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Higher-order derivatives of self-intersection local time for linear fractional stable processes

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Abstract. In this paper, we aim to consider the $k = (k_1, k_2, \cdots, k_d)$-th order derivatives $\beta^{(k)}(T, x)$ of self-intersection local time $\beta(T, x)$ for the linear fractional stable process $X^{\alpha,H}$ in $\mathbb{R}^d$ with indices $\alpha \in (0, 2)$ and $H = (H_1, \cdots, H_d) \in (0, 1)^d$. We first give sufficient condition for the existence and joint Hölder continuity of the derivatives $\beta^{(k)}(T, x)$ using the local nondeterminism of linear fractional stable processes. As a related problem, we also study the power variation of $\beta^{(k)}(T, x)$.

1. Introduction

Let $\{B_s, s \geq 0\}$ be one dimensional Brownian motion with $B_0 = 0$ and $L(t, x)$ be its local time at x up to time t. In connection with stochastic area integrals with respect to local time and the Brownian excursion filtration, Rogers and Walsh [21, 22] studied the space integral of local time. Let

$$A(t, B_t) = \int_0^t \mathbf{1}_{[0,\infty)}(B_t - B_s)ds,$$

they showed that $A(t, B_t)$ was not a semimartingale, and in fact showed that

$$A(t, B_t) - \int_0^t L(s, B_s)dB_s,$$

has finite non-zero $\frac{4}{3}$-variation, where $L(s, x) = \int_0^s \delta(B_r - x)dr$ and $\delta(\cdot)$ is the Dirac delta function.

If one lets $h(x) = \mathbf{1}_{[0,\infty)}(x)$ and then

$$\frac{d}{dx}h(x) = \delta(x), \qquad \frac{d^2}{dx^2}h(x) = \delta'(x),$$

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in the sense of Schwartz's distribution. Rosen [23] developed a new approach to the study of $A(t, B_t)$, and Markowsky [17] proved a Tanaka-style formula as follows,

$$\frac{1}{2}\alpha_t^{'}(y) + \frac{1}{2}\mathrm{sgn}(y)t = \int_0^t L(s, B_s - y)dB_s - \frac{1}{2}\int_0^t \mathrm{sgn}(B_t - B_u - y)du,$$

where

$$\mathrm{sgn}(y) = \begin{cases} -1, & \textit{if } y < 0, \\ 0, & \textit{if } y = 0, \\ 1, & \textit{if } y > 0, \end{cases}$$

and $\alpha_t^{'}(y)$ denotes the derivative of the intersection local time of B. Rosen [23] demonstrated the existence of $\alpha_t^{'}(y)$ and formally defined as

$$\alpha_t^{'}(y) := -\int_0^t \int_0^s \delta^{'}(B_s - B_r - y)drds.$$

Note that the Dirac delta function $\delta(\cdot)$ can be approximated by the heat kernel function

$$f_\varepsilon(x) = \frac{1}{(2\pi\varepsilon)^{\frac{1}{2}}}e^{-\frac{|x|^2}{2\varepsilon}} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\zeta}e^{-\frac{\varepsilon|\zeta|^2}{2}}d\zeta, \qquad x \in \mathbb{R}, \quad i = \sqrt{-1}. \tag{1}$$

Then the derivative of self-intersection local time $\alpha(t, y)$ can be approximated by

$$\alpha_\varepsilon^{'}(t, y) := -\int_0^t \int_0^s f_\varepsilon^{'}(B_s - B_r - y)drds, \quad \varepsilon \to 0.$$

The study of self-intersection local time for Brownian motion has attracted the attention of many scholars. Hu [9] discussed the exact smoothness of the self-intersection local time of Brownian motion in the sense of Meyer-Watanabe. If the Brownian motion is replaced by a more general Gaussian process (fractional Brownian motion), Hu [10] considered the self-intersection local time of fractional Brownian motions via chaos expansion and showed the condition of the existence. In [11], Hu and Nualart proved existence condition of the renormalized self-intersection local time for fractional Brownian motion, and gave two central limit theorems for nonexistence conditions. Also, the regularity in the sense of the Malliavin calculus of the renormalized self-intersection local time of d-dimensional fractional Brownian motion has been studied in Hu and Nualart [12]. Jaramillo and Nualart [13] obtained functional limit theorem for the self-intersection local time of the fractional Brownian motion. Yu et al. [31] studied the self-intersection local time for a class of non-Gaussian process (Rosenblatt process).

It is worth noting that the derivatives of self-intersection local time can be used in the application of Itô formula, they have received much attention recently. For fractional Brownian motion, the corresponding derivative of self-intersection local time was considered in Jung and Markowsky [15]. Yan and Yu [30] considered derivative for self-intersection local time of multidimensional fractional Brownian motion and showed the Bouleau-Yor type identity. Moreover, Jung and Markowsky [16] introduced a new version of this derivative and proved Hölder continuous conditions both in time and space variables. For the condition that the derivative does not exist and its critical condition, there are naturally two central limit theorems, which have been proved in Jaramillo and Nualart [14], and Yu [32], respectively. However, in concrete situations when the Gaussianity is not plausible for the model, one can use, for example, the stable process which have been applied in some scientific areas such as network traffic modeling and finance. This processes form an important subclass of infinitely divisible processes and have attracted a good deal of attention in recent years since their heavy-tailed distributions, self-similarity properties, long memory properties and so on.

Recently, several types of anisotropic stable random fields have arisen in theory and in applications, such as linear fractional stable processes, harmonizable fractional stable processes and so on. In [6], Delbeke and

Abry used a stochastic integral representation of the linear fractional stable motion to describe the wavelet coefficients as α-stable integrals. Stoev and Taqqu [26] presented efficient methods for simulation, using the fast Fourier transform algorithm of the linear fractional stable processes and generated paths of the linear fractional stable process by using Riemann-sum approximations and provided bounds and estimates of the approximation error. Ayahce, Roueff and Xiao [2] obtained an anisotropic uniform and quasi-optimal modules of continuity as well as upper bound for their behavior at infinity and around the coordinate axes of the sample paths for linear fractional stable sheets. Ayachea and Xiao [3] proved that for every $\alpha \in (0, 2)$, the N-parameter harmonizable fractional α-stable field is locally nondeterministic. They also established the joint continuity of the local time for harmonizable fractional α-stable field.

Motivated by the aforementioned works in the field of fractional stable processes, we are absorbed in the case of the linear fractional stable process, (LFSP, in short, $0 < \alpha < 2, H \in (0, 1)^d$) which does not satisfy the Gaussian property unless $\alpha = 2$. Thanks to Xiao [28], we give the definition of d-dimensional LFSP.

Definition 1.1. *For any given $\alpha \in (0, 2)$ and $H = (H_1, \cdots, H_d)$ with $H_l \in (0, 1)$ for $l = 1, \cdots, d$, a d-dimensional LFSP $X^{\alpha,H} = (X_t^{\alpha,H_1}, \cdots, X_t^{\alpha,H_d}), t \geq 0$ is defined by the following integral representation:*

$$X_t^{\alpha,H_l} = \int_{\mathbb{R}} g_{H_l}(t, s) M_\alpha(ds),$$

where M_α is a symmetric α-stable random measure on $\mathbb{R}$ with Lebesgue control measure and

$$g_{H_l}(t, s) = C\{((t - s)_+)^{H_l - 1/\alpha} - ((-s)_+)^{H_l - 1/\alpha}\}. \tag{2}$$

In the above, $t_+ = \max\{0, t\}$ for $t \geq 0$ and $C > 0$ is a normalizing constant.

Note that, when $\alpha = 2$, $X^{\alpha,H}$ is known as multidimensional fractional Brownian motion. When $H_1 = \cdots = H_d = 1/\alpha$, $X^{\alpha,H}$ becomes the symmetric stable process. There has been several interesting studies about sample path properties, local time of the fractional stable fields and even multifractional stable process. For some details on the distributional properties and limiting theorems of this class of processes, we can see Cambanis and Maejima [4]. In [1], Ayache, Roueff and Xiao extended the properties of the local time in the Gaussian case to the symmetric α-stable case. They proved the existence of the local time for linear fractional stable sheets and showed the local time is jointly continuous as well. Shen, Yu and Li [25] obtained the existence of the local times of linear multifractional stable sheets and established its joint continuity and Hölder regularity.

Although there exist many investigations in the literature devoted to studying the derivatives of self-intersection local time for Brownian motion, fractional Brownian motion, as we know, there is little research on the derivative of self-intersection local time for non-Gaussian processes (Rosen [23] considered the α-stable processes with $\alpha \neq 2$), especially the LFSP. Moreover, due to their heavy-tailed distributions, self-similarity properties, long memory properties, they are potentially useful and important for modelling complex systems in diverse areas of applications. This motivates us to carry out the present paper, aiming to study the higher-order derivative of self-intersection local time for the LFSP.

Now, we define the derivative of self-intersection local time for LFSP, which is similar to that in Gaussian processes.

Definition 1.2. *Let $X^{\alpha,H} = (X_t^{\alpha,H_1}, \cdots, X_t^{\alpha,H_d}), t \geq 0$ be a LFSP in $\mathbb{R}^d$ with parameters $\alpha \in (0, 2)$ and $H = (H_1, \cdots, H_d) \in (0, 1)^d$. The self-intersection local time of $X^{\alpha,H}$, denoted by $\beta(T, x)$, is formally defined by*

$$\beta(T, x) := \int_0^T \int_0^s \delta(X_s^{\alpha,H} - X_r^{\alpha,H} - x) dr ds,$$

for all $T \geq 0$ and $x \in \mathbb{R}^d$, where $\delta(\cdot)$ is the Dirac delta function. Intuitively, its higher-order derivative can be defined as

$$\beta^{(k)}(T, x) := \int_0^T \int_0^s \delta^{(k)}(X_s^{\alpha,H} - X_r^{\alpha,H} - x) dr ds,$$

where the order $k = (k_1, k_2, \cdots, k_d)$ *and*

$$\delta^{(k)}(x) = \frac{\partial^{(k)}}{\partial x_1^{k_1} \ldots \partial x_d^{k_d}} \delta(x),$$

where the derivative of the δ *function is in the sense of Schwartz's distribution.*

Since the Dirac delta function can be approximated by the heat kernel function in (1), the similar approximation can be extended in $\mathbb{R}^d$ (see, for example, Guo, Hu and Xiao [7], Hong and Xu [8]) as the following expression

$$f_\varepsilon(x) = \frac{1}{(2\pi\varepsilon)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2\varepsilon}} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x, \zeta \rangle} e^{-\frac{\varepsilon|\zeta|^2}{2}} d\zeta, \qquad x \in \mathbb{R}^d.$$

Thus we can consider approximating $\beta^{(k)}(T, x)$ by

$$\beta_\varepsilon^{(k)}(T, x) := \int_0^T \int_0^s f_\varepsilon^{(k)}(X_s^{\alpha,H} - X_r^{\alpha,H} - x) dr ds, \qquad \text{as} \quad \varepsilon \to 0, \tag{3}$$

where

$$f_\varepsilon^{(k)}(x) = \frac{\partial^{(k)}}{\partial x_1^{k_1} \ldots \partial x_d^{k_d}} f_\varepsilon(x) \equiv \frac{i^{|k|}}{(2\pi)^d} \int_{\mathbb{R}^d} (\prod_{l=1}^d \zeta_l^{k_l}) e^{ix\zeta} e^{-\frac{\varepsilon|\zeta|^2}{2}} d\zeta.$$

If $\beta_\varepsilon^{(k)}(T, x)$ converges to a random variable in L^m spaces with $m \in [1, \infty)$ as $\varepsilon \to 0$, we denote the limit by $\beta^{(k)}(T, x)$ and call it the k-th-order derivative of self-intersection local time for LFSP.

Moreover, the study of p-variation of the local times for non-Gaussian processes has always been a hot topic in the field of stochastic analysis. For example, Marcus and Rosen [18] studied p-variation of the local times of symmetric stable processes in $\mathbb{R}$ both in almost surely case and L^p case. For the derivative case, Yan, Yu and Chen [29] proved the p-variation for the derivative of intersection local time of independent symmetric α-stable processes in $\mathbb{R}$ was zero under certain conditions. As a related problem, the second research object of this paper is the p-variation of $\beta^{(k)}(T, x)$ for LFSP in $\mathbb{R}^d$. For convenience, we will consider location $x = 0$ (because the modulus of $e^{-i\langle \xi, x \rangle}$ equal to one), i.e. the p-variation of $\beta^{(k)}(T, 0)$.

The rest of this paper is organized as follows. Section 2 contains some necessary preliminaries on LFSP and some basic lemmas. Section 3 devotes to discussing the existence and joint Hölder continuity of $\beta^{(k)}(T, x)$. In Section 4, we study the p-variation of $\beta^{(k)}(T, 0)$. Throughout this paper, if not mentioned otherwise, the letter C, with or without a subscript, denotes a generic positive finite constant and may change from line to line.

2. Preliminaries

In this section, we briefly recall the property of LFSP, which is important to the proofs in Section 3 and Section 4 and introduce the local nondeterminism of LFSP.

It is well known that for $\alpha \in (0, 2)$, if $\widetilde{X}_t^\alpha = (\widetilde{X}_t^{\alpha,1}, \cdots, \widetilde{X}_t^{\alpha,d})$ is a d-dimensional stochastic process having the following integral representation:

$$\widetilde{X}_t^{\alpha,l} = \int_{\mathbb{R}} g_l(t, s) M_\alpha(ds),$$

for $l = 1, \cdots, d$ and all $t \geq 0$, where M_α is a symmetric α-stable random measure on a measurable space $(\Lambda, \mathcal{F})$ with control measure m and $g_l(t, \cdot) : \Lambda \to \mathbb{R}$ $(t \in \mathbb{R})$ is a family of measurable functions satisfying

$$\int_\Lambda |g_l(t, x)|^\alpha m(dx) < \infty, \qquad \forall t \in \mathbb{R}.$$

Then for all $a_1, \cdots, a_m \in \mathbb{R}$, the characteristic function of $\widetilde{X}_t^{\alpha,l}$ is given by

$$\mathbb{E}\exp(i\sum_{j=1}^{m} a_j \widetilde{X}_t^{\alpha,l}) = \exp(-\Big\|\sum_{j=1}^{m} a_j \widetilde{X}_t^{\alpha,l}\Big\|_{\alpha}^{\alpha}), \tag{4}$$

where

$$\Big\|\sum_{j=1}^{m} a_j \widetilde{X}_t^{\alpha,l}\Big\|_{\alpha}^{\alpha} := \int_{\mathbb{R}} \Big|\sum_{j=1}^{m} a_l g_l(t,s)\Big|^{\alpha} ds.$$

More details about stable processes can be found in [20], [24] and references therein. In particularly when $\widetilde{X}_t^{\alpha,l} = X_t^{\alpha,H_l}$, the characteristic function of $X_t^{\alpha,H_l}, l = 1, 2, \cdots, d$ is given by

$$\mathbb{E}\exp(i\sum_{j=1}^{m} a_j X_t^{\alpha,H_l}) = \exp(-\Big\|\sum_{j=1}^{m} a_j X_t^{\alpha,H_l}\Big\|_{\alpha}^{\alpha}) \tag{5}$$

for all $a_1, \cdots, a_m \in \mathbb{R}$ and $t \geq 0$, where

$$\Big\|\sum_{j=1}^{m} a_j X_t^{\alpha,H_l}\Big\|_{\alpha}^{\alpha} := \int_{\mathbb{R}} \Big|\sum_{j=1}^{m} a_j g_{H_l}(t,s)\Big|^{\alpha} ds,$$

here g_{H_l} is the kernel function given in (2).

It is easy to find that for $\alpha = 2$, $X_t^{\alpha,H}$ becomes a fractional Brownian motion. Since g_{H_l} is chosen such that $\|X_t^{2,H}\|_2 = 1$, we obtain that when $\alpha = 2$,

$$\mathbb{E}\exp\left\{i\sum_{j=1}^{m} a_j X_t^{2,H_l}\right\} = \exp\left\{-\frac{1}{2}\mathrm{Var}(\sum_{j=1}^{m} a_j X_t^{2,H_l})\right\}.$$

Besides, when $\alpha \neq 2$, $X^{\alpha,H}$ is non-Gaussian and non-Markovian process, yet it is self-similar and has stationary increments in the following sense. For any $a > 0$ and $b > 0$,

$$X_{at}^{\alpha,H} \overset{\text{Law}}{=} (a^{H_1} X_t^{\alpha,H_1}, \cdots, a^{H_d} X_t^{\alpha,H_d}),$$

and

$$X_{t+b}^{\alpha,H_l} - X_b^{\alpha,H_l} \overset{\text{Law}}{=} X_t^{\alpha,H_l} - X_0^{\alpha,H_l}.$$

Next, we introduce one-sided local nondeterminism for the LFSP, which is the key feature leading to the proofs of our main results in Section 3. For more applications of the local nondeterminisim, we can refer to [5], [19], [27] and references therein.

Definition 2.1. *[28] Let $X^{\alpha,H} := (X_t^{\alpha,H_1}, \cdots, X_t^{\alpha,H_d}), t \geq 0$ be d-dimensional LFSP with kernel function $g_{H_l}(\cdot)$ given in* (2)*. Then $X^{\alpha,H}(\cdot)$ is said to have the local nondeterminism on any Borel set $I \in \mathbb{R}_+$ if for every integer $t, s \in I$ with $|t-s|$ sufficiently small,*

$$\|X_t^{\alpha,H_l}\|_{\alpha} > 0, \qquad \|X_t^{\alpha,H_l} - X_s^{\alpha,H_l}\|_{\alpha} > 0,$$

and there exists a constant $C > 0$ such that for every $n \geq 2$ and $t_1, \cdots, t_m \in I$ with $t_j \leq t_m$ for all $j = \{1, \cdots, m-1\}$, we have

$$\|X_{t_m}^{\alpha,H_l} | X_{t_1}^{\alpha,H_l}, \cdots, X_{t_{m-1}}^{\alpha,H_l}\|_{\alpha} \geq C \min_{1 \leq j \leq m-1} (t_m - t_j)^{H_l},$$

where the distance from $X_{t_m}^{\alpha,H_l}$ to span $\{X_{t_1}^{\alpha,H_l}, \cdots, X_{t_{m-1}}^{\alpha,H_l}\}$ is defined by

$$\|X_{t_m}^{\alpha,H_l} | X_{t_1}^{\alpha,H_l}, \cdots, X_{t_{m-1}}^{\alpha,H_l}\|_{\alpha} := \begin{cases} \inf\limits_{a_1,\cdots,a_{m-1} \in \mathbb{R}} \|X_{t_m}^{\alpha,H_l} - \sum\limits_{j=1}^{m-1} a_j X_{t_j}^{\alpha,H_l}\|_{\alpha}, & \alpha \in (0,2), \\ Var(X_{t_m}^{\alpha,H_l} | X_{t_1}^{\alpha,H_l}, \cdots, X_{t_{m-1}}^{\alpha,H_l}), & \alpha = 2. \end{cases}$$

3. Existence and joint Hölder continuity

In this section, we discuss the existence and joint Hölder continuity of higher-order derivatives of self-intersection local time for the LFSP.

Theorem 3.1. *For* $\beta_\varepsilon^{(k)}(T,x)$ *is defined in (3) with* $H=(H_1,\cdots,H_d)\in(0,1)^d$ *and* $k=(k_1,\cdots,k_d)$. *If* $H_l(k_l+\frac{1}{2})<1$ *for all* $l=1,2,\cdots,d$, *then we can get* $\beta^{(k)}(T,x)$ *exists in* L^m *spaces with* $m\in[1,\infty)$. *Moreover, the process* $\beta^{(k)}(T,x)$ *has a modification which is a.s. joint Hölder continuous in* (T,x).

In order to prove the Theorem 3.1, we need the following lemma which can make the proof more clearly. For any $x\in\mathbb{R}^d$ and bounded Borel set $B\subseteq\mathbb{R}^2$, let

$$\beta_\varepsilon^{(k)}(B,x)=\int_B f_\varepsilon^{(k)}(X_s^{\alpha,H}-X_r^{\alpha,H}-x)drds.$$

We use $|B|$ to denote the Lebesgue measure of $B\subseteq\mathbb{R}^2$.

Lemma 3.2. *Under the conditions of Theorem 3.1, for some* $0<\rho_1<\min_{l=1,2,\cdots,d}\{1-H_l(k_l+\frac{1}{2})\}, 0<\rho_2<\min_{l=1,2,\cdots,d}\{1\wedge(\frac{1}{H_l}-k_l-\frac{1}{2})\}$, *we can obtain*

$$\left|\mathbb{E}[(\beta_\varepsilon^{(k)}(B,x))^m]\right|\le C_{m,\rho,H}|B|^{m\rho_1}, \tag{6}$$

and

$$\left|\mathbb{E}[(\beta_\varepsilon^{(k)}(B,x)-\beta_{\varepsilon'}^{(k)}(B,x'))^m]\right|\le C_{m,\rho,H}|(\varepsilon,x)-(\varepsilon',x')|^{m\rho_2} \tag{7}$$

hold for all $m\in[1,\infty)$, $0<\varepsilon,\varepsilon'\le1$, $x,x'\in\mathbb{R}^d$ *and all Borel sets* $B\subseteq D_1^1=:[0,\frac{1}{2}]\times[\frac{1}{2},1]$.

Proof. We first claim that $\beta_\varepsilon^{(k)}(B,x)\in L^m$, for even integer $m\ge1$,

$$\begin{aligned}\left|\mathbb{E}[(\beta_\varepsilon^{(k)}(B,x))^m]\right|=&(\frac{1}{2\pi})^{md}\int_{B^m}\int_{\mathbb{R}^{md}}\mathbb{E}\exp(\sum_{j=1}^m i\langle\zeta_j,X_{s_j}^{\alpha,H}-X_{r_j}^{\alpha,H}-x\rangle)\\&\times\exp(-\frac{\varepsilon}{2}\sum_{j=1}^m|\zeta_j|^2)\prod_{j=1}^m(\prod_{l=1}^d|\zeta_{j,l}|^{k_l})d\zeta dsdr,\end{aligned}$$

where $\langle\zeta_j,X_{s_j}^{\alpha,H}-X_{r_j}^{\alpha,H}\rangle:=\sum_{l=1}^d\zeta_{j,l}(X_{s_j}^{\alpha,H_l}-X_{r_j}^{\alpha,H_l})$ and $|\zeta_j|^2=\sum_{l=1}^d|\zeta_{j,l}|^2$.

Then we can get

$$\begin{aligned}\left|\mathbb{E}[(\beta_\varepsilon^{(k)}(B,x))^m]\right|\le&(\frac{1}{2\pi})^{md}\int_{B^m}\int_{\mathbb{R}^{md}}\mathbb{E}\exp(\sum_{j=1}^m i\langle\zeta_j,X_{\frac{1}{2}}^{\alpha,H}-X_{r_j}^{\alpha,H}\rangle)\\&\times\mathbb{E}\exp(\sum_{j=1}^m i\langle\zeta_j,X_{s_j}^{\alpha,H}-X_{\frac{1}{2}}^{\alpha,H}\rangle)\prod_{j=1}^m(\prod_{l=1}^d|\zeta_{j,l}|^{k_l})d\zeta dsdr.\end{aligned}$$

Fix an ordering of the set $\{r_1,r_2,\cdots,r_m\}$ and let $t_1\le t_2\le\cdots\le t_m$ be a relabeling of the set $\{r_1,r_2,\cdots,r_m\}$. Denote the set $U=\{u:r_u\le t_j,u=1,\cdots,m\}$ satisfying $\xi_j=\sum_{u\in U}\zeta_u$ so that the ξ_j spans $\mathbb{R}^m$. Using the method of variable substitution, it follows that

$$\sum_{j=1}^m\langle\zeta_j,X_{\frac{1}{2}}^{\alpha,H}-X_{r_j}^{\alpha,H}\rangle=\sum_{j=1}^m\langle\xi_j,X_{t_{j+1}}^{\alpha,H}-X_{t_j}^{\alpha,H}\rangle$$

where $\langle \xi_j, X^{\alpha,H}_{t_{j+1}} - X^{\alpha,H}_{t_j}\rangle := \sum_{l=1}^{d} \xi_{j,l}(X^{\alpha,H_l}_{t_{j+1}} - X^{\alpha,H_l}_{t_j})$ and the $t_1,\cdots,t_m$ are the r_i's relabeled so that $t_1 \le \cdots \le t_m \le t_{m+1} = \frac{1}{2}$.

Similarly, we can fix an ordering of the set $\{s_1, s_2, \cdots, s_m\}$ and let $t'_1 \le t'_2 \le \cdots \le t'_m$ be a relabeling of the set $\{s_1, s_2, \cdots, s_m\}$. Denote the set $U' = \{u : s_u \le t'_j, u = 1, \cdots, m\}$ satisfying $\xi'_j = \sum_{u\in U'} \zeta_u$ so that the ξ'_j also spans $\mathbb{R}^m$. Then we can rewrite

$$\sum_{j=1}^{m} \langle \zeta_j, X^{\alpha,H}_{s_j} - X^{\alpha,H}_{\frac{1}{2}}\rangle = \sum_{j=1}^{m} \langle \xi'_j, X^{\alpha,H}_{t'_j} - X^{\alpha,H}_{t'_{j-1}}\rangle$$

with $\langle \xi'_j, X^{\alpha,H}_{t'_j} - X^{\alpha,H}_{t'_{j-1}}\rangle := \sum_{l=1}^{d} \xi'_{j,l}(X^{\alpha,H_l}_{t'_j} - X^{\alpha,H_l}_{t'_{j-1}})$ and $\frac{1}{2} = t'_0 \le t'_1 \le \cdots \le t'_m$. Applying (5) and the local nondeterminism in Definition (2.1), we have

$$\begin{aligned}\Big|\mathbb{E}[(\beta^{(k)}_{\varepsilon}(B,x))^m]\Big| \le (\frac{1}{2\pi})^{md} \int_{B^m}\int_{\mathbb{R}^{md}} & \exp(-\sum_{j=1}^{m}\sum_{l=1}^{d} |\xi_{j,l}|^{\alpha}(t_{j+1}-t_j)^{\alpha H_l}) \\ & \times \exp(-\sum_{j=1}^{m}\sum_{l=1}^{d} |\xi'_{j,l}|^{\alpha}(t'_j - t'_{j-1})^{\alpha H_l}) \prod_{j=1}^{m}(\prod_{l=1}^{d} |\zeta_{j,l}|^{k_l}) d\zeta dt dt', \end{aligned} \tag{8}$$

where $\xi_{j,l} \in \mathbb{R}$ and $|\xi_{j,l}|$ denotes the absolute value of $\xi_{j,l}$.

Then using the simple bound

$$\int_0^1 \exp(-t^{\alpha H_l}|\xi_{j,l}|^{\alpha})dt \le \frac{c}{1+|\xi_{j,l}|^{\frac{1}{H_l}}}, \qquad l = 1, \cdots, d \tag{9}$$

m-times for integrals dt and dt', it follows that

$$\begin{aligned}\Big|\mathbb{E}[(\beta^{(k)}_{\varepsilon}(B,x))^m]\Big| \le & C\int_{\mathbb{R}^{md}} \prod_{j=1}^{m}\prod_{l=1}^{d} \frac{1}{1+|\xi_{j,l}|^{\frac{1}{H_l}}} \prod_{j=1}^{m}\prod_{l=1}^{d} \frac{1}{1+|\xi'_{j,l}|^{\frac{1}{H_l}}} \prod_{j=1}^{m}(\prod_{l=1}^{d} |\zeta_{j,l}|^{k_l}) d\zeta \\ \le & C\int_{\mathbb{R}^{md}} (\prod_{j=1}^{m}\prod_{l=1}^{d} \frac{|\zeta_{j,l}|^{\frac{k_l}{2}}}{1+|\xi_{j,l}|^{\frac{1}{H_l}}})(\prod_{j=1}^{m}\prod_{l=1}^{d} \frac{|\zeta_{j,l}|^{\frac{k_l}{2}}}{1+|\xi'_{j,l}|^{\frac{1}{H_l}}}) d\zeta \\ \le & C\Big\|\prod_{j=1}^{m}\prod_{l=1}^{d} \frac{|\zeta_{j,l}|^{\frac{k_l}{2}}}{1+|\xi_{j,l}|^{\frac{1}{H_l}}}\Big\|_2 \Big\|\prod_{j=1}^{m}\prod_{l=1}^{d} \frac{\prod_{l=1}^{d}|\zeta_{j,l}|^{\frac{k_l}{2}}}{1+|\xi'_{j,l}|^{\frac{1}{H_l}}}\Big\|_2 . \end{aligned}$$

Since $\xi_i = \sum_{u\in U} \zeta_u$, we can obtain that

$$\Big\|\prod_{j=1}^{m}\prod_{l=1}^{d} \frac{|\zeta_{j,l}|^{\frac{k_l}{2}}}{1+|\xi_{j,l}|^{\frac{1}{H_l}}}\Big\|_2^2 \le \int_{\mathbb{R}^{md}} \prod_{j=1}^{m}\prod_{l=1}^{d} \frac{|\zeta_{j,l}|^{k_l}}{1+|\xi_{j,l}|^{\frac{2}{H_l}}} d\zeta \le \int_{\mathbb{R}^{md}} \prod_{j=1}^{m}\prod_{l=1}^{d} \frac{1+|\xi_{j,l}|^{k_l}+|\xi_{j,l}|^{2k_l}}{1+|\xi_{j,l}|^{\frac{2}{H_l}}} d\xi$$

is bounded if $\frac{2}{H_l} - 2k_l > 1$, for all $l = 1, 2, \cdots, d$.

Hence, we get

$$\Big|\mathbb{E}[(\beta^{(k)}_{\varepsilon}(B,x))^m]\Big| < \infty,$$

for all $\varepsilon > 0$, $x \in \mathbb{R}^d$ and Borel sets $B \subseteq [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$ under the condition $\frac{2}{H_l} - 2k_l > 1$.

Now, We first establish (6). Applying the Hölder inequality to (8), for any $\frac{1}{a} + \frac{1}{a'} = 1$, we have

$$\begin{aligned}
&\Big|\mathbb{E}[(\beta_\varepsilon^{(k)}(B,x))^m]\Big| \\
&\leq C_m |B|^{\frac{m}{a}} \int_{\mathbb{R}^{md}} \Big\{\Big(\int_{B^m} \exp(-\sum_{j=1}^m \sum_{l=1}^d |\xi_{j,l}|^\alpha (t_{j+1}-t_j)^{\alpha H_l}) \\
&\qquad \times \exp(-\sum_{j=1}^m \sum_{l=1}^d |\xi'_{j,l}|^\alpha (t'_j - t'_{j-1})^{\alpha H_l}) dt dt'\Big)^{a'}\Big\}^{\frac{1}{a'}} (\prod_{j=1}^m \prod_{l=1}^d |\zeta_{j,l}|^{k_l}) d\zeta.
\end{aligned}$$

Using (9) for integrals with respect to dt and dt', we have

$$\begin{aligned}
&\Big|\mathbb{E}[(\beta_\varepsilon^{(k)}(B,x))^m]\Big| \\
&\leq C_{m,\alpha,H} |B|^{\frac{m}{a}} \int_{\mathbb{R}^{md}} \Big(\prod_{j=1}^m \prod_{l=1}^d \frac{1}{1+|\xi_{j,l}|^{\frac{1}{H_l}}} \cdot \prod_{j=1}^m \prod_{l=1}^d \frac{1}{1+|\xi'_{j,l}|^{\frac{1}{H_l}}} \Big)^{\frac{1}{a'}} (\prod_{j=1}^m \prod_{l=1}^d |\zeta_{j,l}|^{k_l}) d\zeta \\
&\leq C_{m,\alpha,H} |B|^{\frac{m}{a}} \int_{\mathbb{R}^{md}} \Big(\prod_{j=1}^m \prod_{l=1}^d \frac{|\xi_{j,l} - \xi_{j-1,l}|^{\frac{k_l}{2}}}{1+|\xi_{j,l}|^{\frac{1}{a' H_l}}} \Big) \cdot \Big(\prod_{j=1}^m \prod_{l=1}^d \frac{|\xi'_{j,l} - \xi'_{j-1,l}|^{\frac{k_l}{2}}}{1+|\xi'_{j,l}|^{\frac{1}{a' H_l}}} \Big) d\xi \\
&\leq C_{m,\alpha,H} |B|^{\frac{m}{a}} \Big\| \prod_{j=1}^m \prod_{l=1}^d \frac{|\xi_{j,l}|^{k_l}}{1+|\xi_{j,l}|^{\frac{1}{a' H_l}}} \Big\|_2 \Big\| \prod_{j=1}^m \prod_{l=1}^d \frac{|\xi'_{j,l}|^{k_l}}{1+|\xi'_{j,l}|^{\frac{1}{a' H_l}}} \Big\|_2 \\
&\leq C_{m,\alpha,H} |B|^{\frac{m}{a}},
\end{aligned}$$

where $1 < a' < \frac{1}{H_l(k_l+\frac{1}{2})}$ and $H_l(k_l + \frac{1}{2}) < 1$ for all $l = 1, 2, \cdots, d$. If a' is chosen close to 1, and the estimate (6) can be obtained when the Hölder order $\rho_1 = \frac{1}{a}$.

Next, we obtain the estimate (7). To handle the variation in x, we have

$$\begin{aligned}
&\Big|\mathbb{E}[(\beta_\varepsilon^{(k)}(B,x) - \beta_\varepsilon^{(k)}(B,x'))^m]\Big| \\
&= (\frac{1}{2\pi})^{md} \int_{B^m} \int_{\mathbb{R}^{md}} |\exp(-i\sum_{j=1}^m \langle \zeta_j, x\rangle) - \exp(-i\sum_{j=1}^m \langle \zeta_j, x'\rangle)| \\
&\qquad \times \exp(-\sum_{j=1}^m \sum_{l=1}^d |\xi_{j,l}|^\alpha (t_{j+1}-t_j)^{\alpha H_l}) \exp(-\sum_{j=1}^m \sum_{l=1}^d |\xi'_{j,l}|^\alpha (t'_j - t'_{j-1})^{\alpha H_l}) \\
&\qquad \times \prod_{j=1}^m (\prod_{l=1}^d |\zeta_{j,l}|^{k_l}) \exp(-\frac{\varepsilon}{2} \sum_{j=1}^m \sum_{l=1}^d |\zeta_{j,l}|^2) d\zeta ds dr.
\end{aligned}$$

Using the estimate

$$|\exp(-i\langle x, \zeta_j\rangle) - \exp(-i\langle x', \zeta_j\rangle)| \leq C_\rho |\zeta_j|^\lambda |x - x'|^\lambda$$

and

$$|\zeta_j|^\lambda \leq C(|\zeta_{j,1}|^\lambda + \cdots + |\zeta_{j,d}|^\lambda)$$

for any $0 \leq \lambda \leq 1$. Similar to the proof of (6) and replace k_l by $k_l + \lambda$, we can obtain

$$\Big|\mathbb{E}[(\beta_\varepsilon^{(k)}(B,x) - \beta_\varepsilon^{(k)}(B,x'))^m]\Big| < C_{m,\rho,H} |x - x'|^{m\rho_2} \tag{10}$$

for $0 < \rho_2 < \{1 \wedge (\frac{1}{H_l} - k_l - \frac{1}{2})\}$ for all $l = 1, 2, \cdots, d$. Similarly, to handle the variation in ε, by the estimate

$$\left|\widehat{f}(\varepsilon\zeta_j) - \widehat{f}(\varepsilon'\zeta_j)\right| \leq C|\zeta_j|^{\lambda}|\varepsilon - \varepsilon'|^{\lambda}$$

for any $0 \leq \lambda \leq 1$, where $\widehat{f}$ is the Fourier transform of f. Then we can get

$$\left|\mathbb{E}[(\beta^{(k)}_{\varepsilon}(B,x) - \beta^{(k)}_{\varepsilon'}(B,x))^m]\right| < C_{m,\rho,H}|\varepsilon - \varepsilon'|^{m\rho_2}. \tag{11}$$

Thus the estimate (7) can be obtained by (10) and (11). This completes the proof. □

Proof of Theorem 3.1.

Proof. Now, let us give the proof of Theorem 3.1. Let

$$D^n_q = [(2q-2)2^{-n}, (2q-1)2^{-n}] \times [(2q-1)2^{-n}, (2q)2^{-n}],$$

where $q \in \{1, \cdots, 2^{n-1}\}$. Using the scaling $X^{\alpha,H_l}_{\lambda t} \overset{\text{Law}}{=} \lambda^{H_l} X^{\alpha,H_l}_t$ for $l = 1, \cdots, d$ and $f^{(k)}_{\lambda\varepsilon}(x) = \frac{1}{\lambda^{|k|+d}} f^{(k)}_{\varepsilon}(x/\lambda)$ we have

$$\beta^{(k)}_{\varepsilon}(B,x) \overset{\text{Law}}{=} 2^{-n(2-(|k|+d)|H|)}\beta^{(k)}_{2^{n|H|}\varepsilon}(2^n B, 2^{n|H|}x),$$

where $|H| = H_1 + \cdots + H_d$. It follows from (6) and (7), we can get that for all Borel sets $B \subseteq D^{n+1}_q$,

$$\left|\mathbb{E}[(\beta^{(k)}_{\varepsilon}(B,x))^m]\right| \leq C_{m,\rho,H}2^{-nm(2-(|k|+d)|H|-2\rho_1)}|B|^{m\rho_1},$$

and

$$\left|\mathbb{E}[(\beta^{(k)}_{\varepsilon}(B,x) - \beta^{(k)}_{\varepsilon'}(B,x'))^m]\right| \leq C_{m,\rho,H}2^{-nm(2-(|k|+d)|H|-\rho_2|H|)}|(\varepsilon,x) - (\varepsilon',x')|^{m\rho_2}. \tag{12}$$

Then if we choose $0 < \rho_1 < \min_{l=1,2,\cdots,d}\{1 - H_l(k_l + \frac{1}{2})\}$ and $0 < \rho_2 < \min_{l=1,2,\cdots,d}\{1 \wedge (\frac{1}{H_l} - k_l - \frac{1}{2})\}$ we have

$$\left|\mathbb{E}[(\beta^{(k)}_{\varepsilon}(B,x))^m]\right| \leq C|B|^{m\rho_1}, \tag{13}$$

and

$$\left|\mathbb{E}[(\beta^{(k)}_{\varepsilon}(B,x) - \beta^{(k)}_{\varepsilon'}(B,x'))^m]\right| \leq C|(\varepsilon,x) - (\varepsilon',x')|^{m\rho_2}.$$

Let $B_T = \{0 \leq r \leq s \leq T\}$ and set $\beta^{(k)}_{\varepsilon}(T,x) =: \beta^{(k)}_{\varepsilon}(B_T,x)$. If $T, T' \leq M < \infty$, then $|B_T - B_{T'}| \leq M|T - T'|$. In the following discussion, we let $D = \{(s,r) : 0 \leq r \leq s \leq T\}$, $D' = \{(s,r) : 0 \leq r \leq s \leq T'\}$. Without loss of generality, we assume that $T < T'$. Then, by (13), we have

$$\left|\mathbb{E}[(\beta^{(k)}_{\varepsilon}(T,x) - \beta^{(k)}_{\varepsilon}(T',x))^m]\right| \leq C|T' - T|^{m\rho_1}. \tag{14}$$

Combining (14) with (12), for some $\gamma < \min\{\rho_1, \rho_2\}$, we have

$$\left|\mathbb{E}[(\beta^{(k)}_{\varepsilon}(T,x) - \beta^{(k)}_{\varepsilon'}(T',x'))^m]\right| \leq C|(\varepsilon,x,T) - (\varepsilon',x',T')|^{m\gamma}$$

holds locally, which assures us of a locally uniform and continuous limit

$$\beta^{(k)}(T,x) = \lim_{\varepsilon\to 0}\beta^{(k)}_{\varepsilon}(T,x).$$

Hence, these show that $\beta^{(k)}(T,x)$ exists in L^m spaces for all $m \geq 1$ and has a modification which is a.s. joint Hölder continuous in (T,x). This completes the proof of Theorem 3.1. □

Remark 3.3. *Note that if $d = 1$, $k = 1$ and $H = \frac{1}{\alpha}$, the condition for the existence of $\beta^{(k)}(T,x)$ in Theorem 3.1 is consistent with that in Rosen [23]. It is easy to see that if $\alpha = 2$, the LFSP degenerates into fractional Brownian motion. If $d = 1$, $k = 1$ and $\alpha = 2$, the condition for the existence of $\beta^{(k)}(T,x)$ in Theorem 3.1 is consistent with that in Jung and Markowsky [16].*

4. *P*-variation

In this section, as a related problem, we discuss the p-variation of derivatives of self-intersection local time for the LFSP.

For better study of p-variation of $\beta^{(k)}(T,0)$, we recall the definition of p-variation associated with a stochastic process $X = \{X_t, t \geq 0\}$. For fixed $M > 0$ and any partitions $\{0 = t_0 < t_1 < t_2 < \cdots < t_n = M\}$ of $[0, M]$ with $\max_i |t_i - t_{i-1}| \to 0$ ($n \to \infty$), define

$$V_p(X, M) := \lim_{n\to\infty} \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^p, \qquad p > 0,$$

if the limit exists in $L^1(\Omega)$. The process X_t has bounded p-variation if $V_p(X, M) < \infty$ for any $M > 0$, and $V_p(X, M)$ is called the p-variation associated with a stochastic process X_t on $[0, M]$.

Theorem 4.1. *Assume that $H_l(k_l + 1) < 1$ for all $l = 1, 2, \cdots, d$, then the p-variation of $\beta^{(k)}(T,0)$ on $[0, M]$ equals to zero for any $M > 0$ provided $p > \max_{l=1,2,\cdots,d}\{\frac{2}{2-H_l(k_l+1)}\}$.*

Proof. By Theorem 3.1, we have $\beta^{(k)}_\varepsilon(T,0) \to \beta^{(k)}(T,0)$ in L^2, as $\varepsilon \to 0$. Hence

$$\begin{aligned}
&\mathbb{E}\Big[(\beta^{(k)}(T_1,0) - \beta^{(k)}(T_2,0))^2\Big] \\
&= \lim_{\varepsilon\to 0} \mathbb{E}\Big[\Big(\int_0^{T_1}\int_0^s f^{(k)}_\varepsilon(X^{\alpha,H}_s - X^{\alpha,H}_r)drds - \int_0^{T_2}\int_0^s f^{(k)}_\varepsilon(X^{\alpha,H}_s - X^{\alpha,H}_r)drds\Big)^2\Big] \\
&= \lim_{\varepsilon\to 0} \mathbb{E}\Big[\Big(\int_{T_2}^{T_1}\int_0^s f^{(k)}_\varepsilon(X^{\alpha,H}_s - X^{\alpha,H}_r)drds\Big)^2\Big] \\
&=: \Theta(\alpha, H)
\end{aligned}$$

for all $0 < T_2 < T_1 \leq M$. For simplicity, we set $M = 1$.

$$\begin{aligned}
\Theta(\alpha, H) &= \lim_{\varepsilon\to 0} \mathbb{E}\Big[\Big(\int_{T_2}^{T_1}\int_0^s f^{(k)}_\varepsilon(X^{\alpha,H}_s - X^{\alpha,H}_r)drds\Big)^2\Big] \\
&= \frac{1}{(2\pi)^{2d}} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^{2d}} \int_{T_2}^{T_1}\int_0^{s_1}\int_{T_2}^{T_1}\int_0^{s_2} \mathbb{E} e^{i\langle \zeta, (X^{\alpha,H}_{s_1} - X^{\alpha,H}_{r_1})\rangle} e^{i\langle \eta, (X^{\alpha,H}_{s_2} - X^{\alpha,H}_{r_2})\rangle} \\
&\qquad \times e^{-\frac{\varepsilon}{2}\sum_{l=1}^d (|\zeta_l|^2 + |\eta_l|^2)} \Big(\prod_{l=1}^d |\zeta_l|^{k_l}\Big)\Big(\prod_{l=1}^d |\eta_l|^{k_l}\Big) dr_2 ds_2 dr_1 ds_1 d\zeta d\eta,
\end{aligned}$$

where $\zeta = (\zeta_1, \cdots, \zeta_d)$, $\eta = (\eta_1, \cdots, \eta_d)$, $|\zeta_l|$ and $|\eta_l|$ denote the absolute value of ζ_l and η_l respectively. There are six possibilities for the ordering of r_1, s_1 and r_2, s_2. Then by the symmetry of r_1, s_1 and r_2, s_2, we divide them into the following three cases: $\{0 < r_1 < r_2 < s_2 < s_1\}$, $\{0 < r_2 < r_1 < s_2 < s_1\}$ and $\{0 < r_2 < s_2 < r_1 < s_1\}$. This gives

$$\Theta(\alpha, H) = \frac{2}{(2\pi)^{2d}}(\Upsilon_1 + \Upsilon_2 + \Upsilon_3),$$

where

$$\begin{aligned}
\Upsilon_1 := \lim_{\varepsilon\to 0} &\int_{\mathbb{R}^{2d}} \int_{T_2}^{T_1}\int_{T_2}^{s_1}\int_0^{s_2}\int_0^{r_2} \mathbb{E}\Big(e^{i\langle \zeta, (X^{\alpha,H}_{s_1} - X^{\alpha,H}_{r_1})\rangle} e^{i\langle \eta, (X^{\alpha,H}_{s_2} - X^{\alpha,H}_{r_2})\rangle}\Big) e^{-\frac{\varepsilon}{2}\sum_{l=1}^d (|\zeta_l|^2 + |\eta_l|^2)} \\
&\times \Big(\prod_{l=1}^d |\zeta_l|^{k_l}\Big)\Big(\prod_{l=1}^d |\eta_l|^{k_l}\Big) dr_1 dr_2 ds_2 ds_1 d\zeta d\eta,
\end{aligned}$$

$$\begin{aligned}\Upsilon_2 := \lim_{\varepsilon\to 0}\int_{\mathbb{R}^{2d}}\int_{T_2}^{T_1}\int_{T_2}^{s_1}\int_0^{s_2}\int_0^{r_1} &\mathbb{E}\Big(e^{i\langle\zeta,(X^{\alpha,H}_{s_1}-X^{\alpha,H}_{r_1})\rangle}e^{i\langle\eta,(X^{\alpha,H}_{s_2}-X^{\alpha,H}_{r_2})\rangle}\Big)e^{-\frac{\varepsilon}{2}\sum_{l=1}^d(|\zeta_l|^2+|\eta_l|^2)}\\ &\times(\prod_{l=1}^d|\zeta_l|^{k_l})(\prod_{l=1}^d|\eta_l|^{k_l})dr_2dr_1ds_2ds_1d\zeta d\eta,\end{aligned}$$

and

$$\begin{aligned}\Upsilon_3 := \lim_{\varepsilon\to 0}\int_{\mathbb{R}^{2d}}\int_{T_2}^{T_1}\int_{T_2}^{s_1}\int_0^{r_1}\int_0^{s_2} &\mathbb{E}\Big(e^{i\langle\zeta,(X^{\alpha,H}_{s_1}-X^{\alpha,H}_{r_1})\rangle}e^{i\langle\eta,(X^{\alpha,H}_{s_2}-X^{\alpha,H}_{r_2})\rangle}\Big)e^{-\frac{\varepsilon}{2}\sum_{l=1}^d(|\zeta_l|^2+|\eta_l|^2)}\\ &\times(\prod_{l=1}^d|\zeta_l|^{k_l})(\prod_{l=1}^d|\eta_l|^{k_l})dr_2ds_2dr_1ds_1d\zeta d\eta.\end{aligned}$$

For the term Υ_1, we have

$$\begin{aligned}\Upsilon_1 &=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^{2d}}\int_{T_2}^{T_1}\int_{T_2}^{s_1}\int_0^{s_2}\int_0^{r_2}\mathbb{E}\Big(e^{i\langle\zeta,(X^{\alpha,H}_{s_1}-X^{\alpha,H}_{s_2}+X^{\alpha,H}_{s_2}-X^{\alpha,H}_{r_2}+X^{\alpha,H}_{r_2}-X^{\alpha,H}_{r_1})\rangle}\cdot e^{-i\langle\eta,(X^{\alpha,H}_{s_2}-X^{\alpha,H}_{r_2})\rangle}\Big)\\ &\qquad\times e^{-\sum_{l=1}^d\frac{\varepsilon}{2}(|\zeta_l|^2+|\eta_l|^2)}(\prod_{l=1}^d|\zeta_l|^{k_l})(\prod_{l=1}^d|\eta_l|^{k_l})dr_1dr_2ds_2ds_1d\zeta d\eta\\ &=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^{2d}}\int_{T_2}^{T_1}\int_{T_2}^{s_1}\int_0^{s_2}\int_0^{r_2}e^{-\sum_{l=1}^d(s_1-s_2)^{\alpha H_l}|\zeta_l|^\alpha-\sum_{l=1}^d(s_2-r_2)^{\alpha H_l}|\zeta_l+\eta_l|^\alpha-\sum_{l=1}^d(r_2-r_1)^{\alpha H_l}|\zeta_l|^\alpha}\\ &\qquad\times e^{-\sum_{l=1}^d\frac{\varepsilon}{2}(|\zeta_l|^2+|\eta_l|^2)}(\prod_{l=1}^d|\zeta_l|^{k_l})(\prod_{l=1}^d|\eta_l|^{k_l})dr_1dr_2ds_2ds_1d\zeta d\eta\\ &=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^{2d}}\int_{E_1}e^{-\sum_{l=1}^d w_4^{\alpha H_l}|\zeta_l|^\alpha-\sum_{l=1}^d w_3^{\alpha H_l}|\zeta_l+\eta_l|^\alpha-\sum_{l=1}^d w_2^{\alpha H_l}|\zeta_l|^\alpha}\prod_{i=1}^4 dw_i\\ &\qquad\times e^{-\sum_{l=1}^d\frac{\varepsilon}{2}(|\zeta_l|^2+|\eta_l|^2)}(\prod_{l=1}^d|\zeta_l|^{k_l})(\prod_{l=1}^d|\eta_l|^{k_l})d\zeta d\eta,\end{aligned}$$

where the set

$$E_1=\Big\{(w_1,w_2,w_3,w_4): w_1+w_2+w_3\ge T_2, w_1+w_2+w_3+w_4\le T_1, w_1,w_2,w_3,w_4\ge 0\Big\}.$$

By the Hölder inequality with $0<a<1$ and the inequality

$$\int_0^1 e^{-x^{\alpha H_l}|v|^\alpha}dx\le\frac{C_\alpha}{1+|v|^{1/H_l}},$$

$l=1,\cdots,d$, we have

$$\begin{aligned}\int_0^{T_1-w_1-w_2-w_3}e^{-\sum_{l=1}^d w_4^{\alpha H_l}|\zeta_l|^\alpha}dw_4&\le(T_1-T_2)^a\Big(\int_0^1 e^{-\sum_{l=1}^d w_4^{\alpha H_l}|\zeta_l|^\alpha/(1-a)}dw_4\Big)^{1-a}\\ &\le C\frac{(T_1-T_2)^a}{1+\prod_{l=1}^d|\zeta_l|^{(1-a)/H_l}}.\end{aligned}$$

The integral with respect to dw_1 is bounded by

$$\int_{T_2-w_2-w_3}^{T_1-w_2-w_3-w_4}dw_1\le T_1-T_2,$$

for all $(w_1, w_2, w_3, w_4) \in E_1$.

Thus, we can get

$$
\begin{aligned}
\Upsilon_1 \leq & C(T_1 - T_2)^{1+a} \int_{\mathbb{R}^{2d}} \int_{E_1} \frac{1}{1 + \prod_{l=1}^d |\zeta_l|^{(1-a)/H_l}} \cdot e^{-\sum_{l=1}^d w_3^{\alpha H_l} |\zeta_l + \eta_l|^\alpha - \sum_{l=1}^d w_2^{\alpha H_l} |\zeta_l|^\alpha} dw_2 dw_3 \\
& \times (\prod_{l=1}^d |\zeta_l|^{k_l})(\prod_{l=1}^d |\eta_l|^{k_l}) d\zeta d\eta \\
\leq & C(T_1 - T_2)^{1+a} \int_{\mathbb{R}^{2d}} \frac{1}{1 + \prod_{l=1}^d |\zeta_l|^{(1-a)/H_l}} \cdot \frac{1}{1 + \prod_{l=1}^d |\zeta_l + \eta_l|^{1/H_l}} \cdot \frac{1}{1 + |\zeta|^{1/H_l}} \\
& \times \prod_{l=1}^d |\zeta_l|^{k_l} \prod_{l=1}^d |\eta_l|^{k_l} d\zeta d\eta.
\end{aligned}
$$

Let $\zeta_l = x_l$ and $\zeta_l + \eta_l = y_l$, it can be obtained that

$$
\begin{aligned}
& \int_{\mathbb{R}^{2d}} \frac{1}{1 + \prod_{l=1}^d |\zeta_l|^{(1-a)/H_l}} \cdot \frac{1}{1 + \prod_{l=1}^d |\zeta_l + \eta_l|^{1/H_l}} \cdot \frac{1}{1 + \prod_{l=1}^d |\zeta_l|^{1/H_l}} \prod_{l=1}^d |\zeta_l|^{k_l} \prod_{l=1}^d |\eta_l|^{k_l} d\zeta d\eta \\
= & \int_{\mathbb{R}^{2d}} \frac{1}{1 + \prod_{l=1}^d |x_l|^{(1-a)/H_l}} \cdot \frac{1}{1 + \prod_{l=1}^d |y_l|^{1/H_l}} \cdot \frac{1}{1 + \prod_{l=1}^d |x_l|^{1/H_l}} \prod_{l=1}^d |x_l|^{k_l} \prod_{l=1}^d |y_l - x_l|^{k_l} dx dy \\
\leq & \int_{\mathbb{R}^{2d}} \frac{1}{1 + \prod_{l=1}^d |x_l|^{(2-a)/H_l}} \cdot \frac{1}{1 + \prod_{l=1}^d |y_l|^{1/H_l}} \cdot \prod_{l=1}^d |x_l|^{k_l} \prod_{l=1}^d (|x_l|^{k_l} + |y_l|^{k_l}) dx dy \\
\leq & \int_{\mathbb{R}^{2d}} \frac{\prod_{l=1}^d (|x_l|^{2k_l} + |x_l|^{k_l} |y_l|^{k_l})}{(1 + \prod_{l=1}^d |x_l|^{(2-a)/H_l})(1 + \prod_{l=1}^d |y_l|^{1/H_l})} dx dy,
\end{aligned}
$$

which is bounded as long as $\frac{1}{H_l}(2-a) - 2k_l > 1$ and $1 < \frac{1}{H_l} - k_l$, for all $l = 1, 2, \cdots, d$. This gives

$$
0 < a < 2 - H_l(2k_l + 1), \quad H_l(k_l + 1) < 1.
$$

Similar to the proof of Υ_1, for the term Υ_2, we then have

$$
\begin{aligned}
\Upsilon_2 = & \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2d}} \int_{T_2}^{T_1} \int_{T_2}^{s_1} \int_0^{s_2} \int_0^{r_1} \mathbb{E}\Big(e^{i\langle \zeta, (X_{s_1}^{\alpha,H} - X_{s_2}^{\alpha,H} + X_{s_2}^{\alpha,H} - X_{r_1}^{\alpha,H}) \rangle} \cdot e^{i\langle \eta, (X_{s_2}^{\alpha,H} - X_{r_1}^{\alpha,H} + X_{r_1}^{\alpha,H} - X_{r_2}^{\alpha,H}) \rangle} \Big) \\
& \times e^{-\sum_{l=1}^d \frac{\varepsilon}{2}(|\zeta_l|^2 + |\eta_l|^2)} (\prod_{l=1}^d |\zeta_l|^{k_l})(\prod_{l=1}^d |\eta_l|^{k_l}) dr_2 dr_1 ds_2 ds_1 d\zeta d\eta \\
= & \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2d}} \int_{T_2}^{T_1} \int_{T_2}^{s_1} \int_0^{s_2} \int_0^{r_1} e^{-\sum_{l=1}^d (s_1 - s_2)^{\alpha H_l} |\zeta_l|^\alpha - \sum_{l=1}^d (s_2 - r_1)^{\alpha H_l} |\zeta_l + \eta_l|^\alpha - \sum_{l=1}^d (r_1 - r_2)^{\alpha H_l} |\eta_l|^\alpha} \\
& \times e^{-\sum_{l=1}^d \frac{\varepsilon}{2}(|\zeta_l|^2 + |\eta_l|^2)} (\prod_{l=1}^d |\zeta_l|^{k_l})(\prod_{l=1}^d |\eta_l|^{k_l}) dr_2 dr_1 ds_2 ds_1 d\zeta d\eta \\
\leq & \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2d}} \int_{E_1} e^{-\sum_{l=1}^d w_4^{\alpha H_l} |\zeta_l|^\alpha - \sum_{l=1}^d w_3^{\alpha H_l} |\zeta_l + \eta_l|^\alpha - \sum_{l=1}^d w_2^{\alpha H_l} |\eta_l|^\alpha} \prod_{i=1}^4 dw_i (\prod_{l=1}^d |\zeta_l|^{k_l})(\prod_{l=1}^d |\eta_l|^{k_l}) d\zeta d\eta \\
\leq & C(T_1 - T_2)^{1+a} \int_{\mathbb{R}^{2d}} \frac{1}{1 + \prod_{l=1}^d |\zeta_l|^{(1-a)/H_l}} \cdot \frac{1}{1 + \prod_{l=1}^d |\zeta_l + \eta_l|^{1/H_l}} \cdot \frac{1}{1 + \prod_{l=1}^d |\eta_l|^{1/H_l}} \\
& \times \prod_{l=1}^d |\zeta_l|^{k_l} \prod_{l=1}^d |\eta_l|^{k_l} d\zeta d\eta.
\end{aligned}
$$

Using the same variable substitution of $\zeta_l = x_l$ and $\zeta_l + \eta_l = y_l$, we can get

$$
\begin{aligned}
\Upsilon_2 \le & C(T_1 - T_2)^{1+a} \int_{\mathbb{R}^{2d}} \frac{1}{1 + \prod_{l=1}^{d} |x_l|^{(1-a)/H_l}} \cdot \frac{1}{1 + \prod_{l=1}^{d} |y_l|^{1/H_l}} \cdot \frac{1}{1 + \prod_{l=1}^{d} |y_l - x_l|^{1/H_l}} \\
& \times \prod_{l=1}^{d} |x_l|^{k_l} \prod_{l=1}^{d} |y_l - x_l|^{k_l} dxdy \\
\le & C(T_1 - T_2)^{1+a} \int_{\mathbb{R}^{2d}} \frac{\prod_{l=1}^{d} |x_l|^{k_l}}{1 + \prod_{l=1}^{d} |x_l|^{(1-a)/H_l}} \cdot \frac{1}{1 + \prod_{l=1}^{d} |y_l|^{1/H_l}} \cdot \frac{\prod_{l=1}^{d} |y_l - x_l|^{k_l}}{1 + \prod_{l=1}^{d} |y_l - x_l|^{1/H_l}} dxdy,
\end{aligned}
$$

is bounded if $0 < a < 1 - H_l(k_l + 1)$ for all $l = 1, 2, \cdots, d$.

Then for the term Υ_3, it is easier to obtain

$$
\begin{aligned}
\Upsilon_3 = & \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2d}} \int_{T_2}^{T_1} \int_{T_2}^{s_1} \int_{0}^{r_1} \int_{0}^{s_2} e^{-\sum_{l=1}^{d} (s_1 - r_1)^{\alpha H_l} |\zeta_l|^{\alpha} - \sum_{l=1}^{d} (s_2 - r_2)^{\alpha H_l} |\eta_l|^{\alpha}} \cdot e^{-\frac{\varepsilon}{2} \sum_{l=1}^{d} (|\zeta_l|^2 + |\eta_l|^2)} \\
& \times (\prod_{l=1}^{d} |\zeta_l|^{k_l})(\prod_{l=1}^{d} |\eta_l|^{k_l}) dr_2 ds_2 dr_1 ds_1 d\zeta d\eta \\
= & \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{2d}} \int_{E_2} e^{-\sum_{l=1}^{d} w_4^{\alpha H_l} |\zeta_l|^{\alpha} - \sum_{l=1}^{d} w_2^{\alpha H_l} |\eta_l|^{\alpha}} \prod_{i=1}^{4} dw_i \cdot e^{-\frac{\varepsilon}{2} \sum_{l=1}^{d} (|\zeta_l|^2 + |\eta_l|^2)} \\
& \times (\prod_{l=1}^{d} |\zeta_l|^{k_l})(\prod_{l=1}^{d} |\eta_l|^{k_l}) d\zeta d\eta \\
\le & C(T_1 - T_2)^{1+a} \int_{\mathbb{R}^{2d}} \frac{1}{1 + \prod_{l=1}^{d} |\zeta_l|^{(1-a)/H_l}} \cdot \frac{1}{1 + \prod_{l=1}^{d} |\eta_l|^{1/H_l}} \prod_{l=1}^{d} |\zeta_l|^{k_l} \prod_{l=1}^{d} |\eta_l|^{k_l} d\zeta d\eta \\
= & C(T_1 - T_2)^{1+a} \int_{\mathbb{R}^{2d}} \frac{\prod_{l=1}^{d} |x_l|^{k_l}}{1 + \prod_{l=1}^{d} |x_l|^{(1-a)/H_l}} \cdot \frac{\prod_{l=1}^{d} |y_l - x_l|^{k_l}}{1 + \prod_{l=1}^{d} |y_l - x_l|^{1/H_l}} dxdy,
\end{aligned}
$$

where

$$
E_2 = \Big\{ (w_1, w_2, w_3, w_4) : w_1 + w_2 \ge T_2, w_1 + w_2 + w_3 + w_4 \le T_1, w_1, w_2, w_3, w_4 \ge 0 \Big\}
$$

and applying the same variable substitution of $\zeta_l = x_l$ and $\zeta_l + \eta_l = y_l$. Thus Υ_3 is bounded if $0 < a < 1 - H_l(k_l + 1)$ and $H_l(k_l + 1) < 1$ for all $l = 1, 2, \cdots, d$. So we can have

$$
\Theta(\alpha, H) \le C(T_1 - T_2)^{1+a}.
$$

From what we have discussed above, the desired estimate is

$$
\mathbb{E}\Big[(\beta^{(k)}(T_1, 0) - \beta^{(k)}(T_2, 0))^2 \Big] \le C(T_1 - T_2)^{1+a},
$$

for all $0 < T_2 < T_1$ and $0 < a < 1 - H_l(k_l + 1)$. It follows that

$$
\mathbb{E}\Big[(\beta^{(k)}(T_1, 0) - \beta^{(k)}(T_2, 0))^p \Big] \le \Big\{ \mathbb{E}[(\beta^{(k)}(T_1, 0) - \beta^{(k)}(T_2, 0))^2] \Big\}^{\frac{p}{2}} \le C(T_1 - T_2)^{(1+a)\frac{p}{2}},
$$

for all $0 < p \le 2$.

Thus, the p-variation of $\beta^{(k)}(T, 0)$ equals to zero, provided $(1 + a)\frac{p}{2} > 1$ for all $0 < a < 1 - H_l(k_l + 1)$ and $l = 1, 2, \cdots, d$. This gives

$$
p > \max_{l = 1, 2, \cdots, d} \left\{ \frac{2}{2 - H_l(k_l + 1)} \right\}.
$$

This completes the proof. □

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