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Some new neutrosophic normed sequence spaces defined by Jordan totient operator

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Abstract. In 2020 İlkhan et al. [10] defined Jordan's totient matrix operator. In this paper, the Jordan totient matrix operator A^r and $I-convergence$ are used to define two new sequence spaces, $C^I_{0(\Phi,\Psi,\Upsilon)}(A^r)$ and $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$, as the domain of the Jordan totient matrix operator A^r with respect to the neutrosophic norm. Some results about these sequence spaces are also studied.

1. Introduction

L.A. Zadeh defined the Fuzzy set [29] in 1965. Intuitionistic fuzzy set theory was given by Atanassov [2]. In 2004 Park [24] described a metric on an intuitionistic fuzzy set. Further, the concept of intuitionistic fuzzy norm space (IFNS) was studied by Saadati and Park [25]. The neutrosophic set (NS) is a generalization of IFNS and was introduced by Smarandache [26]. Bera and Mahapatra ([3],[4],[5]) studied neutrosophic soft linear spaces. Moreover, Kirisci and Simsek [20] defined neutrosophic normed space.

The idea of statistical convergence of sequences of real numbers was independently developed by Fast [7] and Steinhaus [27]. Then as a generalization of statistical convergence Kostyrko et al. [21] proposed $I-$*Convergence* in 2000. The concept of statistical convergence in probabilistic normed space was investigated by Karakus [14]. Recently, Khan et al. ([15, 16]) looked at ideal convergence for single and double sequences in intuitionistic fuzzy normed space. Kirisci and Simsek [20] studied neutrosophic normed spaces and statistical convergence. In this paper, we define two new sequence spaces, $C^I_{0(\Phi,\Psi,\Upsilon)}(A^r)$ and $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$, and study the concepts given by Kirisci and Simsek [20] in neutrosophic normed space and obtain some interesting results.

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ are the sets of all positive integers and real numbers, respectively. Let ω stand for the collection of all real/complex sequences, i.e.,

$$\omega := \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}, \forall \quad k \in \mathbb{N}\}. \tag{1}$$

For each positive integer r, the Jordan totient function J_r (generalization of Euler totient function [9]) of a positive integer n is an arithmetic function defined as; number of k-tuples of integers $(m_1, m_2, ..., m_k)$ such

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that gcd $(m_1, m_2, ..., m_k) = 1$ and $1 \leq m_i \leq k$, for $i = 1, 2, 3, ..., k$. To know more properties and applications, we refer ([1],[6],[22],[28]). Let $n = p_1^{s_1} p_2^{s_2} ....p_k^{s_k}$ is the unique prime factorization of n, then Jordan toteint function can be define as follows;

$$J_r(n) = n^r \prod_{p|n} \Big(1 - \frac{1}{p^r}\Big), \text{ p is a prime number,}$$

or

$$J_r(n) = n^r \Big(1 - \frac{1}{p_1^r}\Big)\Big(1 - \frac{1}{p_2^r}\Big)\Big(1 - \frac{1}{p_3^r}\Big)...\Big(1 - \frac{1}{p_k^r}\Big).$$

In classical summability theory, the idea behind generalising the convergence of sequences of real or complex numbers is to give divergent sequences some limit by looking at a matrix transform of the sequence instead of the original sequence. Recently, the Jordan totient matrix A^r was used and considered as a compact operator on the space of all absolutely p-summable sequences ℓ_p. Kara et al. did tremendous work in sequence spaces ([11], [12], [13]). Ilkhan et al. [10] define Jordan totient matrix operator $A^r = (a^r_{nk})$ as follows;

$$A^r = (a^r_{nk}) = \begin{cases} \dfrac{J_r(k)}{n^r}, & \text{if } k|n \\ \\ 0, & \text{otherwise.} \end{cases} .$$

Definition 1.1. ([8]) Suppose $H \subset \mathbb{N}$ and symbol $|.|$ denote the cardinality of the set. Then the asymptotic density of H denoted by $d(H)$ is defined as

$$d(H) = \lim_{\lambda\to\infty} \frac{1}{\lambda} |\{\kappa \leq \lambda : \kappa \in H\}|. \tag{2}$$

Definition 1.2. ([8]) A real sequence $\eta = (\eta_k)$ is statistically convergent to the number ξ if, for each $\theta > 0$, the set $H(\theta) = \{k \leq \lambda : |\eta_k - \xi| > \theta\}$ has $d(H) = 0$ i.e.

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} |\{k \leq \lambda : |\eta_k - \xi| \geq \theta\}| = 0. \tag{3}$$

We denote the statistical convergence as $st - lim = \xi$.

Definition 1.3. ([8]) A real sequence $\eta = (\eta_k)$ is statistically Cauchy sequence if, for every $\theta > 0$ there exists a positive integer (depends upon θ) M such that

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \Big|\{k \leq \lambda : |\eta_k - \eta_M| \geq \theta\}\Big| = 0. \tag{4}$$

Definition 1.4. ([21]) Suppose $X \neq \phi$ and $I \subset P(X)$ (power set of X). Then I is an ideal if,
(a) $\phi \in I$,
(b) $H_1, H_2 \in I \implies H_1 \cup H_2, R$
(c) For each $H_1 \in I$ and $H_2 \subset H_1$, we have $H_2 \in I$.
If, $X \notin I$, then I is called non-trivial ideal.

Definition 1.5. ([21]) An ideal I in X is admissible ideal if, $I \neq P(X)$, non trivial and $\{x\} \in I$ for each $x \in X$.

Definition 1.6. ([21]) Suppose X be a non-empty set. Then the non-empty family $F \subset P(X)$ is called the filter on X if and only if,
(a) $\phi \notin F$,
(b) $H_1, H_2 \in F \implies H_1 \cap H_2 \in F$,
(c) for each $H_1 \in F$ and $H_2 \supset H_1$, we have $H_2 \in F$.

Definition 1.7. ([21]) Suppose $I \subset P(X)$ be a non-trivial ideal. Then a class $F(I) = \{H \subset X : H = X - M, \text{for some } M \in I\}$ is a filter on X, called *filter associated* with the ideal I.

Definition 1.8. ([21]) Suppose $I \subset P(\mathbb{N})$ be a non trivial ideal in $\mathbb{N}$. Then a number sequence $\eta = (\eta_k)$ is ideally convergent ($I-$*convergent*) to ξ if, for every $\theta > 0$, the set $\{k \in \mathbb{N} : |\eta_k - \xi| \geq \theta\} \in I$, and we write it as $I - \lim \eta = \xi$.

Definition 1.9. ([23]) Given a binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous $t-$*norm* if:
(a) $*$ is commutative and associative,
(b) $*$ is continuous,
(c) $s * 1 = s, \forall\, s \in [0,1]$,
(d) $s \leq u$ and $p \leq q \Longrightarrow s * p \leq u * q$, for each $s, p, u, q \in [0,1]$.

Definition 1.10. ([23]) Given a binary operation $\bullet : [0,1]\times[0,1] \longrightarrow [0,1]$ is said to be a continuous $t-$*conorm* if:
(a) $\bullet$ is commutative and associative,
(b) $\bullet$ is continuous,
(c) $s \bullet 0 = s\ \forall\, s \in [0,1]$,
(d) $s \leq u$ and $p \leq q \Longrightarrow s \bullet p \leq u \bullet q$ for each $s, p, u, q \in [0,1]$.

Remark 1.11. ([24]) Form above definitions, we note that if we choose $0 < \epsilon_1, \epsilon_2 < 1$ for $\epsilon_1 > \epsilon_2$, then there exist $0 < \epsilon_3, \epsilon_4 < 1$, such that $\epsilon_1 * \epsilon_3 \geq \epsilon_2, \epsilon_1 \geq \epsilon_4 \bullet \epsilon_2$. Further, if we choose $\epsilon_5 \in (0,1)$, then there exist $\epsilon_6, \epsilon_7 \in (0,1)$ such that $\epsilon_6 * \epsilon_6 \geq \epsilon_5$ and $\epsilon_7 \bullet \epsilon_7 \leq \epsilon_5$.

2. Preliminaries

In 2018 Bera, Tuhin, and Nirmal Kumar Mahapatra did great work by defining neutrosophic norm on soft linear space and also study Continuity and Convergence on neutrosophic soft normed linear spaces, sequences in this space [3]. To know more about neutrosophic norm space, I would like to refer articles O. *Kişi* [17],[19].

Definition 2.1. ([20]) Let X be a linear space, $*$, and $\bullet$ be continuous $t-$*norm* and continuous $t-$*conorm*, respectively. A four tuple of the form $\{X, \Phi(z,.), \Psi(z,.), \Upsilon(z,.) : z \in X\}$, is called neutrosophic normed space, where Φ, Ψ, and Υ are fuzzy sets on $X \times \mathbb{R}^+$ which satisfy the following conditions: For all $z, w \in X$, and $a, b \in \mathbb{R}^+$,

1. $\Phi(z,a) + \Psi(z,a) + \Upsilon(z,a) \leq 3, \quad \forall\, a \in \mathbb{R}^+$
2. $\Phi(z,a) = 1 \iff z = 0$,
3. $\Phi(pz,a) = \Phi(z, \frac{a}{|p|})$, for $p \neq 0$,
4. $*\Big(\Phi(z,a), \Phi(w,b)\Big) \leq \Phi(z+w, a+b)$,
5. $\Phi(z,.)$ is continuous non-decreasing function,
6. $\lim_{a\to\infty} \Phi(z,a) = 1$,
7. $\Psi(z,a) = 0 \iff z = 0$,
8. $\Psi(pz,a) = \Psi(z, \frac{a}{|p|})$ for $p \neq 0$,
9. $\bullet\Big(\Psi(z,a), \Psi(w,b)\Big) \geq \Psi(z+w, a+b)$
10. $\Psi(z,.)$ is continuous non-increasing function,
11. $\lim_{a\to\infty} \Psi(z,a) = 0$,
12. $\Upsilon(z,a) = 0$ (for $a > 0$) if and only if $z = 0$,
13. $\Upsilon(pz,a) = \Upsilon(z, \frac{a}{|p|})$ if $p \neq 0$,
14. $\bullet\Big(\Upsilon(z,a), \Upsilon(w,b)\Big) \geq \Upsilon(z+w, a+b)$,

15. $\Upsilon(z,.)$ is continuous non-increasing function,
16. $\lim_{a\to\infty}\Upsilon(z,a)=0$,
17. If $a\leq 0$, then $\Phi(z,a)=0, \Psi(z,a)=1$, and $\Upsilon(z,\mathbf{a})=1$.

Then $\mathcal{N}=\left(\Phi,\Psi,\Upsilon\right)$ is called neutrosophic norm. Throughout the paper we will use usual t-norm and usual t-conorm i.e; $*(g_1,g_2)=min\{g_1,g_2\}$ and $\bullet(g_1,g_2)=max\{g_1,g_2\}$. These three tuples, $\mathcal{N}=\left\langle\Phi,\Psi,\Upsilon\right\rangle$ is called neutrosophic norm (NN).

Example 2.2. ([20]) Suppose $(X,\|\ .\ \|)$ be a norm space. Given operations as $u_1 * u_2 = u_1u_2$ and $u_1\bullet u_2=u_1+u_2-u_1u_2$. For $p>\|u_1\|$,

$$\Phi(u_1,p)=\frac{p}{p+\|u_1\|},\quad \Psi(u_1,p)=\frac{\|u_1\|}{p+\|u_1\|},\quad (u_1,p)=\frac{\|u_1\|}{p} \tag{5}$$

for all $u_1,u_2\in X$, and $p>0$. If we take $p\leq\|u_1\|$, then

$$\Phi(u_1,p)=0,\ \ \Psi(u_1,p)=1 \text{ and } \Upsilon(u_1,p)=1. \tag{6}$$

Hence, $(X,\mathcal{N},*,\bullet)$ is neutrosophic normed space such that Φ,Ψ and Υ are the functions from X $\times(0,\infty)$ to $[0,1]$.

Definition 2.3. ([20]) Let $(X,\mathcal{N},*,\bullet)$ be a neutrosophic normed space. A sequence $\eta=(\eta_k)\in X$ is said to be convergent to ξ with respect to neutrosophic norm $\mathcal{N}=\left\langle\Phi,\Psi,\Upsilon\right\rangle$ if, for every $0<\theta<1$ and $\delta>0$ there exists $M\in\mathbb{N}$ such that $\Phi(\eta_k-\xi,\delta)>1-\theta$, $\Psi(\eta_k-\xi,\delta)<\theta$ and $\Upsilon(\eta_k-\xi,\delta)<\theta$, for all $k\geq M$, we have

$$\lim_{k\to\infty}\Phi(\eta_k-\xi,\delta)=1,\ \ \lim_{k\to\infty}\Psi(\eta_k-\xi,\delta)=0 \text{ and } \lim_{k\to\infty}\Upsilon(\eta_k-\xi,\delta)=0. \tag{7}$$

The convergence of a sequence in $(X,\mathcal{N},*,\bullet)$ is denoted by $\mathcal{N}-\lim\eta_k=\xi$.

Definition 2.4. ([20]) Suppose $(X,\mathcal{N},*,\bullet)$ be a neutrosophic normed space. A sequence $\eta=(\eta_k)\in X$, is said to be *Cauchy* sequence with respect to neutrosophic norm $\mathcal{N}=\left\langle\Phi,\Psi,\Upsilon\right\rangle$ if, for every $0<\theta<1$ and $\delta>0$ there exists $N\in\mathbb{N}$ such that $\Phi(\eta_k-\eta_m,\delta)>1-\theta$, $\Psi(\eta_k-\eta_m,\delta)<\theta$ and $\Upsilon(\eta_k-\eta_m,\delta)<\theta$, for $k,m\geq N$.

Definition 2.5. ([20]) Let $(X,\mathcal{N},*,\bullet)$ be a neutrosophic normed space. Then open ball with center $x\in X$ and radius r is defined as, for $0<r<1$, $x\in X$ and $\delta>0$,

$$\mathcal{B}_x(r,\delta)=\{y\in X:\Phi(x-y,\delta)\ >1-r,\ \ \Psi(x-y,\delta)<r,\ \ \Upsilon(x-y,\delta)<r\}. \tag{8}$$

Definition 2.6. ([20]) Let $(X,\mathcal{N},*,\bullet)$ be a NNS and $Y\subseteq X$. Then Y is said to be open if, for each $y\in Y$, there exist $t>0$, $0<r<1$ such that $B_y(r,t)\subseteq Y$.

Definition 2.7. ([20]) Let $(X,\mathcal{N},*,\bullet)$ be an NNS, a sequence $\eta=(\eta_k)\in X,\ k\in\mathbb{N}$ is said to be *statistical convergent* with respect to neutrosophic norm $\mathcal{N}=\left\langle\Phi,\Psi,\Upsilon\right\rangle$ if, there exists $\xi\in X$, such that the set

$$\mathbb{Q}_\theta:=\{k\leq n\in\mathbb{N}:\Phi(\eta_k-\xi,\delta)\leq 1-\theta,\ \ \Psi(\eta_k-\xi,\delta)\geq\theta,\ \ \Upsilon(\eta_k-\xi,\delta)\geq\theta\}. \tag{9}$$

or equivalently

$$\mathbb{Q}_\theta:=\{k\leq n\in\mathbb{N}:\Phi(\eta_k-\xi,\delta)>1-\theta,\ \ \Psi(\eta_k-\xi,\delta)<\theta,\ \ \Upsilon(\eta_k-\xi,\delta)<\theta\}, \tag{10}$$

has $d(\mathbb{Q}_\theta)=0$, for every $\theta>0$ and $\delta>0$. We write it as $\mathbb{S}_{\mathcal{N}}-\lim\eta_k=\xi$, or $\eta_k\to\xi(\mathbb{S}_{\mathcal{N}})$.

Definition 2.8. ([18]) Let $(X, \mathcal{N}, *, \bullet)$ be an NNS, a sequence $\eta = (\eta_k) \in X$, $k \in \mathbb{N}$ is said to be $I-$*convergent* to ξ with respect to neutrosophic norm $\mathcal{N} = \left\langle \Phi, \Psi, \Upsilon \right\rangle$ if, for every $\theta > 0$ and $\delta > 0$,

$$\{k \in \mathbb{N} : \Phi(\eta_k - \xi, \delta) \leq 1 - \theta, \ \Psi(\eta_k - \xi, \delta) \geq \theta, \ \Upsilon(\eta_k - \xi, \delta) \geq \theta\} \ \in I. \tag{11}$$

Lemma 2.9. ([18]) *Let* $(X, \mathcal{N}, *, \bullet)$ *be an NNS and sequence* $\eta = (\eta_k) \in X$, $k \in \mathbb{N}$*. The following situations are equivalent, for every* $\epsilon > 0$ *and* $\delta > 0$*,*
(a) $I-$*convergent to* ξ *with respect to neutrosophic norm* $\mathcal{N} = \left\langle \Phi, \Psi, \Upsilon \right\rangle$
(b) $\{k \in \mathbb{N} : \Phi(\eta_k - \xi, \delta) \leq 1 - \epsilon\} \in I$, $\{k \in \mathbb{N} : \Psi(\eta_k - \xi, \delta) \geq \epsilon\} \in I$ *and* $\{k \in \mathbb{N} : \Upsilon(\eta_k - \xi, \delta) \geq \epsilon\} \in I$
(c) $\{k \in \mathbb{N} : \Phi(\eta_k - \xi, \delta) > 1 - \epsilon, \ \Psi(\eta_k - \xi, \delta) < \epsilon, \ \Upsilon(\eta_k - \xi, \delta) < \epsilon\} \in F(I)$
(d) $\Phi(\eta_k - \xi, \delta) = 1, \Psi(\eta_k - \xi, \delta) = 0, \Upsilon(\eta_k - \xi, \delta) = 0$, *as* $k \to \infty$*.*

Definition 2.10. ([15]) A sequence $\eta = (\eta_k) \in \omega$ is said to be Jordan $I-$*convergent* to a number $\xi \in \mathbb{R}$ If, for every $\theta > 0$, the set $\{n \in \mathbb{N} : |A_n^r(\eta) - \xi| \geq \theta\} \in I$.

In this paper, we define the sequence $A_n^r(x)$ that will be used as A^r transform of the sequence $x = (x_k)$ as follows;

$$A_n^r(x) = \frac{1}{n^r} \sum_{k|n} J_r(k) x_k.$$

.

3. Main results

Definition 3.1. Let $(X, \mathcal{N}, *, \bullet)$ be a neutrosophic normed space. A sequence $\eta = (\eta_k) \in X$ is said to be neutrosophic Jordan $I-$*convergent* to ξ with respect to neutrosophic norm $\mathcal{N} = \left\langle \Phi, \Psi, \Upsilon \right\rangle$ if, for every $\theta \in (0,1)$ and $\delta > 0$. The set

$$M_1 = \left\{ n \in \mathbb{N} : \Phi\left(A_n^r(\eta) - \xi, \delta\right) \leq 1 - \theta, \Psi\left(A_n^r(\eta) - \xi, \delta\right) \geq \theta, \ \Upsilon\left(A_n^r(\eta) - \xi, \delta\right) \geq \theta \right\} \in I,$$

and we write $I_{(\Phi,\Psi,\Upsilon)}(A^r)$–$\lim(\eta_k) = \xi$.

Example 3.2. Let $X = \mathbb{R}$ be the neutrosophic normed space as defined in example 2.2 and I be a non trivial admissible ideal in $\mathbb{N}$. Sequence $\eta = (\eta_k) = \left(\frac{1}{k}\right) \in \mathbb{R}$ is Jordan $I-$*convergent* to 0.

Example 3.3. Let $X = \mathbb{R}$ be the neutrosophic normed space as defined in example 2.2 and ideal I be the collection of all those subsets of $\mathbb{N}$ whose natural density is 0. Then the sequence $\eta = (\eta_k)$ defined as

$$\eta_k = \begin{cases} 1, & k = \text{prime} \quad \text{number} \\ 0, & \textit{otherwise}. \end{cases}$$

is also Jordan $I-$*convergent*.

Definition 3.4. Let $(X, \mathcal{N}, *, \bullet)$ be a neutrosophic normed space. Then a sequence $\eta = (\eta_k) \in X$ is said to be neutrosophic Jordan $I-$*Cauchy* with respect to neutrosophic norm $\mathcal{N} = \left\langle \Phi, \Psi, \Upsilon \right\rangle$ if, for every $\theta \in (0,1)$ and $\delta > 0$, there exists a $N \in \mathbb{N}$ such that the set

$$M_2 = \left\{ k \in \mathbb{N} : \Phi\left(A_k^r(\eta) - A_N^r(\eta), \delta\right) \leq 1 - \theta, \Psi\left(A_k^r(\eta) - A_N^r(\eta), \delta\right) \geq \theta, \Upsilon\left(A_k^r(\eta) - A_N^r(\eta), \delta\right) \geq \theta \right\} \in I.$$

Example 3.5. Let $X = \mathbb{R}$ be a neutrosophic normed space as in example 2.2 and I be a non trivial admissible ideal in $\mathbb{N}$. Then the null sequence is Jordan $I-$*Cauchy*.

In this section, we introduce the following sequence spaces:

$$C^I_{(\Phi,\Psi,\Upsilon)}(A^r) := \Big\{x = (x_k) \in \omega : \Big\{n \in \mathbb{N} : \text{for some } \ell \in \mathbb{C}\,,\ \Phi\Big(A^r_n(x) - \ell, \delta\Big) \leq 1 - \epsilon, \\ \Psi\Big(A^r_n(x) - \ell, \delta\Big) \geq \epsilon,\ \Upsilon\Big(A^r_n(x) - \ell, \delta\Big) \geq \epsilon\Big\} \in I\Big\}, \tag{12}$$

$$C^I_{0(\Phi,\Psi,\Upsilon)}(A^r) := \Big\{x = (x_k) \in \omega : \Big\{n \in \mathbb{N} : \Phi\Big(A^r_n(x), \delta\Big) \leq 1 - \epsilon,\ \Psi\Big(A^r_n(x), \delta\Big) \geq \epsilon,\ \Upsilon(A^r_n(x), \delta\Big) \geq \epsilon\Big\} \in I\Big\} \tag{13}$$

We define the open ball with respect to Jordan matrix operator A^r, with center at x and radius $\rho > 0$ with respect to neutrosophic norm $\mathcal{N}$ and $\epsilon \in (0, 1)$, as follows:

$$\mathcal{B}_x(\rho, \epsilon)(A^r) := \Big\{y = (y_k) \in \omega : \Big\{n \in \mathbb{N} : \Phi\Big(A^r_n(x) - A^r_n(y), \rho\Big) \leq 1 - \epsilon, \Psi\Big(A^r_n(x) - A^r_n(y), \rho\Big) \geq \epsilon, \\ \Upsilon\Big(A^r_n(x) - A^r_n(y), \rho\Big) \geq \epsilon\Big\} \tag{14}$$

Theorem 3.6. *The spaces $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$ and $C^I_{0(\Phi,\Psi,\Upsilon)}(A^r)$ are linear spaces over $\mathbb{R}$.*

Proof. We will prove the result for $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$. The proof of linearity of the space $C^I_{0(\Phi,\Psi,\Upsilon)}(A^r)$ follows similarly. Let $x = (x_k)$, $y = (y_k) \in C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$. Then there exist ℓ_1, $\ell_2 \in \mathbb{C}$, such that $x = (x_k)$ and $y = (y_k)$ Jordan I–converge to ℓ_1 and ℓ_2 respectively. We will show that for any scalars α_1 and α_2 the sequence $\alpha_1 x_k + \alpha_2 y_k$ Jordan I–converges to $\alpha_1\ell_1 + \alpha_2\ell_2$. For $\delta > 0$ and $0 < \epsilon < 1$ consider the following sets;

$$P = \Big\{n \in \mathbb{N} : \Phi\Big(A^r_n(x) - \ell_1, \frac{\delta}{2|\alpha_1|}\Big) \leq 1 - \epsilon, \Psi\Big(A^r_n(x) - \ell_1, \frac{\delta}{2|\alpha_1|}\Big) \geq \epsilon, \Upsilon\Big(A^r_n(x) - \ell_1, \frac{\delta}{2|\alpha_1|}\Big) \geq \epsilon\Big\} \in I.$$

$$P^c = \Big\{n \in \mathbb{N} : \Phi\Big(A^r_n(x) - \ell_1, \frac{\delta}{2|\alpha_1|}\Big) > 1 - \epsilon,\ \Psi\Big(A^r_n(x) - \ell_1, \frac{\delta}{2|\alpha_1|}\Big) < \epsilon, \Upsilon\Big(A^r_n(x) - \ell_1, \frac{\delta}{2|\alpha_1|}\Big) < \epsilon\Big\} \in F(I).$$

$$Q = \Big\{n \in \mathbb{N} : \Phi\Big(A^r_n(y) - \ell_2, \frac{\delta}{2|\alpha_2|}\Big) \leq 1 - \epsilon,\ \Psi\Big(A^r_n(y) - \ell_2, \frac{\delta}{2|\alpha_2|}\Big) \geq \epsilon, \Upsilon\Big(A^r_n(y) - \ell_2, \frac{\delta}{2|\alpha_2|}\Big) \geq \epsilon\Big\} \in I.$$

$$Q^c = \Big\{n \in \mathbb{N} : \Phi\Big(A^r_n(y) - \ell_2, \frac{t}{2|\alpha_2|}\Big) > 1 - \epsilon,\ \Psi\Big(A^r_n(y) - \ell_2, \frac{\delta}{2|\alpha_2|}\Big) < \epsilon, \Upsilon\Big(A^r_n(y) - \ell_2, \frac{\delta}{2|\alpha_2|}\Big) < \epsilon\Big\} \in F(I).$$

Define the set $D = P \cup Q$ so that $D \in I$. It follows that $\phi \neq D^c \in F(I)$. we will show that for each $x = (x_k), (y) = (y_k) \in C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$,

$$D^c \subset \Big\{n \in \mathbb{N} : \quad \Phi\Big(\Big(\alpha_1 A^r_n(x) - \alpha_2 A^r_n(y)\Big) - \Big(\alpha_1\ell_1 - \alpha_2\ell_2\Big), \delta\Big) > 1 - \epsilon, \\ \Psi\Big(\Big(\alpha_1 A^r_n(x) - \alpha_2 A^r_n(y)\Big) - \Big(\alpha_1\ell_1 - \alpha_2\ell_2\Big), \delta\Big) < \epsilon, \\ \Upsilon\Big(\Big(\alpha_1 A^r_n(x) - \alpha_2 A^r_n(y)\Big) - \Big(\alpha_1\ell_1 - \alpha_2\ell_2\Big), \delta\Big) < \epsilon\Big\}$$

Let $m \in D^c$. In this case,

$$\Phi\Big(A^r_m(x) - \ell_1, \frac{\delta}{2|\alpha_1|}\Big) > 1 - \epsilon, \ \ \Psi\Big(A^r_m(x) - \ell_1, \frac{\delta}{2|\alpha_1|}\Big) < \epsilon, \Upsilon\Big(A^r_m(x) - \ell_1, \frac{\delta}{2|\alpha_1|}\Big) < \epsilon$$

and

$$\Phi\Big(A^r_m(y) - \ell_2, \frac{\delta}{2|\alpha_2|}\Big) > 1 - \epsilon, \ \ \Psi\Big(A^r_m(y) - \ell_2, \frac{\delta}{2|\alpha_2|}\Big) < \epsilon, \Upsilon\Big(A^r_m(y) - \ell_2, \frac{\delta}{2|\alpha_2|}\Big) < \epsilon$$

$$\begin{aligned}\Phi\left(\Big(\alpha_1 A^r_m(x) - \alpha_2 A^r_m(y)\Big) - \Big(\alpha_1\ell_1 - \alpha_2\ell_2\Big), \delta\right) &\geq \Phi\left(\alpha_1 A^r_m(x) - \alpha_1\ell_1, \frac{\delta}{2}\right) * \Phi\Big(\alpha_2 A^r_m(y) - \alpha_2\ell_2, \frac{\delta}{2}\Big)\\ &= \Phi\left(A^r_m(x) - \ell_1, \frac{\delta}{2|\alpha_1|}\right) * \Phi\left(A^r_m(y) - \ell_2, \frac{\delta}{2|\alpha_2|}\right)\\ &> (1-\epsilon) * (1-\epsilon)\\ &= 1 - \epsilon.\end{aligned}$$

This implies that

$$\Phi\left(\Big(\alpha_1 A^r_m(x) - \alpha_2 A^r_m(y)\Big) - \Big(\alpha_1\ell_1 - \alpha_2\ell_2\Big), \delta\right) > 1 - \epsilon.$$

In a similar way,

$$\begin{aligned}\Psi\left(\Big(\alpha_1 A^r_m(x) - \alpha_2 A^r_m(y)\Big) - \Big(\alpha_1\ell_1 - \alpha_2\ell_2\Big), \delta\right) &\leq \Psi\left(\alpha_1 A^r_m(x) - \alpha_1\ell_1, \frac{\delta}{2}\right) \bullet \Psi\Big(\alpha_2 A^r_m(y) - \alpha_2\ell_2, \frac{\delta}{2}\Big)\\ &= \Psi\left(A^r_m(x) - \ell_1, \frac{\delta}{2|\alpha_1|}\right) \bullet \Psi\left(A^r_m(y) - \ell_2, \frac{\delta}{2|\alpha_2|}\right)\\ &< \epsilon \bullet \epsilon\\ &= \epsilon.\end{aligned}$$

This implies that

$$\Psi\left(\Big(\alpha_1 A^r_m(x) - \alpha_2 A^r_m(y)\Big) - \Big(\alpha_1\ell_1 - \alpha_2\ell_2\Big), \delta\right) < \epsilon.$$

And,

$$\begin{aligned}\Upsilon\left(\Big(\alpha_1 A^r_m(x) - \alpha_2 A^r_m(y)\Big) - (\alpha_1\ell_1 - \alpha_2\ell_2), \delta\right) &\leq \Upsilon\left(\alpha_1 A^r_m(x) - \alpha_1\ell_1, \frac{\delta}{2}\right) \bullet \Upsilon\Big(\alpha_2 A^r_m(y) - \alpha_2\ell_2, \frac{\delta}{2}\Big)\\ &= \Upsilon\left(A^r_m(x) - \ell_1, \frac{\delta}{2|\alpha_1|}\right) \bullet \Upsilon\left(A^r_m(y) - \ell_2, \frac{\delta}{2|\alpha_2|}\right)\\ &< \epsilon \bullet \epsilon\\ &= \epsilon.\end{aligned}$$

This implies that

$$\Upsilon\left(\Big(\alpha_1 A^r_m(x) - \alpha_2 A^r_m(y)\Big) - \Big(\alpha_1\ell_1 - \alpha_2\ell_2\Big), \delta\right) < \epsilon.$$

Therefore, we have

$$D^c \subset \Big\{ m \in \mathbb{N} : \Phi\Big(\big(\alpha_1 A^r_m(x) - \alpha_2 A^r_m(y)\big) - \big(\alpha_1 \ell_1 - \alpha_2 \ell_2\big), \delta\Big) > 1 - \epsilon \ ,$$

$$\Psi\Big(\big(\alpha_1 A^r_m(x) - \alpha_2 A^r_m(y)\big) - \big(\alpha_1 \ell_1 - \alpha_2 \ell_2\big), \delta\Big) < \epsilon,$$

$$\Upsilon\Big(\big(\alpha_1 A^r_m(x) - \alpha_2 A^r_m(y)\big) - \big(\alpha_1 \ell_1 - \alpha_2 \ell_2\big), \delta\Big) < \epsilon. \Big\}$$

By the definition of *filter associated* with ideal I, we have

$$\Big\{ m \in \mathbb{N} : \Phi\Big(\big(\alpha_1 A^r_m(x) - \alpha_2 A^r_m(y)\big) - \big(\alpha_1 \ell_1 - \alpha_2 \ell_2\big), \delta\Big) > 1 - \epsilon \ ,$$

$$\Psi\Big(\big(\alpha_1 A^r_m(x) - \alpha_2 A^r_m(y)\big) - \big(\alpha_1 \ell_1 - \alpha_2 \ell_2\big), \delta\Big) < \epsilon,$$

$$\Upsilon\Big(\big(\alpha_1 A^r_m(x) - \alpha_2 A^r_m(y)\big) - \big(\alpha_1 \ell_1 - \alpha_2 \ell_2\big), \delta\Big) < \epsilon \Big\} \in F(I).$$

Hence the sequence $(\alpha_1 x_k + \alpha_2 y_k)$ Jordan I–converges to $\alpha_1 \ell_1 + \alpha_2 \ell_2$. Therefore, $(\alpha_1 x_k + \alpha_2 y_k) \in C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$. Hence $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$ is a linear space. □

Theorem 3.7. *Every open ball $\mathcal{B}_x(\rho, \epsilon)(A^r)$ with center x and radius $\rho > 0$ with respect to neutrosophic norm $\mathcal{N}$ and $0 < \epsilon < 1$, is an open set in $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$.*

Proof. Consider the open ball with center at x and radius $\rho > 0$ with parameter of neutrosophic $0 < \epsilon < 1$,

$$\mathcal{B}_x\big(\rho, \epsilon\big)(A^r) = \Big\{ y = (y_n) \in \omega : \big\{ n \in \mathbb{N} : \Phi\big(A^r_n(x) - A^r_n(y), \rho\big) \leq 1 - \epsilon,\ \Psi\big(A^r_n(x) - A^r_n(y), \rho\big) \geq \epsilon,$$

$$\Upsilon\big(A^r_n(x) - A^r_n(y), \rho\big) \geq \epsilon \big\} \Big\}.$$

or

$$\mathcal{B}_x\big(\rho, \epsilon\big)(A^r) = \Big\{ y = (y_n) \in \omega : \big\{ n \in \mathbb{N} : \Phi\big(A^r_n(x) - A^r_n(y), \rho\big) > 1 - \epsilon,\ \Psi\big(A^r_n(x) - A^r_n(y), \rho\big) < \epsilon,$$

$$\Upsilon\big(A^r_n(x) - A^r_n(y), \rho\big) < \epsilon \big\} \Big\}.$$

Let $y = (y_k) \in \mathcal{B}_x\big(\rho, \epsilon\big)(A^r)$. Then we have, the set such that

$$\Big\{ n \in \mathbb{N} : \Phi\big(A^r_n(x) - A^r_n(y), \rho\big) > 1 - \epsilon,\ \Psi\big(A^r_n(x) - A^r_n(y), \rho\big) < \epsilon, \Upsilon\big(A^r_n(x) - A^r_n(y), \rho\big) < \epsilon \Big\}.$$

$$\Phi\big(A^r_n(x) - A^r_n(y), \rho\big) > 1 - \epsilon, \Psi\big(A^r_n(x) - A^r_n(y), \rho\big) < \epsilon \ \text{ and } \ \Upsilon\big(A^r_n(x) - A^r_n(y), \rho\big) < \epsilon$$

there exists $\rho_0 \in (0, \rho)$ such that,

$$\Phi\Big(A_n^r(x) - A_n^r(y), \rho_0\Big) > 1 - \epsilon, \Psi\Big(A_n^r(x) - A_n^r(y), \rho_0\Big) < \epsilon \text{ and } \Upsilon\Big(A_n^r(x) - A_n^r(y), \rho_0\Big) < \epsilon.$$

Putting $\epsilon_0 = \Phi\Big(A_n^r(x) - A_n^r(y), \rho_0\Big)$, this implies that $\epsilon_0 > 1 - \epsilon$. Then there exist $s \in (0, 1)$ such that $\epsilon_0 > 1 - s > 1 - \epsilon$. For, $\epsilon_0 > 1 - s$, we can have $\epsilon_1, \epsilon_2, \epsilon_3 \in (0, 1)$ such that $\epsilon_0 * \epsilon_1 > 1 - s$, $(1 - \epsilon_0) \bullet (1 - \epsilon_2) < s$ and $(1 - \epsilon_0) \bullet (1 - \epsilon_3) < s$. Let $\epsilon_4 = \max\{\epsilon_1, \epsilon_2, \epsilon_3\}$.
Now consider the open ball $\mathcal{B}_y\Big(\rho - \rho_0, 1 - \epsilon_4\Big)(A^r)$. We will show that

$$\mathcal{B}_y\Big(\rho - \rho_0, 1 - \epsilon_4\Big)(A^r) \subset \mathcal{B}_x\Big(\rho, \epsilon\Big)(A^r).$$

Let $z = (z_k) \in \mathcal{B}_y\Big(\rho - \rho_0, 1 - \epsilon_4\Big)(A^r)$, then we have the set

$$\Big\{n \in \mathbb{N} : \Phi\Big(A_n^r(y) - A_n^r(z), \rho - \rho_0\Big) > \epsilon_4 \,,\ \ \Psi\Big(A_n^r(y) - A_n^r(z), \rho - \rho_0\Big) < 1 - \epsilon_4,$$
$$\Upsilon\Big(A_n^r(y) - A_n^r(z), \rho - \rho_0\Big) < 1 - \epsilon_4\Big\}.$$

Therefore,

$$\begin{aligned}\Phi\Big(A_n^r(x) - A_n^r(z), \rho\Big) &\geq \ \Phi\Big(A_n^r(x) - A_n^r(y), \rho_0\Big) * \Phi\Big(A_n^r(y) - A_n^r(z), \rho - \rho_0\Big)\\ &\geq \ \epsilon_0 * \epsilon_4 \geq \epsilon_0 * \epsilon_1\\ &> \ (1 - s) > (1 - \epsilon).\end{aligned}$$

This implies that $\Phi\Big(A_n^r(x) - A_n^r(z), \rho\Big) > 1 - \epsilon$,

$$\begin{aligned}\Psi\Big(A_n^r(x) - A_n^r(z), \rho\Big) &\leq \Psi\Big(A_n^r(x) - A_n^r(y), \rho_0\Big) \bullet \Psi\Big(A_n^r(y) - A_n^r(z), \rho - \rho_0\Big)\\ &\leq (1 - \epsilon_0) \bullet (1 - \epsilon_4) \leq (1 - \epsilon_0) \bullet (1 - \epsilon_2)\\ &\leq s < \epsilon.\end{aligned}$$

Hence, we get $\Psi\Big(A_n^r(x) - A_n^r(z), \rho\Big) < \epsilon$ and

$$\begin{aligned}\Upsilon\Big(A_n^r(x) - A_n^r(z), \rho\Big) &\leq \Upsilon\Big(A_n^r(x) - A_n^r(y), \rho_0\Big) \bullet \Upsilon\Big(A_n^r(y) - A_n^r(z), \rho - \rho_0\Big)\\ &\leq (1 - \epsilon_0) \bullet (1 - \epsilon_4) \leq (1 - \epsilon_0) \bullet (1 - \epsilon_3)\\ &\leq s < \epsilon.\end{aligned}$$

This implies that,

$$\Upsilon\Big(A_n^r(x) - A_n^r(z), \rho\Big) < \epsilon.$$

Therefore we have the set, such that

$$\Big\{n \in \mathbb{N} : \Phi\Big(A^r_n(x) - A^r_n(z), \rho - \rho_0\Big) > 1 - \epsilon\ ,\ \ \Psi\Big(A^r_n(x) - A^r_n(z), \rho\rho_0\Big) < \epsilon,$$
$$\Upsilon\Big(A^r_n(x) - A^r_n(z), \rho - \rho_0\Big) < \epsilon\Big\}.$$

Hence we get $z = (z_k) \in \mathcal{B}_x\Big(\rho, \epsilon\Big)(A^r)$. This implies that

$$\mathcal{B}_y\Big(\rho - \rho_0, 1 - \epsilon_4\Big)(A^r) \subset \mathcal{B}_x\Big(\rho, \epsilon\Big)(A^r).$$

□

Remark 3.8. The spaces $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$ and $C^I_{0(\Phi,\Psi,\Upsilon)}(A^r)$ are NNS with respect to neutrosophic norm $\mathcal{N} = \Big\langle \Phi, \Psi, \Upsilon\Big\rangle$.

Now define

$$\tau_{(\Phi,\Psi,\Upsilon)}(A^r) = \Big\{W \subset C^I_{(\Phi,\Psi,\Upsilon)}(A^r) : \text{for each } x = (x_k) \in W, \exists\ \ \rho > 0 \text{ and } \epsilon \in (0, 1) \text{ such that } \mathcal{B}_x(\rho, \epsilon)(A^r) \subset W\Big\}.$$

Then $\tau_{(\Phi,\Psi,\Upsilon)}(A^r)$ defines a topology on the sequence space $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$. The collection defined by $\mathcal{B} = \Big\{\mathcal{B}_x(\rho, \epsilon) : x = (x_k) \in C^I_{(\Phi,\Psi,\Upsilon)}(A^r), \rho > 0 \text{ and } \epsilon \in (0, 1)\Big\}$ is a base for the topology $\tau_{(\Phi,\Psi,\Upsilon)}(A^r)$ on the space $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$.

Theorem 3.9. *The topology $\tau_{(\Phi,\Psi,\Upsilon)}(A^r)$ on the space $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$ is first countable.*

Proof. For each $x = (x_i) \in C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$, consider the set $\mathcal{B} = \Big\{\mathcal{B}_x(\frac{1}{n}, \frac{1}{n})(A^r) : n = 1, 2, 3, 4, ...\Big\}$, which is a countable local base at x. Therefore the topology $\tau_{(\Phi,\Psi,\Upsilon)}(A^r)$ on the space $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$ is first countable. □

Theorem 3.10. *The spaces $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$ and $C^I_{0(\Phi,\Psi,\Upsilon)}(A^r)$ are Hausdorff spaces.*

Proof. We shall prove the result only for $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$ and another one follows similarly. Let $x = (x_n)$ and $y = (y_n) \in C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$ such that $x \neq y$. Then for each $i \in \mathbb{N}$ and $\rho > 0$, we have,

$$0 < \Phi\Big(A^r_n(x) - A^r_n(y), \rho\Big) < 1,\ \ 0 < \Psi\Big(A^r_n(x) - A^r_n(y), \rho\Big) < 1 \ \text{ and }\ 0 < \Upsilon\Big(A^r_n(x) - A^r_n(y), \rho\Big) < 1 \tag{15}$$

Putting,

$$\epsilon_1 = \Phi\Big(A^r_n(x) - A^r_n(y), \rho\Big),\ \ \epsilon_2 = \Psi\Big(A^r_n(x) - A^r_n(y), \rho\Big),\ \ \epsilon_3 = \Upsilon\Big(A^r_n(x) - A^r_n(y), \rho\Big). \tag{16}$$

and $\epsilon = \max\{\epsilon_1, 1 - \epsilon_2, 1 - \epsilon_3\}$. Then for each $\epsilon_0 > \epsilon$ there exist $\epsilon_4, \epsilon_5, \epsilon_6 \in (0, 1)$ such that

$$\epsilon_4 * \epsilon_4 \geq \epsilon_0, (1 - \epsilon_5) \bullet (1 - \epsilon_5) \leq (1 - \epsilon_0) \ \text{ and }\ (1 - \epsilon_6) \bullet (1 - \epsilon_6) \leq (1 - \epsilon_0).$$

Again putting $\epsilon_7 = \max\{\epsilon_4, \epsilon_5, \epsilon_6, \}$, consider the open balls $\mathcal{B}_x\Big(1 - \epsilon_7, \frac{\rho}{2}\Big)(A^r)$ and $\mathcal{B}_y\Big(1 - \epsilon_7, \frac{\rho}{2}\Big)(A^r)$ centered at x and y respectively. We show that $\mathcal{B}_x\Big(1 - \epsilon_7, \frac{\rho}{2}\Big)(A^r) \cap \mathcal{B}_y\Big(1 - \epsilon_7, \frac{\rho}{2}\Big)(A^r) = \phi$.

If possible let $z = (z_n) \in \mathcal{B}_x\Big(1-\epsilon_7, \frac{\rho}{2}\Big)(A^r) \cap \mathcal{B}_y\Big(1-\epsilon_7, \frac{\rho}{2}\Big)(A^r)$. Then for the set, $\{k \in \mathbb{N}\}$ we have,

$$\begin{aligned}\epsilon_1 =& \Phi\Big(A^r_n(x) - A^r_n(y), \rho\Big)\\ \geq& \;\Phi\left(A^r_n(x) - A^r_n(z), \frac{\rho}{2}\right) * \Phi\left(A^r_n(z) - A^r_n(y), \frac{\rho}{2}\right)\\ >& \;\epsilon_7 * \epsilon_7 \geq \epsilon_4 * \epsilon_4 \geq \epsilon_0 > \epsilon_1.\end{aligned} \tag{17}$$

$$\begin{aligned}\epsilon_2 =& \Psi\Big(A^r_n(x) - A^r_n(y), \rho\Big)\\ \leq& \;\Psi\left(A^r_n(x) - A^r_n(z), \frac{\rho}{2}\right) \bullet \Psi\left(A^r_n(z) - A^r_n(y), \frac{\rho}{2}\right)\\ <& \;(1-\epsilon_7) \bullet (1-\epsilon_7) \leq (1-\epsilon_5) \bullet (1-\epsilon_5)\\ <& \;(1-\epsilon_0) < \epsilon_2.\end{aligned} \tag{18}$$

And

$$\begin{aligned}\epsilon_3 =& \Upsilon\Big(A^r_n(x) - A^r_n(y), r\Big)\\ \leq& \;\Upsilon\left(A^r_n(x) - A^r_n(z), \frac{\rho}{2}\right) \bullet \Upsilon\left(A^r_n(z) - A^r_n(y), \frac{\rho}{2}\right)\\ <& \;(1-\epsilon_7) \bullet (1-\epsilon_7) \leq (1-\epsilon_6) \bullet (1-\epsilon_6)\\ <& \;(1-\epsilon_0) < \epsilon_3.\end{aligned} \tag{19}$$

From equations (17), (18), and (19) we have a contradiction.Therefore, $\mathcal{B}_x\Big(1-\epsilon_7, \frac{\rho}{2}\Big)(A^r) \cap \mathcal{B}_y\Big(1-\epsilon_7, \frac{\rho}{2}\Big)(A^r) = \phi$. Hence the space $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$ is a Hausdorff space. □

Theorem 3.11. *Let sequence $x = (x_k) \in C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$ be ideal convergent. Then for some $\ell \in \mathbb{C}$, $x_k \to \ell$ if and only if for every $\epsilon \in (0,1)$ and $\delta > 0$, there exist positive integers $K = K(\ell, \epsilon, \delta)$ such that*

$$\left\{K \in \mathbb{N} : \Phi\Big(A^r_K(x) - \ell, \frac{\delta}{2}\Big) > 1-\epsilon,\ \Psi\Big(A^r_K(x) - \ell, \frac{\delta}{2}\Big) < \epsilon, \Upsilon\Big(A^r_K(x) - \ell, \frac{\delta}{2}\Big) < \epsilon\right\} \in F(I).$$

Proof. Suppose $x_k \to \ell$, for some $\ell \in \mathbb{C}$. For, given $\epsilon \in (0,1)$ there exists $\rho \in (0,1)$ such that $(1-\epsilon)*(1-\epsilon) > 1-\rho$ and $\epsilon \bullet \epsilon < \rho$. Since $x_k \to \ell$, for all $\delta > 0$,

$$\mathcal{D} = \left\{i \in \mathbb{N} : \Phi\Big(A^r_i(x) - \ell, \frac{\delta}{2}\Big) \leq 1-\epsilon,\ \Psi\Big(A^r_i(x) - \ell, \frac{\delta}{2}\Big) \geq \epsilon, \Upsilon\Big(A^r_i(x) - \ell, \frac{\delta}{2}\Big) \geq \epsilon\right\} \in I$$

which implies that

$$\mathcal{D}^c = \left\{i \in \mathbb{N} : \Phi\Big(A^r_i(x) - \ell, \frac{\delta}{2}\Big) > 1-\epsilon,\ \Psi\Big(A^r_i(x) - \ell, \frac{\delta}{2}\Big) < \epsilon, \Upsilon\Big(A^r_i(x) - \ell, \frac{\delta}{2}\Big) < \epsilon\right\} \in F(I)$$

Conversely, let us choose $K \in \mathcal{D}^c$. Then

$$\Phi\Big(A^r_K(x) - \ell, \frac{\delta}{2}\Big) > 1-\epsilon,\ \Psi\Big(A^r_K(x) - \ell, \frac{\delta}{2}\Big) < \epsilon, \Upsilon\Big(A^r_K(x) - \ell, \frac{\delta}{2}\Big) < \epsilon.$$

We show that there exists a positive integer $K = K(x, \epsilon, \delta)$ such that

$$\left\{a \in \mathbb{N} : \Phi\Big(A_a^r(x) - A_K^r(x), \delta\Big) \leq 1 - \rho,\ \Psi\Big(A_a^r(x) - A_K^r(x), \delta\Big) \geq \rho, \Upsilon\Big(A_a^r(x) - A_K^r(x), \delta\Big) \geq \rho\right\} \in I$$

So, for $x = (x_k) \in C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$ define

$$\mathbb{E} = \left\{a \in \mathbb{N} : \Phi\Big(A_a^r(x) - A_K^r(x), \delta\Big) \leq 1 - \rho,\ \Psi\Big(A_a^r(x) - A_K^r(x), \delta\Big) \geq \rho, \Upsilon\Big(A_a^r(x) - A_K^r(x), \delta\Big) \geq \rho\right\} \in I$$

We shall show that $\mathcal{D} \subseteq \mathbb{E}$. Let on contrary $\mathbb{E} \nsubseteq \mathcal{D}$ i.e; there exists $b \in \mathbb{E}$ such that $b \notin \mathcal{D}$. Then

$$\Phi\Big(A_b^r(x) - A_K^r(x), \delta\Big) \leq 1 - \rho \text{ or}, \Phi\Big(A_b^r(x) - \ell, \frac{\delta}{2}\Big) > 1 - \epsilon$$

In particular

$$\Phi\Big(A_K^r(x) - \ell, \frac{\delta}{2}\Big) > 1 - \epsilon.$$

Therefore, we have

$$\begin{aligned} 1 - \rho &\geq \Phi\Big(A_b^r(x) - A_K^r(x), \delta\Big) \\ &\geq \Phi\Big(A_b^r(x) - \ell, \frac{\delta}{2}\Big) * \Phi\Big(A_K^r(x) - \ell, \frac{\delta}{2}\Big) \\ &\geq (1 - \epsilon) * (1 - \epsilon) \\ &> 1 - \rho. \end{aligned}$$

which is a contradiction. Similarly,

$$\Psi\Big(A_b^r(x) - A_K^r(x), \delta\Big) \geq \rho \text{ or } \Psi\Big(A_b^r(x) - \ell, \frac{\delta}{2}\Big) < \epsilon$$

In particular

$$\Psi\Big(A_K^r(x) - \ell, \frac{\delta}{2}\Big) < \epsilon.$$

Therefore, we have

$$\begin{aligned} \rho &\leq \Psi\Big(A_b^r(x) - A_K^r(x), \delta\Big) \\ &\leq \Psi\Big(A_b^r(x) - \ell, \frac{\delta}{2}\Big) \bullet \Psi\Big(A_K^r(x) - \ell, \frac{\delta}{2}\Big) \\ &\leq \epsilon \bullet \epsilon \\ &< \rho. \end{aligned}$$

Similarly on the other way

$$\Upsilon\Big(A_b^r(x) - A_K^r(x), \frac{\delta}{2}\Big) \geq \rho \text{ or} \Upsilon\Big(A_b^r(x) - \ell, \frac{\delta}{2}\Big) < \epsilon$$

In particular

$$\Upsilon\Big(A_K^r(x)-\ell,\frac{\delta}{2}\Big)<\epsilon$$

$$\begin{aligned}\rho &\leq \Upsilon\Big(A_b^r(x)-A_K^r(x),\delta\Big)\\ &\leq \Upsilon\Big(A_b^r(x)-\ell,\frac{\delta}{2}\Big)\bullet\Upsilon\Big(A_K^r(x)-\ell,\frac{\delta}{2}\Big)\\ &\leq \epsilon\bullet\epsilon\\ &<\rho.\end{aligned}$$

which is again a contradiction. Hence $\mathcal{D}\subseteq\mathbb{E}$ and since $\mathbb{E}\in I$. This implies that, $\mathcal{D}\in I$. □

Theorem 3.12. *A sequence $x=(x_k)\in\omega$, is neutrosophic Jordan I- convergent with respect to neutrosophic norms $\Big\langle\Phi,\Psi,\Upsilon\Big\rangle$ if and only if it is neutrosophic Jordan I- Cauchy with respect to the same norms.*

Proof. Suppose $x=(x_k)\in\omega$ is neutrosophic Jordan I- convergent with respect to neutrosophic norms $\Big\langle\Phi,\Psi,\Upsilon\Big\rangle$ such that $I_{(\Phi,\Psi,\Upsilon)}(A^r)-\lim(x_k)=\ell$. For given $\epsilon\in(0,1)$ there exists $\epsilon_1\in(0,1)$ such that $(1-\epsilon_1)*(1-\epsilon_1)>1-\epsilon$ and $\epsilon_1\bullet\epsilon_1<\epsilon$. Since $I_{(\Phi,\Psi,\Upsilon)}(A^r)-\lim(x_k)=\ell$, therefore for all $\delta>0$

$$D=\Big\{i\in\mathbb{N}:\Phi\Big(A_i^r(x)-\ell,\delta\Big)\leq 1-\epsilon_1,\ \Psi\Big(A_i^r(x)-\ell,\delta\Big)\geq\epsilon_1,\Upsilon\Big(A_i^r(x)-\ell,\delta\Big)\geq\epsilon_1\Big\}\in I$$

that implies

$$D^c=\Big\{i\in\mathbb{N}:\Phi\Big(A_i^r(x)-\ell,\delta\Big)>1-\epsilon_1,\ \Psi\Big(A_i^r(x)-\ell,\delta\Big)<\epsilon_1,\Upsilon\Big(A_i^r(x)-\ell,\delta\Big)<\epsilon_1\Big\}\in F(I)$$

For $i\in D^c$, we have

$$\Phi\Big(A_i^r(x)-\ell,\delta\Big)>1-\epsilon_1,\ \Psi\Big(A_i^r(x)-\ell,\delta\Big)<\epsilon_1,\Upsilon\Big(A_j^r(y)-\ell,\delta\Big)<\epsilon_1.$$

For fix $k\in D^c$, let

$$\Upsilon=\Big\{i\in\mathbb{N}:\Phi\Big(A_i^r(x)-A_k^r(x),\delta\Big)\leq 1-\epsilon,\ \Psi\Big(A_i^r(x)-A_k(x),\delta\Big)\geq\epsilon,\Upsilon\Big(A_i(x)-A_k(x),\delta\Big)\geq\epsilon\Big\}\in I.$$

We show that $\Upsilon\subset D$. Let $i\in\Upsilon$, we have

$$\Phi\Big(A_i(x)-A_k(x),\delta\Big)\leq 1-\epsilon,\ \Psi\Big(A_i(x)-A_k(x),\delta\Big)\geq\epsilon,\Upsilon\Big(A_i(x)-A_k(x),\delta\Big)\geq\epsilon.$$

We have two possible cases, firstly consider $\Phi\Big(A_i(x)-A_k^r(x),\delta\Big)\leq 1-\epsilon$. Then $\Phi\Big(A_i^r(x)-\ell\frac{\delta}{2}\Big)\leq 1-\epsilon_1$. If possible let $\Phi\Big(A_i^r(x)-\ell,\frac{\delta}{2}\Big)>1-\epsilon_1$. Then

$$\begin{aligned}1-\epsilon &\geq \Phi\Big(A_i^r(x)-A_k^r(x),\delta\Big)\\ &\geq \Phi\Big(A_i^r(x)-\ell,\frac{\delta}{2}\Big)*\Phi\Big(A_k^r(x)-\ell,\frac{\delta}{2}\Big)\\ &>(1-\epsilon_1)*(1-\epsilon_1)\\ &>(1-\epsilon),\end{aligned}$$

which is a contradiction. Similarly, consider $\Psi\Big(A_i^r(x) - A_k^r(x), \delta\Big) \geq \epsilon$, then $\Psi\Big(A_i^r(x) - \ell, \frac{\delta}{2}\Big) \geq \epsilon_1$. If possible suppose $\Psi\Big(A_i^r(x) - \ell, \frac{\delta}{2}\Big) < \epsilon_1$. Hence,

$$\begin{aligned}\epsilon &\leq \Psi\Big(A_i^r(x) - A_k^r(x), \delta\Big)\\ &\leq \Psi\bigg(A_i^r(x) - \ell, \frac{\delta}{2}\bigg) \bullet \Psi\bigg(A_k^r(x) - \ell, \frac{\delta}{2}\bigg)\\ &< \epsilon_1 \bullet \epsilon_1\\ &< \epsilon,\end{aligned}$$

which is again a contradiction.

$$\Upsilon\Big(A_i^r(x) - A_k^r(x), \delta\Big) \geq \epsilon, \text{ then } \Upsilon\Big(A_i^r(x) - \ell, \frac{\delta}{2}\Big) \geq \epsilon_1.$$

If possible suppose $\Upsilon\Big(A_i^r(x) - \ell, \frac{\delta}{2}\Big) < \epsilon_1$. Hence

$$\begin{aligned}\epsilon &\leq \Upsilon\Big(A_i^r(x) - A_k^r(x), \delta\Big)\\ &\leq \Upsilon\bigg(A_i^r(x) - \ell, \frac{\delta}{2}\bigg) \bullet \Upsilon\bigg(A_k^r(x) - \ell, \frac{\delta}{2}\bigg)\\ &< \epsilon_1 \bullet \epsilon_1\\ &< \epsilon,\end{aligned}$$

which is again a contradiction. Therefore for $i \in Y$, we have

$$\Big\{\Phi\Big(A_i^r(x) - \ell, \frac{\delta}{2}\Big) \leq (1-\epsilon_1), \Psi\Big(A_i^r(x) - \ell, \frac{\delta}{2}\Big) \geq \epsilon_1 \text{ and } \Upsilon\Big(A_i^r(x) - \ell, \frac{\delta}{2}\Big) \geq \epsilon_1\Big\} \in I.$$

This implies that $i \in D$. Hence, $Y \subset D$. Since $D \in I$, so consequently, the sequence $x = (x_i)$ is neutrosophic Jordan I- Cauchy with respect to norms $\Big\langle \Phi, \Psi, \Upsilon \Big\rangle$.

Conversely, suppose the sequence $x = (x_i) \in \omega$ is neutrosophic Jordan I- Cauchy with respect to the norms $\Big\langle \Phi, \Psi, \Upsilon \Big\rangle$, denoted by E. Let on contrary the sequence $x = (x_i) \in \omega$ is not neutrosophic Jordan I-convergent, denoted by T. Then there exists $i \in \mathbb{N}$ such that

$$E = \bigg\{i \in \mathbb{N} : \Phi\Big(A_i^r(x) - A_k^r(x), \delta\Big) \leq 1 - \epsilon,\ \Psi\Big(A_i^r(x) - A_k^r(x), \delta\Big) \geq \epsilon, \Upsilon\Big(A_i^r(x) - A_k^r(x), \delta\Big) \geq \epsilon\bigg\} \in I$$

Let on contrary,

$$T = \bigg\{i \in \mathbb{N} : \Phi\Big(A_i^r(x) - \ell, \frac{\delta}{2}\Big) > 1 - \epsilon_1,\ \Psi\Big(A_i^r(x) - \ell, \frac{\delta}{2}\Big) < \epsilon_1, \Upsilon\Big(A_j^r(y) - \ell, \frac{\delta}{2}\Big) < \epsilon_1\bigg\} \notin I.$$

This implies that,

$$\begin{aligned}1 - \epsilon &\geq \Phi\Big(A_i^r(x) - A_k^r(x), \delta\Big)\\ &\geq \Phi\Big(A_i^r(x) - \ell, \frac{\delta}{2}\Big) * \Phi\Big(A_k^r(x) - \ell, \frac{\delta}{2}\Big)\\ &> (1-\epsilon_1) * (1-\epsilon_1)\\ &> 1 - \epsilon,\end{aligned}$$

which is a contradiction. Now,

$$\begin{aligned}\epsilon &\leq \Psi\Big(A_i^r(x) - A_k^r(x), \delta\Big)\\ &\leq \Psi\Big(A_i^r(x) - \ell, \frac{\delta}{2}\Big) \bullet \Psi\Big(A_k^r(x) - \ell, \frac{\delta}{2}\Big)\\ &< \epsilon_1 \bullet \epsilon_1\\ &< \epsilon,\end{aligned}$$

which is again a contradiction. And

$$\begin{aligned}\epsilon &\leq \Upsilon\Big(A_i^r(x) - A_k^r(x), \delta\Big)\\ &\leq \Upsilon\Big(A_i^r(x) - \ell, \frac{\delta}{2}\Big) \diamond \Upsilon\Big(A_k^r(x) - \ell, \frac{\delta}{2}\Big)\\ &< \epsilon_1 \bullet \epsilon_1\\ &< \epsilon,\end{aligned}$$

which is again a contradiction. Therefore $T \in I$ and hence $x = (x_i) \in \omega$ is neutrosophic Jordan I-convergent. □

4. Conclusion

We constructed two sequence spaces $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$ and $C^I_{0(\Phi,\Psi,\Upsilon)}(A^r)$ with the help of neutrosophic norm and defined a topology $\tau_{(\Phi,\Psi,\Upsilon)}(A^r)$ on $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$. We find that these two spaces are Hausdorff spaces and first countable with respect to the defined topology. We get an if and only if relation between Jordan I-convergence and Jordan I- Cauchy i.e. if a sequence in $C^I_{(\Phi,\Psi,\Upsilon)}(A^r)$ is Jordan I- convergence then Jordan I-Cauchy and vice versa. Our research opens a path for researchers to work on these two spaces. Researchers can study more topological properties of these two spaces. These two spaces can be defined for double sequences also.

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