Filomat 37:27 (2023), 9119–9134
https://doi.org/10.2298/FIL2327119S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Milne's rule type inequalities in quantum calculus

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Abstract. The main goal of the current study is to establish some new Milne's rule type inequalities for single-time differentiable convex functions in the setting of quantum calculus. For this, we establish a quantum integral identity and then we prove some new inequalities of Milne's rule type for quantum differentiable convex functions. These inequalities are very important in Open-Newton's Cotes formulas because, with the help of these inequalities, we can find the bounds of Milne's rule for differentiable convex functions in classical or quantum calculus. The method adopted in this work to prove these inequalities are very easy and less conditional compared to some existing results. Finally, we give some mathematical examples to show the validity of newly established inequalities.

1. Introduction

The Hermite–Hadamard inequality is the first result given between convex functions and integrals. This inequality was introduced by Hermite [1] in 1883 which was later proved by Hadamard [2] in 1893. This inequality has the following mathematical form:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

where f is a convex function. This inequality also holds in the reverse direction for concave functions.

This inequality has many advantages, especially in approximation theory, it is used a lot. Because of its great applications, mathematicians started working on it and came up with many new results. For example, Dragomir and Agarwal [3] found the boundaries of the trapezoidal formula by taking the difference between the middle and right parts of this inequality and using differentiable convexity in the whole process. Later, Kirmaci [4] gave the boundaries of the midpoint formula, which were formed from the same inequality, he took the difference of the middle part from the left part and he also derived his results by using differentiable convexity. Qi and Xi [5] took the difference between the middle part of this inequality and the average of the left and right parts to establish a new inequality that we know as Bullen's.

2020 *Mathematics Subject Classification*. 26D10, 26A51, 26D15
Keywords. Hermite–Hadamard inequality, Jensen-inequality, convex interval-valued functions.
Received: 21 February 2023; Accepted: 16 May 2023
Communicated by Dragan S. Djordjević
This research was also partially supported by the National Natural Science Foundation of China 11971241.
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In [6], Milne's type inequality was established for the four times differentiable functions over a real open interval (a, b):

$$\left|\frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{7(b-a)^4}{23040}\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|. \tag{2}$$

On the side which is the main motivation of this paper, Alp et al. [7] used quantum calculus and proved the following new version of Hermite–Hadamard inequality (1):

Theorem 1.1. *For a convex function $f : [a, b] \to \mathbb{R}$, the following inequalities hold for $q \in (0, 1)$:*

$$f\left(\frac{qa+b}{1+q}\right) \le \frac{1}{b-a}\int_a^b f(x)\ {}_a d_q x \le \frac{qf(a)+f(b)}{1+q}. \tag{3}$$

Burmudo et al. [8] used a different approach to prove the following versions in quantum calculus of the Hermite–Hadamard inequality (1):

Theorem 1.2. *For a convex function $f : [a, b] \to \mathbb{R}$, the following inequalities hold for $q \in (0, 1)$:*

$$f\left(\frac{a+qb}{1+q}\right) \le \frac{1}{b-a}\int_a^b f(x)\ {}^b d_q x \le \frac{f(a)+qf(b)}{1+q} \tag{4}$$

and

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2(b-a)}\left[\int_a^b f(x)\ {}_a d_q x + \int_a^b f(x)\ {}^b d_q x\right] \le \frac{f(a)+f(b)}{2}. \tag{5}$$

Alp et al. [7] found the bounds of the midpoint formula in quantum calculus by taking the difference between the middle part and the right part of the inequality (3) and used q-differentiable convexity in the whole process. Later, Noor et al. [9] gave the bounds of the trapezoidal formula in quantum calculus, which was formed from the same inequality (3), they took the difference of the middle part from the left part and he also derived his results by using differentiable convexity. Budak used the same approaches in [10] to find the bounds of midpoint and trapezoidal formulas in quantum calculus using the inequality (4). Some more bounds of midpoint and trapezoidal formulas in quantum calculus were established using the inequality (5) in [11].

After an interesting study of quantum integral inequalities, Ali et al. [12] gave the following version in quantum calculus of Hermite–Hadamard inequality (1):

Theorem 1.3. *If $f : [a, b] \to \mathbb{R}$ be a convex function, then we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\ {}_a d_q x + \int_{\frac{a+b}{2}}^b f(x)\ {}^b d_q x\right] \le \frac{f(a)+f(b)}{2}. \tag{6}$$

Another new version of Hermite–Hadamard inequality (1) in quantum calculus was established by Sitthiwirattham et al. [13]:

Theorem 1.4. *If $f : [a, b] \to \mathbb{R}$ be a convex function, then we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\ {}^{\frac{a+b}{2}} d_q x + \int_{\frac{a+b}{2}}^b f(x)\ {}_{\frac{a+b}{2}} d_q x\right] \le \frac{f(a)+f(b)}{2}. \tag{7}$$

In [12], Ali et al. also found some bounds of midpoint and trapezoidal formulas in quantum calculus using the inequality (6) with the same approaches already discussed. For some bounds of midpoint and trapezoidal formulas in quantum calculus associated with (7), one can consult [13]. For more recent inequalities of Simpson's and Newton's type in quantum calculus, one can consult [14, 15] and references cited therein. For some more interesting inequalities, we suggest to read [16–18].

Inspired by the ongoing studies, we prove some dual Simpson's type inequalities in quantum calculus. For this, we prove a new quantum integral identity and then use it to establish desired inequalities for q-differentiable convex functions in the framework of quantum calculus. These inequalities can be helpful in finding the error bounds for dual Simpson's formula in quantum and classical calculus. As we know, classical Milne's type inequality (2) has already been proved, and we need a function that is four times differentiable, but in the results proved here only the first differentiability of the functions is required, and one can obtain the bounds for Milne's formula.

2. Preliminaries

In this section, some basics of quantum calculus are given for the better understanding of the new readers.

Tariboon and Ntouyas [19] introduced the left or q_a-derivative and integral concepts in 2013. They also discuss their properties, here we recall the following definitions from their work:

Definition 2.1. *[19] Let $f : [a, b] \to R$ be a continuous function. Then the q_a-derivative of f at $x \in (a, b]$ is defined by*

$$_aD_qf(x)=\frac{f(x)-f(a+q(x-a))}{(1-q)(x-a)}.$$

The q_a-integral is defined by

$$\int_a^x f(t)\ _ad_qt = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(a+q^n(x-a)).$$

In 2020, some new definitions of quantum derivative and integral using a different approach were introduced by Bermudo et al. [8], namely right or q^bderivative and integral. They also discussed some basic properties of the given operators, here we recall the following definitions from their work:

Definition 2.2. *[8] Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the q^b-derivative of f at $x \in [a, b)$ is defined by*

$$^bD_qf(x) = \frac{f(b+q(x-b))-f(x)}{(1-q)(b-x)}.$$

The q^b-integral is defined by

$$\int_x^b f(t)\ ^bd_qt = (1-q)(b-x)\sum_{n=0}^{\infty} q^n f(b+q^n(x-b)).$$

In [14] and [15], the authors provide the following formulas of q-integration by parts:

Lemma 2.3. *For continuous functions $h, f : [a, b] \to \mathbb{R}$, the following equality holds:*

$$\int_0^c h(t)_aD_qf(a+t(b-a))\ _0d_qt = \left.\frac{h(t)f(a+t(b-a))}{b-a}\right|_0^c - \frac{1}{b-a}\int_0^c f(a+qt(b-a))\ _0D_qh(t)\ _0d_qt. \tag{8}$$

Lemma 2.4. *For continuous functions $h, f : [a,b] \to \mathbb{R}$, the following equality holds:*

$$\int_0^c h(t)\, {}^bD_q f(b + t(a-b))\, {}_0d_q t$$
$$= \frac{1}{b-a}\int_0^c f(b + qt(a-b))\, {}_0D_q h(t)\, {}_0d_q t - \left.\frac{h(t) f(b + t(a-b))}{b-a}\right|_0^c. \tag{9}$$

3. New Identities for Quantum Integrals

In this section, by using quantum differentiable functions, we prove the main equalities which will help us to obtain the desired results.

Lemma 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a q-differentiable function. If ${}^bD_q f(t)$ is q-integrable on $[a,b]$, then the following equality holds:*

$$\frac{1}{b-a}\int_a^b f(x)^b d_q x - \frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right]$$
$$= (b-a)\left[I_1 + I_2 + I_3 + I_4\right], \tag{10}$$

where

$$I_1 = \int_0^{\frac{1}{4}} qt\, {}^bD_q f(b + t(a-b))\, d_q t$$
$$I_2 = \int_{\frac{1}{4}}^{\frac{1}{2}} \left(qt - \frac{2}{3}\right) {}^bD_q f(b + t(a-b))\, d_q t$$
$$I_3 = \int_{\frac{1}{2}}^{\frac{3}{4}} \left(qt - \frac{1}{3}\right) {}^bD_q f(b + t(a-b))\, d_q t$$
$$I_4 = \int_{\frac{3}{4}}^{1} (qt - 1)\, {}^bD_q f(b + t(a-b))\, d_q t.$$

Proof. By the definition of q-integral, we have

$$\begin{aligned} I_2 &= \int_{\frac{1}{4}}^{\frac{1}{2}} \left(qt - \frac{2}{3}\right) {}^bD_q f(b + t(a-b))\, d_q t \\ &= \int_0^{\frac{1}{2}} \left(qt - \frac{2}{3}\right) {}^bD_q f(b + t(a-b))\, d_q t - \int_0^{\frac{1}{4}} \left(qt - \frac{2}{3}\right) {}^bD_q f(b + t(a-b))\, d_q t \\ &= \int_0^{\frac{1}{2}} qt\, {}^bD_q f(b + t(a-b))\, d_q t - \frac{2}{3}\int_0^{\frac{1}{2}} {}^bD_q f(b + t(a-b))\, d_q t \\ &\quad - \int_0^{\frac{1}{4}} qt\, {}^bD_q f(b + t(a-b))\, d_q t + \frac{2}{3}\int_0^{\frac{1}{4}} {}^bD_q f(b + t(a-b))\, d_q t, \end{aligned}$$

and similarly

$$\begin{aligned} I_3 &= \int_{\frac{1}{2}}^{\frac{3}{4}} \left(qt - \frac{1}{3}\right) {}^bD_q f(b + t(a-b))\, d_q t \\ &= \int_0^{\frac{3}{4}} qt\, {}^bD_q f(b + t(a-b))\, d_q t - \frac{1}{3}\int_0^{\frac{3}{4}} {}^bD_q f(b + t(a-b))\, d_q t \end{aligned}$$

$$-\int_0^{\frac{1}{2}} qt\,{}^bD_qf(b+t(a-b))\,d_qt+\frac{1}{3}\int_0^{\frac{1}{2}} {}^bD_qf(b+t(a-b))\,d_qt$$

and

$$\begin{aligned}I_4 &= \int_{\frac{3}{4}}^1 (qt-1)\,{}^bD_qf(b+t(a-b))\,d_qt\\ &= \int_0^1 qt\,{}^bD_qf(b+t(a-b))\,d_qt-\int_0^1 {}^bD_qf(b+t(a-b))\,d_qt\\ &\quad -\int_0^{\frac{3}{4}} qt\,{}^bD_qf(b+t(a-b))\,d_qt+\int_0^{\frac{3}{4}} {}^bD_qf(b+t(a-b))\,d_qt.\end{aligned}$$

Then it follows that

$$\begin{aligned}&I_1+I_2+I_3+I_4 \qquad (11)\\ =&\int_0^1 qt\,{}^bD_qf(b+t(a-b))\,d_qt+\frac{2}{3}\int_0^{\frac{1}{4}} {}^bD_qf(b+t(a-b))\,d_qt\\ &-\frac{1}{3}\int_0^{\frac{1}{2}} {}^bD_qf(b+t(a-b))\,d_qt+\frac{2}{3}\int_0^{\frac{3}{4}} {}^bD_qf(b+t(a-b))\,d_qt\\ &-\int_0^1 {}^bD_qf(b+t(a-b))\,d_qt.\end{aligned}$$

By Lemma 2.4, we have

$$\begin{aligned}&\int_0^1 qt\,{}^bD_qf(b+t(a-b))\,d_qt \qquad (12)\\ =&\frac{1}{b-a}\int_0^1 q\,{}^bD_qf(qta+(1-qt)b)\,d_qt-\left.\frac{qtf(b+t(a-b))}{b-a}\right|_0^1\\ =&\frac{1-q}{b-a}\sum_{n=0}^{\infty} q^{n+1}f\left(q^{n+1}a+(1-q^{n+1})b\right)-\frac{q}{b-a}f(a)\\ =&\frac{1-q}{b-a}\sum_{n=0}^{\infty} q^nf(q^na+(1-q^n)b)-\frac{1}{b-a}f(a)\\ =&\frac{1}{(b-a)^2}\int_a^b f(x)\,{}^bd_qx-\frac{1}{b-a}f(a).\end{aligned}$$

On the other hand, by Lemma 2.4, we get

$$\begin{aligned}\int_0^{\frac{1}{4}} {}^bD_qf(b+t(a-b))\,d_qt &= -\left.\frac{f(b+t(a-b))}{b-a}\right|_0^{\frac{1}{4}} \qquad (13)\\ &= -\frac{1}{b-a}\left[f\left(\frac{a+3b}{4}\right)-f(b)\right].\end{aligned}$$

Similarly,

$$\int_0^{\frac{1}{2}} {}^bD_qf(b+t(a-b))\,d_qt=-\frac{1}{b-a}\left[f\left(\frac{a+b}{2}\right)-f(b)\right], \qquad (14)$$

$$\int_0^{\frac{3}{4}} {}^bD_qf(b+t(a-b))\,d_qt=-\frac{1}{b-a}\left[f\left(\frac{3a+b}{4}\right)-f(b)\right] \qquad (15)$$

and

$$\int_0^1 {}^bD_q f(b+t(a-b))\,d_qt = -\frac{1}{b-a}\left[f(a)-f(b)\right]. \tag{16}$$

If we substitute the equalities (12)-(16) in (11), then we have

$$\begin{aligned} & I_1+I_2+I_3+I_4 \\ = & \frac{1}{(b-a)^2}\int_a^b f(x)^b d_qx - \frac{1}{3(b-a)}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right] \end{aligned}$$

which completes the proof. □

4. Milne's Inequalities

In this section, we establish some Milne's type inequalities in quantum calculus for differentiable convex functions. For the sake of brevity, we will use the following notation of quantum numbers:

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{k=0}^{n-1} q^k.$$

Theorem 4.1. *Assume that the assumptions of Lemma 3.1 hold. If* $|{}^bD_q f|$ *is convex on* $[a,b]$*, then the following inequality holds:*

$$\begin{aligned} & \left|\frac{1}{b-a}\int_a^b f(x)^b d_qx - \frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]\right| \\ & \le (b-a)\left[|{}^bD_q f(a)|\left(A_1(q)+\frac{36-7q-7q^2}{64[2]_q[3]_q}\right)\right. \\ & \left.+|{}^bD_q f(b)|\left(A_2(q)+\frac{-28q^2+101q-28}{192\,[3]_q}\right)\right], \end{aligned} \tag{17}$$

where

$$A_1(q) = \int_{\frac{1}{2}}^{\frac{3}{4}} t\left|qt-\frac{1}{3}\right| d_qt = \begin{cases} \frac{20-37q-37q^2}{192[2]_q[3]_q}, & 0<q\le\frac{4}{9}, \\ \frac{-340+477q+477q^2}{1728[2]_q[3]_q}, & \frac{4}{9}<q\le\frac{2}{3}, \\ \frac{-20+37q+37q^2}{192[2]_q[3]_q}, & \frac{2}{3}<q<1 \end{cases}$$

and

$$A_2(q) = \int_{\frac{1}{2}}^{\frac{3}{4}} (1-t)\left|qt-\frac{1}{3}\right| d_qt = \begin{cases} \frac{-4+9q+9q^2-44q^3}{192[2]_q[3]_q}, & 0<q\le\frac{4}{9}, \\ \frac{4-195q-195q^2+648q^3}{1728[2]_q[3]_q}, & \frac{4}{9}<q\le\frac{2}{3}, \\ \frac{4-9q-9q^2+44q^3}{192[2]_q[3]_q}, & \frac{2}{3}<q<1. \end{cases}$$

Proof. By Lemma 3.1, we have

$$\begin{aligned} & \left|\frac{1}{b-a}\int_a^b f(x)^b d_qx - \frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]\right| \\ \le & (b-a)\left[\int_0^{\frac{1}{4}} qt\left|{}^bD_q f(b+t(a-b))\right| d_qt\right. \end{aligned} \tag{18}$$

$$+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|qt-\frac{2}{3}\right|\left|{}^{b}D_{q}f\left(b+t\left(a-b\right)\right)\right|d_{q}t$$
$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\frac{1}{3}\right|\left|{}^{b}D_{q}f\left(b+t\left(a-b\right)\right)\right|d_{q}t$$
$$\left.+\int_{\frac{3}{4}}^{1}\left|qt-1\right|\left|{}^{b}D_{q}f\left(b+t\left(a-b\right)\right)\right|d_{q}t\right].$$

Since $|{}^{b}D_{q}f|$ is convex on $[a,b]$, therefore

$$\left|{}^{b}D_{q}f\left(b+t\left(a-b\right)\right)\right|\leq t|{}^{b}D_{q}f(a)|+(1-t)|{}^{b}D_{q}f(b)|. \tag{19}$$

Substituting (19) in (18), we obtain

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)^{b}d_{q}x-\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]\right|$$
$$\leq(b-a)\left[\int_{0}^{\frac{1}{4}}qt\left[t|{}^{b}D_{q}f(a)|+(1-t)|{}^{b}D_{q}f(b)|\right]d_{q}t\right.$$
$$+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|qt-\frac{2}{3}\right|\left[t|{}^{b}D_{q}f(a)|+(1-t)|{}^{b}D_{q}f(b)|\right]d_{q}t$$
$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\frac{1}{3}\right|\left[t|{}^{b}D_{q}f(a)|+(1-t)|{}^{b}D_{q}f(b)|\right]d_{q}t$$
$$\left.+\int_{\frac{3}{4}}^{1}\left|qt-1\right|\left[t|{}^{b}D_{q}f(a)|+(1-t)|{}^{b}D_{q}f(b)|\right]d_{q}t\right]$$
$$=(b-a)\left[|{}^{b}D_{q}f(a)|\left[q\int_{0}^{\frac{1}{4}}t^{2}d_{q}t+\int_{\frac{1}{4}}^{\frac{1}{2}}t\left|qt-\frac{2}{3}\right|d_{q}t\right.\right.$$
$$\left.+\int_{\frac{1}{2}}^{\frac{3}{4}}t\left|qt-\frac{1}{3}\right|d_{q}t+\int_{\frac{3}{4}}^{1}t\left|qt-1\right|d_{q}t\right]$$
$$+|{}^{b}D_{q}f(b)|\left[\int_{0}^{\frac{1}{4}}qt(1-t)d_{q}t+\int_{\frac{1}{4}}^{\frac{1}{2}}(1-t)\left|qt-\frac{2}{3}\right|d_{q}t\right.$$
$$\left.\left.+\int_{\frac{1}{2}}^{\frac{3}{4}}(1-t)\left|qt-\frac{1}{3}\right|d_{q}t+\int_{\frac{3}{4}}^{1}(1-t)\left|qt-1\right|d_{q}t\right]\right].$$

By calculating the quantum integrals, we have

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)^{b}d_{q}x-\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]\right|$$
$$\leq(b-a)\left[|{}^{b}D_{q}f(a)|\left[\frac{q}{64[3]_{q}}+\frac{8+q+q^{2}}{64[2]_{q}[3]_{q}}\right.\right.$$
$$\left.+A_{1}(q)+\frac{28-9q-9q^{2}}{64[2]_{q}[3]_{q}}\right]$$
$$+|{}^{b}D_{q}f(b)|\left[\frac{3q+3q^{2}+4q^{3}}{64[2]_{q}[3]_{q}}+\frac{8+25q+25q^{2}-4q^{3}}{192[2]_{q}[3]_{q}}\right.$$

$$+A_2(q)+\frac{-12+13q+13q^2-12q^3}{64[2]_q[3]_q}\Bigg]\Bigg]$$
$$=b-a)\left[|^bD_qf(a)|\left(A_1(q)+\frac{36-7q-7q^2}{64[2]_q[3]_q}\right)\right.$$
$$\left.+|^bD_qf(b)|\left(A_2(q)+\frac{-28q^2+101q-28}{192\,[3]_q}\right)\right].$$

Here we use the equalities

$$\int_0^{\frac{1}{4}}t^2=\frac{1}{64[3]_q}, \tag{20}$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}}t\left|qt-\frac{2}{3}\right|d_qt=\frac{8+q+q^2}{64[2]_q[3]_q}, \tag{21}$$

$$\int_{\frac{3}{4}}^{1}t\,(1-qt)\,d_qt=\frac{28-9q-9q^2}{64[2]_q[3]_q}, \tag{22}$$

$$\int_0^{\frac{1}{4}}t(1-t)d_qt=\frac{3+3q+4q^2}{64[2]_q[3]_q}, \tag{23}$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}}(1-t)\left|qt-\frac{2}{3}\right|d_qt=\frac{8+25q+25q^2-4q^3}{192[2]_q[3]_q} \tag{24}$$

and

$$\int_{\frac{3}{4}}^{1}(1-t)\,(1-qt)\,d_qt=\frac{-12+13q+13q^2-12q^3}{64[2]_q[3]_q}. \tag{25}$$

□

Corollary 4.2. *If we take the limit as $q\to 1^-$ in Theorem 4.1, then we obtain the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]\right|$$
$$\leq\frac{5\,(b-a)}{48}\left[|f'(a)|+|f'(b)|\right].$$

Theorem 4.3. *Assume that the assumptions of Lemma 3.1 hold. If $|^bD_qf|^r$ is convex on $[a,b]$ and $\frac{1}{p}+\frac{1}{r}=1$ with $p,r>1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)^bd_qx-\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]\right| \tag{26}$$
$$\leq\quad(b-a)\left[\left(\frac{q^p}{4^{p+1}\,[p+1]_q}\right)^{\frac{1}{p}}\left(\frac{|^bD_qf(a)|^r+(3+4q)\,|^bD_qf(b)|^r}{16[2]_q}\right)^{\frac{1}{r}}\right.$$

$$+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|qt-\frac{2}{3}\right|^{p}d_{q}t\right)^{\frac{1}{p}}\left(\frac{3|^{b}D_{q}f(a)|^{r}+(1+4q)\,|^{b}D_{q}f(b)|^{r}}{16[2]_{q}}\right)^{\frac{1}{r}}$$
$$+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\frac{1}{3}\right|^{p}d_{q}t\right)^{\frac{1}{p}}\left(\frac{5|^{b}D_{q}f(a)|^{r}+(-1+4q)\,|^{b}D_{q}f(b)|^{r}}{16[2]_{q}}\right)^{\frac{1}{r}}$$
$$\left.+\left(\int_{\frac{3}{4}}^{1}(1-qt)^{p}d_{q}t\right)^{\frac{1}{p}}\left(\frac{7|^{b}D_{q}f(a)|^{r}+(-3+4q)\,|^{b}D_{q}f(b)|^{r}}{16[2]_{q}}\right)^{\frac{1}{r}}\right].$$

Proof. By using q-Hölder inequality in (18), we have

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)^{b}d_{q}x-\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]\right| \tag{27}$$
$$\leq (b-a)\left[\left(\int_{0}^{\frac{1}{4}}(qt)^{p}d_{q}t\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{4}}\left|^{b}D_{q}f\left(b+t\left(a-b\right)\right)\right|^{r}d_{q}t\right)^{\frac{1}{r}}\right.$$
$$+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|qt-\frac{2}{3}\right|^{p}d_{q}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|^{b}D_{q}f\left(b+t\left(a-b\right)\right)\right|^{r}d_{q}t\right)^{\frac{1}{r}}$$
$$+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\frac{1}{3}\right|^{p}d_{q}t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|^{b}D_{q}f\left(b+t\left(a-b\right)\right)\right|^{r}d_{q}t\right)^{\frac{1}{r}}$$
$$\left.+\left(\int_{\frac{3}{4}}^{1}(1-qt)^{p}d_{q}t\right)^{\frac{1}{p}}\left(\int_{\frac{3}{4}}^{1}\left|^{b}D_{q}f\left(b+t\left(a-b\right)\right)\right|^{r}d_{q}t\right)^{\frac{1}{r}}\right].$$

Since $|^{b}D_{q}f|^{r}$ is convex on $[a,b]$, we have

$$\begin{aligned}\int_{0}^{\frac{1}{4}}\left|^{b}D_{q}f\left(b+t\left(a-b\right)\right)\right|^{r}d_{q}t &\leq \int_{0}^{\frac{1}{4}}\left[t|^{b}D_{q}f(a)|^{r}+(1-t)|^{b}D_{q}f(b)|^{r}\right]d_{q}t \\ &= \frac{1}{16[2]_{q}}|^{b}D_{q}f(a)|^{r}+\frac{3+4q}{16[2]_{q}}|^{b}D_{q}f(b)|^{r}.\end{aligned} \tag{28}$$

Similarly,

$$\int_{\frac{1}{4}}^{\frac{1}{2}}\left|^{b}D_{q}f\left(b+t\left(a-b\right)\right)\right|^{r}d_{q}t\leq\frac{3}{16[2]_{q}}|^{b}D_{q}f(a)|^{r}+\frac{1+4q}{16[2]_{q}}|^{b}D_{q}f(b)|^{r}, \tag{29}$$

$$\int_{\frac{1}{2}}^{\frac{3}{4}}\left|^{b}D_{q}f\left(b+t\left(a-b\right)\right)\right|^{r}d_{q}t\leq\frac{5}{16[2]_{q}}|^{b}D_{q}f(a)|^{r}+\frac{-1+4q}{16[2]_{q}}|^{b}D_{q}f(b)|^{r} \tag{30}$$

and

$$\int_{\frac{3}{4}}^{1}\left|^{b}D_{q}f\left(b+t\left(a-b\right)\right)\right|^{r}d_{q}t\leq\frac{7}{16[2]_{q}}|^{b}D_{q}f(a)|^{r}+\frac{-3+4q}{16[2]_{q}}|^{b}D_{q}f(b)|^{r}. \tag{31}$$

On the other hand, we also have the equality

$$\int_{0}^{\frac{1}{4}}(qt)^{p}d_{q}t=\frac{q^{p}}{4^{p+1}\left[p+1\right]_{q}}. \tag{32}$$

By substituting (28)-(32) in (27), then we obtain the desired result. □

Corollary 4.4. *If we take the limit as $q \to 1^-$ in Theorem 4.3, then we obtain the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right]\right| \\
\leq\ & (b-a)\left[\left(\frac{1}{4^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^r + 7|f'(b)|^r}{32}\right)^{\frac{1}{r}}\right. \\
&+\left(\frac{5^{p+1}}{12^{p+1}(p+1)} - \frac{1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{3|f'(a)|^r + 5|f'(b)|^r}{32}\right)^{\frac{1}{r}} \\
&+\left(\frac{5^{p+1}}{12^{p+1}(p+1)} - \frac{1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{5|f'(a)|^r + 3|f'(b)|^r}{32}\right)^{\frac{1}{r}} \\
&\left.+\left(\int_{\frac{3}{4}}^1 (1-qt)^p\, d_qt\right)^{\frac{1}{p}}\left(\frac{7|f'(a)|^r + |f'(b)|^r}{32}\right)^{\frac{1}{r}}\right].
\end{aligned}$$

Theorem 4.5. *Assume that the assumptions of Lemma 3.1 hold. If $|{}^bD_qf|^r$ is convex on $[a,b]$ for $r \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,{}^bd_qx - \frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right]\right| \qquad (33)\\
\leq\ & (b-a)\left[\left(\frac{q}{16[2]_q}\right)^{1-\frac{1}{r}}\left(\frac{q[2]_q\left|{}^bD_qf(a)\right|^r + \left(3q+3q^2+4q^3\right)\left|{}^bD_qf(b)\right|^r}{64[2]_q[3]_q}\right)^{\frac{1}{r}}\right. \\
&+\left(\frac{8-q}{48[2]_q}\right)^{1-\frac{1}{r}}\left(\frac{3\left(8+q+q^2\right)\left|{}^bD_qf(a)\right|^r + \left(8+25q+25q^2-4q^3\right)\left|{}^bD_qf(b)\right|^r}{192[2]_q[3]_q}\right)^{\frac{1}{r}} \\
&+(A_3(q))^{1-\frac{1}{r}}\left(A_1(q)\left|{}^bD_qf(a)\right|^r + A_2(q)\left|{}^bD_qf(b)\right|^r\right)^{\frac{1}{r}} \\
&\left.+\left(\frac{4-3q}{16[2]_q}\right)^{1-\frac{1}{r}}\left(\frac{\left(3+3q+4q^2\right)\left|{}^bD_qf(a)\right|^r + \left(-12+13q+13q^2-12q^3\right)\left|{}^bD_qf(b)\right|^r}{64[2]_q[3]_q}\right)^{\frac{1}{r}}\right].
\end{aligned}$$

where A_1, A_2 are defined as in Theorem 4.1 and

$$A_3(q) = \int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt - \frac{1}{3}\right| d_qt = \begin{cases} \frac{4-11q}{48[2]_q}, & 0 < q \leq \frac{4}{9}, \\ \frac{-28+57q}{144[2]_q}, & \frac{4}{9} < q \leq \frac{2}{3}, \\ \frac{-4+11q}{48[2]_q}, & \frac{2}{3} < q < 1. \end{cases}$$

Proof. By utilizing q-power mean inequality in (18), we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,{}^bd_qx - \frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right]\right| \qquad (34)\\
\leq\ & (b-a)\left[\left(\int_0^{\frac{1}{4}} qt\,d_qt\right)^{1-\frac{1}{r}}\left(\int_0^{\frac{1}{4}} qt\left|{}^bD_qf(b+t(a-b))\right|^r d_qt\right)^{\frac{1}{r}}\right. \\
&+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|qt - \frac{2}{3}\right| d_qt\right)^{1-\frac{1}{r}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|qt - \frac{2}{3}\right|\left|{}^bD_qf(b+t(a-b))\right|^r d_qt\right)^{\frac{1}{r}}
\end{aligned}$$

$$+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\frac{1}{3}\right|d_qt\right)^{1-\frac{1}{r}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\frac{1}{3}\right|\left|{}^bD_qf\left(b+t\left(a-b\right)\right)\right|^r d_qt\right)^{\frac{1}{r}}$$
$$+\left(\int_{\frac{3}{4}}^{1}\left(1-qt\right)d_qt\right)^{1-\frac{1}{r}}\left(\int_{\frac{3}{4}}^{1}\left(1-qt\right)\left|{}^bD_qf\left(b+t\left(a-b\right)\right)\right|^r d_qt\right)^{\frac{1}{r}}\Bigg].$$

Since $|{}^bD_qf|^r$ is convex on $[a,b]$, then by the equalities (20) and (23), we have

$$\begin{aligned}&\int_0^{\frac{1}{4}}qt\left|{}^bD_qf\left(b+t\left(a-b\right)\right)\right|^r d_qt \\ \leq\ &\int_0^{\frac{1}{4}}qt\left[t\left|{}^bD_qf\left(a\right)\right|^r+\left(1-t\right)\left|{}^bD_qf\left(b\right)\right|^r\right]d_qt \\ =\ &\frac{q[2]_q\left|{}^bD_qf\left(a\right)\right|^r+\left(3+3q+4q^2\right)\left|{}^bD_qf\left(b\right)\right|^r}{64[3]_q}.\end{aligned} \tag{35}$$

Similarly, by convexity of $|{}^bD_qf|^r$ and the equalities obtained in the proof of Theorem 4.1, we have

$$\begin{aligned}&\int_{\frac{1}{4}}^{\frac{1}{2}}\left|qt-\frac{2}{3}\right|\left|{}^bD_qf\left(b+t\left(a-b\right)\right)\right|^r d_qt \\ \leq\ &\frac{3\left(8+q+q^2\right)\left|{}^bD_qf\left(a\right)\right|^r+\left(8+25q+25q^2-4q^3\right)\left|{}^bD_qf\left(b\right)\right|^r}{192[2]_q[3]_q},\end{aligned} \tag{36}$$

$$\begin{aligned}&\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\frac{1}{3}\right|\left|{}^bD_qf\left(b+t\left(a-b\right)\right)\right|^r d_qt \\ \leq\ &A_1(q)\left|{}^bD_qf\left(a\right)\right|^r+A_2(q)\left|{}^bD_qf\left(b\right)\right|^r\end{aligned} \tag{37}$$

and

$$\begin{aligned}&\int_{\frac{3}{4}}^{1}\left(1-qt\right)\left|{}^bD_qf\left(b+t\left(a-b\right)\right)\right|^r d_qt \\ \leq\ &\frac{\left(28-9q-9q^2\right)\left|{}^bD_qf\left(a\right)\right|^r+\left(-12+13q+13q^2-12q^3\right)\left|{}^bD_qf\left(b\right)\right|^r}{64[2]_q[3]_q}.\end{aligned} \tag{38}$$

We also have the following equalities

$$\int_0^{\frac{1}{4}}t=\frac{1}{16[2]_q}, \tag{39}$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}}\left|qt-\frac{2}{3}\right|d_qt=\frac{8-q}{48[2]_q}, \tag{40}$$

and

$$\int_{\frac{3}{4}}^{1}\left(1-qt\right)d_qt=\frac{4-3q}{16[2]_q}. \tag{41}$$

If we substitute (35)-(41) in (34), we have the inequality

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)^b d_q x - \frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right]\right| \qquad (42)\\
\leq\ & (b-a)\left[\left(\frac{q}{16[2]_q}\right)^{1-\frac{1}{r}}\left(\frac{q[2]_q\left|{}^bD_q f(a)\right|^r + \left(3q+3q^2+4q^3\right)\left|{}^bD_q f(b)\right|^r}{64[2]_q[3]_q}\right)^{\frac{1}{r}}\right.\\
&+\left(\frac{8-q}{48[2]_q}\right)^{1-\frac{1}{r}}\left(\frac{3\left(8+q+q^2\right)\left|{}^bD_q f(a)\right|^r + \left(8+25q+25q^2-4q^3\right)\left|{}^bD_q f(b)\right|^r}{192[2]_q[3]_q}\right)^{\frac{1}{r}}\\
&+\left(A_3(q)\right)^{1-\frac{1}{r}}\left(A_1(q)\left|{}^bD_q f(a)\right|^r + A_2(q)\left|{}^bD_q f(b)\right|^r\right)^{\frac{1}{r}}\\
&\left.+\left(\frac{4-3q}{16[2]_q}\right)^{1-\frac{1}{r}}\left(\frac{\left(3+3q+4q^2\right)\left|{}^bD_q f(a)\right|^r + \left(-12+13q+13q^2-12q^3\right)\left|{}^bD_q f(b)\right|^r}{64[2]_q[3]_q}\right)^{\frac{1}{r}}\right].
\end{aligned}$$

This completes the proof. □

Corollary 4.6. *If we take the limit as $q \to 1^-$ in Theorem 4.5, then we obtain the following inequality:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right]\right| \qquad (43)\\
\leq\ & \frac{(b-a)}{32}\left[\left(\frac{2\left|f'(a)\right|^r + 10\left|f'(b)\right|^r}{12}\right)^{\frac{1}{r}} + \left(\frac{10\left|f'(a)\right|^r + 2\left|f'(b)\right|^r}{12}\right)^{\frac{1}{r}}\right]\\
&+\frac{7(b-a)}{96}\left[\left(\frac{5\left|f'(a)\right|^r + 9\left|f'(b)\right|^r}{14}\right)^{\frac{1}{r}} + \left(\frac{9\left|f'(a)\right|^r + 5\left|f'(b)\right|^r}{14}\right)^{\frac{1}{r}}\right].
\end{aligned}$$

5. Examples

Now, we give some examples of our main results to demonstrate our theorems.

Example 5.1. *Let $f : [0,1] \to \mathbb{R}$ be a function defined by $f(x) = x^3$. Then f is q-differentiable. Moreover, for $q = \frac{3}{4}$,*

$$|{}^bD_q f(x)| = |{}^1D_{\frac{3}{4}} f(x)| = \frac{37}{16}x^2 + \frac{5}{8}x + \frac{1}{16}$$

is convex on $[0,1]$. By applying Theorem 4.1 to the function $f(x) = x^3$, we have

$$\begin{aligned}
&\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right]\\
=\ & \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right]\\
=\ & \frac{1}{4} = 0.25
\end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b f(x)^b d_q x = \int_0^1 x^3\ {}^1d_{\frac{3}{4}} x$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\left(1-\left(\frac{3}{4}\right)^n\right)^3$$
$$= \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\left(1-3\left(\frac{3}{4}\right)^n+3\left(\frac{3}{4}\right)^{2n}-\left(\frac{3}{4}\right)^{3n}\right)$$
$$= \frac{1}{4}\left[4-\frac{48}{7}+\frac{192}{37}-\frac{256}{175}\right]$$
$$= 0.2173.$$

Thus, the left-hand side of (17) *is*

$$|0.2173-0.25| = 0.0327.$$

Next, we consider

$$|{}^bD_qf(a)| = |{}^1D_{\frac{3}{4}}f(0)| = \frac{1}{16} = 0.0625$$
$$|{}^bD_qf(b)| = |{}^1D_{\frac{3}{4}}f(1)| = 3$$
$$A_1\left(\frac{3}{4}\right) = 0.0368$$

and

$$A_2\left(\frac{3}{4}\right) = 0.0138.$$

Hence, the right-hand side of (17) *is*

$$(b-a)\left[|{}^bD_qf(a)|\left(A_1(q)+\frac{36-7q-7q^2}{64[2]_q[3]_q}\right)\right.$$
$$\left.+|{}^bD_qf(b)|\left(A_2(q)+\frac{-28q^2+101q-28}{192\,[3]_q}\right)\right]$$
$$= 0.0625\,(0.0368+0.1035)+3\,(0.0138+0.0721)$$
$$= 0.2665.$$

It is clear that

$$0.0327 \leq 0.2665,$$

which demonstrates the result described in Theorem 4.1.

Example 5.2. *Let* $f:[0,1]\to\mathbb{R}$ *be a function defined by* $f(x)=x^3$ *and* $p=r=2$. *Then* f *is q-differentiable. Moreover, for* $q=\frac{3}{4}$,

$$|{}^bD_qf(x)|^2 = |{}^1D_{\frac{3}{4}}f(x)|^2 = \left(\frac{37}{16}x^2+\frac{5}{8}x+\frac{1}{16}\right)^2$$

is convex on $[0,1]$. *By applying Theorem 4.3, we have the left hand side of* (26) *is* 0.0327.
Since

$$|{}^1D_{\frac{3}{4}}f(0)|^2 = 0.0039,$$
$$|{}^1D_{\frac{3}{4}}f(1)|^2 = 9,$$

$$\left(\frac{q^p}{4^{p+1}\left[p+1\right]_q}\right)^{\frac{1}{p}} = \left(\frac{\frac{9}{16}}{16(1+\frac{3}{4}+\frac{9}{16})}\right)^{\frac{1}{2}} = 0.1233,$$

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \left|qt-\frac{2}{3}\right|^p d_qt = \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{9}{16}t^2 - t + \frac{4}{9}\right) d_{\frac{3}{4}}t = 0.0306,$$

$$\int_{\frac{1}{2}}^{\frac{3}{4}} \left|qt-\frac{1}{3}\right|^p d_qt = \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\frac{9}{16}t^2 - \frac{t}{2} + \frac{1}{9}\right) d_{\frac{3}{4}}t = 0.0107$$

and

$$\int_{\frac{3}{4}}^{1} (1-qt)^p d_qt = \int_{\frac{3}{4}}^{1} \left(\frac{9}{16}t^2 - \frac{3}{2}t + 1\right) d_{\frac{3}{4}}t = 0.0156,$$

the right hand side of (26) *is*

$$\begin{aligned}
&(b-a)\Bigg[\left(\frac{q^p}{4^{p+1}\left[p+1\right]_q}\right)^{\frac{1}{p}}\left(\frac{|{}^bD_qf(a)|^r + (3+4q)\,|{}^bD_qf(b)|^r}{16[2]_q}\right)^{\frac{1}{r}}\\
&+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|qt-\frac{2}{3}\right|^p d_qt\right)^{\frac{1}{p}}\left(\frac{3|{}^bD_qf(a)|^r + (1+4q)\,|{}^bD_qf(b)|^r}{16[2]_q}\right)^{\frac{1}{r}}\\
&+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|qt-\frac{1}{3}\right|^p d_qt\right)^{\frac{1}{p}}\left(\frac{5|{}^bD_qf(a)|^r + (-1+4q)\,|{}^bD_qf(b)|^r}{16[2]_q}\right)^{\frac{1}{r}}\\
&+\left(\int_{\frac{3}{4}}^{1}(1-qt)^p d_qt\right)^{\frac{1}{p}}\left(\frac{7|{}^bD_qf(a)|^r + (-3+4q)\,|{}^bD_qf(b)|^r}{16[2]_q}\right)^{\frac{1}{r}}\Bigg]\\
=\;& 0.1233\left(\frac{0.0039+54}{16(1+\frac{3}{4})}\right)^{\frac{1}{2}} + (0.0306)^{\frac{1}{2}}\left(\frac{3\times 0.0039+36}{16(1+\frac{3}{4})}\right)^{\frac{1}{2}}\\
&+(0.0107)^{\frac{1}{2}}\left(\frac{5\times 0.0039+18}{16(1+\frac{3}{4})}\right)^{\frac{1}{2}} + (0.0156)^{\frac{1}{2}}\left(\frac{7\times 0.0039+0}{16(1+\frac{3}{4})}\right)^{\frac{1}{2}}\Bigg]\\
=\;& 0.1712+0.1984+0.0830+0.0039\\
=\;& 0.4565.
\end{aligned}$$

It is clear that

$$0.0327 \le 0.4565,$$

which demonstrates the result described in Theorem 4.3.

Example 5.3. *Let $f:[0,1]\to\mathbb{R}$ be a function defined by $f(x)=x^3$ and $r=2$. Then f is q-differentiable. Moreover, for $q=\frac{3}{4}$,*

$$|{}^bD_qf(x)|^2 = |{}^1D_{\frac{3}{4}}f(x)|^2 = \left(\frac{37}{16}x^2 + \frac{5}{8}x + \frac{1}{16}\right)^2$$

is convex on $[0,1]$. *By applying Theorem 4.5, we have the left hand side of* (33) *is* 0.0327.
Since

$$|{}^1D_{\frac{3}{4}}f(0)|^2 = 0.0039,$$

$$|^1D_{\frac{3}{4}}f(1)|^2 = 9,$$

$$A_1\left(\frac{3}{4}\right) = 0.0368,$$

$$A_2\left(\frac{3}{4}\right) = 0.0138,$$

and

$$A_3\left(\frac{3}{4}\right) = 0.0506,$$

the right hand side of (33) *is*

$$\begin{aligned}
&(b-a)\left[\left(\frac{q}{16[2]_q}\right)^{1-\frac{1}{r}}\left(\frac{q[2]_q\left|{}^bD_qf(a)\right|^r+\left(3q+3q^2+4q^3\right)\left|{}^bD_qf(b)\right|^r}{64[2]_q[3]_q}\right)^{\frac{1}{r}}\right.\\
&+\left(\frac{8-q}{48[2]_q}\right)^{1-\frac{1}{r}}\left(\frac{3\left(8+q+q^2\right)\left|{}^bD_qf(a)\right|^r+\left(8+25q+25q^2-4q^3\right)\left|{}^bD_qf(b)\right|^r}{192[2]_q[3]_q}\right)^{\frac{1}{r}}\\
&+(A_3(q))^{1-\frac{1}{r}}\left(A_1(q)\left|{}^bD_qf(a)\right|^r+A_2(q)\left|{}^bD_qf(b)\right|^r\right)^{\frac{1}{r}}\\
&\left.+\left(\frac{4-3q}{16[2]_q}\right)^{1-\frac{1}{r}}\left(\frac{\left(3+3q+4q^2\right)\left|{}^bD_qf(a)\right|^r+\left(-12+13q+13q^2-12q^3\right)\left|{}^bD_qf(b)\right|^r}{64[2]_q[3]_q}\right)^{\frac{1}{r}}\right]\\
=\;&\left(\frac{\frac{3}{4}}{16\left(1+\frac{3}{4}\right)}\right)^{\frac{1}{2}}\left(\frac{\frac{3}{4}\left(1+\frac{3}{4}\right)\times 0.0039+9\left(\frac{9}{4}+\frac{27}{16}+\frac{27}{16}\right)}{64\left(1+\frac{3}{4}\right)\left(1+\frac{3}{4}+\frac{9}{16}\right)}\right)^{\frac{1}{2}}\\
&\left(\frac{8-\frac{3}{4}}{48\left(1+\frac{3}{4}\right)}\right)^{\frac{1}{2}}\left(\frac{3\left(8+\frac{3}{4}+\frac{9}{16}\right)\times 0.0039+9\left(8+\frac{75}{4}+\frac{225}{16}-\frac{27}{16}\right)}{192\left(1+\frac{3}{4}\right)\left(1+\frac{3}{4}+\frac{9}{16}\right)}\right)^{\frac{1}{2}}\\
&+(0.0506)^{\frac{1}{2}}(0.0368\times 0.0039+9\times 0.0138)^{\frac{1}{2}}\\
&+\left(\frac{4-\frac{9}{4}}{16\left(1+\frac{3}{4}\right)}\right)^{\frac{1}{2}}\left(\frac{\left(3+\frac{9}{4}+\frac{9}{4}\right)\times 0.0039+9\left(-12+\frac{39}{4}+\frac{117}{16}-\frac{81}{16}\right)}{64\left(1+\frac{3}{4}\right)\left(1+\frac{3}{4}+\frac{9}{16}\right)}\right)^{\frac{1}{2}}\Bigg]\\
=\;&0.3273\times 0.4421+0.2938\times 0.6733+0.2249\times 0.3526+0.25\times 0.0106\\
=\;&0.4245.
\end{aligned}$$

It is clear that

$$0.0327 \leq 0.4245,$$

which demonstrates the result described in Theorem 4.5.

6. Conclusion

In this paper, we established some inequalities to find the error bounds for Milne's rule in the framework of quantum and classical calculus. To prove these inequalities, we established a quantum integral identity involving differentiable functions and then used convexity for differentiable functions. These inequalities are very important in Open-Newton's Cotes formulas because with the help of these inequalities, we can find the bounds of Milne's rule for differentiable convex functions in classical or quantum calculus. The method adopted in this work to prove these inequalities is very easy and less conditional compared to some existing results. It is an interesting and new problem that the upcoming researchers can obtain similar inequalities for different integral operators and different kinds of convexity in their future work.

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