



Duals of generalized Orlicz Hilbert sequence spaces and matrix transformations

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Abstract. In this paper, we define the sequence spaces $h_c(\Delta_v^{(m)}, \mathcal{M}, u, p)$, $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and $h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)$ resulting from the infinite Hilbert matrix and the Musielak-Orlicz function. We give some topological properties and inclusion relations of these newly created spaces. We also identified α -, β - and γ -duals of the spaces. Finally, we tried to characterize some matrix transformations between these spaces.

1. Introduction and Preliminaries:

Hilbert defined the Hilbert matrix in 1894. The Hilbert matrix played both several branches of mathematics and computational sciences. The $n \times n$ matrix $H = (h_{i,j}) = \frac{1}{i+j-1}$ ($i, j \in \mathbb{N}$) is a Hilbert matrix [1]. We consider the infinite Hilbert matrix H as follows:

$$H = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \dots \\ 1/2 & 1/3 & 1/4 & \dots & \cdot \\ 1/3 & 1/4 & \dots & \cdot & \cdot \\ 1/4 & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

and it can be showed in integral form as follows:

$$H = (h_{i,j}) = \int_0^1 x^{i+j-2} dx$$

The inverse of Hilbert matrix H^{-1} is defined by

$$H^{-1} = (h_{i,j}^{-1}) = (-1)^{i+j} (i+j-1) \begin{pmatrix} n+i-1 \\ n-j \end{pmatrix} \begin{pmatrix} n+j-1 \\ n-i \end{pmatrix} \begin{pmatrix} i+j-1 \\ i-1 \end{pmatrix}^2$$

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for all $i, j \in \mathbb{N}$.

Let us we denote the space of all real or complex sequence with w . We write the sequence spaces of all convergent, null and bounded sequences by c, c_0 and l_∞ , respectively. Also we will denote the space of all bounded, convergent and absolutely convergent series with bs, cs , and l_1 respectively. Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers and X, Y be subsets of w . We write $A_n(x) = \sum_k a_{nk}x_k$ and $Ax = A_n(x)$ for $n, k \in \mathbb{N}$. For a sequence space X , the matrix domain of an infinite matrix A is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\}$$

which is also a sequence space. We denote with (X, Y) the class of all matrices A such that $A : X \rightarrow Y$.

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. For the triangle matrices A, B and a sequence x , $A(Bx) = (AB)x$ holds. We remark that the triangle matrix A uniquely has an inverse matrix $A^{-1} = B$ and the matrix B is also triangle.

Let X be a normed sequence space. A sequence (b_n) in X is called a Schauder basis for X if for every $x \in X$ there is a unique sequence (α_n) of scalars such that

$$\lim_n \left\| x - \sum_{k=0}^n \alpha_k b_k \right\| = 0.$$

A B -space is a complete normed space. A topological sequence space in which all coordinate functionals $\pi_k, \pi_k(x) = x_k$, are continuous is called a K -space. A BK -space is defined as a K -space which is also a B -space, that is, a BK -space is a Banach space with continuous coordinates. For example, the space $l_p (1 \leq p < \infty)$ is BK -space with $\|x\|_p = \left(\sum_{k=0}^\infty |x_k|^p\right)^{\frac{1}{p}}$ and c, c_0 and l_∞ are BK -space with $\|x\|_\infty = \sup_k |x_k|$.

Kızmaz [2] was firstly introduced the concept of the difference operator in the sequence spaces. Further Et and Çolak [3] generalized the idea of difference sequence spaces of Kızmaz. Besides this topic was studied by many authors ([4], [5]). Now, the difference matrix $\Delta = (\delta_{nk})$ defined by

$$\delta_{nk} = \begin{cases} (-1)^{n-k}, & (n-1 \leq k \leq n) \\ 0, & (0 < n-1 \text{ or } n > k). \end{cases}$$

The difference operator order m is defined $\Delta^{(m)} : w \rightarrow w, (\Delta^{(1)}x)_k = (x_k - x_{k-1})$ and $\Delta^{(m)}x = (\Delta^{(1)}x)_k \circ (\Delta^{(m-1)}x)_k$ for $m \geq 2$.

The triangle matrix $\Delta^{(m)} = (\delta_{nk}^{(m)})$ defined by

$$\delta_{nk}^{(m)} = \begin{cases} (-1)^{n-k} \binom{m}{n-k}, & (\max\{0, n-m\} \leq k \leq n) \\ 0, & (0 \leq k < \max\{0, n-m\} \text{ or } n > k) \end{cases}$$

for all $k, n \in \mathbb{N}$ and for any fixed $m \in \mathbb{N}$.

Let $v = (v_k)$ be any fixed sequence of nonzero complex numbers. We define operators $\Delta^{(m)} : w \rightarrow w$ by $m \in \mathbb{N}, \Delta_v^{(0)}x_k = v_kx_k, \Delta_v x_k = (v_kx_k - v_{k-1}x_{k-1}), \Delta_v^{(m)}x_k = \Delta_v^{(m-1)}x_k - \Delta_v^{(m-1)}x_{k-1}$ and so that

$$\Delta_v^{(m)}x_k = \sum_{i=0}^m (-1)^i \binom{m}{i} v_{k-i}x_{k-i}$$

Polat [6] and Kirisci and Polat [7] have defined some new sequence spaces using Hilbert matrix. Let h_c, h_0 and h_∞ be convergent Hilbert, null convergent Hilbert and bounded Hilbert sequence spaces, respectively.

An Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is continuous, non-decreasing and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. Lindenstrauss and Tzafriri [8] used the idea of Orlicz function to construct the following sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

l_M is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

which is called an Orlicz sequence space. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called the Musielak-Orlicz function (see [9], [10]). For more details on sequence spaces, see ([11], [12], [13]) and the references there in.

Let X be a linear metric space. A function $p : X \rightarrow \mathbb{R}$ is called a paranorm, if

- (P1) $p(x) \geq 0$ for $x \in X$,
- (P2) $p(-x) = p(x)$ for all $x \in X$,
- (P3) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
- (P4) If (λ_n) is a sequence of scalars with $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$, then $p(\lambda_n x_n - \lambda x) \rightarrow 0$ as $n \rightarrow \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see ([9], Theorem 10.4.2, page 183)).

2. Main Results

In this section we define the sequence spaces $h_c(\Delta_v^{(m)}, \mathcal{M}, u, p)$, $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and $h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and give some relations between them. These sequence spaces are linear and BK-spaces. We prove that the new Hilbert sequence spaces $h_c(\Delta_v^{(m)}, \mathcal{M}, u, p)$, $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and $h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)$ are isometrically isomorphic to the space c , c_0 and l_∞ respectively.

Definition 2.1. Let $\mathcal{M}=(M_k)$ be a sequence of Orlicz functions, $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Also, let $p = (p_k)$ and $u = (u_k)$ be the bounded sequence and sequence of positive real numbers, respectively and $H = (h_{i,j})$ be an infinite Hilbert matrix. In the present paper we have defined the following sequence spaces:

$$h_c(\Delta_v^{(m)}, \mathcal{M}, u, p) = \left\{ x = (x_k) \in w : \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} x_k|}{\rho} \right) \right]^{p_k} \text{ exists, for some } \rho > 0 \right\},$$

$$h_0(\Delta_v^{(m)}, \mathcal{M}, u, p) = \left\{ x = (x_k) \in w : \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} x_k|}{\rho} \right) \right]^{p_k} = 0, \text{ for some } \rho > 0 \right\},$$

$$h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p) = \left\{ x = (x_k) \in w : \sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} x_k|}{\rho} \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$

If we take $M_k(x) = x$ for all $k \in \mathbb{N}$ and $(v_k) = (1, 1, \dots)$, $M_k(x) = x$ for all $k \in \mathbb{N}$, we obtain that $h_c(\Delta_v^{(m)}, u, p)$, $h_0(\Delta_v^{(m)}, u, p)$, $h_\infty(\Delta_v^{(m)}, u, p)$ and $h_c(\Delta^{(m)}, u, p)$, $h_0(\Delta^{(m)}, u, p)$, $h_\infty(\Delta^{(m)}, u, p)$, respectively. Also if $(u_k) = (1)$ and $(p_k) = (1)$, for all $k \in \mathbb{N}$, we obtain $h_c(\Delta_v^{(m)}, \mathcal{M})$, $h_0(\Delta_v^{(m)}, \mathcal{M})$ and $h_\infty(\Delta_v^{(m)}, \mathcal{M})$.

We define the sequence $y = (y_n)$ which will be frequently used, as the $H\Delta_v^{(m)}$ -transform of a sequence as follows:

$$\begin{aligned} (y_n) &= (H\Delta_v^{(m)}x)^{(\mathcal{M},u,p)} \\ &= \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \sum_{i=k}^n (-1)^{i-k} \binom{m}{i-k} x_i v_k|}{\rho} \right) \right]^{p_k} \end{aligned} \tag{2.1}$$

for each $k, m, n \in \mathbb{N}$.

We will use the following inequality throughout the paper. If $0 < p_k \leq \sup p_k = H, D = \max(1, 2^{H-1})$, then

$$|a_k + b_k|^{p_k} \leq D \{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{2.2}$$

for all k and $a_k, b_k \in \mathbb{C}$.

Theorem 2.2. Let $\mathcal{M}=(M_k)$ be a sequence of Orlicz functions, $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Also, let $p = (p_k)$ and $u = (u_k)$ be the bounded sequence and sequence of positive real numbers, respectively. Then $h_c(\Delta_v^{(m)}, \mathcal{M}, u, p), h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and $h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)$ are linear spaces over the complex field \mathbb{C} .

Proof. We shall prove the assertion for $h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)$ only and others can be proved similarly. Let $x = (x_k), y = (y_k) \in h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\begin{aligned} \sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} x_k|}{\rho_1} \right) \right]^{p_k} &< \infty, \\ \sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} y_k|}{\rho_2} \right) \right]^{p_k} &< \infty, \end{aligned}$$

for some $\rho_1, \rho_2 > 0$. Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is a non-decreasing and convex, using (2.2) inequality, we have

$$\begin{aligned} &\sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} (\alpha x_k + \beta y_k)|}{\rho_3} \right) \right]^{p_k} \\ &\leq \sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} \alpha x_k|}{\rho_3} + \frac{|u_k \Delta_v^{(m)} \beta y_k|}{\rho_3} \right) \right]^{p_k} \\ &\leq D \sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} x_k|}{\rho_1} \right) \right]^{p_k} \\ &\quad + D \sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} y_k|}{\rho_2} \right) \right]^{p_k} \\ &< \infty. \end{aligned}$$

Thus, it becomes $\alpha x + \beta y \in h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)$. This proves that $h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)$ is linear space. Similarly, it can be proved that $h_c(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ are linear spaces.

Theorem 2.3. Let $\mathcal{M}=(M_k)$ be a sequence of Orlicz functions, $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Also, let $p = (p_k)$ and $u = (u_k)$ be the bounded sequence and sequence of positive real numbers, respectively. Then $h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)$ is a paranorm space with the following paranorm,

$$g(x) = \inf \left\{ (\rho)^{\frac{p_k}{G}} : \left(\sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} x_k|}{\rho} \right) \right]^{p_k} \right)^{\frac{1}{G}} \leq 1, \text{ for some } \rho > 0 \right\},$$

where $0 < p_k \leq \sup p_k = H$ and $G = \max(1, H)$.

Proof. (i) Clearly $g(x) \geq 0$ for $x = (x_k) \in h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)$. Since $M_k(0) = 0$, we get $g(\theta) = 0$.

(ii) $g(-x) = g(x)$ is trivial.

(iii) Let $x, y \in h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)$. Then there exist positive numbers ρ_1, ρ_2 such that

$$\sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} x_k|}{\rho_1} \right) \right]^{p_k} \leq 1$$

$$\sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} y_k|}{\rho_2} \right) \right]^{p_k} \leq 1$$

Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we have

$$\begin{aligned} & \sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} (x_k + y_k)|}{\rho} \right) \right]^{p_k} \\ &= \sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} (x_k + y_k)|}{\rho_1 + \rho_2} \right) \right]^{p_k} \\ &\leq \frac{\rho_1}{\rho_1 + \rho_2} \sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} x_k|}{\rho_1} \right) \right]^{p_k} \\ &\quad + \frac{\rho_2}{\rho_1 + \rho_2} \sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} y_k|}{\rho_2} \right) \right]^{p_k} \\ &\leq 1 \end{aligned}$$

and thus,

$$\begin{aligned} g(x+y) &= \inf \left\{ (\rho)^{\frac{p_k}{G}} : \left(\sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} (x_k + y_k)|}{\rho} \right) \right]^{p_k} \right)^{\frac{1}{G}} \leq 1, \text{ for some } \rho > 0 \right\} \\ &\leq \inf \left\{ (\rho_1)^{\frac{p_k}{G}} : \left(\sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} (x_k + y_k)|}{\rho_1} \right) \right]^{p_k} \right)^{\frac{1}{G}} \leq 1, \text{ for some } \rho > 0 \right\} \\ &\quad + \inf \left\{ (\rho_2)^{\frac{p_k}{G}} : \left(\sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)} (x_k + y_k)|}{\rho_2} \right) \right]^{p_k} \right)^{\frac{1}{G}} \leq 1, \text{ for some } \rho > 0 \right\}. \end{aligned}$$

Therefore, $g(x+y) \leq g(x) + g(y)$.

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$\begin{aligned}
 g(\lambda x) &= \inf \left\{ (\rho)^{\frac{p_k}{G}} : \left(\sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)}(\lambda x_k)|}{\rho} \right) \right]^{p_k} \right)^{\frac{1}{G}} \leq 1, \text{ for some } \rho > 0 \right\} \\
 &= \inf \left\{ (|\lambda|t)^{\frac{p_k}{G}} : \left(\sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{t} \right) \right]^{p_k} \right)^{\frac{1}{G}} \leq 1, \text{ for some } \rho > 0 \right\}
 \end{aligned}$$

where $t = \frac{\rho}{|\lambda|} > 0$. Since $|\lambda|^{p_k} \leq \max(1, |\lambda|^{\sup p_k})$, we have

$$g(\lambda x) \leq \max(1, |\lambda|^{\sup p_k}) \cdot \inf \left\{ (t)^{\frac{p_k}{G}} : \left(\sup_n \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{t} \right) \right]^{p_k} \right)^{\frac{1}{G}} \leq 1, \text{ for some } \rho > 0 \right\}.$$

So the fact that scalar multiplication is continuous is due to the above inequality. This completes the proof of the theorem.

Theorem 2.4. Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $u = (u_k)$ be a sequence of positive real numbers and $v = (v_k)$ be any fixed sequence of non-zero complex numbers. If for each k , $0 \leq p_k \leq q_k < \infty$, $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers, then $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p) \subseteq h_0(\Delta_v^{(m)}, \mathcal{M}, u, q)$.

Proof. Let $x \in h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$. Then

$$\sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{\rho} \right) \right]^{p_k} \rightarrow 0 \text{ as } n \rightarrow \infty$$

This means that

$$\frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{\rho} \right) \right]^{p_k} < 1$$

for large enough k values. Since M_k is increasing and $p_k \leq q_k$, we have as $n \rightarrow \infty$

$$\sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{\rho} \right) \right]^{q_k} \leq \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{\rho} \right) \right]^{p_k} \rightarrow 0.$$

Thus $x \in h_0(\Delta_v^{(m)}, \mathcal{M}, u, q)$. This completes the proof.

Theorem 2.5. Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $v = (v_k)$ be any fixed sequence of non-zero complex numbers and $\varphi = \lim_{t \rightarrow \infty} \frac{M_k(t)}{t} > 0$. Then $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p) \subseteq h_0(\Delta_v^{(m)}, u, p)$.

Proof. Let $\varphi > 0$ to prove that $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p) \subseteq h_0(\Delta_v^{(m)}, u, p)$. From the definition of φ , $M_k(t) \geq \varphi(t)$, for all $t > 0$. Since $\varphi > 0$, we have $t \leq \frac{1}{\varphi} M_k(t)$ for all $t > 0$.

Let $x = (x_k) \in h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$. Thus, we have

$$\sum_{k=1}^n \frac{1}{n+k-1} \left[\frac{|u_k \Delta_v^{(m)}(x_k)|}{\rho} \right]^{p_k} \leq \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{\rho} \right) \right]^{p_k}$$

which means that $x = (x_k) \in h_0(\Delta_v^{(m)}, u, p)$. This completes the proof.

Theorem 2.6. Let $\mathcal{M}' = (M'_k)$ and $\mathcal{M}'' = (M''_k)$ be sequences of Orlicz functions and $v = (v_k)$ be any fixed sequence of non-zero complex numbers, then

$$h_0(\Delta_v^{(m)}, \mathcal{M}', u, p) \cap h_0(\Delta_v^{(m)}, \mathcal{M}'', u, p) \subseteq h_0(\Delta_v^{(m)}, (\mathcal{M}' + \mathcal{M}''), u, p).$$

Proof. Let $x = (x_k) \in h_0(\Delta_v^{(m)}, \mathcal{M}', u, p) \cap h_0(\Delta_v^{(m)}, \mathcal{M}'', u, p)$. Therefore,

$$\sum_{k=1}^n \frac{1}{n+k-1} \left[M'_k \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{\rho} \right) \right]^{p_k} \text{ as } n \rightarrow \infty$$

$$\sum_{k=1}^n \frac{1}{n+k-1} \left[M''_k \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{\rho} \right) \right]^{p_k} \text{ as } n \rightarrow \infty.$$

Then, we have

$$\begin{aligned} & \sum_{k=1}^n \frac{1}{n+k-1} \left[(M'_k + M''_k) \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{\rho} \right) \right]^{p_k} \\ & \leq K \left\{ \sum_{k=1}^n \frac{1}{n+k-1} \left[M'_k \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{\rho} \right) \right]^{p_k} \right\} \\ & \quad + K \left\{ \sum_{k=1}^n \frac{1}{n+k-1} \left[M''_k \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{\rho} \right) \right]^{p_k} \right\} \\ & \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus,

$$\sum_{k=1}^n \frac{1}{n+k-1} \left[(M'_k + M''_k) \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{\rho} \right) \right]^{p_k} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, $x = (x_k) \in h_0(\Delta_v^{(m)}, (\mathcal{M}' + \mathcal{M}''), u, p)$ and this completes the proof.

Theorem 2.7. Let $\mathcal{M}' = (M'_k)$ and $\mathcal{M}'' = (M''_k)$ be sequences of Orlicz functions and $v = (v_k)$ be any fixed sequence of non-zero complex numbers, then

$$h_0(\Delta_v^{(m)}, \mathcal{M}', u, p) \subseteq h_0(\Delta_v^{(m)}, (\mathcal{M}' \circ \mathcal{M}''), u, p)$$

Proof. Let $x = (x_k) \in h_0(\Delta_v^{(m)}, \mathcal{M}', u, p)$. Then we have

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n+k-1} \left[M'_k \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{\rho} \right) \right]^{p_k} = 0.$$

Let $\varepsilon > 0$ and choose $\delta > 0$ with $0 < \delta < 1$ such that $M_k(t) < \varepsilon$, for $0 \leq t \leq \delta$.

Write $y_k = \frac{1}{n+k-1} \left[M'_k \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{\rho} \right) \right]$ and consider

$$\sum_{k=1}^n [M_k(y_k)]^{p_k} = \sum_1 [M_k(y_k)]^{p_k} + \sum_2 [M_k(y_k)]^{p_k}$$

where the first summation is over $y_k \leq \delta$ and the second summation is over $y_k > \delta$. Since M_k is continuous, we have

$$\sum_1 [M_k(y_k)]^{p_k} < \varepsilon^H \tag{2.3}$$

and for $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \leq 1 + \frac{y_k}{\delta}.$$

From the definition, we have for $y_k > \delta$

$$M_k(y_k) < 2M_k(1) \frac{y_k}{\delta}.$$

Hence,

$$\sum_2 [M_k(y_k)]^{p_k} \leq \max \left(1, (2M_k(1) \delta^{-1})^H \right) \sum_1 [y_k]^{p_k} \tag{2.4}$$

From the equation (2.3) and (2.4), we have

$$h_0(\Delta_v^{(m)}, \mathcal{M}, u, p) \subseteq h_0(\Delta_v^{(m)}, (\mathcal{M}' \circ \mathcal{M}''), u, p).$$

Theorem 2.8. Hilbert sequence spaces $h_c(\Delta_v^{(m)}, \mathcal{M}, u, p)$, $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and $h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)$ are isometrically isomorphic to the space c , c_0 and l_∞ respectively, that is, $h_c(\Delta_v^{(m)}, \mathcal{M}, u, p) \cong c$, $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p) \cong c_0$ and $h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p) \cong l_\infty$.

Proof. We'll just do the proof for $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p) \cong c_0$ for the others it can be done similarly. To demonstrate the theorem, we must show that there is linear bijection between the space $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and c_0 . For this, we consider the transformation T defined by the notation (2.1), from $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ to c_0 by $x \rightarrow y = Tx$. The linearity of T is obvious. Moreover, when $Tx = \theta$ it is trivial that $x = \theta = (0, 0, 0, \dots)$ and hence T is injective. Next, let $y = (y_n) \in c_0$ and the sequence $x = (x_n)$ is defined as follows:

$$x_n = v_n^{-1} \sum_{k=1}^n \left[\sum_{k=1}^n \binom{m+n-i-1}{i-k} h_{ik}^{-1} \right] y_k$$

where h_{ik}^{-1} is defined by (2.1). Then,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} (H\Delta_v^{(m)} x)_n^{\mathcal{M},u,p} &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \Delta_v^{(m)}(x_k)|}{\rho} \right) \right]^{p_k} \\
 &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \sum_{i=0}^m (-1)^i \binom{m}{i} x_{k-i} v_{k-i}|}{\rho} \right) \right]^{p_k} \\
 &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \sum_{i=k}^n (-1)^{i-k} \binom{m}{i-k} x_k v_k|}{\rho} \right) \right]^{p_k} \\
 &= \lim_{n \rightarrow \infty} y_n = 0.
 \end{aligned}$$

Thus, $x \in h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$. As a result, it is clear that T is surjective. Since it is linear bijection, $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and c_0 are linear isomorphic. This completes the proof.

Remark 2.9. It is well known that the spaces c, c_0 and l_∞ are BK -spaces. Let us considering the fact that $\Delta_v^{(m)}$ is a triangle, we can say that the Hilbert sequence spaces $h_c(\Delta_v^{(m)}, \mathcal{M}, u, p)$, $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and $h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)$ are BK -spaces with the norm defined by

$$\begin{aligned}
 \|x\|_{\Delta}^{\mathcal{M},u,p} &= \|H\Delta_v^{(m)} x\|_{\infty}^{\mathcal{M},u,p} \\
 &= \sup_n \left| \sum_{k=1}^n \frac{1}{n+k-1} \left[M_k \left(\frac{|u_k \sum_{i=0}^m (-1)^i \binom{m}{i} x_{k-i} v_{k-i}|}{\rho} \right) \right]^{p_k} \right|
 \end{aligned} \tag{2.5}$$

Corollary 2.10. Define the space $d^{(k)} = (d_n^{(k)}(\Delta_v^{(m)}, \mathcal{M}, u, p))_{n \in \mathbb{N}}$

$$d_n^{(k)}(\Delta_v^{(m)}, \mathcal{M}, u, p) = \begin{cases} \sum_{k=1}^n \left[M_k \left(\frac{|u_k \sum_{i=k}^n \binom{m+n-i-1}{n-i} h_{ik}^{-1} v_k|}{\rho} \right) \right]^{p_k}, & n \geq k \\ 0, & n < k \end{cases}$$

for every fixed $k \in \mathbb{N}$. The following statements hold:

(i) The sequence $d_n^{(k)}(\Delta_v^{(m)}, \mathcal{M}, u, p)$ is a basis for the space $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and every $x \in h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ has a unique representation of the form

$$x = \sum_k (H\Delta_v^{(m)} x)_k^{\mathcal{M},u,p} d^{(k)}$$

(ii) The set $\{t, d^{(1)}, d^{(2)}, \dots\}$ is a basis or the space $h_c(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and every $x \in h_c(\Delta_v^{(m)}, \mathcal{M}, u, p)$ has a unique representation of the form

$$x = st + \sum_k \left[(H\Delta_v^{(m)} x)_k^{\mathcal{M},u,p} - s \right] d^{(k)}$$

where $t = t_n(\Delta_v^{(m)}, \mathcal{M}, u, p) = \sum_{k=1}^n M_k \left(\left| \frac{u_k \sum_{i=k}^n \binom{m+n-i-1}{n-i} |h_{ik}^{-1} v_k|}{\rho} \right| \right)^{p_k}$ for all $k \in \mathbb{N}$ and $s = \lim_{k \rightarrow \infty} (H\Delta_v^{(m)} x)_k^{M, u, p}$.

Corollary 2.11. The Hilbert sequence spaces $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and $h_c(\Delta_v^{(m)}, \mathcal{M}, u, p)$ are separable.

3. Characterizations of Matrix Transformation and α -, β - and γ -duals

Let $A = (a_{nk})$ be an infinite matrix of complex numbers, X and Y be subsets of the sequence space w . Let $x = (x_k)$ and $y = (y_k)$ be two sequences. Thus, we can write $xy = (x_k y_k)$, $x^{-1} * Y = \{a \in w : ax \in Y\}$ and $M(X, Y) = \cap_{x \in X^{-1}} * Y = \{a \in w : ax \in Y, \text{ for all } x \in X\}$ for the multiplier space of X and Y . In the special cases of $Y = \{l_1, cs, bs\}$, we write $x^\alpha = x^{-1} * l_1$, $x^\beta = x^{-1} * cs$, $x^\gamma = x^{-1} * bs$ and $X^\alpha = M(X, l_1)$, $X^\beta = M(X, cs)$, $X^\gamma = M(X, bs)$ for the α -dual, β -dual, γ -dual of X . By $A_n = (a_{nk})$ we denote the sequence in the n^{th} -row of A and write $A_n(x) = \sum_{k=1}^\infty a_{nk} x_k \forall n \in \mathbb{N}$ and $A(x) = (A_n(x))$, provided $A_n \in x^\beta$ for all n .

We shall begin with the lemmas due to Stieglitz ve Tietz [15] which will be used in the computation of the β - and γ -duals of the Hilbert sequence spaces.

Lemma 3.1. [16] Let X, Y be any two sequence spaces. $A \in (X : Y_T)$ if and only if $TA \in (X : Y)$, where A is an infinite matrix and T is a triangle matrix.

Lemma 3.2. (i) Let $A_n = (a_{nk})$ be an infinite matrix. Then $A \in (c_0 : l_\infty)$ if and only if

$$\sup_n \sum_k |a_{nk}| < \infty \tag{3.1}$$

(ii) $A \in (c_0 : c)$ if and only if (3.1) holds with

$$\lim_n a_{nk} \text{ exists for all } k. \tag{3.2}$$

(iii) $A \in (c_0 : bs)$ if and only if

$$\sup_n \sum_k \left| \sum_{n=0}^m a_{nk} \right| < \infty. \tag{3.3}$$

(iv) $A \in (c_0 : cs)$ if and only if (3.3) holds with

$$\sum_k a_{nk} \text{ convergent for all } k. \tag{3.4}$$

Lemma 3.3. (i) Let $A_n = (a_{nk})$ be an infinite matrix. Then $A \in (c : c)$ if and only if (3.1) and (3.2) hold with

$$\lim_{n \rightarrow \infty} \sum_k a_{nk} \text{ exists.}$$

(ii) $A \in (l_\infty : c)$ if and only if (3.2) holds with

$$\lim_n \sum_k |a_{nk}| = \sum_k \left| \lim_{n \rightarrow \infty} a_{nk} \right|. \tag{3.5}$$

Lemma 3.4.(i) Let $A_n = (a_{nk})$ be an infinite matrix. Then $A \in (c : cs)$ if and only if (3.3), (3.4) hold and

$$\sum_n \sum_k a_{nk} \text{ convergent.} \tag{3.6}$$

(ii) $A \in (l_\infty : cs)$ if and only if (3.2) holds and

$$\lim_m \sum_k \left| \sum_{n=m}^\infty a_{nk} \right| = 0. \tag{3.7}$$

Lemma 3.5. [17] Let $U = (u_{nk})$ be an infinite matrix of complex numbers for all $n, k \in \mathbb{N}$. Let $B^U = (b_{nk})$ be defined via a sequence $a = (a_k) \in w$ and inverse of the triangle matrix $U = (u_{nk})$ by

$$b_{nk} = \sum_{j=k}^n a_j u_{jk}$$

for all $n, k \in \mathbb{N}$. Then,

$$\begin{aligned} X_U^\alpha &= \{a = (a_k) \in w : B^U \in (X : l_1)\}, \\ X_U^\beta &= \{a = (a_k) \in w : B^U \in (X : c)\} \\ X_U^\gamma &= \{a = (a_k) \in w : B^U \in (X : l_\infty)\}. \end{aligned}$$

Theorem 3.6. The α -, β - and γ -duals of the Hilbert sequence spaces defined as

$$\begin{aligned} [h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\alpha &= \{a = (a_k) \in w : W \in (c_0 : l_1)\}, \\ [h_c(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\alpha &= \{a = (a_k) \in w : W \in (c : l_1)\}, \\ [h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\alpha &= \{a = (a_k) \in w : W \in (l_\infty : l_1)\}, \\ [h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\beta &= \{a = (a_k) \in w : W \in (c_0 : c)\}, \\ [h_c(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\beta &= \{a = (a_k) \in w : W \in (c : c)\}, \\ [h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\beta &= \{a = (a_k) \in w : W \in (l_\infty : c)\}, \\ [h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\gamma &= \{a = (a_k) \in w : W \in (c_0 : l_\infty)\}, \\ [h_c(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\gamma &= \{a = (a_k) \in w : W \in (c : l_\infty)\}, \\ [h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\gamma &= \{a = (a_k) \in w : W \in (l_\infty : l_\infty)\}. \end{aligned}$$

Proof. We shall only compute the α -, β - and γ -duals of $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ sequence space. Let h_n^{-1} is defined by (2.1). Let us take any $a = (a_k) \in w$. We define the matrix $W = (w_{nk})$ by

$$w_{nk} = \sum_{k=1}^n \left[M_k \left(\frac{u_k \left| \sum_{i=k}^n \binom{m+n-i-1}{n-i} h_{ik}^{-1} a_n v_n^{-1} \right| \right)}{\rho} \right]^{pk}.$$

Consider the equation

$$\begin{aligned} \sum_{k=1}^n a_k x_k &= \sum_{k=1}^n (a_k v_k^{-1}) (v_k x_k) \\ &= \sum_{k=1}^n \left[M_k \left(\frac{u_k \left| \sum_{i=1}^k \left\{ \sum_{j=i}^k \binom{m+k-j-1}{k-j} h_{ij}^{-1} \right\} a_k v_k^{-1} y_i \right|}{\rho} \right) \right]^{p_k} \\ &= \sum_{k=1}^n \left[M_k \left(\frac{u_k \left| \sum_{i=1}^k \left\{ \sum_{j=i}^k \binom{m+k-j-1}{k-j} h_{ij}^{-1} a_i v_i^{-1} \right\} y_k \right|}{\rho} \right) \right]^{p_k} \\ &= (Wy)_n. \end{aligned} \tag{3.8}$$

Using (3.8), we have $ax = (a_k x_k) \in cs$ or bs whenever $x = (x_k) \in h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ if and only if $Wy \in l_1, c$ or l_∞ whenever $y = (y_k) \in c_0$. Then, from Lemma 3.1 and Lemma 3.5, we obtain that $a = (a_k) \in [h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\alpha$, $a = (a_k) \in [h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\beta$ or $a = (a_k) \in [h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\gamma$ if and only if $W \in (c_0 : l_1)$, $W \in (c_0 : c)$ or $W \in (c_0 : l_\infty)$, which is required result.

Therefore, the α -, β - and γ -duals of Hilbert sequence spaces will be helpful in the characterization of matrix transformations. Let X and Y be arbitrary subsets of w . We will show that the characterization of the classes $(X : Y_T)$ ve $(X_T : Y)$ can be reduced to (X, Y) , where T is a triangle. Since if the sequence spaces $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and c_0 are linearly isomorphic, then the equivalence class $x \in h_0(\Delta_v^{(m)}, \mathcal{M}, u, p) \Leftrightarrow y \in c_0$ holds. So using Lemma 3.1 and 3.5, we get the desired result.

Theorem 3.7. Let us consider the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$. These matrices get associated with each other by the relations:

$$b_{nk} = \sum_{k=1}^n \left[M_k \left(\frac{u_k \left| \sum_{j=k}^\infty \binom{m+n-j-1}{n-j} h_{jk}^{-1} a_{nj} v_k^{-1} \right|}{\rho} \right) \right]^{p_k} \tag{3.9}$$

for all $k, m, n \in \mathbb{N}$. Then the following statements are true:

- (i) $A \in (h_0(\Delta_v^{(m)}, \mathcal{M}, u, p) : Y)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in [h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\beta$ for all $n \in \mathbb{N}$ and $B \in (c_0, Y)$, where Y be any sequence space;
- (ii) $A \in (h_c(\Delta_v^{(m)}, \mathcal{M}, u, p) : Y)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in [h_c(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\beta$ for all $n \in \mathbb{N}$ and $B \in (c, Y)$, where Y be any sequence space;
- (iii) $A \in (h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p) : Y)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in [h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\beta$ for all $n \in \mathbb{N}$ and $B \in (l_\infty, Y)$, where Y be any sequence space.

Proof. We suppose that the relation in (3.9) holds between $A = (a_{nk})$ and $B = (b_{nk})$. The spaces $h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and c_0 are linearly isomorphic. Let $A \in (h_0(\Delta_v^{(m)}, \mathcal{M}, u, p) : Y)$ and $y = (y_k) \in c_0$. Then $BH\Delta_v^{(m)}$ exists and $(a_{nk}) \in [h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\beta$ for all $k \in \mathbb{N}$, it means that $(b_{nk}) \in c_0$ for all $k, n \in \mathbb{N}$. Hence, By exists for each $y \in c_0$. Thus, if we take $m \rightarrow \infty$ in the equality,

$$\sum_{k=1}^m a_{nk}x_k = \sum_{k=1}^m \left[M_k \left(\frac{\left| u_k \left[\sum_{i=1}^k \sum_{j=i}^k \binom{m+k-j-1}{k-j} h_{ij}^{-1} \right] a_{nk} v_k^{-1} \right|}{\rho} \right) \right]^{p_k} = \sum_k b_{nk}y_k$$

for all $m, n \in \mathbb{N}$ which conclude that $B \in (c_0, Y)$. On the contrary, let $(a_{nk})_{k \in \mathbb{N}} \in [h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)]^\beta$ for each $k \in \mathbb{N}$ and $B \in (c_0, Y)$ and $x = (x_k) \in h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$. Then it is clear that Ax exists. Thus, we attain from the following equality for all $n \in \mathbb{N}$

$$\sum_k b_{nk}y_k = \sum_k a_{nk}x_k$$

as $m \rightarrow \infty$ that $Ax = By$ and it is easy to show that $A \in (h_0(\Delta_v^{(m)}, \mathcal{M}, u, p) : Y)$. This completes the proof.

Theorem 3.8. Let us assume that components of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the following relation

$$e_{nk} = \sum_{k=1}^n \sum_{j=k}^n \frac{1}{n+j-1} \left[M_k \left(\frac{\left| u_k \sum_{j=k}^n (-1)^{j-k} \binom{m}{j-k} a_{jk} v_k^{-1} \right|}{\rho} \right) \right]^{p_k} \tag{3.10}$$

for all $m, n \in \mathbb{N}$ and X be any given sequence space. Then the following statements are true:

- (i) $A = (a_{nk}) \in (X : h_0(\Delta_v^{(m)}, \mathcal{M}, u, p))$ if and only if $E \in (X : c_0)$;
- (ii) $A = (a_{nk}) \in (X : h_c(\Delta_v^{(m)}, \mathcal{M}, u, p))$ if and only if $E \in (X : c)$;
- (iii) $A = (a_{nk}) \in (X : h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p))$ if and only if $E \in (X : l_\infty)$.

Proof. Let us suppose that $z = (z_k) \in X$. Using the relation (3.10), we have

$$\sum_{k=1}^m e_{nk}z_k = \sum_{k=1}^m \left[\sum_{k=1}^n \sum_{j=k}^n \frac{1}{n+j-1} \left[M_k \left(\frac{\left| u_k \left[\sum_{j=k}^n (-1)^{j-k} \binom{m}{j-k} a_{jk} v_k^{-1} \right] z_k \right|}{\rho} \right) \right]^{p_k} \right] \tag{3.11}$$

for all $m, n \in \mathbb{N}$. Then, for $m \rightarrow \infty$ equation (3.11) gives us that $(Ez)_n = \{H\Delta_v^{(m)}(Az)\}_n$. Thus, we can obtain that $Az \in h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$ if and only if $Ez \in c_0$. This completes the proof.

Now, we give some conditions:

$$\lim_k a_{nk} = 0 \text{ for all } n, \tag{3.12}$$

$$\lim_{n \rightarrow \infty} \sum_k |a_{nk}| = 0, \tag{3.13}$$

$$\lim_{n \rightarrow \infty} \sum_k |a_{nk} - a_{n,k+1}| = 0, \tag{3.14}$$

$$\sup_n \sum_k |a_{nk} - a_{n,k+1}| < \infty, \tag{3.15}$$

$$\lim_k (a_{nk} - a_{n,k+1}) \text{ exists for all } k, \quad (3.16)$$

$$\lim_{n \rightarrow \infty} \sum_k |a_{nk} - a_{n,k+1}| = \sum_k \left| \lim_{n \rightarrow \infty} (a_{nk} - a_{n,k+1}) \right|, \quad (3.17)$$

$$\sup_n \left| \lim_{n \rightarrow \infty} a_{nk} \right| < \infty, \quad (3.18)$$

Corollary 3.9. Let $A = (a_{nk})$ be an infinite matrix and $X = h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$, $Y = h_c(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and $Z = h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)$. Then, the following statements hold:

- (a) $A = (a_{nk}) \in (X, l_\infty)$ if and only if (3.1) holds with b_{nk} instead of a_{nk} ;
- (b) $A = (a_{nk}) \in (X, bs)$ if and only if (3.3) holds with b_{nk} instead of a_{nk} ;
- (c) $A = (a_{nk}) \in (Y, cs)$ if and only if (3.3), (3.4) and (3.6) hold with b_{nk} instead of a_{nk} ;
- (d) $A = (a_{nk}) \in (Z, c)$ if and only if (3.2) and (3.5) hold with b_{nk} instead of a_{nk} ;
- (e) $A = (a_{nk}) \in (Z, cs)$ if and only if (3.7) holds with b_{nk} instead of a_{nk} ;
- (f) $A = (a_{nk}) \in (X, c)$ if and only if (3.1) and (3.2) hold with b_{nk} instead of a_{nk} ;
- (g) $A = (a_{nk}) \in (X, cs)$ if and only if (3.3) and (3.4) hold with b_{nk} instead of a_{nk} .

Corollary 3.10. Let $A = (a_{nk})$ be an infinite matrix and $X = h_0(\Delta_v^{(m)}, \mathcal{M}, u, p)$, $Y = h_c(\Delta_v^{(m)}, \mathcal{M}, u, p)$ and $Z = h_\infty(\Delta_v^{(m)}, \mathcal{M}, u, p)$. Then, the following statements hold:

- (a) $A = (a_{nk}) \in (l_\infty, X)$ if and only if (3.13) holds with e_{nk} instead of a_{nk} ;
- (b) $A = (a_{nk}) \in (bs, X)$ if and only if (3.12) and (3.14) hold with e_{nk} instead of a_{nk} ;
- (c) $A = (a_{nk}) \in (bs, Y)$ if and only if (3.12), (3.16) and (3.17) hold with e_{nk} instead of a_{nk} ;
- (d) $A = (a_{nk}) \in (cs, Y)$ if and only if (3.15) and (3.2) hold with e_{nk} instead of a_{nk} ;
- (e) $A = (a_{nk}) \in (bs, Z)$ if and only if (3.12) and (3.15) hold with e_{nk} instead of a_{nk} ;
- (f) $A = (a_{nk}) \in (cs, Z)$ if and only if (3.15) and (3.18) hold with e_{nk} instead of a_{nk} ;
- (g) $A = (a_{nk}) \in (cs, X)$ if and only if (3.2) and (3.15) hold with e_{nk} instead of a_{nk} .

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