Filomat 37:26 (2023), 8983–8993
https://doi.org/10.2298/FIL2326983E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The multiplicity and asymptotic forms of eigenvalues of vectorial diffusion equations with some certain assumptions

Abdullah Ergün[a]

[a]*Sivas Vocational School of Technical Sciences, Sivas Cumhuriyet University, 58140, Sivas, Turkey*

Abstract. The main motivation point of this study is to obtain some novel results on the multiplicity of eigenvalues of diffusion equations. A diffusion equation with some boundary and jump conditions have been analyzed and integral equations have been obtained for the solution under certain initial conditions. Later, integral representations of these solutions have been provided. Finally, asymptotic formulas of eigenvalues with zeros of the characteristic have been considered. A brief conclusion has been given.

1. Introduction

Spectral theory of differential operators is an important field that has many applications in physics, mechanics, geophysics, electronics, mathematics and engineering, which is divided into two parts as direct and reverse spectral problems. For example, in mechanics, learning the density distribution in an inhomogeneous arc according to the given wavelengths, determining the interaction forces between the particles according to the energy levels of the particles; finding field potentials according to scattering data in quantum physics; determination of underground mines according to distribution characteristics of underground elements in geophysics; can be given as examples of inverse problems.
In addition to classical boundary value problems, discontinuous boundary value problems are important problems that both provide solutions to new concrete problems of mathematical physics and contribute to the development of theoretical mathematics. Boundary value problems with discontinuity in the interior of the interval are frequently encountered problems in mathematics, physics, geophysics and many branches of natural sciences in [2–4]. In general, these problems are associated with discontinuous material properties. For example; In electronics, the discontinuous inverse problem is applied to determine the parameters of the power line. Another example is geophysical models for the earth's oscillation. The discontinuity here is related to the reflection of shear waves at the base of the earth's crust. Now, an important diffusion problem in this direction is presented as follows.
We will start with the m-dimensional vectorial diffusion equation given by:

$$-y'' + [2\vartheta\hbar(x) + \Upsilon(x)]\,y = \vartheta^2 y \tag{1}$$

with the boundary conditions:

$$y'(0) = \theta \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 34K08, 34L05, 34K06; Secondary 34E05, 34L10
Keywords. Asymptotic forms, Vectorial diffusion operator, Eigenvalues.
Received: 28 March 2023; Revised: 06 April 2023; Accepted: 16 April 2023
Communicated by Maria Alessandra Ragusa
Email address: aergun@cumhuriyet.edu.tr (Abdullah Ergün)

$$y'(\pi) = \theta \tag{3}$$

and the jump conditions:

$$y(a+0) = \nu y(a-0)\ ,\ y'(a+0) = \nu^{-1} y'(a-0) \tag{4}$$

where ϑ is the spectral parameter and $y = (y_1, y_2, ...y_m)^T$ is an m-dimensional vector function. Hence, we assume that $\nu > 0\ ,\ \nu \neq 1$ and $a \in (0, \pi)$. The potential matrix $\hbar(x)$ and $\Upsilon(x)$ are $m \times m$ real symmetric matrix functions. Let us recall that $\hbar(x) \in W_2^1[0, \pi]$, $\Upsilon(x) \in L_2[0, \pi]$. Besides, without loss of generality, we denote the matrix function as $\Upsilon(x)$ that is defined positively and the m-dimensional zero vector as θ.
Several researchers have been studied on scalar and vectorial Sturm-Liouville problems extensively in the papers $[1-19]$. Vectorial Sturm-Liouville equations have wide applications in many areas. In [23], the author have investigated the importance of multi-particle problems in quantum mechanics. In [7], the authors have focused these important problems in terms of the hydrogen molecular ion. Also, several novel results have been considered on the diffusion equation in [16–19]. In [19], Mukhtarov and Yakubov have given some new findings on boundary value problems with discontinuous conditions in varied assortment of physical transfer problems. In [9], the author has investigated the problem that is given in $(1)-(4)$ in scaler case. It is not easy to extend all classical Sturm-Liouville theories and inverse spectral theories to the vectorial case. The origin of the difficulties arises from the multiplicity of the eigenvalues. In [20], Shen and Shies studied the multiplicity of eigenvalues of the m-dimensional vectorial Sturm-Liouville problem

$$y'' + \Upsilon(x)y = \vartheta y\ ,\ y(0) = y(1) = \theta$$

where Υ is continuous and $m \times m$ Jacobi matrix-valued function defined on $0 \leq x \leq 1$. Then, in [15], the author generalized the case when Υ is real symmetric. In [22], the authors extended the findings of the paper [20] to the Sturm-Liouville equations with a weight function, a leading coefficient and general seperation conditions. On all of these brief historical background, we know that there are no such result for the discontinuous problem $(1)-(4)$.

The paper is organized as follows: Firstly, we define the characteristic function of the eigenvalues of the problem given in $(1)-(4)$. We provide the eigenvalues of the problem overlap with the zeros of characteristic function. Then, we demonstrate the asymptotic forms of the solutions and obtain novel results on the multiplicity of the eigenvalues. Finally, we give a brief conclusion.

2. Characteristic function and asymptotics of solutions

We will start with remembering the Hilbert space $H = L^2(I, \mathbb{C}^m)$ with the scalar product

$$(f, g) = \int_0^a g_l^* f_l dx + \int_a^\pi g_r^* f_r dx = \int_0^\pi g^* f dx$$

where $f = (f_1, f_2, ..., f_m)^T$, $g = (g_1, g_2, ...g_m)^T$ and $f_i, g_i \in L^2(I)$,
$f_l(x) = f(x)\big|_{(0,a)}$ and $f_r(x) = f(x)\big|_{(a,\pi)}$. L is an operator associated with the problem $(1)-(4)$ on H as following:

$$Ly := -y'' + [2\vartheta\hbar(x) + \Upsilon(x)]y,\ y \in D(L),$$
$$D(L) = \Big\{y \in H; y, y' \in AC[I, \mathbb{C}^m],\ Ly \in L^2[I, \mathbb{C}^m], y'(0) = y'(\pi) = \theta,$$
$$y(a+0) = \nu y(a-0),\ y'(a+0) = \nu^{-1} y'(a-0)\Big\}.$$

Lemma 2.1 The eigenvalues of the problem given by $(1)-(4)$ are real, nonzero, and simple.
Proof. We can define the operator $L_0 y = -y'' + \Upsilon(x)y$ in $D(L_0)$ as:

$$D(L_0) = \Big\{y(x) \in W_2^2; L_0 y \in [I, \mathbb{C}^m],\ y'(0) = y'(\pi) = 0,$$
$$y(a+0) = \nu y(a-0),\ y'(a+0) = \nu^{-1} y'(a-0)\Big\} .$$

$\Upsilon(x)$ defined positively and symmetric (Hermitian matrix function). It is clear that the operator L_0 is a self-adjoint (see [21]). Hence, it is obvious that $(L_0 y, y) > 0$ for $\forall y(x) \in D(L_0)$. Namely, L_0 is positively defined.
Assume that ϑ is the eigenvalue of the problem $(1)-(4)$ and $y(x)$ is the eigenfunction corresponding to ϑ and holds the condition $(y, y) = 1$. In this case, from (1) we conclude that:

$$\vartheta^2 - 2\vartheta(\hbar y, y) - (L_0 y, y) = 0$$

Moreover, we have

$$\vartheta = (\hbar y, y) \pm \sqrt{(\hbar y, y)^2 + (L_0 y, y)}.$$

From the facts that $(L_0 y, y) > 0$ and $\hbar(x)$ is real and symmetric, we haveachieved to show that ϑ is real and nonzero.
Now we are in a position that to show the eigenvalue ϑ is simple. Let's assume for a moment that the reverse is true. Suppose that $y_1(x)$ and $y_2(x)$ are the linear independent eigenfunctions corresponding to the eigenvalue ϑ. Then, due to each $y(x)$ solution of equation (1) will be a linear combination of functions $y_1(x)$ and $y_2(x)$, it must satisfy the boundary conditions $(2)-(3)$ and discontinuity conditions (4). However, this is not true. Thus, the eigenvalues ϑ are simple. This completes the proof.
E_m will be $m \times m$ identify matrix and θ_m will be $m \times m$ zero matrix, in this case we investigate the problem $(0, a)$ and (a, π), respectively. The matrix initial value problem is given by

$$\begin{cases} -Y'' + (2\vartheta\hbar(x) + \Upsilon(x))Y = \vartheta^2 Y, \ x \in (0, a) \\ Y(0, \vartheta) = E_m, Y'(0, \vartheta) = \theta_m \end{cases} \tag{5}$$

on $(0, a)$ has a unique solution as $\kappa_1(x, \vartheta)$. Furthermore, $\kappa_1(x, \vartheta)$ is an entire matrix function in ϑ for any fixed $x \in (0, a)$ $(see\,[1], p17)$. By changing of the constants, we obtain

$$\kappa_1(x, \vartheta) = \cos\vartheta x E_m + \frac{1}{\vartheta}\int_0^x \sin\vartheta(x-t)(2\vartheta\hbar(t) + \Upsilon(t))\kappa_1(t, \vartheta)\,dt \tag{6}$$

on (a, π). Then, the matrix initial value problem

$$\begin{cases} -Y'' + (2\vartheta\hbar(x) + \Upsilon(x))Y = \vartheta^2 Y, \ x \in (a, \pi) \\ Y(a+0, \vartheta) = \nu Y(a-0, \vartheta) \\ Y'(a+0, \vartheta) = \nu^{-1} Y'(a-0, \vartheta) \end{cases} \tag{7}$$

has a unique solution $\kappa_2(x, \vartheta)$. Again, $\kappa_2(x, \vartheta)$ is an entire matrix function in ϑ for any fixed $x \in (a, \pi)$. By a similar way, changing of the constants, we get

$$\begin{aligned} \kappa_2(x, \vartheta) &= \tfrac{1}{2}\left(\nu + \tfrac{1}{\nu}\right) cos\vartheta x E_m + \tfrac{1}{2}\left(\nu - \tfrac{1}{\nu}\right) cos\vartheta(2a - x) E_m \\ &+ \tfrac{1}{2}\left(\nu + \tfrac{1}{\nu}\right)\int_0^a \tfrac{\sin\vartheta(x-t)}{\vartheta}(2\vartheta\hbar(t) + \Upsilon(t))\kappa_1(t, \vartheta)\,dt \\ &+ \tfrac{1}{2}\left(\nu - \tfrac{1}{\nu}\right)\int_0^a \tfrac{\sin\vartheta(x+t-2a)}{\vartheta}(2\vartheta\hbar(t) + \Upsilon(t))\kappa_1(t, \vartheta)\,dt \\ &+ \int_a^x \tfrac{\sin\vartheta(x-t)}{\vartheta}(2\vartheta\hbar(t) + \Upsilon(t))\kappa_2(t, \vartheta)\,dt \end{aligned} \tag{8}$$

or

$$\begin{aligned} \kappa_2(x, \vartheta) &= \nu\cos\vartheta(x-a)\kappa_1(a-0, \vartheta)E_m + \tfrac{\nu^{-1}}{\vartheta}\sin\vartheta(x-a)\kappa_1'(a-0, \vartheta)E_m \\ &+ \tfrac{1}{\vartheta}\int_a^x \sin\vartheta(x-t)(2\vartheta\hbar(t) + \Upsilon(t))\kappa_2(t, \vartheta)\,dt. \end{aligned} \tag{9}$$

As a consequence, one can write

$$\kappa(x, \vartheta) = \begin{cases} \kappa_1(x, \vartheta), \ x \in (0, a) \\ \kappa_2(x, \vartheta), x \in (a, \pi) \end{cases}$$

Then, any solution of the equations (1) under the conditions that are given in (2) and (4) can be expressed as

$$y(x,\vartheta)=\kappa(x,\vartheta)c_1=\begin{cases}\kappa_1(x,\vartheta)c_1, & x\in(0,a)\\ \kappa_2(x,\vartheta)c_1, & x\in(a,\pi)\end{cases} \tag{10}$$

where c_1 is an arbitrary m-dimensional constant vector. If ϑ is an eigenvalue of the problem (1) − (4), then $c_1\neq\theta$ and $y(x,\vartheta)$ holds the boundary condition at $x=\pi$, that is,

$$y'(\pi,\vartheta)=\kappa'(\pi,\vartheta)c_1=\kappa_2'(\pi,\vartheta)c_1=\theta.$$

Hence, we have

$$\det\left(\kappa_2'(\pi,\vartheta)\right)=0.$$

Similarly, on (a,π), consider the matrix initial value problem given by

$$\begin{cases}-Y''+(2\vartheta\hbar(x)+\Upsilon(x))Y=\vartheta^2Y\ ,\ x\in(a,\pi)\\ Y(\pi,\vartheta)=E_m\ ,Y'(\pi,\vartheta)=\theta_m.\end{cases} \tag{11}$$

The problem (11) has a unique solution $\psi_2(x,\vartheta)$. What's more, for any fixed $x\in(a,\pi)$, $\tau_2(x,\vartheta)$ is an entire matrix function in ϑ. Consider the matrix initial value problem

$$\begin{cases}-Y''+(2\vartheta\hbar(x)+\Upsilon(x))Y=\vartheta^2Y\ ,\ x\in(0,a)\\ Y(a-0,\vartheta)=\nu^{-1}Y(a+0,\vartheta)\\ Y'(a-0,\vartheta)=\nu Y'(a+0,\vartheta).\end{cases} \tag{12}$$

The problem (12) has a unique solution $\tau_1(x,\vartheta)$. What's more, for any fixed $x\in(0,a)$, $\tau_1(x,\vartheta)$is an entire matrix function in ϑ. We can deduce

$$\tau(x,\vartheta)=\begin{cases}\tau_1(x,\vartheta)\ ,\ x\in(0,a)\\ \tau_2(x,\vartheta)\ ,x\in(a,\pi)\end{cases}$$

In this stage, any solution of the equations (1) under the conditions (3) and (4) can be stated as

$$y(x,\vartheta)=\tau(x,\vartheta)c_2=\begin{cases}\tau_1(x,\vartheta)c_2, & x\in(0,a)\\ \tau_2(x,\vartheta)c_2, & x\in(a,\pi)\end{cases} \tag{13}$$

where c_2 is an arbitrary m-dimensional constant vector. If ϑ is an eigenvalue of the problem (1) − (4), then $c_2\neq\theta$ and $y(x,\vartheta)$ holds the boundary condition at $x=0$, namely,

$$y'(0,\vartheta)=\tau'(0,\vartheta)c_2=\tau_1'(0,\vartheta)c_2=\theta.$$

Then, one obtain

$$\det\left(\tau_1'(0,\vartheta)\right)=0.$$

Let $\Delta_j(\vartheta)=W\left(\kappa_j(x,\vartheta),\tau_j(x,\vartheta)\right)$ be the Wronskian of solution matrices $\kappa_j(x,\vartheta)$ and $\tau_j(x,\vartheta)$, $j=1,2$, that is,

$$\Delta_1(\vartheta)=\begin{vmatrix}\kappa_1(x,\vartheta) & \tau_1(x,\vartheta)\\ \kappa_1'(x,\vartheta) & \tau_1'(x,\vartheta)\end{vmatrix}.\ \Delta_2(\vartheta)=\begin{vmatrix}\kappa_2(x,\vartheta) & \tau_2(x,\vartheta)\\ \kappa_2'(x,\vartheta) & \tau_2'(x,\vartheta)\end{vmatrix}. \tag{14}$$

Lemma 2.2 For any $\vartheta\in C$, one can write $\Delta_1(\vartheta)=\Delta_2(\vartheta)$.

Proof. Because, the Wronskian of the solution matrices

$$\Delta_2(\vartheta) = \Delta_2(\vartheta)|_{x=a+0} =$$

$$= \begin{vmatrix} \kappa_2(a+0,\vartheta) & \tau_2(a+0,\vartheta) \\ \kappa_2'(a+0,\vartheta) & \tau_2'(a+0,\vartheta) \end{vmatrix} = \begin{vmatrix} \nu\kappa_1(a-0,\vartheta) & \nu\tau_1(a-0,\vartheta) \\ \nu^{-1}\kappa_1'(a-0,\vartheta) & \nu^{-1}\tau_1'(a-0,\vartheta) \end{vmatrix}$$

$$= \begin{vmatrix} \kappa_1(a-0,\vartheta) & \tau_1(a-0,\vartheta) \\ \kappa_1'(a-0,\vartheta) & \tau_1'(a-0,\vartheta) \end{vmatrix} = \begin{vmatrix} \kappa_1(x,\vartheta) & \tau_1(x,\vartheta) \\ \kappa_1'(x,\vartheta) & \tau_1'(x,\vartheta) \end{vmatrix}_{x=a-0} = \Delta_1(\vartheta).$$

This completes the proof.
We can denote $\Delta(\vartheta) = \Delta_1(\vartheta) = \Delta_2(\vartheta)$, we can write the following lemma.
Lemma 2.3 ϑ is an eigenvalue of (1) − (4) if any only if $\Delta(\vartheta) = 0$.
Proof. Necessity. Assume that ϑ_0 is an eigenvalue of (1) − (4). $y(x,\vartheta_0)$ is the eigenfunctions corresponding to ϑ_0, than by (8), we obtain

$$y(x,\vartheta_0) = \kappa(x,\vartheta_0)c_3 = \begin{cases} \kappa_1(x,\vartheta_0)c_3, & x\in(0,a) \\ \kappa_2(x,\vartheta_0)c_3, & x\in(a,\pi) \end{cases} \tag{15}$$

$$y(x,\vartheta_0) = \tau(x,\vartheta_0)c_4 = \begin{cases} \tau_1(x,\vartheta_0)c_4, & x\in(0,a) \\ \tau_2(x,\vartheta_0)c_4, & x\in(a,\pi) \end{cases} \tag{16}$$

c_3, c_4 are m-dimensional nonzero constant vector. So, from (15) and (16), we have

$$\left.\begin{matrix} \kappa_1(x,\vartheta_0)c_3 = \tau_1(x,\vartheta_0)c_4 \\ \kappa_1'(x,\vartheta_0)c_3 = \tau_1'(x,\vartheta_0)c_4 \end{matrix}\right\} x\in(0,a).$$

By simplfying the result, we provide

$$\begin{pmatrix} \kappa_1(x,\vartheta_0) & -\tau_1(x,\vartheta_0) \\ \kappa_1'(x,\vartheta_0) & -\tau_1'(x,\vartheta_0) \end{pmatrix} \cdot \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} \theta \\ \theta \end{pmatrix}.$$

Due to $c_3, c_4 \neq 0$, the coefficient determinant of above linear system of equations can be written as

$$\begin{vmatrix} \kappa_1(x,\vartheta_0) & -\tau_1(x,\vartheta_0) \\ \kappa_1'(x,\vartheta_0) & -\tau_1'(x,\vartheta_0) \end{vmatrix} = (-1)^m \begin{vmatrix} \kappa_1(x,\vartheta_0) & \tau_1(x,\vartheta_0) \\ \kappa_1'(x,\vartheta_0) & \tau_1'(x,\vartheta_0) \end{vmatrix} = (-1)^m \Delta_1(\vartheta_0).$$

With the help of the previous lemma, we still have $\Delta_2(\vartheta_0) = \Delta_1(\vartheta_0) = \Delta(\vartheta_0) = 0$.
Sufficiency. If for some $\vartheta_0 \in \mathrm{C}, \Delta(\vartheta_0) = 0$. Then, the linear systems of equations

$$\begin{pmatrix} \kappa_1(x,\vartheta_0) & \tau_1(x,\vartheta_0) \\ \kappa_1'(x,\vartheta_0) & \tau_1'(x,\vartheta_0) \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \theta \\ \theta \end{pmatrix} \tag{17}$$

$$\begin{pmatrix} \kappa_2(x,\vartheta_0) & \tau_2(x,\vartheta_0) \\ \kappa_2'(x,\vartheta_0) & \tau_2'(x,\vartheta_0) \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \theta \\ \theta \end{pmatrix} \tag{18}$$

have nonzero solutions. By a direct computation, we get

$$\left.\begin{matrix} \kappa_1(x,\vartheta_0)c_1 = -\tau_1(x,\vartheta_0)c_2 \\ \kappa_1'(x,\vartheta_0)c_1 = -\tau_1'(x,\vartheta_0)c_2 \end{matrix}\right\} x\in(0,a)$$

and

$$\left.\begin{matrix} \kappa_2(x,\vartheta_0)c_1 = -\tau_2(x,\vartheta_0)c_2 \\ \kappa_2'(x,\vartheta_0)c_1 = -\tau_2'(x,\vartheta_0)c_2 \end{matrix}\right\} x\in(a,\pi)$$

We can denote

$$y(x,\vartheta_0)=\begin{cases}\kappa_1(x,\vartheta_0)c_1=-\tau_1(x,\vartheta_0)c_2, & x\in(0,a)\\ \kappa_2(x,\vartheta_0)c_1=-\tau_2(x,\vartheta_0)c_2, & x\in(a,\pi)\end{cases}.$$

We conclude that $y(x,\vartheta_0)$ holds the conditions $(2),(3)$ and jump condition (4). Namely, $y(x,\vartheta_0)$ is the eigenfunctions corresponding to ϑ_0. Thus, ϑ_0 is eigenvalue of the problem $(1)-(4)$.

Remark 2.1 As two especial case

$$\Delta(\vartheta)=\begin{vmatrix}\kappa_1(x,\vartheta_0) & \tau_1(x,\vartheta_0)\\ \kappa_1'(x,\vartheta_0) & \tau_1'(x,\vartheta_0)\end{vmatrix}_{x=0}=\begin{vmatrix}E_m & \tau_1(0,\vartheta_0)\\ \theta_m & \tau_1'(0,\vartheta_0)\end{vmatrix}=\det\left(\tau_1'(0,\vartheta)\right)$$

$$\Delta(\vartheta)=\begin{vmatrix}\kappa_2(x,\vartheta_0) & \tau_2(x,\vartheta_0)\\ \kappa_2'(x,\vartheta_0) & \tau_2'(x,\vartheta_0)\end{vmatrix}_{x=\pi}=\begin{vmatrix}\kappa_2(\pi,\vartheta_0) & E_m\\ \kappa_2'(\pi,\vartheta_0) & \theta_m\end{vmatrix}=(-1)^m\det\left(\kappa_2'(\pi,\vartheta)\right).$$

Definition 2.1 The characteristic function of the eigenvalues of the problem given by $(1)-(4)$ is $\Delta(\vartheta)$.

Definition 2.2 The order of ϑ as a zero of $\Delta(\vartheta)$ is the algebraic multiplicity of an eigenvalue ϑ. The number of linearly independent solutions of the boundary value problem is the geometric multiplicity of ϑ as an eigenvalue of the problem $(1)-(4)$.

The geometric multiplicity of ϑ_0 is equal to number of linear independent solutions of (17) or (18). If we show $2m\times 2m$ matrices

$$A(x,\vartheta_0)=\begin{pmatrix}\kappa_1(x,\vartheta_0) & \tau_1(x,\vartheta_0)\\ \kappa_1'(x,\vartheta_0) & \tau_1'(x,\vartheta_0)\end{pmatrix}, B(x,\vartheta_0)=\begin{pmatrix}\kappa_2(x,\vartheta_0) & \tau_2(x,\vartheta_0)\\ \kappa_2'(x,\vartheta_0) & \tau_2'(x,\vartheta_0)\end{pmatrix}$$

and the ranks are given as $R(A(x,\vartheta_0))$, $R(B(x,\vartheta_0))$, respectively.

Obviously, we have the following corollary.

Corollary 2.1 The geometric multiplicity of ϑ_0 as an eigenvalue of the problem $(1)-(4)$ is equal to $2m-R(A(x,\vartheta_0))$ or $2m-R(B(x,\vartheta_0))$.

Remark 2.2 $R(A(x,\vartheta_0))$ or $R(B(x,\vartheta_0))$ is at least equal to m, so the geometric multiplicity of ϑ_0 changes from 1 to m. When the geometric multiplicity of an eigenvalue is m , also considering Corollary 2.1, it is clear that the eigenvalue has maximal (full) multiplicity. In the sequel of the paper, the multiplicity will imply the geometric multiplicity.

An entire function of non-integer order has an infinite set of zeros. The zeros of an analytic function which does not vanish identically are isolated [4] . $\tau_1'(0,\vartheta)$ and $\kappa_2'(\pi,\vartheta)$ are entire function of order $\frac{1}{2}$ matrices. The sums and products of such functions are entire of order not exceeding $\frac{1}{2}$. Thus, the determinants of $\tau_1'(0,\vartheta)$ and $\kappa_2'(\pi,\vartheta)$, that is, the characteristic functions are also non-integer.

Lemma 2.4 The boundary value problem of the system given by $(1)-(4)$ has a countable number of eigenvalues that grow infinite, when they will put in order in terms of their absolute value.

$A(x)=\left(a_{ij}\right)_{i,j=1}^{m}:I\to M_{mxm}^{\mathrm{R}}$,, for any $x\in I$, the norm of $A(x)$ may be taken as

$$\|A(x)\|=\max_{1\le i\le m}\sum_{j=1}^{m}\left|a_{ij}\right| \tag{19}$$

Suppose that $\vartheta=\sigma+i\tau$, $\sigma,\tau\in\mathrm{R}$. The following results can be obtained.

Lemma 2.5 When $|\vartheta|\to\infty$, the following asymptotic formulas hold on $0\le x<a$,

$$\kappa_1(x,\vartheta)=\cos\left(\vartheta x-\beta^+(x)\right)E_m+O\left(|\vartheta|^{-1}e^{|\tau|x}\right) \tag{20}$$

$$\kappa_1'(x,\vartheta)=(\vartheta-\hbar(x))\sin\left(\vartheta x-\beta^+(x)\right)E_m+O\left(e^{|\tau|x}\right) \tag{21}$$

Proof.
As in the paper [12], $\hbar(x) \in W_2^1[0,\pi]$ and $\Upsilon(x) \in L_2[0,\pi]$ hold the conditions $\kappa_1(0,\vartheta) = E_m$ and $\kappa_1'(0,\vartheta) = \theta_m$ and the in this case the following representation can be obtained:

$$\kappa_1(x,\vartheta) = \cos(\vartheta x - \beta^+(x)) E_m + \int_{-x}^{x} A(x,\vartheta) e^{i\vartheta t} \tag{22}$$

$$\beta^+(x) = \int_0^x \hbar(t)\,dt, \quad A_\Upsilon(x,-x) = \frac{1}{2}\omega_\Upsilon \hbar(0) e^{\omega_\Upsilon \beta^+(x)} \tag{23}$$

$$A_\Upsilon(x,x) = \frac{1}{2}\left\{\omega_\Upsilon \hbar(x) + \int_0^x \left[\Upsilon(t) + \hbar^2(t)\right] dt\right\} e^{-\omega_\Upsilon \beta^+(x)}, (\Upsilon = 1,2\ ,\ \omega_1 = i,\ \omega_2 = -i) \tag{24}$$

Besides, $A_\Upsilon(x,t)$ have the derivatives as $A'_{\Upsilon t}(x,t)$ and $A'_{\Upsilon x}(x,t)$ in $L_2[-\pi,\pi]$. Here, by applying integration by parts to the right hand side of equation (22) and by using the equations (23) and (24), we obtain

$$\begin{aligned}&\kappa_1(x,\vartheta) = \cos(\vartheta x - \beta^+(x)) E_m + \tfrac{1}{2}[\hbar(x) - \hbar(0)] \tfrac{\cos(\vartheta x - \beta^+(x))}{\vartheta} + \\ &+ \tfrac{1}{2}\left(\int_0^x \left[\Upsilon(t) + \hbar^2(t)\right] dt\right) \tfrac{\sin(\vartheta x - \beta^+(x))}{\vartheta} - \tfrac{1}{2i\vartheta}\int_{-x}^{x} A_t'(x,t) e^{i\vartheta t} dt\end{aligned}$$

In this case, equation (20) is correct. If we take the derivative of (22) equation with respect to x and by taking into account the equations (23) and (24), the equality (21) is provided. Which is the desired result.
Lemma 2.6 When $|\vartheta| \to \infty$, $\kappa_2(x,\vartheta)$ and $\kappa_2'(x,\vartheta)$ have the following asymptotic formulas on $a < x < \pi$,

$$\kappa_2(x,\vartheta) = \frac{\nu^+}{2} \exp(-i(\vartheta x - \beta^+(x))) E_m \left(1 + O\left(\frac{1}{\vartheta}\right)\right) \tag{25}$$

$$\kappa_2'(x,\vartheta) = -\frac{i\nu^+}{2}(\vartheta - \hbar(x)) \exp(-i(\vartheta x - \beta^+(x))) E_m + O(1) \tag{26}$$

where $\nu^\pm = \frac{1}{2}\left(\nu \pm \frac{1}{\nu}\right)$ and $\beta^\pm(x) = \int_{(a \mp a)_{/2}}^{x} \hbar(t)\,dt$.
Proof. Since $\kappa_2(x,\vartheta)$ is the solution of initial value problem (7) , we obtain

$$\kappa_2(x,\vartheta) = \nu^+ \cos(\vartheta x - \beta^+(x)) E_m + \nu^- \cos(\vartheta(2a - x) - \beta^-(x)) E_m + O\left(\frac{1}{\vartheta} e^{|\tau|x}\right).$$

We have

$$\begin{aligned}\kappa_2(x,\vartheta) = &\tfrac{\nu^+}{2} e^{i\left[\vartheta x - \beta^+(x)\right]} E_m + \tfrac{\nu^+}{2} e^{-i\left[\vartheta x - \beta^+(x)\right]} E_m + \tfrac{\nu^-}{2} e^{i\left[\vartheta(2a-x) - \beta^-(x)\right]} E_m \\ &+ \tfrac{\nu^-}{2} e^{-i\left[\vartheta(2a-x) - \beta^-(x)\right]} E_m + O\left(\tfrac{1}{\vartheta} e^{|\tau|x}\right)\end{aligned} \tag{27}$$

Suppose that $f(x,\vartheta) := O\left(\frac{1}{\vartheta} e^{|\tau|x}\right)$ and

$$\kappa_2(x,\vartheta) = \frac{\nu^+}{2} exp{-i}\left[\vartheta x - \beta^+(x)\right] E_m + (1 + g(x,\vartheta)).$$

By a simple computation for (27), we get

$$\begin{aligned}&g(x,\vartheta) = e^{2i\left[\vartheta x - \beta^+(x)\right]} E_m + \tfrac{\nu^-}{\nu^+} e^{i\left[2\vartheta a - \beta^-(x) - \beta^+(x)\right]} E_m \\ &+ \tfrac{\nu^-}{\nu^+} e^{i\left[-\vartheta a + \beta^-(x) + \beta^+(x)\right]} E_m + \tfrac{2}{\nu^+ e^{-i\left[\vartheta x - \beta^+(x)\right]}} f(x,\vartheta) E_m.\end{aligned}$$

Let's examine $g(x,\vartheta) = O\left(\frac{1}{\vartheta}\right)$ accuracy.

$$\left|g(x,\vartheta)\right| \le \left|e^{2i\left[\vartheta x-\beta^{+}(x)\right]}E_m\right| + \left|\frac{\nu^-}{\nu^+}e^{i\left[2\vartheta a-\beta^-(x)-\beta^+(x)\right]}E_m\right|$$
$$+\left|\frac{\nu^-}{\nu^+}e^{i\left[-\vartheta a+\beta^-(x)+\beta^+(x)\right]}E_m\right| + \left|\frac{1}{\frac{\nu^+}{2}e^{-i\left[\vartheta x-\beta^+(x)\right]}}f(x,\vartheta)E_m\right|$$
$$\le e^{-2|\tau|x}E_m + \left|\frac{\nu^-}{\nu^+}\right|e^{-2|\tau|x}E_m + \left|\frac{\nu^-}{\nu^+}\right|e^{-2|\tau|a}E_m + \frac{c}{\vartheta}e^{|\tau|x}e^{|\tau|x}E_m$$

Furthermore, $|\tau| > \varepsilon|\vartheta|, \varepsilon > 0$ in D. Thus, $-|\tau| < -\varepsilon|\vartheta|$ and $e^{-2|\tau|x} < e^{-\varepsilon|\tau|x}$.
Since $\frac{x}{e^x} \to 0$, $x < ce^x$ $(c > 0)$. Thus, $e^{-2|\tau|x} < \frac{c}{\varepsilon|\vartheta|x}$. We get
$g(x,\vartheta) = O\left(\frac{1}{\vartheta}\right)$ $\vartheta \to \infty$. Hence,

$$\kappa_2(x,\vartheta) = \frac{\nu^+}{2}\exp\left(-i\left(\vartheta x-\beta^+(x)\right)\right)E_m\left(1+O\left(\frac{1}{\vartheta}\right)\right), \vartheta \to \infty.$$

By taking derivative the both sides of (25) and using the first formula of (27), one can get the formula of (26). This completes the proof.

3. Multiplicities of eigenvalues of the vectorial problem

Theorem 3.1 Let $m \ge 2$. Assume that, for some $i, j \in \{1,2,...,m\}$ with $i \ne j$
either

$$(i)\quad \begin{cases} \int_0^a p_{ij}(x)\,dx + \frac{(\nu^+)^2}{2}\int_a^\pi p_{ij}(x)\,dx \ne 0 \\ \int_0^a q_{ij}(x)\,dx + \frac{(\nu^+)^2}{2}\int_a^\pi q_{ij}(x)\,dx \ne 0 \end{cases} \tag{28}$$

or

$$(ii)\quad \begin{cases} \int_0^a \left[p_{ii}(x)-p_{jj}(x)\right]dx + \frac{(\nu^+)^2}{2}\int_a^\pi \left[p_{ii}(x)-p_{jj}(x)\right]dx \ne 0 \\ \int_0^a \left[q_{ii}(x)-q_{jj}(x)\right]dx + \frac{(\nu^+)^2}{2}\int_a^\pi \left[q_{ii}(x)-q_{jj}(x)\right]dx \ne 0 \end{cases} \tag{29}$$

Then, with finite number of exceptions the multiplicities of the eigenvalues of the problem (1) − (4) are at most $m-1$.
Proof . (i) Let (28) be holds. Assume that there exists a sequence of eigenvalues $\{\vartheta_n\}_{n=1}^{\infty}$ whole multiplicities are all m. Clearly, we can write $\vartheta_n \to \infty$ as $n \to \infty$. By (7) and denoting $\kappa_2(x,\vartheta) = \left\{y_{ij}^+(x)\right\}_{i,j=1}^m$, when $\vartheta = \vartheta_n$ for $n = 1,2,...,$ we obtain

$$\left(y_{ii}^+\right)''(x) + \left(\vartheta - \left(2\vartheta p_{ii}(x) + q_{ii}(x)\right)\right)y_{ii}^+(x) - \sum_{k\ne i}\left(2\vartheta p_{ii}(x) + q_{ii}(x)\right)y_{ki}^+(x) = 0 \tag{30}$$

and

$$\left(y_{ij}^+\right)''(x) + \left(\vartheta - \left(2\vartheta p_{ii}(x) + q_{ii}(x)\right)\right)y_{ij}^+(x) - \sum_{k\ne j}\left(2\vartheta p_{ii}(x) + q_{ii}(x)\right)y_{kk}^+(x) = 0 \tag{31}$$

Multiplying (30) and (31) by $y_{ij}^+(x)$ and $y_{ii}^+(x)$ respectively, then by subtracting and using (25), nothing that the eigenvalues of the problem are all real, we have

$$\left(\left(y_{ii}^+\right)'(x)\,y_{ij}^+(x) - y_{ii}^+(x)\left(y_{ij}^+\right)'(x)\right)'$$
$$= \sum_{k\ne i}\left(2\vartheta p_{ik}(x) + q_{ik}(x)\right)\left(y_{ki}^+(x)\,y_{ij}^+(x) - y_{ii}^+(x)\,y_{kj}^+(x)\right)$$
$$= \left(2\vartheta p_{ij}(x) + q_{ij}(x)\right)\left[y_{ij}^+(x)\,y_{ji}^+(x) - y_{ii}^+(x)\,y_{ij}^+(x)\right]$$
$$+ \sum_{k\ne i}\left(2\vartheta p_{ij}(x) + q_{ij}(x)\right)\left(y_{ki}^+(x)\,y_{ij}^+(x) - y_{ii}^+(x)\,y_{kj}^+(x)\right)$$

thus,

$$\left(\left(y_{ii}^{+}\right)'(x)\, y_{ij}^{+}(x) - y_{ii}^{+}(x)\left(y_{ij}^{+}\right)'(x)\right)' = \\ -\left(2\vartheta p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\nu^{+})^{2}}{2}\cos^{2}\left(\vartheta x - \beta^{+}(x)\right)\right] + O\left(1 + \frac{1}{\vartheta}\right). \tag{32}$$

Similarly, by (5) and denoting $\kappa_1(x,\vartheta) = \left\{y_{ij}^{-}(x)\right\}_{i,j=1}^{m}$, we have

$$\left(\left(y_{ii}^{-}\right)'(x)\, y_{ij}^{-}(x) - y_{ii}^{-}(x)\left(y_{ij}^{-}\right)'(x)\right)' = \\ = -\left(2\vartheta p_{ij}(x) + q_{ij}(x)\right)\left[\cos^{2}(\vartheta x)\right] + O\left(\frac{1}{\vartheta}\right). \tag{33}$$

When ϑ is an eigenvalue with multiplicity m, we have

$$\kappa_2'(\pi,\vartheta) = \theta_m .$$

By integrating both sides of (32) over a to π, by taking into account $\vartheta_n \to \vartheta$ and $n \to \infty$, we obtain

$$\left(\left(y_{ii}^{+}\right)'(x)\, y_{ij}^{+}(x) - y_{ii}^{+}(x)\left(y_{ij}^{+}\right)'(x)\right) \\ = \int_a^{\pi}\left[-\left(2\vartheta p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\nu^{+})^{2}}{2}\cos^{2}\left(\vartheta x - \beta^{+}(x)\right)\right] + O\left(\frac{1}{\vartheta}\right)\right]dx. \tag{34}$$

By integrating both sides of (33) over 0 to a and applying the boundary condition $\kappa_1'(0,\vartheta) = \theta_m$, we obtain, for $\vartheta_n \to \vartheta$ and $n \to \infty$,

$$\left(\left(y_{ii}^{-}\right)'(x)\, y_{ij}^{-}(x) - y_{ii}^{-}(x)\left(y_{ij}^{-}\right)'(x)\right) \\ = -\int_0^{a}\left[\left(2\vartheta p_{ij}(x) + q_{ij}(x)\right)\left[\cos^{2}(\vartheta x)\right] + O\left(\frac{1}{\vartheta}\right)\right]dx \tag{35}$$

By adding (34) and (35) and using the initial conditions at point $x = a$, we get

$$0 = -\int_0^{a}\left[\left(2\vartheta p_{ij}(x) + q_{ij}(x)\right)\left[\cos^{2}(\vartheta x)\right] + O\left(\frac{1}{\vartheta}\right)\right]dx \\ -\int_a^{\pi}\left[\left(2\vartheta p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\nu^{+})^{2}}{2}\cos^{2}\left(\vartheta x - \beta^{+}(x)\right)\right] + O\left(\frac{1}{\vartheta}\right)\right]dx.$$

By a simple computation, we deduce

$$\int_0^{a}\left(2\vartheta p_{ij}(x) + q_{ij}(x)\right)dx + \frac{(\nu^{+})^{2}}{2}\int_a^{\pi}\left(2\vartheta p_{ij}(x) + q_{ij}(x)\right)dx = \\ = -\int_0^{a}\left[\left(2\vartheta p_{ij}(x) + q_{ij}(x)\right)\left[\cos 2(\vartheta x)\right] + O\left(\frac{1}{\vartheta}\right)\right]dx \\ -\int_a^{\pi}\left[\left(2\vartheta p_{ij}(x) + q_{ij}(x)\right)\left[\frac{(\nu^{+})^{2}}{2}\cos 2\left(\vartheta x - \beta^{+}(x)\right)\right] + O\left(\frac{1}{\vartheta}\right)\right]dx \tag{36}$$

$$= -2\vartheta\int_0^{a} p_{ij}(x)\cos(2\vartheta x)\,dx - \int_0^{a} q_{ij}(x)\cos(2\vartheta x)\,dx \\ -2\vartheta\frac{(\nu^{+})^{2}}{2}\int_a^{\pi} p_{ij}(x)\cos(2\vartheta x)\cos 2\beta^{+}(x)\,dx \\ -2\vartheta\frac{(\nu^{+})^{2}}{2}\int_a^{\pi} p_{ij}(x)\sin(2\vartheta x)\sin 2\beta^{+}(x)\,dx \\ -\frac{(\nu^{+})^{2}}{2}\int_a^{\pi} q_{ij}(x)\cos(2\vartheta x)\cos 2\beta^{+}(x)\,dx \\ -\frac{(\nu^{+})^{2}}{2}\int_a^{\pi} q_{ij}(x)\sin(2\vartheta x)\sin 2\beta^{+}(x)\,dx + O\left(\frac{1}{\vartheta}\right)$$

Then, for $\vartheta_n \to \infty$ and $n \to \infty$, we get

$$= -2\int_0^{a} p_{ij}(x)\cos(2\vartheta x)\,dx - (\nu^{+})^{2}\int_a^{\pi} p_{ij}(x)\cos(2\vartheta x)\cos 2\beta^{+}(x)\,dx \\ -(\nu^{+})^{2}\int_a^{\pi} p_{ij}(x)\sin(2\vartheta x)\sin 2\beta^{+}(x)\,dx + O\left(\frac{1}{\vartheta}\right)$$

by Riemann-Lebesgue Lemma, the right side of (36) converges to 0 as $\vartheta_n = \vartheta$ and $n \to \infty$.

This implies that

$$\begin{array}{l} \int_0^a p_{ij}(x)\,dx + \frac{(\nu^+)^2}{2}\int_a^\pi p_{ij}(x)\,dx = 0 \\ \int_0^a q_{ij}(x)\,dx + \frac{(\nu^+)^2}{2}\int_a^\pi q_{ij}(x)\,dx = 0 \end{array}$$

This is a contradiction. The conclusion for this case is proved.
(ii) Next, suppose that
$\int_0^a \left(2\vartheta p_{ij}(x) + q_{ij}(x)\right)dx + \frac{(\nu^+)^2}{2}\int_a^\pi \left(2\vartheta p_{ij}(x) + q_{ij}(x)\right)dx = 0$, for $\forall i \neq j$
or
$\int_0^a s_{ij}(x)\,dx + \frac{(\nu^+)^2}{2}\int_a^\pi s_{ij}(x)\,dx = 0$, for $\forall i \neq j$,where $s_{ij}(x) = \left(2\vartheta p_{ij}(x) + q_{ij}(x)\right)$.
and

$$\int_0^a \left[s_{ii}(x) - s_{jj}(x)\right]dx + \frac{(\nu^+)^2}{2}\int_a^\pi \left[s_{ii}(x) - s_{jj}(x)\right]dx \neq 0, \exists i \neq j.$$

Without loss of generality, we assume that for $i = 1, j = 2$

$$\int_0^a \left[s_{11}(x) - s_{22}(x)\right]dx + \frac{(\nu^+)^2}{2}\int_a^\pi \left[s_{11}(x) - s_{22}(x)\right]dx \neq 0.$$

Let

$$T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & & & \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

and $y = Tz$. Then, the system given by $(1) - (4)$ becomes

$$\left.\begin{array}{l} z'' + (\vartheta E_m - R(x))\,z = 0 \\ z'(0) = z'(\pi) = 0 \\ z(a+0) = \nu z(a-0) \\ z'(a+0) = \nu^{-1} z'(a-0) \end{array}\right\} \tag{37}$$

where $R(x) = T^{-1}S(x)\,T$. By a simple computation, we get

$$R(x) = \begin{bmatrix} \frac{1}{2}(s_{11} + s_{22}) + s_{12} & \frac{1}{2}s\ldots & & & \\ * & * & \vdots & \ddots & \\ * & * & & \cdots & q_{mm} \end{bmatrix}(x).$$

We note that the two problems $(1) - (4)$ and (37) have exactly the same spectral structure. Let us denote $R(x) = \left\{r_{ij}(x)\right\}_{i,j=1}^m$. Since

$$\begin{array}{l} \int_0^a r_{12}(x)\,dx + \frac{(\nu^+)^2}{2}\int_a^\pi r_{12}(x)\,dx = \\ = \frac{1}{2}\left(\int_0^a \left[s_{11}(x) - s_{22}(x)\right]dx + \frac{(\nu^+)^2}{2}\int_a^\pi \left[s_{11}(x) - s_{22}(x)\right]dx\right) \neq 0 \end{array}$$

By part (i), the conclusion of the theorem holds for the problem (37), and hence holds for the problem $(1) - (4)$.□

Corollary 3.1 Assume that $S(x) = [2\vartheta\hbar(x) + \Upsilon(x)] \equiv [2\vartheta\hbar + \Upsilon] \equiv S$ is a constant real symmetric matrix on $(0, a) \cup (a, \pi)$. Then the following are equivalent:

(i) there is an infinite number of eigenvalues of the problem $(1)-(4)$ with multiplicity m;
(ii) the multiplicity of all eigenvalues of the problem $(1)-(4)$ is m.
Corollary 3.2 Let $m=2$, and assume that
either (i) $\int_0^a \left(2\vartheta p_{12}(x)+q_{12}(x)\right)dx+\frac{(\nu^+)^2}{2}\int_a^\pi \left(2\vartheta p_{12}(x)+q_{12}(x)\right)dx\neq 0$
or (ii) $\int_0^a \left[s_{11}(x)-s_{22}(x)\right]dx+\frac{(\nu^+)^2}{2}\int_a^\pi \left[s_{11}(x)-s_{22}(x)\right]dx\neq 0.$
Then, with finitely many exceptions, all eigenvalue of the problem $(1)-(4)$ are simple.

4. Conclusion

In this study, the solutions of the diffusion problem presented with the (1)-(4) equation system, which satisfy certain initial and discontinuity conditions, are investigated. In addition, the important properties of the eigenvalues and eigenfunctions of the boundary value problem have been investigated, as well as various findings that are very useful for the solution of this problem. Researchers interested in the subject can examine similar problems under different boundary values and conditions. In addition, it is thought that the methodology followed is motivating in the use of integral notation in systems of equations with inverse problems and inverse problems.

References

[1] Agranovich Z. S., Marchenco V. A. The inverse problem of scattering theory, Gordon and Breach Science Publisher, New York-London (1963).
[2] Akdemir A.O., Ersoy M.T., Furkan H., Ragusa M.A., Some functional sections in topological sequence spaces, Journal of Function Spaces, vol.2022, art.n.6449630 (2022).
[3] Aksoy Yildrim N., Sariahmet E., On the Stability of Finite Difference Scheme for the Schrödinger Equation Including Momentum Operator, Turkish Journal of Science, 7 (2): 107-115 (2022).
[4] Baglan I., Canel T. Analysis of Inverse Euler-Bernoulli Equation with Periodic Boundary Conditions, Turkish Journal of Science 7(3): 146-156 (2022),
[5] Bellmann, R. Introduction to matrix analysis (2nd ed.) McGraw-Hill, (1970).
[6] Boas, R. P. Entire functions, Academic press, New York, (1954)
[7] Carvert, J. M., Davison, W. D. Oscillation theory and computational procedures for matrix Sturm-Liouville eigenvalue problems with an application to the hydrogen molecular ion, Journal of Physics A Mathematical and General., 23: 278-292 (1969).
[8] Ergun, A. The multiplicity of eigenvalues of a vectorial diffusion equations with discontinuous function inside a finite interval, Turkish Journal of Science, 5 (2): 73-85 (2020).
[9] Ergun, A. Inverse problem for singular diffusion operator, Miskolc Mathematical Notes, 22(1): 173-192 (2021).
[10] Gasymov, M.G., Guseinov, G.Sh. Determination diffusion operator on spectral data, SSSR Dokl., 37: 19-23 (1981).
[11] Guseinov, G. Sh. On the spectral analysis of a quadratic pencıl of Sturm-Liouville operators, Doklayd Akad. Nauk SSSR, 285, 1292-1296, English translation, Soviet Mathematics Doklayd., 32: 859-862 (1985).
[12] Guseinov, G. Sh. Inverse spectral problems for a quadratic pencıl of Sturm-Liouville operators on a fınıte interval, Spectral Theory of Operators and ıts Applications, Elm, Baku, Azerbaijann, 51-101 (1986)
[13] Jaulent, M. and Jean, C., The Inverse Problem for the one-dimensional Schrödinger equation with an energy-dependent potential. Annales de L'Institut Henri Poincare A, 25, 2,105-137, 1976.
[14] Kauffman, R. M., Zhang, H. K. A class of ordinary differential operators with jump baundary conditions, Evolution equations, Lecture notes in Pure and Appl. Math., Dekkar, New York, 234: 253-274 (2003).
[15] Kong, Q. Multiplicities of eigenvalues of a vector-valued Sturm-Liouville Problem, Mathematica., 49: 119-127 (2002).
[16] Koyunbakan H., Panakhov E.S. Half inverse problem for diffusion operators on the finite interval, *J. Math. Anal. Appl.* 326: 1024-1030 (2007).
[17] Levitan, B. M., Sargsyan, I. S. Introduction to Spectral Theory, Amer. Math. Soc. (1975).
[18] Marchenko, V. A. Sturm-Liouville Operators and Applications, AMS: Chelsea Publishing (1986).
[19] Mukhtarov, O., Yakubov, S. Problem for ordinary differential equations with transmission conditions, Appl. Anal, 81: 1033-1064 (2002).
[20] Shen C. L, Shieh C. On the multiplicity of eigenvalues of a vectorial Sturm-Liouville differential equations and some related spectral problems, Proc. Amer. Math. Soc. 127: 2943-2952 (1999).
[21] Wang, A. P., Sun, J., Zettl, A. Two-interval Sturm-Liouville operators in modified Hilbert spaces, Journal of Mathematical Analysis and Applications, 328: 390-399 (2007).
[22] Yang, C. F ,Huang Z Y, Yang X P. The multiplicity of spectra of a vectorial Sturm- Liouville differential equation of dimension two and some applications, Rocky Mountain journal of Mathematics, 37: 1379-1398 (2007).
[23] Yurko, V. A. Introduction to the Theory of Inverse Spectral Problems, Fizmatlit, Russian (2007).