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Tempered fractional Dirac type systems

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Abstract. In this research, we present a boundary value problem for a Dirac system with tempered fractional derivatives. Firstly, the definitions and properties of tempered fractional derivatives and tempered fractional integrals are given. Next, it is shown that the operator of the corresponding eigenvalue problem is a self-adjoint operator, that the eigenfunctions are orthogonal concerning different eigenvalues, and in which case the eigenvalue is simple.

Fractional derivative and fractional integral, which is a sub-branch of mathematical analysis, is the extended form of derivative and integral to non-integer orders [1–3]. In the fractional derivative and integral fields, which are known to have emerged towards the end of the 17th century; Many researchers such as Leibnitz, Riemann, Liouville, Weyl, Euler, Lagrange, Fourier, Greenwald, Letnikov, Laplace, Abel, Holmgren, Heaviside, Hadamard, Lacroix, and Caputo have done many studies. Mathematical models created with fractional differential equations have obtained more successful results than classical integer differential equations. Fractional calculus has been used frequently in the modeling and applications of problems in the fields of science and engineering in recent years. Fractional calculation technique with the discovery of its wide application area; transmission line theory, signal processing, chemical analysis, heat transfer, dam hydraulics, materials science, temperature field problems, oil layers, diffusion problems, fractal equation, waves in liquids and gases, Schrödinger equation, fluids, physics, and control theory, analytical and numerical methods are used in many fields such as earthquake sciences. With the spread of fractional calculations, many scientists have worked in this field [1–8].

Caputo defined the fractional derivative of Caputo in 1967 to eliminate the problem of calculating the initial values and measuring experimentally arises in the applications of the Laplace transform of the Riemann-Liouville definition. The difference between fractional analysis from the classical analysis is that there is more than one derivative definition. Since the existence of more than one definition allows using the most appropriate definition according to the type of problem, the best solution to the problem is obtained. Main fractional derivative definitions; Riemann-Liouville, Grünwald-Letnikov, and Caputo are fractional derivatives.

The tempered fractional derivative is an expanded form of the fractional derivative, and the tempered fractional derivative is obtained by multiplying the classical fractional derivative with an exponential function. The fractional operator is dependent on a parameter l, and if this parameter is equal to zero, classical Caputo and Riemann-Liouville fractional derivatives are obtained [9]. Tempered fractional derivatives are used in finance, groundwater hydrology, geophysical flow, and poroelasticity calculations. Tempered fractional derivatives; It is used in the modeling of price fluctuations in finance, to provide slow convergence in

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diffusion. In the study [10], discretized collocation methods on piecewise polynomial spaces are proposed for the solution of equations by examining a class of tempered fractional differential equations with extreme value problems. In addition, regularity results are built on weighted spaces and the order of convergence is examined, examples are given and compared with other methods. In the study [13], they investigated the existence of infinite eigenvalues and eigenfunctions for the tempered fractional Sturm-Liouville problem, showed that the set of eigenfunctions for different eigenvalues was orthogonal and presented an example where eigenvalue bounds were obtained for classical and tempered cases. In the study, [14], by defining the tempered $\Psi-$Caputo fractional derivative, the Cauchy problem for fractional differential equations, and some existence and uniqueness results are examined. A Henry-Gronwall type inequality is presented for the tempered fractional integral and integral inequality. Many mathematicians have worked on tempered fractional calculation [11, 12, 18–21]. The Dirac equation was found in the first quarter of the 20th century while searching for a relative covariant wave equation of the Schrödinger form, and it has an important place today. When the literature is reviewed, it is seen that there is a need for new studies involving the fractional Dirac system ([23, 24]). In the study [15], they investigated the properties of a regular q-fractional Dirac type system and gave the existence and uniqueness condition of eigenfunctions using the fixed point theorem. In the study [16], they dealt with the exponential Dirac system in the sense of Riemann-Liouville and Caputo and the fractional Dirac system with Mittag-Leffler core and obtained the representations of the solutions for Dirac systems by Laplace transforms. In the study [22], they investigated one-dimensional fractional Dirac-type systems containing the Caputo and Riemann-Liouville fractional derivatives. In this study, we examined the Dirac type system, which includes Riemann-Liouville tempered fractional integrals and their derivatives, and the Dirac type system, which includes Caputo tempered fractional integrals and derivatives. We show that the operator generated by the defined fractional Dirac type system is self-adjoint, and its eigenfunctions are orthogonal. We also proved the existence and uniqueness theorem of the defined system.

1. Preliminaries

In this chapter, we gave definitions and properties of tempered fractional derivatives and tempered fractional integrals which are used in the other chapter. The definitions and the properties play the most important role in the other chapter.

Definition 1.1. *([18]) Let $\alpha \in (0,1), \sigma \geq 0$ be a fix parameter and $\nu(t) \in L^p(a,b) \quad (p \in [1,\infty))$. The left and right Riemann-Liouville tempered fractional integrals of the order α are defined as*

$$\left({}_aI_t^{\alpha,\sigma}\right)\nu(t) = (e^{-\sigma t}\,{}_aI_t^{\alpha} e^{\sigma t})\nu(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{e^{-\sigma(t-s)}\nu(s)}{(t-s)^{1-\alpha}}ds, \qquad t > a, \tag{1}$$

and

$$\left({}_tI_b^{\alpha,\sigma}\right)\nu(t) = (e^{\sigma t}\,{}_tI_b^{\alpha} e^{-\sigma t})\nu(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \frac{e^{-\sigma(s-t)}\nu(s)}{(s-t)^{1-\alpha}}ds, \qquad t < b, \tag{2}$$

where $\Gamma(\alpha)$ is a Euler's gamma function, the expressions ${}_aI_t^{\alpha}\nu(x)$ and ${}_tI_b^{\alpha}\nu(x)$ are Riemann-Liouville fractional integrals (([17])).

Definition 1.2. *([18]) Let $\alpha \in (0,1), \sigma \geq 0$ be a fix parameter and $\nu(t) \in L^p(a,b)(p \in [1,\infty))$. The left and right Riemann-Liouville tempered fractional derivatives of order α are defined as*

$$\left({}_aD_t^{\alpha,\sigma}\right)\nu(t) = (e^{-\sigma t}\,{}_aD_t^{\alpha} e^{\sigma t})\nu(t) = \frac{e^{-\sigma t}}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{e^{\sigma s}\nu(s)}{(t-s)^{\alpha}}ds, \qquad t > a, \tag{3}$$

and

$$\left({}_tD_b^{\alpha,\sigma}\right)\nu(t) = (e^{\sigma t}\,{}_tD_b^{\alpha} e^{-\sigma t})\nu(t) = \frac{-e^{\sigma t}}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b \frac{e^{-\sigma s}\nu(s)}{(s-t)^{\alpha}}ds, \qquad t < b. \tag{4}$$

The tempered fractional derivatives (3) and (4) are the Riemann-Liouville fractional derivatives ${}_aD_t^{\alpha,\sigma}\nu(t)$ and ${}_tD_b^{\alpha,\sigma}\nu(t)$ for $\sigma = 0$ ([17]).

Definition 1.3. *([18]) Let $\alpha \in (0,1), \sigma \geq 0$ be a fix parameter and $\nu(t) \in L^p(a,b)(p \in [1,\infty))$. The left and right Caputo tempered fractional derivatives of order α are defined as*

$$\left({}_a^CD_t^{\alpha,\sigma}\right)\nu(t) = (e^{-\sigma t}\,{}_a^CD_t^{\alpha}e^{\sigma t})\nu(t) = \frac{e^{-\sigma t}}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{[e^{\sigma s}\nu(s)]'}{(t-s)^{\alpha}}ds, \qquad t > a, \tag{5}$$

and

$$\left({}_t^CD_b^{\alpha,\sigma}\right)\nu(t) = (e^{\sigma t}\,{}_t^CD_b^{\alpha}e^{-\sigma t})\nu(t) = \frac{-e^{\sigma t}}{\Gamma(1-\alpha)}\frac{d}{dt}\int_x^b \frac{[e^{-\sigma s}\nu(s)]'}{(-s-t)^{\alpha}}ds, \qquad t < b. \tag{6}$$

The tempered fractional derivatives (5) and (6) are the Caputo fractional derivatives ${}_a^CD_t^{\alpha,\sigma}\nu(t)$ and ${}_t^CD_b^{\alpha,\sigma}\nu(t)$ for $\sigma = 0$ ([17]).

Proposition 1.4 ([19, 20]). *Let $\alpha > 0$ and $\nu(t) \in L^p(a,b)(p \in [1,\infty))$.*

1. For $\alpha_1, \alpha_2 > 0$ and $\sigma \geq 0$, the tempered fractional integral operators satisfy following

$${}_aI_t^{\alpha_1,\sigma}\,{}_aI_t^{\alpha_2,\sigma}\nu(t) = {}_aI_t^{\alpha_1+\alpha_2,\sigma}\nu(t)$$
$${}_tI_b^{\alpha_1,\sigma}\,{}_tI_b^{\alpha_2,\sigma}\nu(t) = {}_tI_b^{\alpha_1+\alpha_2,\sigma}\nu(t).$$

2. For $\alpha > 0$, $\sigma \geq 0$ and any function $\nu(t) \in L^p(a,b)$,

$${}_aD_t^{\alpha,\sigma}\,{}_aI_t^{\alpha,\sigma}\nu(t) = \nu(t)$$
$${}_tD_b^{\alpha,\sigma}\,{}_tI_b^{\alpha,\sigma}\nu(t) = \nu(t).$$

3. If $\nu(t)$ is continuous on $[a,b]$ then for $\alpha > 0$ and $\sigma \geq 0$,

$${}_a^CD_t^{\alpha,\sigma}\,{}_aI_t^{\alpha,\sigma}\nu(t) = \nu(t)$$
$${}_t^CD_b^{\alpha,\sigma}\,{}_tI_b^{\alpha,\sigma}\nu(t) = \nu(t).$$

4. Let $\alpha \in (0,1)$ and $\sigma \geq 0$, for $\nu(t) \in AC[a,b]$ (where $AC[a,b]$ stands for absolutely continuous functions on $[a,b]$),

$${}_aI_t^{\alpha,\sigma}\,{}_a^CD_t^{\alpha,\sigma}\nu(t) = \nu(t) - e^{-\sigma(t-a)}\nu(a) \tag{7}$$
$${}_tI_b^{\alpha,\sigma}\,{}_t^CD_b^{\alpha,\sigma}\nu(t) = v(t) - e^{-\sigma(t-b)}\nu(b). \tag{8}$$

5. Let $\alpha_1 > \alpha_2 > 0$, $\sigma \geq 0$ and $t \in [a,b]$ then

$${}_aD_t^{\alpha_2,\sigma}\,{}_aI_t^{\alpha_1,\sigma}\nu(t) = {}_aI_t^{\alpha_1-\alpha_2,\sigma}\nu(t)$$
$${}_tD_b^{\alpha_2,\sigma}\,{}_tI_b^{\alpha_1,\sigma}\nu(t) = {}_tI_b^{\alpha_1-\alpha_2,\sigma}\nu(t)$$

6. Let $\nu(t) \in AC[a,b]$ and $v(t) \in L^p(a,b)$ $\quad (p \in [1,\infty))$, then for $\alpha \in (0,1)$ and for $\sigma \geq 0$,

$$\int_a^b \nu(t){}_aD_t^{\alpha,\sigma}v(t)dt = \int_a^b v(t){}_t^CD_b^{\alpha,\sigma}\nu(t)dt + \nu(t){}_aI_t^{1-\alpha,\sigma}v(t)|_{t=a}^{t=b} \tag{9}$$

7. For $\alpha > 0$, $\sigma \geq 0$ and $p \geq 1$, ${}_aI_t^{\alpha,\sigma}$ is bounded on $L^p(a,b)$, i.e.

$$\left\|{}_aI_t^{\alpha,\sigma}\nu(t)\right\|_{L^p} \leq M_\alpha\left\|e^{\sigma t}\nu(t)\right\|_{L^p}, \qquad M_\alpha = \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}. \tag{10}$$

2. Tempered Fractional Dirac Type System

In this chapter, including $\alpha \in (0,1)$; we examined the Dirac type system which includes α order right and left Riemann-Liouville tempered fractional integrals and derivatives, and the Dirac type system which includes α order right and left Caputo tempered fractional integrals and derivatives.

Let $p(t)$ and $r(t)$ are real-valued continuous functions defined on $[a,b]$, and let

$$\begin{aligned} Fy &= \begin{pmatrix} 0 & {}_aD^{\alpha,\sigma} \\ {}^CD_b^{\alpha,\sigma} & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} p(t) & 0 \\ 0 & r(t) \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \\ &= \begin{pmatrix} {}_aD^{\alpha,\sigma} y_2(t) + p(t) y_1(t) \\ {}^CD_b^{\alpha,\sigma} y_1(t) + r(t) y_2(t) \end{pmatrix}. \end{aligned}$$

The fractional Dirac system is:

$$Fy_\lambda = \lambda \omega y_\lambda, \ a \le t \le b < \infty. \tag{11}$$

where λ is a complex spectral parameter, $y_\lambda(t) = \begin{pmatrix} y_{\lambda 1}(t) \\ y_{\lambda 2}(t) \end{pmatrix}$, and $\omega(t) = \begin{pmatrix} \omega_1(t) & 0 \\ 0 & \omega_2(t) \end{pmatrix}$ are real-valued continuous functions defined on $[a,b]$ and $\omega_i(t) > 0, \forall t \in [a,b], (i = 1,2)$. We consider the boundary conditions

$$c_{11} y(a) + c_{12\,a} I^{1-\alpha,\sigma} y(a) = 0, \tag{12}$$

$$c_{21} y(b) + c_{22\,a} I^{1-\alpha,\sigma} y(b) = 0, \tag{13}$$

with $c_{11}^2 + c_{12}^2 \neq 0$ and $c_{21}^2 + c_{22}^2 \neq 0$. This problem is worked out at ([22]) for $\sigma = 0$. Now, we introduce convenient Hilbert space $L^2_\omega((a,b);E)$ $\left(E := \mathbb{C}^2\right)$ of vector-valued functions using the inner product

$$\langle y, z \rangle = \int_a^b y_1(t)\overline{z_1}(t)\omega_1(t)dt + \int_a^b y_2(t)\overline{z_2}(t)\omega_2(t)dt,$$

where

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \qquad z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix},$$

and y_i, z_i $(i = 1,2)$ are real-valued continuous functions defined on $[a,b]$.

Theorem 2.1. *The operator $\Pi = \omega^{-1}F$ generated by fractional Dirac type system defined by (11)-(13) is formally self-adjoint on $L^2_\omega((a,b);E)$.*

Proof. Let $y(.), z(.) \in L^2_\omega((a,b);E)$. Then we get

$$\begin{aligned} \langle \Pi y, z \rangle - \langle y, \Pi z \rangle &= \int_a^b \left({}_aD^{\alpha,\sigma} y_1(t) + r(t) y_2(t)\right) \overline{z_2(t)} dt \\ &+ \int_a^b \left({}^CD_b^{\alpha,\sigma} y_2(t) + p(t) y_1(t)\right) \overline{z_1(t)} dt \\ &- \int_a^b y_1(t) \left(\overline{{}^CD_b^{\alpha,\sigma} z_2(t) + p(t) z_1(t)}\right) dt \\ &- \int_a^b y_2(t) \left(\overline{{}_aD^{\alpha,\sigma} z_1(t) + r(t) z_2(t)}\right) dt \\ &= \int_a^b \left({}_aD^{\alpha,\sigma} y_1(t)\right) \overline{z_2(t)} dt + \int_a^b r(t) y_2(t) \overline{z_2(t)} dt \end{aligned}$$

$$+\int_a^b \left({}^{C}D_b^{\alpha,\sigma} y_2(t)\right) z_1(t)dt + \int_a^b p(t)\, y_1(t)\overline{z_1(t)}dt$$
$$-\int_a^b y_1(t)\overline{\left({}^{C}D_b^{\alpha,\sigma} z_2(t)\right)}dt - \int_a^b p(t)\, y_1(t)\overline{z_1(t)}dt$$
$$-\int_a^b y_2(t)\overline{({}_aD^{\alpha,\sigma} z_1(t))}dt - \int_a^b r(t)\, y_2(t)\overline{z_2(t)}dt$$
$$=\int_a^b ({}_aD^{\alpha,\sigma} y_1(t))\,\overline{z_2(t)}dt + \int_a^b \left({}^{C}D_b^{\alpha,\sigma} y_2(t)\right)\overline{z_2(t)}dt$$
$$-\int_a^b y_1(t)\overline{\left({}^{C}D_b^{\alpha,\sigma} z_2(t)\right)}dt - \int_a^b y_2(t)\overline{({}_aD^{\alpha,\sigma} z_1(t))}dt.$$

Since

$$\int_a^b \left({}^{C}D_b^{\alpha,\sigma} y_2(t)\right) z_1(t)dt = \int_a^b y_2(t)\left(\overline{{}_aD^{\alpha,\sigma} z_1(t)}\right)dt - y_2(t)\overline{{}_aI^{1-\alpha,\sigma} z_1(t)}|_a^b$$

and

$$\int_a^b y_1(t)\overline{\left({}^{C}D_b^{\alpha,\sigma} z_2(t)\right)}dt = \int_a^b ({}_aD^{\alpha,\sigma} y_1(t))\,\overline{z_2(t)}dx - \overline{z_2(t)}{}_aI^{1-\alpha,\sigma} y_1(t)|_a^b,$$

we get

$$\langle \Pi y, z\rangle - \langle y, \Pi z\rangle = [y,z]_b - [y,z]_a\,, \tag{14}$$

where

$$[y,z]_t = y_2(t){}_aI^{1-\alpha,\sigma}\overline{z_1(t)}|_a^b - \overline{z_2(t)}{}_aI^{1-\alpha,\sigma} y_1(t)|_a^b.$$

The equality $\langle \Pi y, z\rangle = \langle y, \Pi z\rangle$ for any $y(.)\,,\ z(.) \in L^2_\omega((a,b);E)$. We have $[y,z]_b = 0$ and $[y,z]_a = 0$ from the boundary (12)-(13). Consequently, we get

$$\langle \Pi y, z\rangle = \langle y, \Pi z\rangle. \tag{15}$$

□

Theorem 2.2. *All eigenvalues of the problem (11)-(13) are real.*

Proof.

$$\langle f, Fy\rangle = \int_a^b f_1(t)\left({}^{C}D_b^{\alpha,\sigma} y_2(t) + p(t)\, y_1(t)\right)dt + \int_a^b f_2(t)\left({}_aD^{\alpha,\sigma} y_1(t) + r(t)\, y_2(t)\right)dt$$

$$= \int_a^b f_1(t){}^{C}D_b^{\alpha,\sigma} y_2(t)dt + \int_a^b f_1(t)p(t)\, y_1(t)dt + \int_a^b f_2(t){}_aD^{\alpha,\sigma} y_1(t)dt + \int_a^b f_2(t)r(t)\, y_2(t)dt$$

$$= \int_a^b y_2(t)\_aD^{\alpha,\sigma} f_1(t)dt - f_1(t){}_aI^{1-\alpha,\sigma} y_2(t)|_a^b + \int_a^b f_1(t)p(t)\, y_1(t)dt + \int_a^b y_1(t){}^{C}D_b^{\alpha,\sigma} f_2(t)dt$$
$$+ f_2(t){}_aI^{1-\alpha,\sigma} y_1(t)|_a^b + \int_a^b f_2(t)r(t)\, y_2(t)dt$$

Let λ be an eigenvalue of (11)-(13) and $y(t) = (y_1(t), y_2(t))^T$ be the corresponding eigenfunction. Then y and its complex conjugate $\overline{y(t)} = \left(\overline{y_1(t)}, \overline{y_2(t)}\right)^T$ satisfy

$$Fy(t) = \lambda\omega(t)y(t) \tag{16}$$

$$c_{11}y_1(a) + c_{12}\ {}_aI^{1-\alpha,\sigma}y_2(a) = 0 \tag{17}$$

$$c_{21}y_1(b) + c_{22}\ {}_aI^{1-\alpha,\sigma}y_2(b) = 0 \tag{18}$$

and

$$F\overline{y(t)} = \lambda\omega(t)\overline{y(t)} \tag{19}$$

$$c_{11}\overline{y_1(a)} + c_{12}\ {}_aI^{1-\alpha,\sigma}\overline{y_2(a} = 0 \tag{20}$$

$$c_{21}\overline{y_1(b)} + c_{22}\ {}_aI^{1-\alpha,\sigma}\ \overline{y_2(b)} = 0 \tag{21}$$

with $c_{11}^2 + c_{12}^2 \neq 0$ and $c_{21}^2 + c_{22}^2 \neq 0$. Then we obtain

$$\begin{aligned}(\lambda - \overline{\lambda})\langle y, y\rangle &= \langle \lambda y, y\rangle - \langle \overline{\lambda} y, y\rangle \\ &= \langle Fy, y\rangle - \langle F\overline{y}, y\rangle \\ &= y(b)\left[\begin{array}{c} c_{22}\ {}_aI^{1-\alpha,\sigma}y\,(b)\,c_{21}\overline{y\,(b)} \\ -c_{22}\ {}_aI^{1-\alpha,\sigma}\overline{y\,(b)}c_{21}y\,(b)\end{array}\right] \\ +y(a)&\left[\begin{array}{c} c_{12a}I^{1-\alpha,\sigma}\overline{y\,(a)}c_{11}y\,(a) \\ -c_{12a}I^{1-\alpha,\sigma}y\,(a)\,c_{11}\overline{y\,(a)}\end{array}\right].\end{aligned}$$

From boundary conditions (17), (18), (20) and (21), we have

$$(\lambda - \overline{\lambda})\langle y, y\rangle = 0$$

since $y(t)$ is nontrivial and $\omega > 0$, we have $\lambda = \overline{\lambda}$. □

Lemma 2.3. *If λ_1 and λ_2 are two different eigenvalues of the Fractional Dirac system defined by (11)-(13), then the corresponding eigenfunctions y_{λ_1} and y_{λ_2} are orthogonal in the space $L^2_\omega\left((a,b)\,;E\right)$.*

Proof. Assume λ_1 and λ_2 are distinct eigenvalues of (11)-(13) and y_{λ_1} and y_{λ_2} be the eigenfunctions. Then we have

$$Fy_{\lambda_1}(t) = \lambda_1\omega(t)y_{\lambda_1}(t) \tag{22}$$

$$c_{11}y_{\lambda_{11}}(a) + c_{12\,a}I^{1-\alpha,\sigma}y_{\lambda_{12}}(a) = 0 \tag{23}$$

$$c_{21}y_{\lambda_{11}}(b) + c_{22}\ {}_aI^{1-\alpha,\sigma}y_{\lambda_{12}}(b) = 0 \tag{24}$$

and

$$Fy_{\lambda_2}(t) = \lambda_2\omega(t)y_{\lambda_2}(t) \tag{25}$$

$$c_{11}y_{\lambda_{21}}(a) + c_{12\,a}I^{1-\alpha,\sigma}y_{\lambda_{22}}(a) = 0 \tag{26}$$

$$c_{21}y_{\lambda_{21}}(b) + c_{22}\ {}_aI^{1-\alpha,\sigma}y_{\lambda_{22}}(b) = 0 \tag{27}$$

Then we obtain,

$$(\lambda_1 - \lambda_2)\langle y_{\lambda_1}, y_{\lambda_2}\rangle = y(b)\left[\begin{array}{c} c_{22a}I^{1-\alpha,\sigma}y_{\lambda_2}(b)\,c_{21}y_{\lambda_1}(b) \\ -c_{22a}I^{1-\alpha,\sigma}y_{\lambda_1}(b)c_{21}y_{\lambda_2}(b)\end{array}\right]$$

$$+\, y(a)\begin{bmatrix} c_{12a}I^{1-\alpha,\sigma}y_{\lambda_1}(a)\, c_{11}y_{\lambda_2}(a) \\ -c_{12a}I^{1-\alpha,\sigma}y_{\lambda_2}(a)\, c_{11}y_{\lambda_1}(a) \end{bmatrix}.$$

From the boundary conditions (23), (24), (26) and (27), we have that

$$(\lambda_1 - \lambda_2)\langle y_{\lambda_1}, y_{\lambda_2}\rangle = 0$$

and hence as $\lambda_1 \neq \lambda_2$, $\langle y_{\lambda_1}(t), y_{\lambda_2}(t)\rangle = 0$. □

Theorem 2.4. *The Wronskian of any solution of Eq. (11) is independent of t.*

Proof. Assume that $v(t)$ and $\omega(t)$ be two solutions of Eq. (11). From Green's formula (14), we get

$$\langle \Pi y, \omega\rangle - \langle y, \Pi\omega\rangle = [y,\omega](b) - [y,\omega](a).$$

$Fv = \lambda v$ and $F\omega = \lambda\omega$, we get

$$\langle \lambda v, \omega\rangle - \langle v, \lambda\omega\rangle = [v,\omega](b) - [v,\omega](a),$$
$$(\lambda - \overline{\lambda})(v,\omega) = [v,\omega](b) - [v,\omega](a).$$

Since $\lambda \in \mathbb{R}$, we have $[v,\omega](b) = [v,\omega](a) = W(v,\overline{\omega})(a)$, i.e. the Wronskian is independent of t. □

Theorem 2.5. *Any two solutions of the Eq. (11) are linearly dependent if and only if their Wronskian is zero.*

Proof. Let $y(t)$ and $\omega(t)$ be two linearly dependent solutions of Eq. (11). Thus, for the constant $\eta > 0$, we infer that $y(t) = \eta.\omega(t)$. Then, we have

$$W(y,z) = \begin{vmatrix} y_1(t) & {}_aI^{1-\alpha,\sigma}y_2(t) \\ \omega_1(t) & {}_aI^{1-\alpha,\sigma}\omega_2(t) \end{vmatrix} = \begin{vmatrix} \eta\omega_1(t) & \eta\, {}_aI^{1-\alpha,\sigma}\omega_2(t) \\ \omega_1(t) & {}_aI^{1-\alpha,\sigma}\omega_2(t) \end{vmatrix} = 0.$$

Conversely, if the Wronskian $W(y,\omega)(t)$ is zero for some t in $[a,b]$ then we see that $y(t) = \eta.\omega(t)$. From this, it can be seen that $y(t)$ and $\omega(t)$ are linearly dependent on $[a,b]$.

Further, the general solution of the equation $F\,\psi = 0$, i.e.,

$$\begin{pmatrix} 0 & {}^C D_b^\alpha \\ {}_aD^\alpha & 0 \end{pmatrix}\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = \begin{pmatrix} {}^C D_b^\alpha \psi_2(t) \\ {}_aD^\alpha \psi_1(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is given by

$$\psi(t) = \begin{pmatrix} \varsigma_1 e^{-\sigma t}\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} \\ -\varsigma_2 e^{\sigma t} \end{pmatrix}$$

where

$$\Psi(\alpha, a, t) = e^{\sigma t}\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} \tag{28}$$

□

Lemma 2.6. *Let*

$$\Delta = c_{11}c_{22} - c_{21}c_{12}$$

and

$$Y_\lambda(y) = \{\lambda\omega - \vartheta)y_\lambda \tag{29}$$

where $\vartheta(t) = \begin{pmatrix} p(t) & 0 \\ 0 & r(t) \end{pmatrix}$. Assume $\Delta \neq 0$. Then, on the space $C[a,b]$, the fractional Dirac system defined by (11)-(13) is equivalent to the integral equation

$$y_\lambda(t) = -NY_\lambda(t) + Q(t)P + S(t)R,$$

where the matrix N is $\begin{pmatrix} 0 & {}_aI^{\alpha,\sigma} \\ I_b^{\alpha,\sigma} & 0 \end{pmatrix}$.

Proof. Using fractional composition rules and (29), we can rewrite the equation (25) as follows:

$$F\left[y_\lambda(t) + NY_\lambda(y)\right] = 0.$$

Then, we obtain

$$y_\lambda(t) + NY_\lambda(y) = \begin{pmatrix} \varsigma_1 e^{-\sigma x}\Psi(\alpha,a,t) \\ \varsigma_2 e^{\sigma t} \end{pmatrix}$$

$$y_\lambda(t) = -NY_\lambda(y) + \begin{pmatrix} \varsigma_1 e^{-\sigma t}\Psi(\alpha,a,t) \\ \varsigma_2 e^{\sigma t} \end{pmatrix} \tag{30}$$

If we rewrite ς_i $(i = 1,2)$ to the values a_{ij} $(i,j = 1,2)$ in the boundary conditions (12)-(13), from Eq. (30), we get

$$\Phi y_\lambda(t) = -\Phi NY_\lambda(y) + \Phi\begin{pmatrix} \varsigma_1 e^{-\sigma t}\Psi(\alpha,a,t) \\ \varsigma_2 e^{\sigma t} \end{pmatrix}$$

where $\Phi = \begin{pmatrix} {}_aI^{1-\alpha,\sigma} & 0 \\ 0 & 1 \end{pmatrix}$. Then we have

$$\begin{pmatrix} {}_aI^{1-\alpha}y_{\lambda_1}(t) \\ y_{\lambda_2}(t) \end{pmatrix} = \begin{pmatrix} -{}_aI^1Y_{\lambda_2}(y) \\ -I_b^1Y_{\lambda_1}(y) \end{pmatrix} + \begin{pmatrix} \varsigma_1 e^{-\sigma t} \\ \varsigma_2 e^{\sigma t} \end{pmatrix}$$

By (26) and (27), we conclude that

$$\begin{aligned} {}_aI^{1-\alpha}y_{\lambda_1}(a) &= -\varsigma_1 e^{-\sigma t} \\ {}_aI^{1-\alpha}y_{\lambda_1}(b) &= -{}_aI^1Y_{\lambda_2}(y)|_{t=b} + -\varsigma_1 e^{-\sigma t} \\ y_{\lambda_2}(a) &= -I_b^{\alpha}Y_{\lambda_1}(y)|_{t=a} + \varsigma_2 e^{\sigma t} \\ y_{\lambda_2}(b) &= \varsigma_2 e^{\sigma t}. \end{aligned}$$

From the above equations, the following system is obtained:

$$\begin{aligned} c_{11}\varsigma_1 e^{-\sigma t} + c_{12}\varsigma_2 e^{\sigma t} &= c_{12}I_b^{\alpha}Y_{\lambda_1}(y) \\ c_{21}\varsigma_1 e^{-\sigma t} + c_{22}\ \varsigma_2 e^{\sigma t} &= c_{21a}I^1Y_{\lambda_2}(y). \end{aligned}$$

Let $I_b^{\alpha}Y_{\lambda_1}(y) = P$ and ${}_aI^1Y_{\lambda_2}(y) = R$. Then

$$c_{11}\varsigma_1 e^{-\sigma t} + c_{12}\varsigma_2 e^{\sigma t} = c_{12}P$$

$$c_{21}\varsigma_1 e^{-\sigma t} + c_{22}\ \varsigma_2 e^{\sigma t} = c_{21}R.$$

Since $c_{11}c_{22} - c_{21}c_{12} \neq 0$, the solution for coefficients ς_i $(i = 1, 2)$ is unique;

$$\varsigma_1 = \frac{e^{\sigma t}c_{12}(c_{22}P - c_{21}R)}{c_{11}c_{22} - c_{21}c_{12}} = \frac{e^{\sigma t}c_{12}(c_{22}P - c_{21}R)}{\Delta}$$
$$\varsigma_2 = \frac{c_{21}(c_{11}R - c_{12}P)}{e^{\sigma t}(c_{11}c_{22} - c_{21}c_{12})} = \frac{e^{-\sigma t}c_{21}(c_{11}R - c_{12}P)}{\Delta}$$

Now, we shall prove the existence and uniqueness of the eigenfunction of the regular fractional Dirac system defined by (11)-(13).

Let

$$\varphi(t) = \begin{pmatrix} ({}_aI^{\alpha}1)(t) \\ (I_b^{\alpha}1)(t) \end{pmatrix},\ Q = \|Q(t)\|_C\ \ ,\quad S = \|S(t)\|_C\,,\quad G_\varphi = \left\|\varphi(t)\right\|_C,$$

where

$$Q(t) = \begin{pmatrix} \frac{e^{\sigma t}c_{12}c_{22}\Psi(\alpha,a,t)}{\Delta} \\ -\frac{e^{\sigma t}c_{12}c_{21}}{\Delta} \end{pmatrix},$$

$$S(t) = \begin{pmatrix} -\frac{e^{-\sigma t}c_{21}c_{12}\Psi(\alpha,a,t)}{\Delta} \\ \frac{e^{-\sigma t}c_{21}c_{11}}{\Delta} \end{pmatrix}.$$

and $\|.\|_C$ denotes the supremum norm on the space $C([a.b, E])$. □

Theorem 2.7. *Let $\alpha \in (0, 1)$ and assume that $\Delta \neq 0$. Then unique continuous function y_λ for the regular fractional Dirac system defined by (11)-(13) corresponding to each eigenvalue obeying*

$$\|\lambda\omega - \vartheta\|_C \leq \frac{1}{G_\varphi + Q\left\|\varphi(a)\right\|_C + S(b-a)} \tag{31}$$

exists and such eigenvalue is simple.

Proof. Let us define the mapping $L : C([a.b], E) \to C([a.b], E)$ by the formula

$$L\rho = -NY_\lambda(\rho) + Q(t)P(\rho) + S(t)R(\rho)$$
$$L\tau = -NY_\lambda(\tau) + Q(t)P(\tau) + S(t)R(\tau)$$

□

Now, we show that Eq. (11) can be interpreted as a fixed point condition on the space $C([a.b], E)$. Using the following estimate

$$\left\|Y_\lambda(\rho) - Y_\lambda(\tau)\right\|_C \leq \left\|\rho - \tau\right\|_C \|\lambda\omega - \vartheta\|_C\,,$$

we conclude that

$$\left\|L\rho - L\tau\right\| = -N\left(Y_\lambda(\tau) - Y_\lambda(\rho)\right) - Q(t)\left(P(\tau) - P(\rho)\right) - S(t)(R(\tau) - R(\rho))$$

$$\leq \left\|\rho - \tau\right\|_C \|\lambda\omega - \vartheta\|_C\ G_\varphi + Q\left\|\rho - \tau\right\|_C \left\|\varphi(a)\right\|_C \|\lambda\omega - \vartheta\|_C + S(b-a)\left\|\rho - \tau\right\|_C \|\lambda w - \vartheta\|_C$$

$$= \left\|\rho - \tau\right\|_C \|\lambda\omega - \vartheta\|_C \left(G_\varphi + Q \left\|\varphi(a)\right\|_C + S(b-a)\right)$$

$$= \Lambda \left\|\rho - \tau\right\|_C,$$

where

$$\Lambda = \|\lambda w - \vartheta\|_C \left(G_\varphi + Q \left\|\varphi(a)\right\|_C + S(b-a)\right).$$

By the condition (31), the mapping L is a contraction on the space $C([a.b], E)$ so it has a unique fixed point. Therefore, such an eigenvalue is simple.

3. Conclusion

This study investigated the Dirac-type system including tempered fractional integrals and their derivatives. First, the definitions and properties of tempered fractional derivatives and tempered fractional integrals are given. Then, it is shown that the operator of the related eigenvalue problem is a self-adjoint operator and its eigenfunctions are orthogonal. Finally, the existence and uniqueness theorem of the defined system is proved.

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