ux) + d(T_k x, Vx)) + \frac{1}{2}\sum_{k=1}^{n} \left|\beta^k - \alpha^k\right| (d(T_{n+1}x, Ux) + d(T_{n+1}x, Vx)).$$

Let $z \in \bigcap_{k=1}^{\infty} \mathcal{F}(T_k) \subset \bigcap_{k=1}^{n+1} \mathcal{F}(T_k) \subset \bigcap_{k=1}^{n} \mathcal{F}(T_k)$. By (a) of Theorem 2.3, mappings U and V are nonexpansive.

Then we get

$$\begin{aligned}
d(Ux, Vx) &\leq \frac{1}{2}\sum_{k=1}^{n}\left|\beta^k - \alpha^k\right|(d(T_kx, Ux) + d(T_kx, Vx)) + \frac{1}{2}\sum_{k=1}^{n}\left|\beta^k - \alpha^k\right|(d(T_{n+1}x, Ux) + d(T_{n+1}x, Vx)) \\
&\leq \frac{1}{2}\sum_{k=1}^{n}\left|\beta^k - \alpha^k\right|(d(T_kx, z) + d(z, Ux) + d(z, Vx) + d(T_{n+1}x, z)) \\
&\quad + \frac{1}{2}\sum_{k=1}^{n}\left|\beta^k - \alpha^k\right|(d(T_{n+1}x, z) + d(z, Ux) + d(z, Vx) + d(T_{n+1}x, z)) \\
&\leq 4d(x, z)\sum_{k=1}^{n}\left|\beta^k - \alpha^k\right|
\end{aligned}$$

and thus we get desired result. □

Lemma 3.2. *Let X be a Hadamard space, C a nonempty bounded subset of X, T_k a nonexpansive mapping of X into itself for $k \in \mathbb{N}$ with $\bigcap_{k=1}^{\infty}\mathcal{F}(T_k) \neq \emptyset$ and, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset [0, 1]$ such that $\sum_{k=1}^{n}\alpha_n^k = 1$ for $n \in \mathbb{N}$. Let*

$$U_nx = \operatorname*{Argmin}_{y\in X}\sum_{k=1}^{n}\alpha_n^k d(T_kx, y)^2$$

for all $x \in X$ and $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty}\sum_{k=1}^{n}\left|\alpha_{n+1}^k - \alpha_n^k\right| < \infty$, then

$$\sum_{n=1}^{\infty}\sup_{x\in C} d(U_{n+1}x, U_nx) < \infty.$$

Proof. Let $x \in C$. By Lemma 3.1, we get

$$d(U_nx, U_{n+1}x) \leq 4d(x, z)\sum_{k=1}^{n}\left|\alpha_{n+1}^k - \alpha_n^k\right| \leq 4M\sum_{k=1}^{n}\left|\alpha_{n+1}^k - \alpha_n^k\right|$$

for all $z \in \bigcap_{k=1}^{\infty}\mathcal{F}(T_k)$, where $M = \sup_{x\in C} d(x, z)$. Since $\sum_{n=1}^{\infty}\sum_{k=1}^{n}\left|\alpha_{n+1}^k - \alpha_n^k\right| < \infty$, we get

$$\sum_{n=1}^{\infty}\sup_{x\in C} d(U_nx, U_{n+1}x) < \infty.$$

Consequently, we complete the proof. □

By Lemma 3.2, we can prove the following corollary easily.

Corollary 3.3. *Let X be a Hadamard space, T_k a nonexpansive mapping of X into itself with $\bigcap_{k=1}^{\infty}\mathcal{F}(T_k) \neq \emptyset$ and, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset [0, 1]$ such that $\sum_{k=1}^{n}\alpha_n^k = 1$ for $n \in \mathbb{N}$. Let*

$$U_nx = \operatorname*{Argmin}_{y\in X}\sum_{k=1}^{n}\alpha_n^k d(T_kx, y)^2$$

for all $x \in X$ and $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty}\sum_{k=1}^{n}\left|\alpha_{n+1}^k - \alpha_n^k\right| < \infty$, then

$$\sum_{n=1}^{\infty} d(U_{n+1}x, U_nx) < \infty$$

and $\{U_nx\}$ is a Cauchy sequence for each $x \in X$.

By Corollary 3.3, there exists a limit of $\{U_n x\}$. In the following lemma, we consider the properties of it.

Lemma 3.4. *Let X be a Hadamard space, C a nonempty bounded subset of X, T_k a nonexpansive mapping of X into itself for $k \in \mathbb{N}$ with $\bigcap_{k=1}^{\infty} \mathcal{F}(T_k) \neq \emptyset$ and, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset [0,1]$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for $n \in \mathbb{N}$. Let*

$$U_n x = \operatorname*{Argmin}_{y \in X} \sum_{k=1}^{n} \alpha_n^k d(T_k x, y)^2$$

for all $x \in X$ and $n \in \mathbb{N}$. Suppose the following conditions hold:

(a) $\lim_{n\to\infty} \alpha_n^k > 0$ *for $k \in \mathbb{N}$;*
(b) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$.

Put $Ux = \lim_{n\to\infty} U_n x$ for each $x \in X$. Then, the following conditions hold:

(i) $\lim_{n\to\infty} \sup_{x\in C} d(U_n x, Ux) = 0$;
(ii) *U is nonexpansive ;*
(iii) $\mathcal{F}(U) = \bigcap_{k=1}^{\infty} \mathcal{F}(T_k)$.

Proof. (i) Let $m, n \in \mathbb{N}$ such that $n \leq m$ and $x \in X$. Then, we get

$$\begin{aligned} d(U_m x, U_n x) &\leq d(U_m x, U_{n+1} x) + d(U_{n+1} x, U_n x) \\ &\leq d(U_m x, U_{n+2} x) + d(U_{n+2} x, U_{n+1} x) + d(U_{n+1} x, U_n x) \\ &\leq \cdots \\ &\leq \sum_{l=n}^{m-1} d(U_l x, U_{l+1} x) \leq \sum_{l=n}^{\infty} d(U_l x, U_{l+1} x) \end{aligned}$$

and hence

$$d(U_m x, U_n x) \leq \sum_{l=n}^{\infty} d(U_l x, U_{l+1} x). \tag{3}$$

By (3) and Corollary 3.3, letting $m \to \infty$, we get

$$\sup_{x\in C} d(Ux, U_n x) \leq \sum_{l=n}^{\infty} \sup_{x\in C} d(U_l x, U_{l+1} x).$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} \sup_{x\in C} d(Ux, U_n x) = 0$.

(ii) Let $x, y \in X$. Since U_n is nonexpansive for $n \in \mathbb{N}$, we get

$$d(Ux, Uy) = \lim_{n\to\infty} d(U_n x, U_n y) \leq \lim_{n\to\infty} d(x, y) = d(x, y)$$

and hence U is a nonexpansive mapping of X into itself.

(iii) Let $z \in \bigcap_{k=1}^{\infty} \mathcal{F}(T_k) \subset \bigcap_{k=1}^{n} \mathcal{F}(T_k) = \mathcal{F}(U_n)$ for $n \in \mathbb{N}$. Then, we get

$$Uz = \lim_{n\to\infty} U_n z = \lim_{n\to\infty} z = z$$

and thus $z \in \mathcal{F}(U)$. On the other hand, let $z \in \mathcal{F}(U)$ and $w \in \bigcap_{k=1}^{\infty} \mathcal{F}(T_k) \subset \bigcap_{k=1}^{n} \mathcal{F}(T_k) = \mathcal{F}(U_n)$ for $n \in \mathbb{N}$. By (c) of Theorem 2.3, we get

$$\begin{aligned} \sum_{k=1}^{n} \alpha_n^k d(T_k z, U_n z)^2 &\leq \sum_{k=1}^{n} \alpha_n^k d(T_k z, U_n w)^2 - d(U_n z, U_n w)^2 \\ &= \sum_{k=1}^{n} \alpha_n^k d(T_k z, w)^2 - d(U_n z, w)^2 \\ &\leq d(z, w)^2 - d(U_n z, w)^2. \end{aligned}$$

Fix $j \in \mathbb{N}$ arbitrarily. Then, we have

$$0 \leq \alpha_n^j d(T_j z, U_n z)^2 \leq \sum_{k=1}^{n} \alpha_n^k d(T_k z, U_n z)^2 \leq d(z,w)^2 - d(U_n z, w)^2$$

By (a), letting $n \to \infty$, we get $\lim_{n\to\infty} d(T_j z, U_n z) = 0$. Then, it follows that

$$d(T_j z, z) = d(T_j z, Uz) = \lim_{n\to\infty} d(T_j z, U_n z) = 0$$

and hence $z \in \mathcal{F}(T_j)$. Since $j \in \mathbb{N}$ is arbitrary, we get $z \in \bigcap_{k=1}^{\infty} \mathcal{F}(T_k)$. Therefore we get $\mathcal{F}(U) = \bigcap_{k=1}^{\infty} \mathcal{F}(T_k)$ and complete the proof. □

The following result was mentioned in [3] without proof. For the sake of completeness, we give the proof.

Theorem 3.5. *Let X be a Hadamard space, T_k a nonexpansive mapping of X into itself for $k \in \mathbb{N}$ such that $\bigcap_{k=1}^{\infty} \mathcal{F}(T_k) \neq \emptyset$, $\{\alpha_n^k \mid n, k \in \mathbb{N}, k \leq n\} \subset [0,1]$ such that $\sum_{k=1}^{n} \alpha_n^k = 1$ for all $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset [0,1]$. Let*

$$U_n x = \operatorname*{Argmin}_{y\in X} \sum_{k=1}^{n} \alpha_n^k d(T_k x, y)^2$$

for all $x \in X$ and $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ by $u, x_1 \in X$ and

$$x_{n+1} = \delta_n u \oplus (1-\delta_n) U_n x_n$$

for each $n \in \mathbb{N}$. Suppose the following conditions hold:

(a) $\lim_{n\to\infty} \alpha_n^k > 0$ *for $k \in \mathbb{N}$ and* $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| < \infty$;
(b) $\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^{\infty} \delta_n = \infty$ *and* $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.

Then, $\{x_n\}$ is convergent to $P_{\bigcap_{k=1}^{\infty} \mathcal{F}(T_k)} u$, where $P_{\bigcap_{k=1}^{\infty} \mathcal{F}(T_k)}$ is the metric projection of X onto $\bigcap_{k=1}^{\infty} \mathcal{F}(T_k)$.

Proof. Let $z \in \bigcap_{k=1}^{\infty} \mathcal{F}(T_k) \subset \bigcap_{k=1}^{n} \mathcal{F}(T_k) = \mathcal{F}(U_n)$ for $n \in \mathbb{N}$. Then, we get

$$\begin{aligned}
d(x_{n+1}, z) &\leq \delta_n d(u,z) + (1-\delta_n) d(U_n x_n, z) \\
&\leq \delta_n d(u,z) + (1-\delta_n) d(x_n, z) \\
&\leq \max\{d(u,z), d(x_n,z)\} \\
&\leq \max\{d(u,z), d(x_1,z)\}.
\end{aligned}$$

and hence $\{x_n\}$ and $\{U_n x_n\}$ are bounded for all $n \in \mathbb{N}$. Put $M = \max\{d(u,z), d(x_1,z)\}$. Let C be a bounded subset of X including $\{x_n\}$. Then, we get

$$\begin{aligned}
d(x_{n+2}, x_{n+1}) &= d(\delta_{n+1} u \oplus (1-\delta_{n+1}) U_{n+1} x_{n+1}, \delta_n u \oplus (1-\delta_n) U_n x_n) \\
&\leq d(\delta_{n+1} u \oplus (1-\delta_{n+1}) U_{n+1} x_{n+1}, \delta_n u \oplus (1-\delta_n) U_{n+1} x_{n+1}) \\
&\quad + d(\delta_n u \oplus (1-\delta_n) U_{n+1} x_{n+1}, \delta_n u \oplus (1-\delta_n) U_n x_n) \\
&\leq |\delta_{n+1} - \delta_n| d(U_{n+1} x_{n+1}, u) + (1-\delta_n) d(U_{n+1} x_{n+1}, U_n x_n) \\
&\leq |\delta_{n+1} - \delta_n| d(U_{n+1} x_{n+1}, u) + (1-\delta_n)(d(U_{n+1} x_{n+1}, U_n x_{n+1}) + d(U_n x_{n+1}, U_n x_n)) \\
&\leq |\delta_{n+1} - \delta_n| d(U_{n+1} x_{n+1}, u) + (1-\delta_n) d(x_{n+1}, x_n) + (1-\delta_n) d(U_{n+1} x_{n+1}, U_n x_{n+1}) \\
&\leq |\delta_{n+1} - \delta_n| d(U_{n+1} x_{n+1}, u) + (1-\delta_n) d(x_{n+1}, x_n) + d(U_{n+1} x_{n+1}, U_n x_{n+1}) \\
&\leq (1-\delta_n) d(x_{n+1}, x_n) + |\delta_{n+1} - \delta_n| d(U_{n+1} x_{n+1}, u) + \sup_{x\in C} d(U_{n+1} x, U_n x) \\
&\leq (1-\delta_n) d(x_{n+1}, x_n) + 2M|\delta_{n+1} - \delta_n| + \sup_{x\in C} d(U_{n+1} x, U_n x)
\end{aligned}$$

for all $n \in \mathbb{N}$. By Lemma 3.2, we get $\sum_{n=1}^{\infty} \sup_{x \in C} d(U_n x, U_{n+1} x) < \infty$. Using Lemma 2.4, we get $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Further, we get

$$\begin{aligned} d(x_n, U_n x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, U_n x_n) &= d(x_n, x_{n+1}) + d(\delta_n u \oplus (1-\delta_n) U_n x_n, U_n x_n) \\ &= d(x_n, x_{n+1}) + \delta_n d(u, U_n x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we get $\lim_{n\to\infty} d(x_n, U_n x_n) = 0$. By Corollary 3.3, we get $\{U_n x\}$ is a Cauchy sequence for each $x \in X$. Put $Ux = \lim_{n\to\infty} U_n x$. By Lemma 3.4, U is nonexpansive and $\mathcal{F}(U) = \bigcap_{k=1}^{\infty} \mathcal{F}(T_k)$. Put

$$\gamma_n = d\left(u, P_{\bigcap_{k=1}^{\infty} \mathcal{F}(T_k)} u\right)^2 - (1-\delta_n) d(u, U_n x_n)^2.$$

We next show $\limsup_{n\to\infty} \gamma_n \leq 0$. We can take a subsequence $\{\gamma_{n_i}\}$ of $\{\gamma_n\}$ such that

$$\lim_{i\to\infty} \gamma_{n_i} = \limsup_{n\to\infty} \gamma_n.$$

Further, since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \stackrel{\Delta}{\rightharpoonup} x_0 \in X$. We get

$$0 \leq d\left(x_{n_{i_j}}, U x_{n_{i_j}}\right) \leq d\left(x_{n_{i_j}}, U_{n_{i_j}} x_{n_{i_j}}\right) + d\left(U_{n_{i_j}} x_{n_{i_j}}, U x_{n_{i_j}}\right) \leq d\left(x_{n_{i_j}}, U_{n_{i_j}} x_{n_{i_j}}\right) + \sup_{x \in C} d\left(U_{n_{i_j}} x, U x\right).$$

By (i) of Lemma 3.4, letting $j \to \infty$, we obtain $\lim_{j\to\infty} d(x_{n_{i_j}}, U x_{n_{i_j}}) = 0$. Since U is Δ-demiclosed, we have $x_0 \in \mathcal{F}(U) = \bigcap_{k=1}^{\infty} \mathcal{F}(T_k)$. It follows that

$$\begin{aligned} \left|\gamma_n - \left(d\left(u, P_{\bigcap_{k=1}^{\infty} \mathcal{F}(T_k)} u\right)^2 - d(u, x_n)^2\right)\right| &= \left|d(u, x_n)^2 - d(u, U_n x_n)^2 + \delta_n d(u, U_n x_n)^2\right| \\ &\leq \left|d(u, x_n)^2 - d(u, U_n x_n)^2\right| + \delta_n d(u, U_n x_n)^2 \\ &\leq (d(u, x_n) + d(u, U_n x_n)) d(x_n, U_n x_n) + \delta_n d(u, U_n x_n)^2 \to 0. \end{aligned}$$

By Lemma 2.2, letting $n \to \infty$, we get

$$\begin{aligned} \limsup_{n\to\infty} \gamma_n = \lim_{i\to\infty} \gamma_{n_i} = \lim_{j\to\infty} \gamma_{n_{i_j}} &= \lim_{j\to\infty} \left(d\left(u, P_{\bigcap_{k=1}^{\infty} \mathcal{F}(T_k)} u\right)^2 - d\left(u, x_{n_{i_j}}\right)^2 \right) \\ &= d\left(u, P_{\bigcap_{k=1}^{\infty} \mathcal{F}(T_k)} u\right)^2 - \lim_{j\to\infty} d\left(u, x_{n_{i_j}}\right)^2 \\ &\leq d\left(u, P_{\bigcap_{k=1}^{\infty} \mathcal{F}(T_k)} u\right)^2 - d(u, x_0)^2 \\ &\leq 0. \end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned} d\left(x_{n+1}, P_{\bigcap_{k=1}^{\infty} \mathcal{F}(T_k)} u\right)^2 &\leq \delta_n d\left(u, P_{\bigcap_{k=1}^{\infty} \mathcal{F}(T_k)} u\right)^2 + (1-\delta_n) d\left(U_n x_n, P_{\bigcap_{k=1}^{\infty} \mathcal{F}(T_k)} u\right)^2 - \delta_n (1-\delta_n) d(u, U_n x_n)^2 \\ &= (1-\delta_n) d\left(x_n, P_{\bigcap_{k=1}^{\infty} \mathcal{F}(T_k)} u\right)^2 + \delta_n \gamma_n. \end{aligned}$$

Using Lemma 2.4, we get $\lim_{n\to\infty} d\left(x_n, P_{\bigcap_{k=1}^{\infty} \mathcal{F}(T_k)} u\right) = 0$. Consequently, we get the desired result. □

4. New type of the projection method

In this section, we propose *the combining projection method of balanced type* and prove a strong convergence theorem using Theorem 3.5.

Theorem 4.1. *Let X be a Hadamard space. Let T a nonexpansive mapping of X into itself such that $\mathcal{F}(T)$ is nonempty, $\{\alpha_n \mid n \in \mathbb{N}\} \subset [0,1]$, $\{\beta_n^k \mid n,k \in \mathbb{N}, k \leq n\} \subset [0,1]$ such that $\sum_{k=1}^{n} \beta_n^k = 1$ for all $n \in \mathbb{N}$, and $\{\delta_n \mid n \in \mathbb{N}\} \subset [0,1]$. Define sequences $\{x_n\}$ and $\{y_n\}$ of X, a sequence $\{C_n\}$ of subset of X, and mappings $\{U_n\}$ by $u \in X$, $x_1 \in X$ and*

$$\begin{aligned} y_n &= \alpha_n x_n \oplus (1-\alpha_n)Tx_n; \\ C_n &= \left\{z \in X \,\middle|\, d(y_n, z) \leq d(x_n, z)\right\}; \\ U_n x_n &= \operatorname*{Argmin}_{y \in X} \sum_{k=1}^{n} \beta_n^k d\left(P_{C_k} x_n, y\right)^2; \\ x_{n+1} &= \delta_n u \oplus (1-\delta_n) U_n x_n \end{aligned}$$

for each $n \in \mathbb{N}$, where P_K is the metric projection of X onto a nonempty closed convex subset K of X. Suppose the following conditions hold:

(a) *$\{z \in X \mid d(z,v) \leq d(z,v')\}$ is convex for all $v, v' \in X$;*
(b) *$\liminf_{n\to\infty} \alpha_n < 1$;*
(c) *$\lim_{n\to\infty} \beta_n^k > 0$ for $k \in \mathbb{N}$, $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty$;*
(d) *$\lim_{n\to\infty} \delta_n = 0$, $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.*

Then, $\{x_n\}$ is convergent to $P_{\mathcal{F}(T)} u$.

Proof. Let $z \in \mathcal{F}(T)$. Since T is nonexpansive, we get

$$d(y_n, z) \leq \alpha_n d(x_n, z) + (1-\alpha_n) d(Tx_n, z) \leq d(x_n, z)$$

and hence $\mathcal{F}(T) \subset C_n$ for all $n \in \mathbb{N}$. Since C_n is a nonempty closed convex set, the metric projection P_{C_n} is well-defined for $n \in \mathbb{N}$. Then, we get

$$\bigcap_{k=1}^{\infty} \mathcal{F}(P_{C_k}) = \bigcap_{k=1}^{\infty} C_k \supset \mathcal{F}(T) \neq \emptyset.$$

Since P_{C_k} is nonexpansive for all $k \in \mathbb{N}$, we obtain U_n is nonexpansive. By Theorem 3.5, we get $x_n \to P_{\bigcap_{n=1}^{\infty} C_n} u$. Put $x_0 = P_{\bigcap_{n=1}^{\infty} C_n} u$. Since $x_0 \in \bigcap_{n=1}^{\infty} C_n$, letting $n \to \infty$, we get $y_n \to x_0$. By (b), there exists a subsequence $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$ such that $\lim_{i\to\infty} \alpha_{n_i} \in [0,1[$. Then, we get

$$d\left(x_{n_i}, Tx_{n_i}\right) = \frac{1}{1-\alpha_{n_i}} d\left(x_{n_i}, y_{n_i}\right) \leq \frac{1}{1-\alpha_{n_i}} \left(d\left(x_{n_i}, x_0\right) + d\left(x_0, y_{n_i}\right)\right).$$

Letting $i \to \infty$, we get $\lim_{i\to\infty} d\left(x_{n_i}, Tx_{n_i}\right) = 0$. Further, we get

$$d(x_0, Tx_0) \leq d(x_0, x_{n_i}) + d(x_{n_i}, Tx_{n_i}) + d(Tx_{n_i}, Tx_0)$$

and hence $x_0 \in \mathcal{F}(T)$. Therefore we get $x_n \to P_{\mathcal{F}(T)} u$ and complete the proof. □

If we consider Theorem 1.2 with $N = 1$, we obtain a convergence theorem for a single mapping. This result is a special case of Theorem 4.1.

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