



# On the growth of the modulus of the derivative of algebraic polynomials in bounded and unbounded domains with cusps

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**Abstract.** In this present work, we study the behavior of the derivatives of algebraic polynomials in the bounded and unbounded regions bounded by piecewise smooth curve with interior and exterior cusps.

## 1. Introduction and main results

Let  $\mathbb{C}$  denote the complex plane;  $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ ;  $G \subset \mathbb{C}$ ,  $0 \in G$ , be a finite Jordan domain;  $\Gamma := \partial G$ ,  $\Omega := \bar{\mathbb{C}} \setminus \bar{G}$ . For  $R > 0$  and  $t \in \mathbb{C}$ , let us set:  $\Delta(t, R) := \{w : |w - t| > R\}$  (with respect to  $\bar{\mathbb{C}}$ ). Let  $w = \Phi(z)$ ,  $\Phi : \Omega \rightarrow \Delta(0, 1)$ , be the univalent conformal mapping with normalization:  $\Phi(\infty) = \infty$ ,  $\lim_{z \rightarrow \infty} \frac{\Phi(z)}{z} > 0$ . For any  $R > 1$ , we denote by  $\Gamma_R := \{z : |\Phi(z)| = R\}$ , exterior level curve for the  $\Gamma$  and let  $G_R := \text{int}\Gamma_R$ ,  $\Omega_R := \text{ext}\Gamma_R$ .

Denote by  $\wp_n$  the class of algebraic polynomials  $P_n(z)$ ,  $\deg P_n \leq n$ ,  $n \in \mathbb{N}$ .

For the Jordan domain  $G$  and a given weight function  $h(z)$ , we introduce:

$$\|P_n\|_{A_p(h, G)} := \left( \iint_G h(z) |P_n(z)|^p d\sigma_z \right)^{1/p}, \quad 0 < p < \infty, \quad (1)$$
$$\|P_n\|_{A_\infty(1, G)} := \max_{z \in \bar{G}} |P_n(z)|, \quad p = \infty,$$

where  $\sigma$  is the two-dimensional Lebesgue measure, and when  $\Gamma$  is rectifiable:

$$\|P_n\|_p := \|P_n\|_{\mathcal{L}_p(h, \Gamma)} := \left( \int_\Gamma h(z) |P_n(z)|^p |dz| \right)^{1/p}, \quad 0 < p < \infty, \quad (2)$$
$$\|P_n\|_\infty := \|P_n\|_{\mathcal{L}_\infty(1, \Gamma)} := \max_{z \in \Gamma} |P_n(z)|; \quad \mathcal{L}_p(1, \Gamma) := \mathcal{L}_p(\Gamma).$$

In the theory of approximation of functions by algebraic polynomials, one of the important auxiliary results is the so-called Bernstein-Walsh lemma [41], which states that for any  $P_n \in \wp_n$

$$|P_n(z)| \leq |\Phi(z)|^n \|P_n\|_{C(\bar{G})}, \quad z \in \Omega \quad (3)$$

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is true. Hence, for the points  $z \in \overline{G}_R$  from (3), we have:

$$\|P_n\|_{C(\overline{G}_R)} \leq R^n \|P_n\|_{C(\overline{G})}. \quad (4)$$

This implies that the uniform norm (C–norm) of the polynomial  $P_n$ , over the domain  $G$  increases by at most a constant when the domain  $G$  is extended to the domain  $G_{1+(1/n)}$ .

Further, in 1937, in [31] it was proved that:

$$\|P_n\|_{\mathcal{L}_p(\Gamma_R)} \leq R^{n+\frac{1}{p}} \|P_n\|_{\mathcal{L}_p(\Gamma)}, \quad p > 0. \quad (5)$$

This shows that if the C–norm of the polynomial  $P_n$  is replaced by the " $\mathcal{L}_p(\Gamma)$ –norm" (2), then this fact remains valid.

Let us give a weight analogue of (5). For this, we define the weight function  $h(z) \neq 1$  to be used throughout the work as follows.

Let  $\{z_j\}_{j=1}^m \in \Gamma$  be a fixed system of distinct points. The generalized Jacobi weight function  $h(z)$  is defined as follows:

$$h(z) = h_0(z) \prod_{j=1}^m |z - z_j|^{\gamma_j}, \quad z \in G_{R_0}, \quad R_0 > 1, \quad (6)$$

where  $\gamma_j > -1$ , for all  $j = 1, 2, \dots, m$ , and  $h_0$  is uniformly separated from zero in  $G_{R_0}$ , i.e., there exists a constant  $c_1(G) > 0$  such that for all  $z \in G_{R_0}$ ,  $h_0(z) \geq c_1(G) > 0$ .

The weighted analog of the inequality (5) has been proved in [11, Lemma 2.4] for weight function  $h(z)$  as defined in (6) as follows:

$$\|P_n\|_{\mathcal{L}_p(h, \Gamma_R)} \leq R^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h, \Gamma)}, \quad \gamma^* = \max\{0; \gamma_j : 1 \leq j \leq m\}. \quad (7)$$

The analog of the estimates (3) and (7) for the " $A_p(h, G)$ –norm" for the domains with quasiconformal boundary (see: Definition 2.1) and the same weight function (6) with  $\gamma_j > -2$ ,  $j = 1, 2, \dots, m$ , was given in [2] as follows:

$$\|P_n\|_{A_p(h, G_R)} \leq c_2 R^{n+\frac{1}{p}} \|P_n\|_{A_p(h, G)}, \quad p > 0,$$

where  $R^* := 1 + c_3(R - 1)$  and  $c_3 > 0$ ,  $c_2 := c_2(G, p, c_3) > 0$ , are constants independent from  $n$  and  $R$ . Further, for arbitrary Jordan domain  $G$ , any  $P_n \in \wp_n$ ,  $R_1 = 1 + \frac{1}{n}$ , in [6, Theorem 1.1] it was obtained that

$$\|P_n\|_{A_p(G_R)} \leq cR^{n+\frac{2}{p}} \|P_n\|_{A_p(G_{R_1})}, \quad p > 0,$$

is true for arbitrary  $R > R_1 = 1 + \frac{1}{n}$ , where  $c = \left(\frac{2}{p-1}\right)^{\frac{1}{p}} \left[1 + O\left(\frac{1}{n}\right)\right]$ ,  $n \rightarrow \infty$ , is asymptotically exact constant.

In [39] N. Stylianopoulos showed that if the curve  $\Gamma$  is rectifiable and quasiconformal, then there exists a constant  $C = C(\Gamma) > 0$  depending only on  $\Gamma$  such that

$$|P_n(z)| \leq C \frac{\sqrt{n}}{d(z, \Gamma)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega,$$

where  $d(z, \Gamma) := \inf\{|\zeta - z| : \zeta \in \Gamma\}$ , holds for every  $P_n \in \wp_n$ . In this paper, firstly we study the following type estimation for the derivatives  $|P'_n(z)|$ :

In this work, our goal is to study the behavior of the  $|P'_n(z)|$  on the whole complex plane. To do this, we first find estimates for  $|P'_n(z)|$  in an unbounded domain  $\Omega$  as the following type:

$$|P'_n(z)| \leq \eta_n \|P_n\|_p, \quad z \in \Omega; \quad (8)$$

and, on a bounded domain  $\bar{G}$  as the following type:

$$\|P'_n\|_\infty \leq \mu_n \|P_n\|_p, \tag{9}$$

where  $\eta_n := \eta_n(\Gamma, h, p, z) \rightarrow \infty$  and  $\mu_n := \mu_n(\Gamma, h, p) \rightarrow \infty$  as  $n \rightarrow \infty$ , are constants depending on the properties of the  $\Gamma$  and  $h$ .

Estimates analogously to (8) and (9) for arbitrary  $P_n \in \wp_n$ , a different weight function  $h$  and for some norms were obtained in [7, 17, 18, 20, 21, 30, 33, 36, 39] for an unbounded domain, and in [2, 5, 16, 19, 24–28, 30, 32–35, 37, 40] (also reference therein and others) for a bounded domain.

The objects of study in this work will be piecewise smooth regions with interior and exterior zero angles. For this, we give some definitions and notations that will be used later in the text.

Let  $S$  be a rectifiable Jordan curve or an arc and  $z = z(s)$ ,  $s \in [0, |S|]$ ,  $|S| := \text{mes } S$  (linear measure of  $S$ ), denote the natural representation of  $S$ .

**Definition 1.1.** We say that a Jordan curve or an arc  $S \in C_\theta$ , if  $S$  has a continuous tangent  $\theta(z) := \theta(z(s))$  at every point  $z(s)$ .

Throughout this work we denote by  $c, c_0, c_1, c_2, \dots$  are positive and  $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$  are sufficiently small positive constants (in general, different in different relations), which depend on  $G$  in general. Let for any  $j = 1, 2, \dots$  and sufficiently small  $\varepsilon_1 > 0$ ,  $f_j : [0, \varepsilon_1] \rightarrow \mathbb{R}$  denoted the twice differentiable functions, such that  $f_j(0) = 0$ ,  $f_j^{(k)}(x) > 0$ ,  $x > 0$  and  $k = 0, 1, 2$ . For any  $k \geq 0$  and  $m > k$ , notations  $j = \overline{k, m}$  denotes  $j = k, k+1, \dots, m$ .

**Definition 1.2.** We say that a Jordan domain  $G \in C_\theta(\lambda_i; f_j)$ ,  $0 < \lambda_i \leq 2$ ,  $i = \overline{1, m_1}$ ,  $f_j = f_j(x)$ ,  $j = \overline{m_1 + 1, m}$ , if  $\Gamma = \partial G$  consists of a union of finite number of  $C_\theta$ -arcs  $\{\Gamma_j\}_{j=0}^m$ , connecting at the points  $\{z_j\}_{j=0}^m \in \Gamma$  such that  $\Gamma$  is locally smooth at  $z_0 \in \Gamma \setminus \{z_j\}_{j=1}^m$  and:

a) for every  $z_i \in \Gamma$ ,  $i = \overline{1, m_1}$ ,  $m_1 \leq m$ , the domain  $G$  has exterior (with respect to  $\bar{G}$ ) angles  $\lambda_i \pi$ ,  $0 < \lambda_i \leq 2$ , at the corner  $z_i$ ;

b) for every  $z_j \in \Gamma$ ,  $j = \overline{m_1 + 1, m}$ , in the local co-ordinate system  $(x, y)$  with origin at  $z_j$  the following conditions are satisfied

$$b_1) \{z = x + iy : |z| < \varepsilon_1, c_1 f_j(x) \leq y \leq c_2 f_j(x), 0 \leq x \leq \varepsilon_1\} \subset \bar{\Omega},$$

$$b_2) \{z = x + iy : |z| < \varepsilon_1, |y| \geq \varepsilon_2 x, 0 \leq x \leq \varepsilon_1\} \subset \bar{G},$$

for some constants  $-\infty < c_1 < c_2 < +\infty$ ,  $0 < \varepsilon_i < 1$ ,  $i = 1, 2$ .

Thus, Definition 1.2 show that each domain  $G \in C_\theta(\lambda_i; f_j)$  may have exterior non zero angles with opening  $\lambda_i \pi$ ,  $0 < \lambda_i < 2$ , interior zero angles for  $\lambda_i = 2$  at the points  $z_i \in \Gamma$ ,  $i = \overline{1, m_1}$ , and exterior zero angles at the points  $z_j \in \Gamma$ ,  $j = \overline{m_1 + 1, m}$ , at which the boundary arcs are touching with  $f_j(x)$ -speed. If  $m = 0$ , then the domain  $G$  does not have such angles, and in this case we will write:  $G \in C_\theta$ ; if  $m_1 = m \geq 1$ , then  $G$  has only  $\lambda_i \pi$ ,  $0 < \lambda_i \leq 2$ ,  $i = \overline{1, m_1}$ , exterior angles (when  $\lambda_i = 2$  –interior zero angles) and in this case we will write:  $G \in C_\theta(\lambda_i; 0)$ ; if  $m_1 = 0$  and  $m \geq 1$ , then  $G$  has only exterior zero angles and in this case we will write:  $G \in C_\theta(1; f_j)$ .

We assume that the points  $\{z_j\}_{j=1}^m \in \Gamma$  defined in (6) and Definition 1.2 are the same. Also we assume that these points on the curve  $\Gamma = \partial G$  are located in the positive direction such that,  $G$  has  $\lambda_j \pi$ ,  $0 < \lambda_j \leq 2$ ,  $j = \overline{1, m_1}$ , exterior angles (when  $\lambda_j = 2$  –interior zero angles) at the points  $\{z_j\}_{j=1}^{m_1}$ ,  $m_1 \leq m$ , and has exterior zero angles on the points  $\{z_j\}_{j=m_1+1}^m$  and  $w_j := \Phi(z_j)$ .

Before stating our main results, we introduce some notation. For clarity of results, we will consider  $m_1 = 1$ ,  $m = 2$ , i.e. the pieewise smooth curve  $\Gamma$  has two singular points  $z_1 \in \Gamma$  and  $z_2 \in \Gamma$ . The reasoning for the cases  $m_1 > 1$ ,  $m > 2$  is carried out in exactly the same way. For  $0 < \delta_j < \delta_0 := \frac{1}{4} \min \{|z_i - z_j| : i, j = 1, 2, \dots, \Gamma, i \neq j\}$ , let  $\Omega(z_j, \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$ ;  $\delta := \min_{1 \leq j \leq \Gamma} \delta_j$ . For  $\Gamma = \partial G$ , we put:

$U_\infty(\Gamma, \delta) := \bigcup_{\zeta \in \Gamma} U(\zeta, \delta)$  - infinite open cover of the curve  $\Gamma$ ;  $U_N(\Gamma, \delta) := \bigcup_{j=1}^N U_j(\Gamma, \delta) \subset U_\infty(\Gamma, \delta)$  - finite open cover of the curve  $\Gamma$ . For any  $t \geq 1$  we put:  $\Omega_t(\delta) := \Omega(\Gamma_t, \delta) := \Omega_t \cap U_N(\Gamma_t, \delta)$ ,  $\widehat{\Omega}_t := \Omega_t \setminus \Omega_t(\delta)$ .

Now we proceed to the formulation of new results. Throughout this work we will assume that  $p > 1$  and we put:

$$\begin{aligned} \widetilde{\lambda}_k &:= \begin{cases} \max\{1, \lambda_k\} + \varepsilon, & 0 < \lambda_k < 2, \\ 2, & \lambda_k = 2, \end{cases} \quad k = 1, 2. ; \quad \widetilde{\lambda} := \max\{\widetilde{\lambda}_1, \widetilde{\lambda}_2\}; \quad \widetilde{\gamma} := \max\{\gamma_1, \gamma_2\}, \quad \gamma_k^* := \max\{0, \gamma_k\}, \\ \widetilde{\gamma}^* &:= \max\{\gamma_1^*, \gamma_2^*\}; \quad \alpha_* := \min\{\alpha_1, \alpha_2\}; \quad \alpha^* := \max\{\alpha_1, \alpha_2\}. \end{aligned}$$

**Theorem 1.3.** Assume that  $G \in C_\theta(\lambda_1; c\alpha^{1+\alpha_2})$  for some  $0 < \lambda_1 \leq 2, \alpha_2 > 0$  and  $h(z)$  is defined by (6) for  $j = 2$ . Then, for any  $P_n \in \wp_n, n \in \mathbb{N}$ ,

$$|P'_n(z)| \leq c_1 \frac{|\Phi^{2(n+1)}(z)|}{d(z, \Gamma)} \|P_n\|_p \begin{cases} D_n^{(1)} + D_n^{(2)}, & z \in \Omega_R(\delta), \\ E_n, & z \in \widehat{\Omega}_R(\delta), \end{cases} \tag{10}$$

is true, where  $c_1 = c_1(\Gamma, p) > 0$  is a constant independent of  $z$  and  $n$ ,  $c$  is defined as in Definition 1.2;

$$\begin{aligned} D_n^{(1)} &:= \begin{cases} n^{\frac{(\gamma_1+1)\widetilde{\lambda}_1}{p}}, & \gamma_1 \geq \frac{\gamma_2+1}{(1+\alpha_2)\lambda_1} - 1, \gamma_2 \geq (1+\alpha_2)\widetilde{\lambda}_1 - 1, \\ n^{\frac{(\gamma_2+1)+\varepsilon}{p(1+\alpha_2)}}, & 0 < \gamma_1 < \frac{\gamma_2+1}{(1+\alpha_2)\lambda_1} - 1, \gamma_2 \geq (1+\alpha_2)\widetilde{\lambda}_1 - 1, \\ n^{\frac{1}{p}\widetilde{\lambda}_1}, & -1 < \gamma_1 < 0, -1 < \gamma_2 < (1+\alpha_2)\widetilde{\lambda}_1 - 1; \end{cases} \\ D_n^{(2)} &:= \begin{cases} n^{\frac{(\gamma_1+1)\widetilde{\lambda}_1}{p}}, & \gamma_1 > p-1 + \frac{\gamma_2+1-p}{(1+\alpha_2)\lambda_1}, \gamma_2 \geq p-1, \\ n^{\widetilde{\lambda}_1 + \frac{(\gamma_2+1-p)}{p} - 1 + \varepsilon}, & p-1 < \gamma_1 \leq p-1 + \frac{\gamma_2+1-p}{(1+\alpha_2)\lambda_1}, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \gamma_1, \gamma_2 = p-1, \\ 1, & -1 < \gamma_1, \gamma_2 < p-1; \end{cases} \\ E_n &:= \begin{cases} n^{\frac{(\gamma_1+1)\widetilde{\lambda}_1}{p} - 1} + n^{\frac{(\gamma_2+1-p)}{p} - 1 + \varepsilon}, & \gamma_1, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \begin{cases} \text{if } \gamma_1 = p-1, -1 < \gamma_2 \leq p-1, \\ \text{or } -1 < \gamma_1 \leq p-1, \gamma_2 = p-1, \end{cases} \\ 1, & -1 < \gamma_1, \gamma_2 < p-1. \end{cases} \end{aligned}$$

Let us consider separate cases when the domain  $G$  has only one type of singular points on the boundary  $\Gamma$ : an exterior non-zero (interior zero) angle or an exterior zero angle. In these cases, from Theorem 1.3 we get the following:

**Corollary 1.4.** Assume that  $G \in C_\theta(\lambda_1; \lambda_2)$ , for some  $0 < \lambda_1, \lambda_2 \leq 2$  and  $h(z)$  is defined by (6) for  $j = 2$ . Then, for any  $P_n \in \wp_n, n \in \mathbb{N}$ ,

$$|P'_n(z)| \leq c_2 \frac{|\Phi^{2(n+1)}(z)|}{d(z, \Gamma)} \|P_n\|_p \begin{cases} D_{n,1}^{(1)} + D_{n,2}^{(1)}, & z \in \Omega_R(\delta), \\ E_{n,1}, & z \in \widehat{\Omega}_R(\delta), \end{cases} \tag{11}$$

is true, where  $c_2 = c_2(\Gamma, p) > 0$  is a constant independent of  $z$  and  $n$ ;

$$\begin{aligned} D_{n,1}^{(1)} &:= n^{\frac{(\widetilde{\gamma}+1)\widetilde{\lambda}}{p}}; \quad D_{n,2}^{(1)} := \begin{cases} n^{\frac{\widetilde{\gamma}+1}{p}\widetilde{\lambda}}, & \gamma_1, \gamma_2 > p-1, \\ n^{\widetilde{\lambda}} (\ln n)^{1-\frac{1}{p}}, & \begin{cases} \text{if } \gamma_1 = p-1, -1 < \gamma_2 \leq p-1, \\ \text{or } -1 < \gamma_1 \leq p-1, \gamma_2 = p-1, \end{cases} \\ n^{\widetilde{\lambda}}, & -1 < \gamma_1, \gamma_2 < p-1; \end{cases} \\ E_{n,1} &:= \begin{cases} n^{\frac{(\widetilde{\gamma}+1)\widetilde{\lambda}}{p} - 1}, & \gamma_1, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \begin{cases} \text{if } \gamma_1 = p-1, -1 < \gamma_2 \leq p-1, \\ \text{or } -1 < \gamma_1 \leq p-1, \gamma_2 = p-1, \end{cases} \\ 1, & -1 < \gamma_1, \gamma_2 < p-1. \end{cases} \end{aligned}$$

**Corollary 1.5.** Assume that  $G \in C_\theta(cx^{1+\alpha_1}; cx^{1+\alpha_2})$ , for some  $\alpha_1, \alpha_2 > 0$  and  $h(z)$  is defined by (6) for  $j = 2$ . Then, for any  $P_n \in \wp_n$ ,  $n \in \mathbb{N}$ ,

$$|P'_n(z)| \leq c_3 \frac{|\Phi^{2(n+1)}(z)|}{d(z, \Gamma)} \|P_n\|_p \begin{cases} D_{n,1}^{(2)} + D_{n,2}^{(2)} & z \in \Omega_R(\delta), \\ E_{n,2} & z \in \widehat{\Omega}_R(\delta), \end{cases} \tag{12}$$

is true, where  $c_3 = c_3(\Gamma, p) > 0$  is a constant independent of  $z$  and  $n$ ,  $c$  is defined as in Definition 1.2;

$$D_{n,1}^{(2)} := n^{\frac{(\overline{\gamma}_2^*+1)}{p(1+\alpha_*)}+\varepsilon}, \quad D_{n,2}^{(2)} := \begin{cases} n^{\frac{\overline{\gamma}+1}{p(1+\alpha_*)}+\varepsilon}, & \gamma_1, \gamma_2 > p-1, \\ n^{\frac{1}{1+\alpha_*}+\varepsilon} (\ln n)^{1-\frac{1}{p}}, & \text{if } \gamma_1 = p-1, -1 < \gamma_2 \leq p-1, \\ & \text{or } -1 < \gamma_1 \leq p-1, \gamma_2 = p-1, \\ n^{\frac{1}{1+\alpha_*}+\varepsilon}, & -1 < \gamma_1, \gamma_2 < p-1; \end{cases}$$

$$E_{n,2} := \begin{cases} n^{\left(\frac{\overline{\gamma}+1}{p}-1\right)\frac{1}{1+\alpha_*}+\varepsilon}, & \gamma_1, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \begin{cases} \text{if } \gamma_1 = p-1, -1 < \gamma_2 \leq p-1, \\ \text{or } -1 < \gamma_1 \leq p-1, \gamma_2 = p-1, \end{cases} \\ 1, & -1 < \gamma_1, \gamma_2 < p-1. \end{cases}$$

Now we give a theorem that gives an estimate for  $|P'_n(z)|$  on  $\overline{G}$ .

**Theorem 1.6.** Assume that  $G \in C_\theta(\lambda_1; cx^{1+\alpha_2})$ , for some  $0 < \lambda_1 \leq 2$ ,  $\alpha_2 > 0$  and  $h(z)$  is defined by (6) for  $j = 2$ . Then, for any  $P_n \in \wp_n$ ,  $n \in \mathbb{N}$ ,

$$\|P'_n\|_\infty \leq c_4 \|P_n\|_p [F_{n,1} + F_{n,2}], \tag{13}$$

is true, where  $c_4 = c_4(\Gamma, p) > 0$  is a constant independent of  $z$  and  $n$ ,  $c$  is defined as in Definition 1.2;

$$F_{n,1} := n^{\left(\frac{\overline{\gamma}_1^*+1}{p}+1\right)\overline{\lambda}_1}; \quad F_{n,2} := \begin{cases} n^{\frac{\overline{\gamma}_2^*+1}{p(1+\alpha_2)}+\frac{\alpha_2}{p(1+\alpha_2)}+\overline{\lambda}_1+\varepsilon}, & 1 < p < 2 + \frac{\overline{\gamma}_2^*}{1+\alpha_2}, \\ n^{\overline{\lambda}_1+1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 2 + \frac{\overline{\gamma}_2^*}{1+\alpha_2}, \\ n^{\overline{\lambda}_1+1-\frac{1}{p}}, & p > 2 + \frac{\overline{\gamma}_2^*}{1+\alpha_2}. \end{cases}$$

Accordingly, as in Theorem 1.3, from here we will write out the consequences related to individual cases.

**Corollary 1.7.** Assume that  $G \in C_\theta(\lambda_1; \lambda_2)$ , for some  $0 < \lambda_1, \lambda_2 \leq 2$  and  $h(z)$  is defined by (6) for  $j = 2$ . Then, for any  $P_n \in \wp_n$ ,  $n \in \mathbb{N}$ ,

$$\|P'_n\|_\infty \leq c_5 n^{\left(\frac{\overline{\gamma}^*+1}{p}+1\right)\overline{\lambda}} \|P_n\|_p, \tag{14}$$

is true, where  $c_5 = c_5(\Gamma, p) > 0$  is a constant independent of  $z$  and  $n$ .

**Corollary 1.8.** Assume that  $G \in C_\theta(cx^{1+\alpha_1}; cx^{1+\alpha_2})$ , for some  $\alpha_1, \alpha_2 > 0$  and  $h(z)$  is defined by (6) for  $j = 2$ . Then, for any  $P_n \in \wp_n$ ,  $n \in \mathbb{N}$ ,

$$\|P'_n\|_\infty \leq c_6 \|P_n\|_p \begin{cases} n^{\frac{\overline{\gamma}^*+1}{p(1+\alpha_*)}+\frac{\alpha_*^*}{p(1+\alpha_*)}+\varepsilon}, & 1 < p < 2 + \frac{\overline{\gamma}^*}{1+\alpha_2}, \\ n^{2-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 2 + \frac{\overline{\gamma}_1^*}{1+\alpha_2} = 2 + \frac{\overline{\gamma}_2^*}{1+\alpha_2}, \\ n^{2-\frac{1}{p}}, & p > 2 + \frac{\overline{\gamma}^*}{1+\alpha_2}, \end{cases} \tag{15}$$

is true, where  $c_6 = c_6(\Gamma, p) > 0$  is a constant independent of  $z$  and  $n$ ,  $c$  is defined as in Definition 1.2.

**Remark 1.9.** The estimates (13) and, consequently, (14), (15) are sharp.

Therefore, combining Theorems 1.3, 1.6, we obtain an estimation of the growth of  $|P'_n(z)|$  on the whole complex plane:

**Theorem 1.10.** Assume that  $G \in C_\theta(\lambda_1; cx^{1+\alpha_2})$ , for some  $0 < \lambda_1 \leq 2, \alpha_2 > 0$  and  $h(z)$  is defined as in (6). Then, for any  $P_n \in \wp_n, n \in \mathbb{N}$ ,

$$|P'_n(z)| \leq c_7 \|P_n\|_p \begin{cases} \frac{|\Phi^{2(n+1)}(z)|}{d(z,\Gamma)} \begin{cases} D_n^{(1)} + D_n^{(2)}, & z \in \Omega_R(\delta), \\ E_n, & z \in \widehat{\Omega}_R(\delta), \\ F_{n,1} + F_{n,2}, & z \in \overline{G}_R, \end{cases} \end{cases}$$

is true, where  $c_7 = c_7(G, p) > 0$  is the constant independent of  $z$  and  $n$ ;  $c$  is defined as in Definition 1.2;  $D_n^{(k)}, E_n; F_{n,k}, k = 1, 2$ , are defined as in (10) and (13), respectively.

Also combining the Corollaries 1.4-1.7 and 1.5-1.8, we can give estimates separately for the cases  $G \in C_\theta(\lambda_1; \lambda_2), 0 < \lambda_1, \lambda_2 \leq 2$ , and  $G \in C_\theta(cx^{1+\alpha_1}; cx^{1+\alpha_2}), \alpha_1, \alpha_2 > 0$ , in the whole plane.

**2. Some auxiliary results**

For the nonnegative functions  $a > 0$  and  $b > 0$ , we shall use the notations “ $a \leq b$ ” (order inequality), if  $a \leq cb$  and “ $a \asymp b$ ” are equivalent to  $c_1a \leq b \leq c_2a$  for some constants  $c, c_1, c_2$  (independent of  $a$  and  $b$ ) respectively.

**Definition 2.1.** ([38]) The Jordan curve (or arc)  $\Gamma$  is called  $K$ -quasiconformal ( $K \geq 1$ ), if there is a  $K$ -quasiconformal mapping  $f$  of the domain  $D \supset \Gamma$  such that  $f(\Gamma)$  is a circle (or line segment). Let  $F(\Gamma)$  denote the set of all sense preserving plane homeomorphisms  $f$  of the domain  $D \supset \Gamma$  such that  $f(\Gamma)$  is a line segment (or circle) and let

$$K_\Gamma := \inf \{K(f) : f \in F(\Gamma)\},$$

where  $K(f)$  is the maximal dilatation of a such mapping  $f$ . Then  $\Gamma$  is a quasiconformal curve, if  $K_\Gamma < \infty$ , and  $\Gamma$  is a  $K$ -quasiconformal curve, if  $K_\Gamma \leq K$ .

According to [29], there exists quasiconformal curves which are not rectifiable. Also, according to the “three-point” criterion [22, p.100], every piecewise smooth curve (without cusps) is is quasiconformal.

On the other hand, from [38], we have:

**Corollary 2.2.** If  $S \in C_\theta$ , then  $S$  is  $(1 + \varepsilon)$ -quasiconformal for arbitrary small  $\varepsilon > 0$ .

**Corollary 2.3.** If  $S$  is an analytic curve or an arc, then  $S$  is 1-quasiconformal.

**Lemma 2.4.** ([1]) Let  $\Gamma$  be a  $K$ -quasiconformal curve,  $z_1 \in \Gamma, z_2, z_3 \in \Omega \cap \{z : |z - z_1| \leq d(z_1, \Gamma_{r_0})\}; w_j = \Phi(z_j), (z_2, z_3 \in G \cap \{z : |z - z_1| \leq d(z_1, \Gamma_{R_0})\}; w_j = \varphi(z_j)), j = 1, 2, 3$ . Then,

a) The statements  $|z_1 - z_2| \leq |z_1 - z_3|$  and  $|w_1 - w_2| \leq |w_1 - w_3|$  are equivalent.

So  $|z_1 - z_2| \asymp |z_1 - z_3|$  and  $|w_1 - w_2| \asymp |w_1 - w_3|$  also are equivalent;

b) If  $|z_1 - z_2| \leq |z_1 - z_3|$ , then

$$\left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{K-2} \leq \left| \frac{z_1 - z_3}{z_1 - z_2} \right| \leq \left| \frac{w_1 - w_3}{w_1 - w_2} \right|^{K^2},$$

where  $0 < r_0 < 1, R_0 := r_0^{-1}$  are constants, depending on  $G$ .

**Corollary 2.5.** Under the assumptions of Lemma 2.4, if  $z_3 \in \Gamma_{R_0}$ , then

$$|w_1 - w_2|^{K^2} \leq |z_1 - z_2| \leq |w_1 - w_2|^{K^{-2}}.$$

**Corollary 2.6.** If  $\Gamma \in C_\theta$ , then

$$|w_1 - w_2|^{1+\varepsilon} \leq |z_1 - z_2| \leq |w_1 - w_2|^{1-\varepsilon},$$

for all  $\varepsilon > 0$ .

Recall that for  $0 < \delta_j < \delta_0 := \frac{1}{4} \min \{|z_i - z_j| : i, j = 1, 2, \dots, m, i \neq j\}$ , we put  $\Omega(z_j, \delta_j) := \Omega \cap \{z : |z - z_j| \leq \delta_j\}$ ;  $\delta := \min_{1 \leq j \leq m} \delta_j$ ,  $\Omega(\delta) := \bigcup_{j=1}^m \Omega(z_j, \delta)$ ,  $\widehat{\Omega} := \Omega \setminus \Omega(\delta)$ . Additionally, let  $\Delta_j := \Phi(\Omega(z_j, \delta))$ ,  $\Delta(\delta) := \bigcup_{j=1}^m \Phi(\Omega(z_j, \delta))$ ,  $\widehat{\Delta}(\delta) := \Delta \setminus \Delta(\delta)$ . Let  $w_j := \Phi(z_j)$  and for  $\varphi_j := \arg w_j$ ,  $j = 1, 2, \dots, m$ , we put  $\Delta'_j := \{t = Re^{i\theta} : R > 1, \frac{\varphi_{j-1} + \varphi_j}{2} \leq \theta < \frac{\varphi_j + \varphi_{j+1}}{2}\}$ , where  $\varphi_0 \equiv \varphi_m$ ,  $\varphi_1 \equiv \varphi_{m+1}$ ;  $\Omega_j := \Psi(\Delta'_j)$ ,  $\Gamma^j := \Gamma \cap \overline{\Omega}_j$ ,  $i = 1, 2, \dots, m$ . Clearly,  $\Omega = \bigcup_{j=1}^m \Omega_j \cdot \Gamma_R^j := \Gamma_R \cap \overline{\Omega}^j$ .  $F^i := \Phi(\Gamma^i) = \overline{\Delta}_i \cap \{\tau : |\tau| = 1\}$ ,  $F_R^i := \Phi(\Gamma_R^i) = \overline{\Delta}_i' \cap \{\tau : |\tau| = R\}$ ,  $i = \overline{1, m}$ .

**Lemma 2.7.** ([9]) Let  $G \in C_\theta(\lambda_1, \dots, \lambda_m)$ ,  $0 < \lambda_j < 2$ ,  $j = 1, 2, \dots, m$ . Then

i) for any  $w \in \Delta_j$ ,  $|w - w_j|^{\lambda_j + \varepsilon} < |\Psi(w) - \Psi(w_j)| < |w - w_j|^{\lambda_j - \varepsilon}$ ,  $|w - w_j|^{\lambda_j - 1 + \varepsilon} < |\Psi'(w)| < |w - w_j|^{\lambda_j - 1 - \varepsilon}$ ,

ii) for any  $w \in \overline{\Delta} \setminus \Delta_j$ ,  $(|w| - 1)^{1 + \varepsilon} < d(\Psi(w), \Gamma) < (|w| - 1)^{1 - \varepsilon}$ ,  $(|w| - 1)^\varepsilon < |\Psi'(w)| < (|w| - 1)^{-\varepsilon}$ .

Recall that,  $\{z_j\}_{j=1}^m$  be a fixed system of the points on  $\Gamma$  and the weight function  $h(z)$  is defined as in(6).

**Lemma 2.8.** ([9]) Let  $\Gamma$  be a rectifiable Jordan curve;  $h(z)$  as defined in (6). Then, for arbitrary  $P_n(z) \in \wp_n$ , any  $R > 1$  and  $n \in \mathbb{N}$ , we have

$$\|P_n\|_{\mathcal{L}_p(h, \Gamma_R)} \leq R^{n + \frac{1 + \gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h, \Gamma)}, \quad p > 0, \tag{16}$$

where  $\gamma^* = \max \{\gamma_k : k \leq m\}$ .

### 3. Proofs

*Proof.* [Proof of Theorem 1.3] Suppose that  $G \in C_\theta(\lambda_1; f_2)$ , for some  $0 < \lambda_1 \leq 2$ ,  $f_2(x) = cx^{1+\alpha_2}$ ,  $\alpha_2 > 0$  and  $h(z)$  is defined as in (6). Let us put:

$$T_n(z) := \frac{P_n(z)}{\Phi^{n+1}(z)}, \quad z \in \Omega. \tag{17}$$

Hence, we obtain:

$$P'_n(z) = \Phi^{n+1}(z) \left[ T'_n(z) - P_n(z) \left( \frac{1}{\Phi^{n+1}(z)} \right)' \right], \quad z \in \Omega. \tag{18}$$

By Cauchy integral representation for the unbounded domain  $\Omega$ , for  $T_n(z)$  and  $\left( \frac{1}{\Phi^{n+1}(z)} \right)'$ , respectively, we have:

$$T'_n(z) = -\frac{1}{2\pi i} \int_{\Gamma} \frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)} \frac{d\zeta}{(\zeta - z)^2}, \quad \left( \frac{1}{\Phi^{n+1}(z)} \right)' = -\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\Phi^{n+1}(\zeta)} \frac{d\zeta}{(\zeta - z)^2}, \quad z \in \Omega_R.$$

Putting them in (18), we find:

$$\begin{aligned}
 |P'_n(z)| &\leq \frac{|\Phi^{n+1}(z)|}{2\pi} \left[ \int_{\Gamma} \left| \frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)} \right| \frac{|d\zeta|}{|\zeta - z|^2} + |P_n(z)| \int_{\Gamma} \frac{|d\zeta|}{|\Phi^{n+1}(\zeta)| |\zeta - z|^2} \right] \\
 &\leq |\Phi^{n+1}(z)| \left[ \int_{\Gamma} \frac{|P_n(\zeta)| |d\zeta|}{|\zeta - z|^2} + |P_n(z)| \int_{\Gamma} \frac{|d\zeta|}{|\zeta - z|^2} \right] \\
 &\leq |\Phi^{n+1}(z)| \left[ \frac{1}{d(z, \Gamma)} \int_{\Gamma} \frac{|P_n(\zeta)| |d\zeta|}{|\zeta - z|} + |P_n(z)| \int_{\Gamma} \frac{|d\zeta|}{|\zeta - z|^2} \right],
 \end{aligned}
 \tag{19}$$

since  $|\Phi(\zeta)| = 1$ , for  $\zeta \in \Gamma$ .

Denoted the last integrals by

$$A_n(z) := \int_{\Gamma} \frac{|P_n(\zeta)| |d\zeta|}{|\zeta - z|}; \quad B_n(z) := \int_{\Gamma} \frac{|d\zeta|}{|\zeta - z|^2},
 \tag{20}$$

and start evaluating them separately.

Multiplying the numerator and denominator of the first integral by  $h^{\frac{1}{p}}(\zeta)$  and applying Hölder inequality, we obtain:

$$A_n(z) \leq \|P_n\|_p \left( \int_{\Gamma} \frac{|d\zeta|}{h^{\frac{q}{p}}(\zeta) |\zeta - z|^q} \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1.
 \tag{21}$$

Denote by  $J_n(z)$  the last integral, we get:

$$\begin{aligned}
 [J_n(z)]^q &:= \int_{\Gamma} \frac{|d\zeta|}{h^{q-1}(\zeta) |\zeta - z|^q} \\
 &\leq \int_{\Gamma^1} \frac{|d\zeta|}{|\zeta - z_1|^{(q-1)\gamma_1} |\zeta - z|^q} + \int_{\Gamma^2} \frac{|d\zeta|}{|\zeta - z_2|^{(q-1)\gamma_2} |\zeta - z|^q} \\
 &=: J_{n,1}^1(z) + J_{n,2}^2(z).
 \end{aligned}
 \tag{22}$$

To make the following calculations easier, we put:  $z_1 = -1, z_2 = 1; (-1, 1) \subset G$  and let local coordinate axis in Definition 1.2 be parallel to natural axis  $OX$  and  $OY$  in the coordinate system  $XOY$ ;  $\Gamma = \Gamma^+ \cup \Gamma^-$ , where  $\Gamma^+ := \{z \in \Gamma : \text{Im}z \geq 0\}, \Gamma^- := \{z \in \Gamma : \text{Im}z < 0\}$ ;  $w^{\pm} := \{w = e^{i\theta} : \theta = \frac{\varphi_1 \pm \varphi_2}{2}\}, z^{\pm} \in \Psi(w^{\pm})$  and let  $\Gamma_i^{\pm}(z_i, z^{\pm})$ –denoted the arcs, connected the points  $z_i$  which  $z^{\pm}$ , respectively;  $|\Gamma_i^{\pm}| := \text{mes } \Gamma_i^{\pm}(z_i, z^{\pm}), i = 1, 2$ . Let  $z_0 \in \Gamma^+$  be taken as an arbitrary point (or  $z_0 \in \Gamma^-$  subject to the chosen direction). Then, from (21) and (22), we have:

$$A_n(z) \leq \|P_n\|_p \left\{ [J_{n,1}^1(z)]^{\frac{1}{q}} + [J_{n,2}^2(z)]^{\frac{1}{q}} \right\}.
 \tag{23}$$

Let us introduce some notations:  $R = 1 + \frac{1}{n}$ ;  $d_{i,R} := d(z_i, \Gamma_R); \mathcal{F}_1^{1,\pm} := \{\zeta \in \Gamma^1 : |\zeta - z_1| < c_1 d_{1,R}\}, \mathcal{F}_2^{1,\pm} := \{\zeta \in \Gamma^1 : c_1 d_{1,R} \leq |\zeta - z_1| \leq |\Gamma_1^{\pm}|\}, \mathcal{F}_1^{2,\pm} := \{\zeta \in \Gamma^2 : |\zeta - z_2| < c_2 d_{2,R}\}, \mathcal{F}_2^{2,\pm} := \{\zeta \in \Gamma^2 : c_2 d_{2,R} \leq |\zeta - z_2| \leq |\Gamma_2^{\pm}|\}$ ;

$$Y_{n,k}^{i,\pm}(z) := \int_{\mathcal{F}_k^{i,\pm}} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i(q-1)} |\zeta - z|^q}; \quad i, k = 1, 2.$$



Then, under the these notations, (23) can be written as:

$$A_n(z) \leq \|P_n\|_p \sum_{i=1}^2 \left[ Y_{n,1}^{i,\pm}(z) + Y_{n,2}^{i,\pm}(z) \right]^{\frac{1}{q}}, \quad i = 1, 2. \tag{24}$$

Let us start to estimate the integrals  $Y_{n,k}^{i,\pm}$  for each  $i, k = 1, 2$ .

1. Let  $z \in \Omega_R(\delta)$ . Denote by

$$\begin{aligned} (\mathcal{F}_k^{i,\pm})_1 &:= \{ \zeta \in \mathcal{F}_k^{i,\pm} : |\zeta - z_i| < |\zeta - z| \}, \quad (\mathcal{F}_k^{i,\pm})_2 := \mathcal{F}_k^{i,\pm} \setminus (\mathcal{F}_k^{i,\pm})_1; \\ [Y_{n,k}^{i,\pm}(z)]_1 &:= \int_{(\mathcal{F}_k^{i,\pm})_1} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i(q-1)+q}}; \quad [Y_{n,k}^{i,\pm}(z)]_2 := \int_{(\mathcal{F}_k^{i,\pm})_2} \frac{|d\zeta|}{|\zeta - z|^{\gamma_i(q-1)+q}}. \end{aligned}$$

1.1. Let  $\gamma_1, \gamma_2 \geq 0$ . We get:

$$\begin{aligned} [Y_{n,1}^{1,\pm}(z)]_1 &\leq \int_0^{c_1 d_{1,R}} \frac{ds}{s^{\gamma_1(q-1)+q}} \leq d_{1,R}^{-(\gamma_1+1)(q-1)}; \\ [Y_{n,1}^{1,\pm}(z)]_2 &\leq d_{1,R}^{-(\gamma_1+1)+q} \cdot \text{mes}(\mathcal{F}_1^{1,\pm}) \leq d_{1,R}^{-(\gamma_1+1)(q-1)}; \\ Y_{n,1}^{1,\pm}(z) &= [Y_{n,1}^{1,\pm}(z)]_1 + [Y_{n,1}^{1,\pm}(z)]_2 \leq d_{1,R}^{-(\gamma_1+1)(q-1)}. \end{aligned} \tag{25}$$

$$\begin{aligned} [Y_{n,2}^{1,\pm}(z)]_1 &\leq \int_{c_1 d_{1,R}}^{|\Gamma_1^\pm|} \frac{ds}{s^{\gamma_1(q-1)+q}} \leq d_{1,R}^{-(\gamma_1+1)(q-1)}; \quad [Y_{n,2}^{1,\pm}(z)]_2 \leq d_{1,R}^{-(\gamma_1+1)(q-1)}; \\ Y_{n,2}^{1,\pm}(z) &= [Y_{n,2}^{1,\pm}(z)]_1 + [Y_{n,2}^{1,\pm}(z)]_2 \leq d_{1,R}^{-(\gamma_1+1)(q-1)}. \end{aligned}$$

Analogously, for the  $J_{n,2}^2(z)$  in neighborhood of the point  $z_2$ , we have:

$$\begin{aligned} [Y_{n,1}^{2,\pm}(z)]_1 &\leq \int_0^{c_2 d_{2,R}} \frac{ds}{s^{\gamma_2(q-1)+q}} \leq d_{2,R}^{-(\gamma_2+1)(q-1)}; \\ [Y_{n,1}^{2,\pm}(z)]_2 &\leq \int_{(\mathcal{F}_1^{2,\pm})_2} \frac{|d\zeta|}{|\zeta - z|^{\gamma_2(q-1)+q}} \leq d_{2,R}^{-(\gamma_2+1)+q} \cdot \text{mes}(\mathcal{F}_1^{2,\pm}) \leq d_{2,R}^{-(\gamma_2+1)(q-1)}; \\ Y_{n,1}^{2,\pm}(z) &= [Y_{n,1}^{2,\pm}(z)]_1 + [Y_{n,1}^{2,\pm}(z)]_2 \leq d_{2,R}^{-(\gamma_2+1)(q-1)}. \end{aligned} \tag{26}$$

$$\begin{aligned} [Y_{n,2}^{2,\pm}(z)]_1 &\leq \int_{(\mathcal{F}_2^{2,\pm})_1} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2(q-1)+q}} \leq \int_{c_2 d_{2,R}}^{|\Gamma_2^\pm|} \frac{ds}{s^{\gamma_2(q-1)+q}} \leq d_{2,R}^{-(\gamma_2+1)(q-1)}; \\ [Y_{n,2}^{2,\pm}(z)]_2 &\leq \int_{(\mathcal{F}_2^{2,\pm})_2} \frac{|d\zeta|}{|\zeta - z|^{\gamma_2(q-1)+q}} \leq \int_{c_2 d_{2,R}}^{|\Gamma_2^\pm|} \frac{ds}{s^{\gamma_2(q-1)+q}} \leq d_{2,R}^{-(\gamma_2+1)(q-1)}; \\ Y_{n,2}^{2,\pm}(z) &= [Y_{n,2}^{2,\pm}(z)]_1 + [Y_{n,2}^{2,\pm}(z)]_2 \leq d_{2,R}^{-(\gamma_2+1)(q-1)}. \end{aligned}$$

Let  $\gamma_1, \gamma_2 < 0$ . Then, analogously to the (25) and (26), we obtain:

$$Y_{n,1}^{1,\pm}(z) = \int_{\mathcal{F}_1^{1,\pm}} \frac{|\zeta - z_1|^{-\gamma_1(q-1)} |d\zeta|}{|\zeta - z|^\beta} \leq d_{1,R}^{-\gamma_1(q-1)-q} \text{mes} \mathcal{F}_1^{1,\pm} \leq d_{1,R}^{(-\gamma_1-1)(q-1)}; \tag{27}$$

$$Y_{n,2}^{1,\pm}(z) \leq \int_{\mathcal{F}_2^{1,\pm}} \frac{|\zeta - z_1|^{-\gamma_1(q-1)} |d\zeta|}{|\zeta - z|^q} \leq \int_{c_1 d_{1,R}}^{\Gamma_1^\pm} \frac{ds}{s^q} \leq d_{1,R}^{-(q-1)};$$

and

$$Y_{n,1}^{2,\pm}(z) \leq \int_{\mathcal{F}_1^{2,\pm}} \frac{|\zeta - z_2|^{-\gamma_2(q-1)} |d\zeta|}{|\zeta - z|^q} \leq d_{2,R}^{(-\gamma_2)(q-1)} \text{mes} \mathcal{F}_1^{2,\pm} \leq d_{2,R}^{(-\gamma_2-1)(q-1)};$$

$$Y_{n,2}^{2,\pm}(z) \leq \int_{\mathcal{F}_2^{2,\pm}} \frac{|\zeta - z_2|^{-\gamma_2(q-1)} |d\zeta|}{|\zeta - z|^q} \leq \int_{c_2 d_{2,R}}^{\Gamma_2^\pm} \frac{ds}{s^q} \leq d_{2,R}^{-(q-1)}.$$

Combining (24) - (27), in this case, we get:

$$A_n(z) \leq \|P_n\|_p \left[ d_{1,R}^{-\frac{(\gamma_1^*+1)}{p}} + d_{2,R}^{-\frac{(\gamma_2^*+1)}{p}} \right], \quad \gamma_i^* := \max \{0; \gamma_i, i = 1, 2\}. \tag{28}$$

Let us estimate the  $B_n(z)$ . By notations from (23),  $\Gamma^\pm = \mathcal{F}_1^{1,\pm} \cup \mathcal{F}_2^{1,\pm} \cup \mathcal{F}_1^{2,\pm} \cup \mathcal{F}_2^{2,\pm}$  and, so:

$$B_n(z) = \sum_{i,k=1}^2 \int_{\mathcal{F}_k^{i,+} \cup \mathcal{F}_k^{i,-}} \frac{|d\zeta|}{|\zeta - z|^2} =: M(\mathcal{F}_k^{i,+}) + M(\mathcal{F}_k^{i,-}). \tag{29}$$

The integrals  $M(\mathcal{F}_k^{i,+})$  and  $M(\mathcal{F}_k^{i,-})$ ,  $i, k = 1, 2$ , are estimated to be similar, then we will estimate only  $M(\mathcal{F}_k^{i,+})$ .

$$M(\mathcal{F}_1^{1,+}) = \int_{|z_1-z|}^{c_1 d_{1,R}} \frac{ds}{s^2} \leq \frac{1}{d_{1,R}}; \quad M(\mathcal{F}_2^{1,+}) = \int_{d_{1,R}}^{\Gamma_1^\pm} \frac{ds}{s^2} \leq \frac{1}{d_{1,R}};$$

$$M(\mathcal{F}_1^{2,+}) = \int_0^{|z_2-z_2^*|} \frac{ds}{s^2} \leq \int_0^{c_2 d_{2,R}} \frac{ds}{s^2} \leq \frac{1}{d_{2,R}}; \quad M(\mathcal{F}_2^{2,+}) = \int_{|z_2-z_2^*|}^{\Gamma_2^\pm} \frac{ds}{s^2} \leq \int_{c_2 d_{2,R}}^{\Gamma_2^\pm} \frac{ds}{s^2} \leq \frac{1}{d_{2,R}}.$$

Then, from (29), we have:

$$B_n(z) \leq d_{1,R}^{-1} + d_{2,R}^{-1}. \tag{30}$$

Comparing (19), (20), (28) and (30), we get:

$$|P'_n(z)| \leq |\Phi^{n+1}(z)| \left[ \frac{1}{d(z, \Gamma)} \|P_n\|_p \left( d_{1,R}^{-\frac{(\gamma_1^*+1)}{p}} + d_{2,R}^{-\frac{(\gamma_2^*+1)}{p}} \right) + |P_n(z)| (d_{1,R}^{-1} + d_{2,R}^{-1}) \right]. \tag{31}$$

Using [9, Cor.1.3.], for the  $|P_n(z)|$  we have the following:

$$|P_n(z)| \leq c \frac{B_{n,1}}{d(z, \Gamma)} \|P_n\|_p |\Phi(z)|^{n+1}, \tag{32}$$

where  $c = c(G, p, \gamma_i) > 0$  is a constant independent of  $n$  and  $z$ , and

$$B_{n,1} := \begin{cases} n^{\frac{(\gamma_1+1-p)\tilde{\lambda}_1}{p} + n^{\frac{\gamma_2+1-p}{p(1+\alpha_2)} + \varepsilon}, & \gamma_1, \gamma_2 > p - 1, \\ (\ln n)^{1-\frac{1}{p}}, & \gamma_1 = p - 1, -1 < \gamma_2 \leq p - 1 \\ & \text{or } -1 < \gamma_1 \leq p - 1, \gamma_2 = p - 1, \\ 1, & -1 < \gamma_1, \gamma_2 < p - 1. \end{cases} \tag{33}$$

Then, from (32) and (31), we obtain:

$$|P'_n(z)| \leq \frac{|\Phi^{2(n+1)}(z)|}{d(z, \Gamma)} \|P_n\|_p \left[ \left( d_{1,R}^{-\frac{(\gamma_1^*+1)}{p}} + d_{2,R}^{-\frac{(\gamma_2^*+1)}{p}} \right) + B_{n,1} \left( d_{1,R}^{-1} + d_{2,R}^{-1} \right) \right]. \tag{34}$$

According to Lemma 2.7, for the  $d_{1,R}$ , we get:

$$d_{1,R} \geq n^{-\tilde{\lambda}_1}, \forall \varepsilon > 0. \tag{35}$$

For the estimate  $d_{2,R}$ , let's set:  $z_R \in \Gamma_R$  such that  $d_{2,R} = |z_2 - z_R|$ ,  $\zeta^\pm \in \Gamma^\pm$  such that  $d(z_R, \Gamma^2 \cap \Gamma^\pm) := d(z_R, \Gamma^+)$ ;  $z_2^\pm := \zeta \in \Gamma^2 : |\zeta - z_2| = c_2 d_{2,R}$ . Under this notations, from Lemma 2.4, we obtain:

$$d_R^\pm := d(z_R, \Gamma^2 \cap \Gamma^\pm) \asymp |z_R - z_2^\pm| \asymp d_{2,R}^{1+\alpha_2}. \tag{36}$$

In this case,  $d_{2,R} = \left( d_R^\pm \right)^{\frac{1}{1+\alpha_2}}$ . On the other hand, according to Lemma 2.7 and [23, Corollary 2], we get:  $d_R^\pm \geq n^{-1-\varepsilon}$ . Therefore,

$$d_{2,R} \geq n^{-\frac{1-\varepsilon}{1+\alpha_2}}. \tag{37}$$

From (34)-(37), we get:

$$\begin{aligned} |P'_n(z)| &\leq \frac{|\Phi^{2(n+1)}(z)|}{d(z, \Gamma)} \|P_n\|_p \left[ \left( n^{\frac{(\gamma_1^*+1)\tilde{\lambda}_1}{p} + n^{\frac{(\gamma_2^*+1)}{p(1+\alpha_2)} + \varepsilon} \right) + B_{n,1} \left( n^{\tilde{\lambda}_1} + n^{\frac{1}{1+\alpha_2} + \varepsilon} \right) \right] \\ &\leq \frac{|\Phi^{2(n+1)}(z)|}{d(z, \Gamma)} \|P_n\|_p \left[ \begin{cases} n^{\frac{(\gamma_1+1)\tilde{\lambda}_1}{p}}, & \gamma_1 \geq \frac{\gamma_2+1}{(1+\alpha_2)\tilde{\lambda}_1} - 1, \gamma_2 \geq (1+\alpha_2)\tilde{\lambda}_1 - 1, \\ n^{\frac{(\gamma_2+1)}{p(1+\alpha_2)} + \varepsilon}, & 0 < \gamma_1 < \frac{\gamma_2+1}{(1+\alpha_2)\tilde{\lambda}_1} - 1, \gamma_2 \geq (1+\alpha_2)\tilde{\lambda}_1 - 1, \\ n^{\frac{1}{p}\tilde{\lambda}_1}, & -1 < \gamma_1 < 0, -1 < \gamma_2 < (1+\alpha_2)\tilde{\lambda}_1 - 1, \\ & + n^{\tilde{\lambda}_1 + \varepsilon} \cdot B_{n,1} \end{cases} \right] \\ &\leq \frac{|\Phi^{n+1}(z)|}{d(z, \Gamma)} \|P_n\|_p \left[ \begin{cases} n^{\frac{(\gamma_1+1)\tilde{\lambda}_1}{p}}, & \gamma_1 \geq \frac{\gamma_2+1}{(1+\alpha_2)\tilde{\lambda}_1} - 1, \gamma_2 \geq (1+\alpha_2)\tilde{\lambda}_1 - 1, \\ n^{\frac{(\gamma_2+1)}{p(1+\alpha_2)} + \varepsilon}, & 0 < \gamma_1 < \frac{\gamma_2+1}{(1+\alpha_2)\tilde{\lambda}_1} - 1, \gamma_2 \geq (1+\alpha_2)\tilde{\lambda}_1 - 1, \\ n^{\frac{1}{p}\tilde{\lambda}_1}, & -1 < \gamma_1 < 0, -1 < \gamma_2 < (1+\alpha_2)\tilde{\lambda}_1 - 1, \\ n^{\frac{\gamma_1+1}{p}\tilde{\lambda}_1} + n^{\frac{\gamma_2+1-p}{p(1+\alpha_2)} + \tilde{\lambda}_1}, & \gamma_1, \gamma_2 > p - 1, \\ (\ln n)^{1-\frac{1}{p}}, & \gamma_1 = p - 1, -1 < \gamma_2 \leq p - 1 \\ & \text{or } -1 < \gamma_1 \leq p - 1, \gamma_2 = p - 1, \\ 1, & -1 < \gamma_1, \gamma_2 < p - 1, \end{cases} \right] \end{aligned} \tag{38}$$

$$\leq \frac{|\Phi^{2(n+1)}(z)|}{d(z, \Gamma)} \|P_n\|_p \left[ \begin{array}{l} \left\{ \begin{array}{l} n^{\frac{\gamma_1+1}{p}\tilde{\lambda}_1}, \quad \gamma_1 \geq \frac{\gamma_2+1}{(1+\alpha_2)\tilde{\lambda}_1} - 1, \gamma_2 \geq (1+\alpha_2)\tilde{\lambda}_1 - 1, \\ n^{\frac{(\gamma_2+1)}{p(1+\alpha_2)}+\varepsilon}, \quad 0 < \gamma_1 < \frac{\gamma_2+1}{(1+\alpha_2)\tilde{\lambda}_1} - 1, \gamma_2 \geq (1+\alpha_2)\tilde{\lambda}_1 - 1, \\ n^{\frac{1}{p}\tilde{\lambda}_1}, \quad -1 < \gamma_1 < 0, \quad -1 < \gamma_2 < (1+\alpha_2)\tilde{\lambda}_1 - 1, \\ n^{(\frac{\gamma_1+1}{p})\tilde{\lambda}_1}, \quad \gamma_1 > p-1 + \frac{\gamma_2+1-p}{(1+\alpha_2)\tilde{\lambda}_1}, \gamma_2 > p-1, \\ n^{\tilde{\lambda}_1 + (\frac{\gamma_2+1}{p}-1)\frac{1}{1+\alpha_2} + \varepsilon}, \quad p-1 < \gamma_1 \leq p-1 + \frac{\gamma_2+1-p}{(1+\alpha_2)\tilde{\lambda}_1}, \gamma_2 > p-1, \\ n^{(\frac{\gamma_1+1}{p})\tilde{\lambda}_1}, \quad \gamma_1 > p-1 + \frac{\gamma_2+1-p}{(1+\alpha_2)\tilde{\lambda}_1}, \gamma_2 = p-1, \\ (\ln n)^{1-\frac{1}{p}}, \quad -1 < \gamma_1 \leq p-1 + \frac{\gamma_2+1-p}{(1+\alpha_2)\tilde{\lambda}_1}, \gamma_2 = p-1, \\ (\ln n)^{1-\frac{1}{p}}, \quad \gamma_1 = p-1, -1 < \gamma_2 < p-1, \\ 1, \quad -1 < \gamma_1 < p-1, \quad -1 < \gamma_2 < p-1, \end{array} \right. \\ + \left\{ \begin{array}{l} n^{\frac{(\gamma_1+1)}{p}\tilde{\lambda}_1}, \quad \gamma_1 \geq \frac{\gamma_2+1}{(1+\alpha_2)\tilde{\lambda}_1} - 1, \gamma_2 \geq (1+\alpha_2)\tilde{\lambda}_1 - 1, \\ n^{\frac{(\gamma_2+1)}{p(1+\alpha_2)}+\varepsilon}, \quad 0 < \gamma_1 < \frac{\gamma_2+1}{(1+\alpha_2)\tilde{\lambda}_1} - 1, \gamma_2 \geq (1+\alpha_2)\tilde{\lambda}_1 - 1, \\ n^{\frac{1}{p}\tilde{\lambda}_1}, \quad -1 < \gamma_1 < 0, \quad -1 < \gamma_2 < (1+\alpha_2)\tilde{\lambda}_1 - 1, \\ n^{(\frac{\gamma_1+1}{p})\tilde{\lambda}_1}, \quad \gamma_1 > p-1 + \frac{\gamma_2+1-p}{(1+\alpha_2)\tilde{\lambda}_1}, \gamma_2 \geq p-1, \\ n^{\tilde{\lambda}_1 + (\frac{\gamma_2+1}{p}-1)\frac{1}{1+\alpha_2} + \varepsilon}, \quad p-1 < \gamma_1 \leq p-1 + \frac{\gamma_2+1-p}{(1+\alpha_2)\tilde{\lambda}_1}, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, \quad -1 < \gamma_1 \leq p-1, -1 < \gamma_2 \leq p-1, \\ 1, \quad -1 < \gamma_1 < p-1, \quad -1 < \gamma_2 < p-1. \end{array} \right. \end{array} \right]$$

Therefore, we get:

$$|P'_n(z)| \leq \frac{|\Phi^{2(n+1)}(z)|}{d(z, \Gamma)} \|P_n\|_p [D_n^{(1)} + D_n^{(2)}], \tag{39}$$

where

$$D_n^{(1)} : = \begin{cases} n^{\frac{(\gamma_1+1)}{p}\tilde{\lambda}_1}, & \gamma_1 \geq \frac{\gamma_2+1}{(1+\alpha_2)\tilde{\lambda}_1} - 1, \gamma_2 \geq (1+\alpha_2)\tilde{\lambda}_1 - 1, \\ n^{\frac{(\gamma_2+1)}{p(1+\alpha_2)}+\varepsilon}, & 0 < \gamma_1 < \frac{\gamma_2+1}{(1+\alpha_2)\tilde{\lambda}_1} - 1, \gamma_2 \geq (1+\alpha_2)\tilde{\lambda}_1 - 1, \\ n^{\frac{1}{p}\tilde{\lambda}_1}, & -1 < \gamma_1 < 0, \quad -1 < \gamma_2 < (1+\alpha_2)\tilde{\lambda}_1 - 1; \end{cases}$$

$$D_n^{(2)} : = \begin{cases} n^{(\frac{\gamma_1+1}{p})\tilde{\lambda}_1}, & \gamma_1 > p-1 + \frac{\gamma_2+1-p}{(1+\alpha_2)\tilde{\lambda}_1}, \gamma_2 \geq p-1, \\ n^{\tilde{\lambda}_1 + (\frac{\gamma_2+1}{p}-1)\frac{1}{1+\alpha_2} + \varepsilon}, & p-1 < \gamma_1 \leq p-1 + \frac{\gamma_2+1-p}{(1+\alpha_2)\tilde{\lambda}_1}, \\ & \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \gamma_1, \gamma_2 = p-1, \\ 1, & -1 < \gamma_1, \gamma_2 < p-1, \end{cases}$$

and we complete the proof for the points  $z \in \Omega(\delta)$ .

2. Let us now  $z \in \tilde{\Omega}_R(\delta)$ .

2.1. Let  $\gamma_1, \gamma_2 \geq 0$ . We get:

$$Y_{n,1}^{\pm}(z) \leq \int_0^{c_1 d_{1,R}} \frac{ds}{s^{\gamma_1(q-1)}} \leq \begin{cases} d_{1,R}^{1-\gamma_1(q-1)}, & \gamma_1(q-1) > 1, \\ \ln \frac{1}{d_{1,R}}, & \gamma_1(q-1) = 1, \\ 1, & \gamma_1(q-1) < 1; \end{cases} \quad Y_{n,2}^{\pm}(z) \leq \int_{c_1 d_{1,R}}^{|\Gamma_1^+|} \frac{ds}{s^{\gamma_1(q-1)}} \leq \begin{cases} d_{1,R}^{1-\gamma_1(q-1)}, & \gamma_1(q-1) > 1, \\ \ln \frac{1}{d_{1,R}}, & \gamma_1(q-1) = 1, \\ 1, & \gamma_1(q-1) < 1; \end{cases}$$

$$Y_{n,1}^{1,\pm}(z) + Y_{n,2}^{1,\pm}(z) \leq \begin{cases} d_{1,R}^{1-\gamma_1(q-1)}, & \gamma_1(q-1) > 1, \\ \ln \frac{1}{d_{1,R}}, & \gamma_1(q-1) = 1, \\ 1, & \gamma_1(q-1) < 1; \end{cases} \tag{40}$$

$$Y_{n,1}^{2,\pm}(z) \leq \int_0^{c_2 d_{2,R}} \frac{ds}{s^{\gamma_2(q-1)}} \leq \begin{cases} d_{2,R}^{1-\gamma_2(q-1)}, & \gamma_2(q-1) > 1, \\ \ln \frac{1}{d_{2,R}}, & \gamma_2(q-1) = 1, \\ 1, & \gamma_2(q-1) < 1; \end{cases}$$

$$Y_{n,2}^{2,\pm}(z) \leq \int_{c_2 d_{2,R}}^{|\Gamma_2^\pm|} \frac{ds}{s^{\gamma_2(q-1)}} \leq \begin{cases} d_{2,R}^{1-\gamma_2(q-1)}, & \gamma_2(q-1) > 1, \\ \ln \frac{1}{d_{2,R}}, & \gamma_2(q-1) = 1, \\ 1, & \gamma_2(q-1) < 1; \end{cases}$$

$$Y_{n,1}^2(z) + Y_{n,2}^2(z) \leq \begin{cases} d_{2,R}^{1-\gamma_2(q-1)}, & \gamma_2(q-1) > 1, \\ \ln \frac{1}{d_{2,R}}, & \gamma_2(q-1) = 1, \\ 1, & \gamma_2(q-1) < 1. \end{cases} \tag{41}$$

Let  $\gamma_1, \gamma_2 < 0$ . Then, analogously to the estimates (27), we get:

$$Y_{n,1}^{1,\pm}(z) \leq d_{1,R}^{(-\gamma_1)(q-1)} \text{mes}\mathcal{F}_1^1 \leq 1, \quad Y_{n,2}^{1,\pm}(z) \leq |\Gamma_1^\pm|^{(-\gamma_1)(q-1)+1} \leq 1;$$

$$Y_{n,1}^1(z) + Y_{n,2}^1(z) \leq 1; \tag{42}$$

$$Y_{n,1}^{2,\pm}(z) \leq d_{2,R}^{(-\gamma_2)(q-1)} \text{mes}\mathcal{F}_1^{2,\pm} \leq 1, \quad Y_{n,2}^{2,\pm}(z) \leq |\Gamma_2^\pm|^{(-\gamma_2)(q-1)+1} \leq 1,$$

$$Y_{n,1}^2(z) + Y_{n,2}^2(z) \leq 1. \tag{43}$$

Combining estimates (24) - (43), in this case, we find:

$$A_n(z) \leq \|P_n\|_p \begin{cases} d_{1,R}^{1-\frac{\gamma_1+1}{p}} + d_{2,R}^{1-\frac{\gamma_2+1}{p}}, & \gamma_1, \gamma_2 > p-1, \\ \left(\ln \frac{1}{d_{1,R}}\right)^{1-\frac{1}{p}} + \left(\ln \frac{1}{d_{2,R}}\right)^{1-\frac{1}{p}}, & \gamma_1, \gamma_2 = p-1, \\ 1, & \gamma_1, \gamma_2 < p-1. \end{cases} \tag{44}$$

$$B_n(z) = \int_\Gamma \frac{|d\zeta|}{|\zeta - z|^2} \leq \int_\Gamma |d\zeta| \leq 1. \tag{45}$$

Comparing (19), (20), (29) and (30), we have:

$$|P'_n(z)| \leq |\Phi^{n+1}(z)| \left[ \frac{\|P_n\|_p}{d(z, \Gamma)} B_{n,1} + |P_n(z)| \right], \tag{46}$$

where  $B_{n,1}$  defined as in (33)

According to (33), from (46), we have:

$$|P'_n(z)| \leq \frac{|\Phi^{2(n+1)}(z)|}{d(z, \Gamma)} \|P_n\|_p B_{n,1}. \tag{47}$$

Therefore, we complete the proof for the points  $z \in \widehat{\Omega}_R(\delta)$ , and so, the proof of the Theorem 1.3 is complete.  $\square$

*Proof.* [Proof of Corollaries 1.4 and 1.5]

1) Under the conditions of Corollary 1.4, for the points  $z \in \Omega_R(\delta)$ , from (34), (33) and (35), we get:

$$|P'_n(z)| \leq \frac{|\Phi^{2(n+1)}(z)|}{d(z, \Gamma)} \|P_n\|_p \left[ n^{\frac{(\gamma_1^*+1)\bar{\lambda}_1}{p}} + n^{\frac{(\gamma_2^*+1)\bar{\lambda}_2}{p}} + B_{n,1} \left( n^{\bar{\lambda}_1} + n^{\bar{\lambda}_2} \right) \right]$$

$$\leq \frac{|\Phi^{2(n+1)}(z)|}{d(z, \Gamma)} \|P_n\|_p \left[ n^{\frac{(\bar{\gamma}^*+1)\bar{\lambda}+\varepsilon}{p}} + \begin{cases} n^{\frac{\bar{\gamma}+1}{p}\bar{\lambda}}, & \gamma_1, \gamma_2 > p-1, \\ n^{\bar{\lambda}} (\ln n)^{1-\frac{1}{p}}, & \text{if } \gamma_1 = p-1, -1 < \gamma_2 \leq p-1, \\ & \text{or } -1 < \gamma_1 \leq p-1, \gamma_2 = p-1, \\ n^{\bar{\lambda}}, & -1 < \gamma_1, \gamma_2 < p-1, \end{cases} \right]$$

where  $\bar{\lambda} := \max\{\bar{\lambda}_1; \bar{\lambda}_2\}$ ,  $\bar{\gamma}^* := \max\{\gamma_1^*; \gamma_2^*\}$ ;  $\bar{\gamma} := \max\{\gamma_1; \gamma_2\}$ , and, for the points  $z \in \widehat{\Omega}_R(\delta)$ , from (47) and (33), we obtain:

$$|P'_n(z)| \leq \frac{|\Phi^{2(n+1)}(z)|}{d(z, \Gamma)} \|P_n\|_p \begin{cases} n^{\frac{(\bar{\gamma}+1-1)\bar{\lambda}}{p}}, & \gamma_1, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \text{if } \gamma_1 = p-1, -1 < \gamma_2 \leq p-1, \\ & \text{or } -1 < \gamma_1 \leq p-1, \gamma_2 = p-1, \\ 1, & -1 < \gamma_1, \gamma_2 < p-1. \end{cases}$$

2) Under the conditions of Corollary 1.5, from (34), (33) and (37), we have:

$$|P'_n(z)| \leq \frac{|\Phi^{2(n+1)}(z)|}{d(z, \Gamma)} \|P_n\|_p \left[ n^{\frac{(\gamma_1^*+1)}{p(1+\alpha_1)}} + n^{\frac{(\gamma_2^*+1)}{p(1+\alpha_2)}+\varepsilon} + B_{n,1} \left( n^{\frac{1}{1+\alpha_1}} + n^{\frac{1}{1+\alpha_2}} \right) \right]$$

$$\leq \frac{|\Phi^{2(n+1)}(z)|}{d(z, \Gamma)} \|P_n\|_p \left[ n^{\frac{(\bar{\gamma}^*+1)}{p(1+\alpha_*)+\varepsilon}} + \begin{cases} n^{\frac{\bar{\gamma}+1}{p(1+\alpha_*)}}, & \gamma_1, \gamma_2 > p-1, \\ n^{\frac{1}{1+\alpha_*}+\varepsilon} (\ln n)^{1-\frac{1}{p}}, & \left\{ \begin{array}{l} \text{if } \gamma_1 = p-1, -1 < \gamma_2 \leq p-1, \\ \text{or } -1 < \gamma_1 \leq p-1, \gamma_2 = p-1, \end{array} \right. \\ n^{\frac{1}{1+\alpha_*}+\varepsilon}, & -1 < \gamma_1, \gamma_2 < p-1, \end{cases} \right]$$

where  $\alpha_* := \min\{\alpha_1; \alpha_2\}$ , and, for the points  $z \in \widehat{\Omega}_R(\delta)$ , from (47) and (33), we obtain:

$$|P'_n(z)| \leq \frac{|\Phi^{2(n+1)}(z)|}{d(z, \Gamma)} \|P_n\|_p \begin{cases} n^{\frac{(\bar{\gamma}+1-1)\frac{1}{1+\alpha_*}+\varepsilon}{p}}, & \gamma_1, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \left\{ \begin{array}{l} \text{if } \gamma_1 = p-1, -1 < \gamma_2 \leq p-1 \\ \text{or } -1 < \gamma_1 \leq p-1, \gamma_2 = p-1, \end{array} \right. \\ 1, & -1 < \gamma_1, \gamma_2 < p-1. \end{cases}$$

□

*Proof.* [Proof of Theorem 1.6] Assume that  $G \in C_\theta(\lambda_1; cx^{1+\alpha_2})$ , for some  $0 < \lambda_1 \leq 2$ , and  $\alpha_2 > 0$ . Let  $z \in \Gamma$  arbitrary fixed point and let us  $B(z, d(z, \Gamma_R)) := \{t : |t - z| < d(z, \Gamma_R)\}$ . Using the Cauchy integral representation for derivatives  $P'_n(z)$  in a bounded domain, we obtain:

$$|P'_n(z)| = \left| \frac{1}{2\pi i} \int_{\partial B(z, d(z, \Gamma_R))} \frac{P_n(\zeta) d\zeta}{(\zeta - z)^2} \right| \leq \frac{1}{2\pi} \int_{\partial B(z, d(z, \Gamma_R))} |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|^2} \leq \max_{z \in \bar{G}_R} |P_n(\zeta)| \sup_{z \in \Gamma} \left\{ \frac{1}{d(z, \Gamma_R)} \right\}.$$

According [9, Cor.1.3.] and applying (4), we have:

$$\max_{z \in \bar{G}_R} |P_n(\zeta)| \leq \|P_n\|_{C(\bar{G})} \leq \|P_n\|_p \left[ n^{\frac{(\gamma_1^*+1)\bar{\lambda}_1}{p}} + \begin{cases} n^{\frac{\gamma_2^*+1}{p(1+\alpha_2)} + \frac{\alpha_2}{p(1+\alpha_2)} + \varepsilon}, & 1 < p < 2 + \frac{\gamma_2^*}{1+\alpha_2}, \\ (n \ln n)^{1-\frac{1}{p}}, & p = 2 + \frac{\gamma_2^*}{1+\alpha_2}, \\ n^{1-\frac{1}{p}}, & p > 2 + \frac{\gamma_2^*}{1+\alpha_2}. \end{cases} \right] \tag{48}$$

From (35) and (37), we have:

$$\sup_{z \in \Gamma} \left\{ \frac{1}{d(z, \Gamma_R)} \right\} \asymp \sup_{z \in \Gamma^1} \left\{ \frac{1}{d(z, \Gamma_R)} \right\}; \sup_{z \in \Gamma^2} \left\{ \frac{1}{d(z, \Gamma_R)} \right\} \leq n^{\bar{\lambda}_1}$$

and, therefore,

$$|P'_n(z)| \leq \|P_n\|_p \left[ n^{\left(\frac{\gamma_1^*+1}{p}+1\right)\bar{\lambda}_1} + \begin{cases} n^{\frac{\gamma_2^*+1}{p(1+\alpha_2)} + \frac{\alpha_2}{p(1+\alpha_2)} + \bar{\lambda}_1 + \epsilon}, & 1 < p < 2 + \frac{\gamma_2^*}{1+\alpha_2}, \\ n^{\bar{\lambda}_1+1-\frac{1}{p}} (\ln n)^{1-\frac{1}{p}}, & p = 2 + \frac{\gamma_2^*}{1+\alpha_2}, \\ n^{\bar{\lambda}_1+1-\frac{1}{p}}, & p > 2 + \frac{\gamma_2^*}{1+\alpha_2}, \end{cases} \right]$$

and we completed the proof of Theorem 1.6.  $\square$

### 3.0.1. Proof of Remark

*Proof.* The proof of the accuracy of the inequalities (13) - (15) can be divided into two parts: (i) :  $\|P'_n\|_\infty \leq n \|P_n\|_\infty$ ; (ii):  $\|P_n\|_\infty \leq \mu_n \|P_n\|_p$ . The first inequality- the well known sharp Markov inequality. The sharpness of the inequality (ii) can be verified in the following examples (see, for exp., [17], [18, Rm 2.9]). Let  $T_n(z) = 1 + z + \dots + z^n$ ,  $h^*(z) = h_0(z)$ ,  $h^{**}(z) = |z - 1|^\gamma$ ,  $\gamma > 0$ ,  $\Gamma := \{z : |z| = 1\}$ . Then, for any  $n \in \mathbb{N}$ , there exist  $c_1 = c_1(h^*, p) > 0$ ,  $c_2 = c_2(h^{**}, p) > 0$  such that:

- a)  $\|T\|_\infty \geq c_1 n^{\frac{1}{p}} \|T\|_{\mathcal{L}_p(h^*, \Gamma)}$ ,  $p > 1$ ;
- b)  $\|T\|_\infty \geq c_2 n^{\frac{\gamma+1}{p}} \|T\|_{\mathcal{L}_p(h^{**}, \Gamma)}$ ,  $p > \gamma + 1$ .

$\square$

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