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On projectively flat Finsler space with a cubic (α, β) metric

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Abstract. In the present paper, we have considered a cubic (α, β) metric which is an special class of p-power Finsler metric, and obtained the conditions under which the Finsler space with such special metric will be projectively flat. Further, we also obtain in which case this Finsler space will be a Berwald space and Douglas space.

1. INTRODUCTION

Let M^n be an n-dimensional C^∞ manifold and T_xM denotes the tangent space of M^n at x. The tangent bundle of M^n is the union of tangent spaces $TM = \cup_{x\epsilon M} T_xM$. We denote the elements of TM by (x, y), where $y \,\epsilon\, T_xM$. Let $TM_0 = TM - \{0\}$.

Definition 1.1. *A Finsler metric on M^n is a function $L : TM \to [0, \infty)$ with the following properties:*

1. *L is C^∞ on TM_0,*
2. *L is positively 1-homogeneous on the fibers of tangent bundle TM, and*
3. *the Hessian of L^2 with element $g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ is regular on TM_0, i.e., $\det\left(g_{ij}\right) \neq 0$.*

The pair (M^n, L) is then called a Finsler space. L is called fundamental function and g_{ij} is called fundamental tensor.

An n-dimensional Finsler space $F^n = (M^n, L)$ is known as a locally Minkowski space [15] if the manifold M^n is covered by coordinate neighbourhood system (x^i) in each of which the metric L is a function of y^i only. Further the Finsler space F^n is known as projectively flat if F^n is projective to a locally Minkowski space. Further Matsumoto [16] gave a condition for a Finsler space with Randers metric and Kropina metric to be projectively flat. The projective flatness property for the Finsler space with various important $(\alpha, \beta)-$ metric was studied by various authors [6, 8, 9, 14, 17, 18, 20, 21] and obtained interesting results in field of Finsler geometry.

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A Finsler space is called Berwald space if the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ is linear. On the basis of berwald connection Basco and Matsumoto [19] introduced the notion of Douglas space as a generalization of the Berwald space from the view point of geodesic equations in the year 1997. The condition for a Finsler space with an (α, β)-metric to be Douglas type was studied by many authors [2, 3, 7, 17].

The concept and importance of (α, β) metric in Finsler space was introduced and explained by Matsumoto [12] in details and the metric $L = L(\alpha, \beta)$ in a n-dimensional manifold M^n is defined as a positively homogeneous function of degree one in α and β, where α is a regular Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^iy^j}$, i.e., $det(a_{ij}) \neq 0$ and β is a one-form $\beta = b_i(x)y^i$ which was an generalization of Rander's metric $L = \alpha + \beta$. Other than Rander's metric there are several important (α, β)-metrics, namely Kropina metric $L = \frac{\alpha^2}{\beta}$, Matsumoto metric $\frac{\alpha^2}{\alpha-\beta}$, generalized Kropina metric $L = \frac{\alpha^{n+1}}{\beta^n}$ and Z. Shen's square metric $L = \frac{(\alpha+\beta)^2}{\alpha}$, infinite series metric $\frac{\beta^2}{\beta-\alpha}$ and many more metric was studied by many authors [2, 6, 10, 13–15, 21] and obtained variours landmarks in the field of Finsler geometry. Further, a generalized form of an (α, β)-metric on an n-dimensional manifold M^n defined as

$$L = \alpha(1 + \frac{\beta}{\alpha})^p, \tag{1}$$

is known as the class p-power (α, β)- metrics [5], where $p \neq 0$ is a real constant. If $p = 1$ then equation (1) reduces to Rander's metric which has important and interesting curvature properties and firstly introduced by Ingarden in 1957. If $p = 2$ then it becomes square metric and it also known as Z. Shen's square metric. If $p = -1$ it reduces to Matsumoto type metric which can be used in measurement of slope of a mountain. If $p = 1/2$ it reduces to square root metric i.e. $L = \sqrt{\alpha(\alpha + \beta)}$, this metric detailed studied by [5] and so on.

In the present paper, we considered $p = 3$ in equation (1) and get special class of (α, β)- metric in the form of

$$L = \frac{(\alpha + \beta)^3}{\alpha^2}, \tag{2}$$

and named as cubic (α, β)- metric in an n-dimensional manifold M^n and a n-dimensional Finsler F^n space equipped with cubic (α, β)- metric is known as Finsler space with cubic (α, β)- metric. Further, we investigate projectively flatness property of the Finsler space F^n and under which conditions it will be Berwald and Douglas space.

2. Projectively flat Finsler space with cubic (α, β)-metric

In present section, we have obtained the condition for a Finsler space F^n with a cubic (α, β)-metric defined in equation (2) to be projectively flat.

Matsumoto [14] considered γ^i_{jk} represent the Christoffel symbols in the Riemannian space (M^n, α)-metric and defined following notations which are used by us in the present paper are given below:

$$r_{ij} = \frac{1}{2}\left\{b_{i;j} + b_{j;i}\right\}, \quad r^i_j = a^{ih}r_{hj}, \quad r_j = b_i r^i_j,$$

$$s_{ij} = \frac{1}{2}\left\{b_{i;j} - b_{j;i}\right\}, \quad s^i_j = a^{ih}s_{hj}, \quad s_j = b_i s^i_j, \quad b^i = a^{ih}b_h, \quad b^2 = b^i b_i.$$

where $b_{i;j}$, is the covariant derivative of the vector field b_i with respect to Riemannian connection γ^i_{jk}, i.e., $b_{i;j} = \frac{\partial b_i}{\partial x^j} - b_k\gamma^k_{ij}$.

Further he proved that a Finsler space $F^n = (M^n, L)$ with an (α, β)-metric $L(\alpha, \beta)$ is projectively flat if and only if for any point of the manifold M^n there exists local coordinate neighbourhoods containing the point such that the Christoffel symbols γ^i_{jk} in the Riemannian space (M^n, α) satisfies:

$$(\gamma^i_{00} - \gamma_{000}y^i/\alpha^2)/2 + (\alpha L_\beta/L_\alpha)s^i_0 + (L_{\alpha\alpha}/L_\alpha)(C + \alpha r_{00}/2\beta)(\alpha^2 b^i/\beta - y^i) = 0, \tag{3}$$

where a subscript '0' means a contraction by y^i and C is given by

$$C + (\alpha^2 L_\beta/\beta L_\alpha)s_0 + (\alpha L_{\alpha\alpha}/\beta^2 L_\alpha)(\alpha^2 b^2 - \beta^2)(C + \alpha r_{00}/2\beta) = 0. \tag{4}$$

By the homogeneity of L, we know that $\alpha^2 L_{\alpha\alpha} = \beta^2 L_{\beta\beta}$, therefore equation (4) can be rewritten as

$$\left\{1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2)\right\}(C + \alpha r_{00}/2\beta) = (\alpha/2\beta)\left\{r_{00} - (2\alpha L_\beta/L_\alpha)s_0\right\}. \tag{5}$$

If $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2) \neq 0$, then we can eliminate $(C + \alpha r_{00}/2\beta)$ in (3) and it is written as in the form

$$\begin{aligned}&\left\{1 + L_{\beta\beta}(\alpha^2 b^2 - \beta^2)/\alpha L_\alpha\right\}\left\{(\gamma^i_{00} - \gamma_{000}y^i/\alpha^2)/2 + (\alpha L_\beta/L_\alpha)s^i_0\right\}\\ &\quad + (L_{\alpha\alpha}/L_\alpha)(\alpha/2\beta)\left\{r_{00} - (2\alpha L_\beta/L_\alpha)s_0\right\}(\alpha^2 b^i/\beta - y^i) = 0.\end{aligned} \tag{6}$$

Thus we have [6]

Theorem 2.1. *If* $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2) \neq 0$, *then a Finsler space* F^n *with an* (α, β)*-metric is projectively flat if and only if (6) is satisfied.*

It is known that if α^2 contains β as a factor, then the dimension is equal to two and $b^2 = 0$ [18]. Throughout this paper, we assume that the dimension is more than two and $b^2 \neq 0$, that is, $\alpha^2 \not\equiv 0(mod\beta)$. The partial derivative with respect to α and β of (2) are given by

$$L_\alpha = \frac{(\alpha+\beta)^2(\alpha - 2\beta)}{\alpha^3}, \quad L_\beta = \frac{3(\alpha+\beta)^2}{\alpha^2}, \quad L_{\alpha\alpha} = \frac{6\beta^2(\alpha+\beta)}{\alpha^4}, \quad L_{\beta\beta} = \frac{6(\alpha+\beta)}{\alpha^2}. \tag{7}$$

If $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 b^2 - \beta^2) = 0$, then we have $\left\{\alpha^2(1 + 6b^2) - \alpha\beta - 8\beta^2\right\} = 0$, which leads a contradiction. Thus theorem 2.1 can be applied.
Substituting (7) into (6), we get

$$\begin{aligned}&(\alpha^2(1 + 6b^2) - \alpha\beta - 8\beta^2)\left\{(\alpha^2\gamma^i_{00} - \gamma_{000}y^i)(\alpha - 2\beta) + 6\alpha^4 s^i_0\right\}\\ &\qquad + 6\alpha^2\left\{r_{00}(\alpha - 2\beta) - 6\alpha^2 s_0\right\}(\alpha^2 b^i - y^i\beta) = 0.\end{aligned} \tag{8}$$

The above equation can be rewritten as polynomial of degree six in α expressed as

$$p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + \alpha(p_5\alpha^4 + p_3\alpha^2 + p_1) = 0, \tag{9}$$

where
$p_6 = 6\left\{s^i_0(1 + 6b^2) - 6s_0 b^i\right\}$,
$p_5 = \alpha^4\gamma^i_{00} + 6\alpha^4 b^2\gamma^i_{00} - 6\alpha^4\beta s^i_0 + 6\alpha^4 r_{00}b^i$,
$p_4 = -3\alpha^4\beta\gamma^i_{00} - 12\alpha^4\beta b^2\gamma^i_{00} - 48\alpha^4\beta^2 s^i_0 - 12\alpha^4\beta r_{00}b^i + 36\alpha^4\beta y^i$,
$p_3 = -\alpha^2\gamma_{00}y^i - 6\alpha^2 b^2\gamma_{000}y^i - 6\alpha^2\beta^2\gamma^i_{00} - 6\alpha^2\beta r_{00}y^i$,
$p_2 = 3\alpha^2\beta\gamma_{000}y^i + 12\alpha^2\beta b^2\gamma_{000}y^i + 16\alpha^2\beta^3\gamma^i_{00} + 12\alpha^2\beta^2 r_{00}y^i$,
$p_1 = 6\beta^2\gamma_{000}y^i$,
$p_0 = -16\beta^3\gamma_{000}y^i$
.

Since $p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0$ and $p_5\alpha^4 + p_3\alpha^2 + p_1$ are rational and α is irrational in y^i, we have

$$p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0 = 0, \tag{10}$$

$$p_5\alpha^4 + p_3\alpha^2 + p_1 = 0. \tag{11}$$

The term which does not contain β in (10) is $p_6\alpha^6$. Therefore there exist a homogeneous polynomial V_6 of degree six in y^i such that

$$6\left\{s^i_0(1 + 6b^2) - 6s_0b^i\right\}\alpha^6 = \beta V_6.$$

Since $\alpha^2 \not\equiv 0(mod\beta)$, we must have a function $u^i = u^i(x)$ satisfying

$$6\left\{s^i_0(1 + 6b^2) - 6s_0b^i\right\} = u^i\beta. \tag{12}$$

Transvecting (12) by b_i, we have

$$6s_0 = u^i\beta b_i, \tag{13}$$

that is, $6s_j = u^ib_ib_j$. Furthermore transvecting this equation by b^j, we have $u^ib_ib^2 = 0$, that is, $u^ib_i = 0$. Substituting this equation into (13), we have $s_0 = 0$, Therefore from (12), we get

$$6(1 + 6b^2)s_{ij} = u_ib_j, \tag{14}$$

which implies $u_ib_j + u_jb_i = 0$. Transvection this equation by b^j, we have $u_ib^2 = 0$ by virtue $u_jb^j = 0$. Therefore we get $u_i = 0$. Hence, from (14), we have $s_{ij} = 0$.

On the other hand, from (11), we have 1-form $v_0 = v_i(x)y^i$ such that

$$\gamma_{000} = v_0\alpha^2. \tag{15}$$

Substituting $s_0 = 0, s^i_0 = 0$ and (15) into (8) , we have

$$\left\{\alpha^2(1 + 6b^2) - \alpha\beta - 8\beta^2\right\}(\gamma^i_{00} - v_0y^i) + 6r_{00}(\alpha^2b^i - y^i\beta) = 0, \tag{16}$$

by virtue of $(\alpha - 2\beta) \neq 0$. Then the equation (16) is written in the form $P\alpha + Q = 0$, where

$$P = -\beta(\gamma^i_{00} - v_0y^i),$$

$$Q = \left\{\alpha^2(1 + 6b^2) - 8\beta^2\right\}(\gamma^i_{00} - v_0y^i) + 6r_{00}(\alpha^2b^i - y^i\beta).$$

Since P and Q are rational and α is irrational in y^i, we have $P = 0$ and $Q = 0$.

First it follows from $P = 0$ that

$$\gamma^i_{00} - v_0y^i = 0, \tag{17}$$

that is,

$$2\gamma^i_{jk} = v_j\delta^i_k + v_k\delta^i_j, \tag{18}$$

which shows that the associated Riemannian space (M^n, α) is projectively flat.

Next, from $Q = 0$ and (17), we have

$$6r_{00}(\alpha^2b^i - \beta y^i) = 0. \tag{19}$$

Transvecting (19) by b_i, we have $6r_{00}(\alpha^2b^2 - \beta^2) = 0$, from which we get $r_{00} = 0$ that is $r_{ij} = 0$. From $s_{ij} = 0$ and $r_{ij} = 0$ we have $b_{i;j} = 0$.
Conversely, if $b_{i;j} = 0$, then we have $r_{00} = s^i_0 = s_0 = 0$. So (8) is a consequence of (17). Thus we have

Theorem 2.2. *A Finsler space F^n equipped with a cubic (α, β)-metric is projectively flat if and only if $b_{i;j} = 0$ and the associated Riemannian space (M^n, α) is projectively flat.*

3. Berwald Space for cubic (α, β)-metric

In the present section, we obtained the condition that a Finsler space F^n with cubic (α, β)-metric to be a Berwald space.

A Finsler space is called Berwald space if the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ is linear. In [10], the function G^i of a Finsler space with an (α, β)-metric are given by $2G^i = \gamma^i_{00} + 2B^i$, then we have $G^i = \gamma^i_{0j} + B^i_j$ and $G^i_{jk} = \gamma^i_{jk} + B^i_{jk}$ where $B^i_{jk} = \dot{\partial}_k B^i_j$ and $B^i_j = \dot{\partial}_j B^i$. Thus a Finsler space with an (α, β)-metric is a Berwald space iff $G^i_{jk} = G^i_{jk}(x)$ equivalently $B^i_{jk} = G^i_{jk}(x)$. Moreover on account of [13] B^i_j is determined by

$$L_\alpha B^t_{ji} y^j y^t + \alpha L_\beta (B^t_{ji} b_t - b_{j;i}) y^j = 0, \quad where \quad y^k = a_{ik} y^i. \tag{20}$$

Substituting (7) in (20) we have,

$$(\alpha - 2\beta) B^t_{ji} y^j y_t + 3\alpha^2 (B^t_{ji} b_t - b_{j;i}) y^j = 0. \tag{21}$$

Assume that F^n is a Berwald space i.e., $B^i_{jk} = B^i_{jk}(x)$. Separating (21) in rational and irrational terms of y^i as

$$3\alpha^2 (B^t_{ji} b_t - b_{j;i}) y^j - 2\beta B^t_{ji} y^j y_t + \alpha B^t_{ji} y^j y_t = 0, \tag{22}$$

which yields two equations

$$3\alpha^2 (B^t_{ji} b_t - b_{j;i}) y^j - 2\beta B^t_{ji} y^j y_t = 0, \tag{23}$$

and

$$B^t_{ji} y^j y_t = 0. \tag{24}$$

Substituting (24) in (23), we have

$$3\alpha^2 (B^t_{ji} b_t - b_{j;i}) y^j = 0. \tag{25}$$

Case (i) If $(B^t_{ji} b_t - b_{j;i}) y^j = 0$, we have

$$B^t_{ji} b_t - b_{j;i} = 0, \tag{26}$$

which gives $b_{i;j} = 0$.
Case (ii) If $3\alpha^2 = 0$
$\implies \alpha = 0$, which is a contradiction. Conversely, If $b_{i;j} = 0$, then $B^t_{ji} = 0$ are uniquely determined from (21). Hence we conclude the following

Theorem 3.1. *A Finsler space F^n equipped with a cubic (α, β)-metric is a Berwald space if and only if $b_{i;j} = 0$.*

4. Douglas Space for cubic (α, β)-metric

In this section, we obtained the condition for a Finsler space F^n with a cubic (α, β)-metric to be of Douglas type.

Douglas [11] introduced a curvature which always vanishes for a Riemannian metrics and this curvature known as Douglas curvature. Douglas metric is a Finsler metric in which Douglas curvature vanishes and the space equipped with Douglas metric is known as Douglas space. Matsumoto and Basco characterizes [19] that a Douglas space as a Finsler space for which $B^{ij} = B^i y^j - B^j y^i$ are homogeneous polynomials of degree 3, in short we write B^{ij} is hp(3).

In view of [13], if $\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0$, then the function $G^i(x, y)$ of F^n with an (α, β)-metric is written in the form

$$2G^i = \gamma^i_{00} + 2B^i,$$

where

$$B^i = \frac{\alpha L_\beta s^i_0}{L_\alpha} + C^*\left\{\frac{\beta L_\beta y^i}{\alpha L} - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left(\frac{y^i}{\alpha} - \frac{\alpha b^i}{\beta}\right)\right\}$$

$$C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2s_0\alpha L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})},$$

and

$$\gamma^2 = b^2\alpha^2 - \beta^2.$$

The vector $B^i(x, y)$ is called the difference vector. Hence B^{ij} is written as

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*(b^i y^j - b^j y^i).$$

Substituting (7) in above equation, we get

$$\begin{aligned}&[\alpha^2(1 + 6b^2) - \alpha\beta - 8\beta^2][(\alpha - \beta)B^{ij} - 3\alpha^2(s^i_0 y^j - s^j_0 y^i)]\\ &\quad -3\alpha^2[r_{00}(\alpha - 2\beta) - 6s_0\alpha^2](b^i y^j - b^j y^i) = 0.\end{aligned} \tag{27}$$

If F^n is a Douglas space, then B^{ij} are homogeneous polynomials of degree 3. i.e., B^{ij} is $hp(3)$. Arranging the rational and irrational terms of equation (27), we have

$$\begin{aligned}&[3\alpha^2\beta + 12\alpha^2 b^2\beta - 16\beta^3]B^{ij} + 3\alpha^2[\alpha^2(1 + 6b^2) - 8\beta^2](s^i_0 y^j - s^j_0 y^i)\\ &\quad -[6\alpha^2\beta r_{00} + 18\alpha^4 s_0](b^i y^j - b^j y^i)\\ &+\alpha[\alpha^2(1 + 6b^2) - 6\beta^2]B^{ij} + 3\alpha^2\beta(s^i_0 y^j - s^j_0 y^i) - 3\alpha^2 r_{00}(b^i y^j - b^j y^i) = 0.\end{aligned} \tag{28}$$

Separating rational and irrational terms of y^i in (28), we have the following two equations

$$\begin{aligned}&[3\alpha^2\beta + 12\alpha^2 b^2\beta - 16\beta^3]B^{ij} + 3\alpha^2[\alpha^2(1 + 6b^2) - 8\beta^2](s^i_0 y^j - s^j_0 y^i)\\ &\quad -[6\alpha^2\beta r_{00} + 18\alpha^4 s_0](b^i y^j - b^j y^i) = 0,\end{aligned} \tag{29}$$

$$[\alpha^2(1 + 6b^2) - 6\beta^2]B^{ij} + 3\alpha^2\beta(s^i_0 y^j - s^j_0 y^i) - 3\alpha^2 r_{00}(b^i y^j - b^j y^i) = 0. \tag{30}$$

Eliminating B^{ij} from (29) and (30), we have

$$P(s^i_0 y^j - s^j_0 y^i) + Q(b^i y^j - b^j y^i) = 0, \tag{31}$$

where

$$P = \alpha^2(-51\beta^2 - 18\beta^2 b^2 + 3\alpha^2 + 18\alpha^2 b^2 + 108\beta^2 b^4 - 108\beta^2 b^2) + 192\beta^4, \tag{32}$$

$$Q = \alpha^2[3\beta r_{00} - s_0(18\alpha^2 - 108\alpha^2 b^2 - 108\beta^2)] - 12\beta^3 r_{00}. \tag{33}$$

Contracting of (31) by $b_i y_j$ leads to

$$Ps_0\alpha^2 + Q(b^2\alpha^2 - \beta^2) = 0. \tag{34}$$

Since the term $-12\beta^5 r_{00}$ of (34) seemingly do not contain α^2, we must have $hp(5)v_5$ such that

$$-12\beta^5 r_{00} = \alpha^2 v_5. \tag{35}$$

Let us discuss the following two cases :
(i) $v_5 = 0$
(ii) $v_5 \neq 0$, $\alpha^2 \not\equiv 0(mod\beta)$.
Case (i) : $v_5 = 0$
In this case of $v_5 = 0$, we have $r_{00} = 0$ from (35), and (34) is reduced to

$$s_0 \left\{P + Q_1(b^2\alpha^2 - \beta^2)\right\} = 0, \tag{36}$$

where,

$$Q_1 = -(18\alpha^2 - 108\alpha^2 b^2 - 108\beta^2). \tag{37}$$

If $P + (b^2\alpha^2 - \beta^2) = 0$ in (36) then the term of (36) which does not contain α^2 is $84\beta^4$. Therefore there exist a $hp(2)v_2$, such that $84\beta^4 = \alpha^2 v_2$. Since we suppose $\alpha^2 \not\equiv 0(mod\beta)$, we have $v_2 = 0$ which leads to contradicton. Therefore $P + Q_1(b^2\alpha^2 - \beta^2) \neq 0$ and we obtain $s_0 = 0$ from (36). Substituting $s_0 = 0$ and $r_{00} = 0$ in (31), we have

$$P(s^i_0 y^j - s^j_0 y^i) = 0. \tag{38}$$

If $P = 0$, (32) implies

$$\alpha^2(-51\beta^2 - 18\beta^2 + 3\alpha^2 + 18\alpha^2 b^2 + 108\beta^2 b^4 - 108\beta^2 b^2) + 192\beta^4 = 0. \tag{39}$$

The term of (39) which seemingly does not contain α^2 is $192\beta^4$. Thus there exist $hp(2)v_2$, such that $192\beta^4 = \alpha^2 v_2$. In this case we have $v_2 = 0$ which leads to contradiction. Therefore $P \neq 0$ and we otain from (38),

$$(s^i_0 y^j - s^j_0 y^i) = 0. \tag{40}$$

Contracting above equation by y_j, we get $s^i_0 = 0$ which implies $r_{ij} = s_{ij} = 0$, i.e., $b_{i;j} = 0$.
Case (ii) : $v_5 \neq 0$, $\alpha^2 \not\equiv 0(mod\beta)$.
From (35), there exists a function $h = h(x)$ such that

$$r_{00} = h\alpha^2. \tag{41}$$

Substituting (41) in (34), we have

$$Ps_0 + [(3\alpha^2\beta - 12\beta^3)h - s_0(18\alpha^2 - 108\alpha^2 b^2 - 108\beta^2)](b^2\alpha^2 - \beta^2) = 0. \tag{42}$$

In (42), the term which seemingly do not contain α^2 are $84\beta^4 s_0 - 12\beta^5 h$. Therefore there exists a $hp(3)v_3$ such that $12\beta^4(7s_0 - \beta h) = \alpha^2 v_3$. Since $\alpha^2 \not\equiv 0(mod\beta)$,we have $v_3 = 0$. Thus

$$7s_0 - \beta h = 0, \tag{43}$$

which implies $7s_i - hb_i = 0$. By contracting this by b^i, we get $b^2 = 0$.
For $b^2 = 0$, (42) is reduced to

$$s_0(-33\alpha^2\beta^2 + 3\alpha^4 + 174\beta^4) - h(3\alpha^2\beta^3 + 15\beta^5) = 0, \tag{44}$$

which implies $s_0 = 0$ and $h = 0$. Thus we have $s_0 = r_{00} = 0$.
Therefore for both the cases $v_5 = 0$ and $v_5 \neq 0$, we have $s_0 = r_{00} = 0$. Hence (31) is reduced to $P(s^i_0 y^j - s^j_0 y^i) = 0$. Since $P \neq 0$, we have $s^i_0 y^j - s^j_0 y^i = 0$ and contraction by y_j gives $s^i_0 = 0$, which implies $r_{ij} = s_{ij} = 0$. i.e, $b_{i;j} = 0$. Conversely, if $b_{i;j} = 0$, then we get $B^{ij} = 0$ from (31). Hence F^n is Douglas space. Thus we have

Theorem 4.1. *A Finsler space F^n equipped with a cubic (α, β)-metric is a Douglas space if and only if $b_{i;j} = 0$.*

5. Conclusion

In the present paper, we have characterize a special (α, β)-metric which is an special class of p-power Finsler metric in the form of $L = \frac{(\alpha+\beta)^3}{\alpha^2}$ and named as cubic (α, β)-metric. Further, we examine the projective flatness property and prove that in theorem (2.2) that the Finsler space F^n equipped with cubic (α, β)-metric will be projectively flat iff $b_{i;j} = 0$ and the associated Riemannian space (M^n, α) is projectively flat. In the last two article we have found the condition which stated in theorem (3.1) and (4.1) for the Finsler space F^n to be a Berwald and Douglas space respectively. In future, we can also find the condition for Weakly Berwald /Landsberg space and results related to conformal change in the Finsler space F^n with the cubic (α, β) - metric which is defined in equation (2).

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