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Some properties of s-paratopological groups

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Abstract. A paratopological group G is called an *s-paratopological group* if every sequentially continuous homomorphism from G to a paratopological group is continuous. For every paratopological groups (G,τ), there is an s-coreflection $(G,\tau_{S(G,\tau)})$, which is an s-paratopological group. A characterization of s-coreflection of (G,τ) is obtained, i.e., the topology $\tau_{S(G,\tau)}$ is the finest paratopological group topology on G whose open sets are sequentially open in τ. We prove that the class of Abelian s-paratopological groups is closed with open subgroups. The class of s-paratopological groups being determined by PT-sequences is particularly interesting. We show that this class of paratopological groups is closed with finite product, and give a characterization that two T-sequences define the same paratopological group topology in Abelian groups. The s-sums of Abelian s-paratopological groups are defined. As applications, using s-sums we give characterizations of Abelian s-paratopological groups and Hausdorff Abelian s-paratopological groups, respectively.

1. Introduction

We denote by $\mathbb{N}$ the set of all positive integers, $\mathbb{Z}$ the set of all integers, and $\omega = \{0\} \cup \mathbb{N}$. Readers may consult [1, 9] for notations and terminology not given here. All spaces considered are assumed to be T_1.

A *paratopological group* G is a group endowed with a topology such that the multiplication operation on G is jointly continuous. A *topological group* is a paratopological group G such that the inverse operation on G is continuous. Denote by $\mathcal{N}_G$ the family of open neighborhoods of the unit e_G (briefly, e) of a paratopological group G.

Let $\mathbf{u} = \{u_n\}_{n\in\omega}$ be a non-trivial sequence in a group G. The following very important question has been studied by many authors, such as Graev [14], Nienhuys [19], Protasov and Zelenyuk [21, 25] et al.

Question 1.1. *Is there a group topology τ on G such that $u_n \to e$ in (G,τ)?*

Protasov and Zelenyuk [21] obtained a criterion that gives the complete answer to this question for Abelian groups [21, Theorem 2.1.3] and countable groups [21, Theorem 3.1.4]. Following [21], we say that a sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ in a group G is a *T-sequence* if there is a group topology on G in which $\mathbf{u}$ converges to e.

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Recall that a mapping $f : X \to Y$ between topological spaces X and Y is said to be *sequentially continuous* if $\{f(x_n)\}_{n\in\omega}$ converges to $f(x)$ in Y whenever a sequence $\{x_n\}_{n\in\omega}$ converges to x in X. It is well known that the sequential continuity of a mapping is in general far too weak to imply its continuity. The following important notion was introduced by Noble [20], more results and historical remarks about s-groups can be found in [2, 12, 13, 15, 23, 24] etc.

Definition 1.2. ([20]) A topological group G is called an *s-group* if each sequentially continuous homomorphism from G to a topological group is continuous.

S.S. Gabriyelyan considered the following question, which is a generalisation of Question 1.1.

Question 1.3. ([12]) *Let G be a group and S be a set of sequences in G. Is there a group topology τ on G in which every sequence of S converges to the unit e?*

To answer Question 1.3, S.S. Gabriyelyan defined T_S-set of sequences.

Definition 1.4. ([12]) Let G be a group and S be a set of sequences in G. The set S is called a T_S-set of sequences if there is a group topology on G in which all sequences of S converge to e. The finest group topology with this property is denoted by τ_S.

Many properties are obtained in [12]. Especially, a topological group (G, τ) is an s-group if and only if there is a T_S-set S in G such that $\tau = \tau_S$, and every non-discrete s-group can be described as quotient of Graev free topological group over a sequential Tychonoff space.

By analogy with s-groups, the authors in [8] defined the s-paratopological groups and PT-sets of sequences.

Definition 1.5. ([8]) A paratopological group G is called an *s-paratopological group* if every sequentially continuous homomorphism from G to a paratopological group is continuous.

Definition 1.6. ([8]) Let G be a group and S be a set of sequences in G. The set S is called a *paratopologized set* (briefly, *PT-set*) in G if there is a paratopological group topology on G in which all sequences of S converge to the unit e of G. The finest paratopological group topology on G with this property is denoted by τ_S.

They established that a paratopological group (G, τ) is an s-paratopological group if and only if there is a PT-set S in G such that $\tau = \tau_S$, and G is an s-paratopological group if and only if it is topologically isomorphic to a quotient group of a free paratopological group on a sequential space.

Recently, F. Lin defined PT-sequence in Abelian groups [16].

Definition 1.7. ([16]) A sequence $\{a_n\}_{n\in\omega}$ of elements of group G is called a *PT-sequence* if there is a paratopological group topology on G in which $\{a_n\}_{n\in\omega}$ converges to 0. Denote by $P(G|\{a_n\}_{n\in\omega})$ the group G endowed with the finest paratopological group topology in which $\{a_n\}_{n\in\omega}$ converges to 0. We say that a paratopological group τ on G is determined by a PT-sequence $\{a_n\}_{n\in\omega}$ if $(G, \tau) = P(G|\{a_n\}_{n\in\omega})$.

Note that we can also define PT-sequence in non-Abelian groups which is a PT-set containing a single sequence. If $\mathbf{u}$ is a PT-sequence on a group G, we denote by $\tau_{\mathbf{u}}$ the finest paratopological group topology on G in which $\mathbf{u} = \{a_n\}_{n\in\omega}$ converges to e. It follows from definitions that every T-sequence is a PT-sequence. And if $\mathbf{u}$ is a T-sequence, then $\tau_{\mathbf{u}}$ is Hausdorff. It is clear that, if S is a PT-set, then S' is a PT-set for every non-empty subset S' of S, and every sequence $\mathbf{u} \in S$ is a PT-sequence. Evidently, $\tau_S \subseteq \tau_{S'}$. Also, if S contains only trivial sequences, then S is a PT-set and τ_S is discrete. By definition, $\tau_{\mathbf{u}}$ is finer than τ_S for every $\mathbf{u} \in S$. Thus, if U is open in τ_S, then it is open in $\tau_{\mathbf{u}}$ for every $\mathbf{u} \in S$. So, by definition, we obtain that $\tau_S \subseteq \bigwedge_{\mathbf{u}\in S} \tau_{\mathbf{u}}$, where $\bigwedge_{\mathbf{u}\in S} \tau_{\mathbf{u}}$ denotes the intersection of the topologies $\tau_{\mathbf{u}}$, i.e., U is open in $\bigwedge_{\mathbf{u}\in S} \tau_{\mathbf{u}}$ if and only if $U \in \tau_{\mathbf{u}}$ for every $\mathbf{u} \in S$.

Many properties of Abelian paratopological groups being determined by PT-sequences are obtained [16]. He proved that if G is an Abelian paratopological group, which is endowed with the finest paratopological group topology being determined by a PT-sequence, then (1) G is a sequential non-Fréchet-Urysohn space; and (2) G does not admit a T_1-complementary Hausdorff paratopological group topology on G. The class of countable paratopological groups (not necessary being Abelian) which is determined by a PT-sequence is also discussed in [16].

In this paper, using the methods established in [12, 13] for s-topological groups, we investigate properties of s-paratopological groups, which is also a continuous work of [8] and [16]. Let (G, τ) be a paratopological group, and $S(G, \tau) = \{\mathbf{u} = \{u_n\}_{n\in\omega} : u_n \to e \text{ in } \tau\}$. It is worth noting that $S(G, \tau)$ is a PT-set. We call the paratopological group $(G, \tau_{S(G,\tau)})$ is the *s-coreflection* of (G, τ). In Section 3, some basic properties of s-paratopological groups, which are not considered in [8], are established. A characterization of s-coreflection of $S(G, \tau)$ is obtained, i.e., the topology $\tau_{S(G,\tau)}$ is the finest paratopological group topology on G whose open sets are sequentially open in τ. We also show that the class of s-paratopological groups is closed with open subgroups.

In Section 4, we consider some properties of the class of s-paratopological groups being determined by PT-sequences. We mainly show that this class of s-paratopological groups is closed with finite product, and give a characterization that two T-sequences define the same paratopological group topology in Abelian groups.

In Section 5, the s-sums of Abelian s-paratopological groups are discussed. As applications, using s-sums we give characterizations of Abelian s-paratopological groups and Hausdorff Abelian s-paratopological groups, respectively. More precisely, (G, τ) is an s-paratopological group if and only if every continuous sequence-covering homomorphism from an s-paratopological group onto (G, τ) is quotient; and a Hausdorff paratopological group (G, τ) is an s-paratopological group if and only if (G, τ) is a quotient group of the s-sum of a nonempty family of copies of $(\mathbb{Z}_0^\omega, \tau_{\mathbf{e}})$.

2. Notation and terminology

Let X be a space. For every $P \subseteq X$, the set P is a *sequential neighborhood* of x in X if every sequence converging to x is eventually in P. The set P is a *sequentially open* subset of X if P is a sequential neighborhood of each point in P. The set P is a *sequentially closed* subset of X if $X \setminus P$ is sequentially open. A space X is said to be a *sequential space* [10] if each sequentially open subset is open in X. For each space (X, τ) the *sequential coreflection* [11] of (X, τ), denoted (X, σ_τ) or σX, is given by $U \in \sigma_\tau$ if and only if U is sequentially open in (X, τ). As it is well known, σX is a sequential space [11, p. 52]; also, X and σX have the same convergent sequences [5, p. 678].

The following description of a neighborhood base at the identity of a paratopological group is well known.

Lemma 2.1. *Let G be a paratopological group and $\mathcal{N}$ be a base at the identity e of G. Then the family $\mathcal{N}$ has the following five properties.*

(1) for every $U, V \in \mathcal{N}$, there exists $W \in \mathcal{N}$ with $W \subseteq U \cap V$;

(2) for every $U \in \mathcal{N}$, there exists $V \in \mathcal{N}$ such that $VV \subseteq U$;

(3) for every $U \in \mathcal{N}$ and $g \in U$, there exists $V \in \mathcal{N}$ such that $gV \subseteq U$;

(4) for every $U \in \mathcal{N}$ and $g \in G$, there exists $V \in \mathcal{N}$ such that $gVg^{-1} \subseteq U$;

(5) $\{e\} = \bigcap \mathcal{N}$.

Conversely, if $\mathcal{N}$ is a family of subsets of an abstract group G containing the identity e of G and satisfying (1)-(5), then G admits the unique topology τ that makes it a paratopological group with $\mathcal{N}$ being a base at e.

- For every $n \in \mathbb{N}$, $\mathbb{S}_n$ denotes the group of all permutations on the set $\{0, 1, ..., n-1\}$.

Let G be a group.

- By $\mathcal{F}(G)$ we denote the set of all functions f from $\omega \times G$ into ω which satisfy the condition:

$$f(k, g) < f(k+1, g), \forall k \in \omega, \forall g \in G.$$

- If $\{A_m : m \leq n\}$ is a family of non-empty subsets of G for $n \in \mathbb{N}$. $A_1...A_n$ denotes the set $\{a_1 \cdots a_n : a_m \in A_m, m \leq n\}$.

- Let $\{A_n\}_{n\in\omega}$ be a sequence of non-empty subsets of G. Following [21, Definition 3.1.3], we write

$$SP_{m\leq n}A_m = \bigcup_{\sigma\in\mathbf{S}_{n+1}} A_{\sigma(0)}A_{\sigma(1)}...A_{\sigma(n)}$$

and

$$SP_{n\in\omega}A_n = \bigcup_{n\in\omega} SP_{m\leq n}A_m = \bigcup_{n\in\omega}\bigcup_{\sigma\in\mathbf{S}_{n+1}} A_{\sigma(0)}A_{\sigma(1)}...A_{\sigma(n)}.$$

- If $\{a_n\}_{n\in\omega}$ is a sequence of elements of G. For each $n \in \omega$, put $A_n = \{a_m : m \geq n\}$, $A_n^* = \{a_m : m \geq n\} \cup \{e\}$ and

$$A(k, m) = \{g_0 g_1 \cdots g_k : \ g_0, g_1, ..., g_k \in A_m^*\}.$$

If G is Abelian, for an increasing sequence $0 \leq n_0 < n_1 < \ldots$ one puts

$$\sum_{k\in\omega} A_{n_k} = \bigcup_{k\in\omega}(A_{n_0} + A_{n_1} + \cdots + A_{n_k}).$$

3. Basic properties of *s*-paratopological groups

By categorical methods, B. Batíková and M. Hušek proved that the product of non-sequentially many of *s*-paratopological groups is an s-paratopological group [6, Corollary 14]. In this section, we consider some basic properties of *s*-paratopological groups. We first give an internal characterization of *s*-coreflection of a paratopological group (G, τ).

The following two results will be frequently used.

Lemma 3.1. ([8, Lemma 2.5]) *Let $S = \{S_i : i \in I\}$ be a PT-set of sequences in a group G, where $S_i = \{x_n^i\}_{n\in\omega}$ for each $i \in I$, and let p be a homomorphism from (G, τ_S) to a paratopological group H. Then p is continuous if and only if the sequence $p(S_i) = \{p(x_n^i)\}_{n\in\omega}$ converges to the identity e_H in H for each $i \in I$.*

Theorem 3.2. ([8, Theorem 2.8]) *Let S be a PT-set of sequences in a group G, H be a closed normal subgroup of (G, τ_S) and let π be the natural projection from G onto the quotient group G/H. Then $\pi(S)$ is a PT-set of sequences in G/H and $(G, \tau_S)/H \cong (G/H, \tau_{\pi(S)})$.*

Proposition 3.3. *Let S be a PT-set of sequences in a group G. Then $\tau_S = \tau_{S(G,\tau_S)}$. In particular, if (G, τ) is an s-paratopological group, then $\tau = \tau_{S(G,\tau)}$.*

Proof. Since $S \subseteq S(G, \tau_S)$, it follows from the definition of the topology τ_S that $\tau_S \supseteq \tau_{S(G,\tau_S)}$. Let $id_G : (G, \tau_{S(G,\tau_S)}) \to (G, \tau_S)$ be the identity map. For every $\mathbf{u} = \{u_n\}_{n\in\omega} \in S(G, \tau_S)$ we have that $id_G(u_n) = u_n \to e$ in τ_S. By Lemma 3.1, id_G is continuous. Then $\tau_S \subseteq \tau_{S(G,\tau_S)}$. Thus $\tau_S = \tau_{S(G,\tau_S)}$. □

Lemma 3.4. *Let (G, τ) be a paratopological group. Then*

(1) $S(G, \tau_{S(G,\tau)}) = S(G, \tau)$;

(2) A set U is sequentially open in $\tau_{S(G,\tau)}$ if and only if U is sequentially open in τ, i.e., $\sigma_{\tau_{S(G,\tau)}} = \sigma_\tau$.

Proof. (1) Since $\tau \subseteq \tau_{S(G,\tau)}$, it follows that $S(G, \tau_{S(G,\tau)}) \subseteq S(G, \tau)$. Conversely, if $\mathbf{u} \in S(G, \tau)$, then $\mathbf{u} \in S(G, \tau_{S(G,\tau)})$ by the definition of $\tau_{S(G,\tau)}$. Therefore, $S(G, \tau_{S(G,\tau)}) = S(G, \tau)$.

(2) Since $\tau \subseteq \tau_{S(G,\tau)}$, it suffices to prove that $\sigma_{\tau_{S(G,\tau)}} \subseteq \sigma_\tau$. Suppose that U is sequentially open in $\tau_{S(G,\tau)}$, and a sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ converges to $g \in U$ in τ. Since (G, τ) is a paratopological group, the sequence $g^{-1}\mathbf{u} = \{g^{-1}u_n\}_{n\in\omega}$ converges to e. Thus $g^{-1}\mathbf{u} \in S(G, \tau)$. By (1), $g^{-1}\mathbf{u} \in S(G, \tau_{S(G,\tau)})$. Note that the translation $l_{g^{-1}} : G \to G$ defined by $l_{g^{-1}}(x) = g^{-1}x$ is a homeomorphism. Thus $g^{-1}U$ is also sequentially open in $\tau_{S(G,\tau)}$. Hence there is $n_0 \in \omega$ such that $g^{-1}u_n \in g^{-1}U$ for all $n > n_0$. Therefore, $u_n \in U$ for all $n > n_0$. Hence U is sequentially open in τ, and then $\sigma_{\tau_{S(G,\tau)}} \subseteq \sigma_\tau$. □

Now we can give a characterization of s-coreflection of a paratopological group (G, τ) using sequentially open sets.

Theorem 3.5. *Let (G, τ) be a paratopological group. Then the topology $\tau_{S(G,\tau)}$ is the finest paratopological group topology on G whose open sets are sequentially open in τ.*

Proof. By Lemma 3.4 (2), we have to show only the minimality of $\tau_{S(G,\tau)}$. Let τ' be an arbitrary paratopological group topology on G whose open sets are sequentially open in τ. According to the definition of $\tau_{S(G,\tau)}$, it is enough to prove that any $\mathbf{u} = \{u_n\}_{n\in\omega} \in S(G, \tau)$ converges to the unit in τ'. Assume the converse, then there is an open neighborhood U of the unit in τ' that does not contain infinitely many terms $\{u_{n_k}\}_{k\in\omega}$ of some $\mathbf{u}$. Let $\mathbf{v} = \{u_{n_k}\}_{k\in\omega}$, then $\mathbf{v} \in S(G, \tau)$ and $\mathbf{v} \cap U = \emptyset$. Hence U is not sequentially open in τ, which is a contradiction. Therefore, $\tau' \subseteq \tau_{S(G,\tau)}$. The proof is completed. □

A space X is called a *k-space* [9, p. 152] if, for every $A \subseteq X$, the set A is closed in X if and only if the intersection of A with any compact subspace K of the space X is relatively closed in K.

Proposition 3.6. *Every non-discrete paratopological group (G, τ) without infinite compact subsets is neither an s-paratopological group nor a k-space and $\tau_{S(G,\tau)}$ is discrete.*

Proof. Since every compact subset in (G, τ) is finite, every convergent sequence in (G, τ) is trivial. Thus the topology $\tau_{S(G,\tau)}$ is discrete, and (G, τ) is not an s-paratopological group.

Assuming that (G, τ) is a k-space. Let A be an arbitrary subset of G, then for every compact subset K of (G, τ) the intersection $A \cap K$ is finite and hence closed in K. Since (G, τ) is a k-space, it follows that A is closed in (G, τ). Note that A is arbitrary, hence (G, τ) is discrete, which contradicts the assumption of the proposition. Thus (G, τ) is not a k-space. □

We will assume that all groups are Abelian in the rest of this section.

Theorem 3.7. ([16, Theorem 5.14]) *For each PT-sequence $\{a_n\}_{n\in\omega}$ on any group G, the paratopological group $P(G|\{a_n\}_{n\in\omega})$ is sequential.*

Lemma 3.8. *Let (G, τ) be a paratopological group, then $\sigma_\tau = \bigwedge_{\mathbf{u}\in S(G,\tau)} \tau_{\mathbf{u}}$.*

Proof. Let $U \in \bigwedge_{\mathbf{u}\in S(G,\tau)} \tau_{\mathbf{u}}$. We will show that U is sequentially open in τ. Suppose that a sequence $\{u_n\}_{n\in\omega}$ converges to $g \in U$ in τ. Since (G, τ) is a paratopological group, $g^{-1}u_n \to e \in g^{-1}U$. Therefore, $\mathbf{v} = \{g^{-1}u_n\}_{n\in\omega} \in S(G, \tau)$. It follows that $U \in \tau_{\mathbf{v}}$. Note that$(G, \tau_{\mathbf{v}})$ is a paratopological group. Thus the translation $l_{g^{-1}} : G \to G$ defined by $l_{g^{-1}}(x) = g^{-1}x$ is a homeomorphism. Thus $g^{-1}U$ is also open in $\tau_{\mathbf{v}}$, and then there is $n_0 \in \omega$ such that $g^{-1}u_n \in g^{-1}U$ for all $n > n_0$. Therefore all but finitely many members of $\{u_n\}_{n\in\omega}$ are contained in U, which shows that U is sequentially open in τ.

Conversely, let U be sequentially open in τ. Then U is sequentially open in $\tau_{\mathbf{u}}$ for each $\mathbf{u} \in S(G, \tau)$. In fact, if $v_n \to g \in U$ in $\tau_{\mathbf{u}}$, then $v_n \to g \in U$ in τ. Since U is sequentially open in τ, almost all v_n are contained in U. Thus U is sequentially open in $\tau_{\mathbf{u}}$. By Theorem 3.7, U is open in $\tau_{\mathbf{u}}$. □

It is worth mentioning that the class of all sequential paratopological groups is not stable under finite products [4, Theorem 6], On the other hand, the class of s-paratopological groups is stable under finite products. Thus there is an s-paratopological group which is not sequential. However, we have the following result.

Theorem 3.9. *Let (G, τ) be a paratopological groups. The following statements are equivalent:*

(1) $(G, \tau_{S(G,\tau)})$ *is sequential;*

(2) $\tau_{S(G,\tau)} = \bigwedge_{\mathbf{u}\in S(G,\tau)} \tau_{\mathbf{u}}$*;*

(3) σ_τ *is a paratopological group topology.*

Proof. (1) ⇒ (2). Let $(G, \tau_{S(G,\tau)})$ be sequential. By the definition of sequential spaces and Lemma 3.8, we have that $\tau_{S(G,\tau)} = \sigma_{\tau_{S(G,\tau)}} = \bigwedge_{\mathbf{u}\in S(G,\tau)} \tau_{\mathbf{u}}$.

(2) ⇒ (3). It follows from Lemma 3.8 and the hypothesis that $\sigma_\tau = \tau_{S(G,\tau)}$, which shows that σ_τ is a paratopological group topology.

(3) ⇒ (1). By Theorem 3.5 and the hypothesis, $\tau_{S(G,\tau)} = \sigma_\tau$. According to Lemma 3.6 (2), we have $\sigma_\tau = \sigma_{\tau_{S(G,\tau)}}$. Therefore, $\tau_{S(G,\tau)} = \sigma_{\tau_{S(G,\tau)}}$, which shows that $(G, \tau_{S(G,\tau)})$ is sequential. □

To conclude this section, we show that the class of *s*-paratopological groups is closed with open subgroups.

Theorem 3.10. *Let (G, τ) be a paratopological group, and H be an open subgroup of G. Then (G, τ) is an s-paratopological group if and only if so is H.*

Proof. Assume that (G, τ) is an *s*-paratopological group. Put $S_1 = S(G, \tau), S_2 = S(H, \tau|_H)$, then $\tau_{S_2} \supseteq \tau|_H$. We will show that $\tau_{S_2} \subseteq \tau|_H$. By Lemma 2.1, $\mathcal{N}_H$ satisfies the conditions of Lemma 2.1. Since G is Abelian, $\mathcal{N}_H$ is also satisfies the conditions of Lemma 2.1 in G. Therefore, G admits the unique topology τ_1 that makes it a paratopological group with $\mathcal{N}_H$ being a base at e. For each $\mathbf{u} = \{u_n\}_{n\in\omega} \in S_1$, since H is an open subgroup of G, there is $n_0 \in \omega$ such that $\mathbf{v} = \{u_{n_0+n}\}_{n\in\omega} \in S_2$. Thus $\mathbf{u}$ is convergent in τ_1. Note that (G, τ) is an *s*-paratopological group, we have that $\tau \supseteq \tau_1$. It follows that $\tau_{S_2} = \tau_1|_H \subseteq \tau|_H$. So $\tau_{S_2} = \tau|_H$, which shows that $(H, \tau|_H)$ is an *s*-paratopological group.

Conversely, suppose that $(H, \tau|_H)$ is an *s*-paratopological group. We will show that $\tau_{S_1} \subseteq \tau$. For each $\mathbf{u} \in S_2$, it is clear that $\mathbf{u} \in S_1$. Therefore, $\mathbf{u}$ is convergent in $(H, \tau_{S_1}|_H)$. By hypothesis that $(H, \tau|_H)$ is an *s*-paratopological group, it follows that $\tau|_H = \tau_{S_1}|_H$. Since H is an open subgroup of G, we can conclude that $id : (G, \tau) \to (G, \tau_{S_1})$ is continuous at e. Thus $id_G : (G, \tau) \to (G, \tau_{S_1})$ is continuous. Therefore, $\tau_{S_1} \subseteq \tau$. Hence $\tau = \tau_{S_1}$, which shows that (G, τ) is an *s*-paratopological group. □

However, the following question is unknown.

Question 3.11. *Let (G, τ) be an s-paratopological group. Which closed subgroups of (G, τ) are s-paratopological groups as well?*

4. *s*-paratopological groups determined by *PT*-sequences

In this section, we consider a special class of *s*-paratopological groups, that is the *s*-paratopological groups which are determined by *PT*-sequences. Firstly, We first show that this class of *s*-paratopological groups is closed with finite product. Then we consider the following interesting question in the rest of this section.

Question 4.1. *Let G be an Abelian group. When do two PT-sequences define the same paratopological group topologies on G?*

Note that the corresponding question for groups in fact is formulated in Exercise 2.1.2 of [21]. We give a characterization that two *T*-sequences define the same paratopological group topologies in Abelian groups, which give a partial answer to Question 4.1.

By Lemma 3.2, we have the following corollary.

Corollary 4.2. *Let $\mathbf{u} = \{u_n\}_{n\in\omega}$ be a PT-sequence in a group G, H be a closed normal subgroup of $(G, \tau_{\mathbf{u}})$ and let π be the natural projection from G onto the quotient group G/H. Then $\pi(\mathbf{u})$ is a PT-sequence in G/H and $G/H \cong (G/H, \tau_{\pi(\mathbf{u})})$.*

A criterion for a set to be a *PT*-set in an abstract group was given in [8, Theorem 2.4]. For the case of a *PT*-sequence, that is a *PT*-set of one sequence, by [8, Theorem 2.4] we have the following corollary.

Corollary 4.3. *Let $\mathbf{u} = \{u_n\}_{n\in\omega}$ be a sequence in a group G. Then the following statements (a), (b), and (c) are equivalent.*

(a) The topology $\tau_{\mathbf{u}}$ on G exists;

(b) $\mathbf{u}$ is a PT-sequence in G;

(c) $\bigcap_{f\in\mathcal{F}(G)} SP_{n\in\omega}A_n(f) = \{e\}$.

Moreover, if one of the statements (a), (b) or (c) holds, then the family $\{SP_{n\in\omega}A_n(f) : f \in \mathcal{F}(G)\}$ is a base at the identity e in $(G, \tau_{\mathbf{u}})$.

Theorem 4.4. *Let $\mathbf{u} = \{u_n\}_{n\in\omega}$ and $\mathbf{v} = \{v_n\}_{n\in\omega}$ be PT-sequences in groups G and H, respectively. Set $\mathbf{d} = \{d_n\}_{n\in\omega}$, where $d_{2n+1} = (u_n, e_H)$ and $d_{2n} = (e_G, v_n)$. Then $\mathbf{d}$ is a PT-sequences in $G \times H$ and $\tau_{\mathbf{d}} = \tau_{\mathbf{u}} \times \tau_{\mathbf{v}}$.*

Proof. It is clear that $\mathbf{d}$ converges to the unit in $(G \times H, \tau_{\mathbf{u}} \times \tau_{\mathbf{v}})$. So $\mathbf{d}$ is a *PT*-sequence in $G \times H$ and $\tau_{\mathbf{u}}\times\tau_{\mathbf{v}} \subseteq \tau_{\mathbf{d}}$. To prove that $\tau_{\mathbf{u}}\times\tau_{\mathbf{v}} = \tau_{\mathbf{d}}$, by Corollary 4.3 it is enough to show that every basic neighborhood $W = SP_{n\in\omega}A_n(f)$, $f \in \mathcal{F}(G \times H)$, of the unit in $\tau_{\mathbf{d}}$ contains a set of the form $W_{\mathbf{u}} \times W_{\mathbf{v}}$, where $W_{\mathbf{u}} \in \tau_{\mathbf{u}}$ and $W_{\mathbf{v}} \in \tau_{\mathbf{v}}$.

For each $f \in \mathcal{F}(G\times H)$, put $f^{\mathbf{u}}(k, g) = f(2k, (g, e_H))$, $f^{\mathbf{v}}(k, h) = f(2k+1, (e_G, h))$ for every $k \in \omega, g \in G, h \in H$. Then $f^{\mathbf{u}} \in \mathcal{F}(G)$ and $f^{\mathbf{v}} \in \mathcal{F}(H)$, and

$$\begin{aligned} A^{\mathbf{u}}_{f^{\mathbf{u}}(k,g)} \times \{e_H\} &= \{e_G, u_{f^{\mathbf{u}}(k,g)}, \cdots\} \times \{e_H\} = \{(e_G, e_H), (u_{f^{\mathbf{u}}(k,g)}, e_H), \cdots\} \\ &= \{(e_G, e_H), (u_{f(2k,(g,h))}, e_H), \cdots\} \subseteq A^{\mathbf{d}}_{f(2k,(g,e_H))}, \end{aligned}$$

$$\begin{aligned} \{e_G\} \times A^{\mathbf{v}}_{f^{\mathbf{v}}(k,h)} &= \{e_G\} \times \{e_H, v_{f^{\mathbf{v}}(k,h)}, \cdots\}\times = \{(e_G, e_H), (e_H, v_{f^{\mathbf{v}}(k,g)}), \cdots\} \\ &= \{(e_G, e_H), (e_G, v_{f(2k+1,(e_G,h))}), \cdots\} \subseteq A^{\mathbf{d}}_{f(2k+1,(e_G,h))}. \end{aligned}$$

Thus

$$A_k(f^{\mathbf{u}}) \times \{e_H\} \subseteq A_{2k}(f) \text{ and } \{e_G\} \times A_k(f^{\mathbf{v}}) \times \{e_H\} \subseteq A_{2k+1}(f).$$

For every $n \in \omega$ and $\sigma', \sigma'' \in \mathbb{S}(n + 1)$ put

$$\sigma(k) = 2\sigma'(k) \text{ and } \sigma(n + 1 + k) = 2\sigma''(k) + 1, 0 \le k \le n.$$

Then $\sigma \in \mathbb{S}(2n + 1)$ and

$$\begin{aligned} &(A_{\sigma'(0)}(f^{\mathbf{u}}) \cdots A_{\sigma'(n)}(f^{\mathbf{u}})) \times (A_{\sigma''(0)}(f^{\mathbf{v}}) \cdots A_{\sigma''(n)}(f^{\mathbf{v}})) \\ = \;& (A_{\sigma'(0)}(f^{\mathbf{u}}) \times \{e_G\}) \cdots (A_{\sigma'(n)}(f^{\mathbf{u}}) \times \{e_G\}) \cdot (\{e_G\} \times A_{\sigma''(0)}(f^{\mathbf{v}})) \cdots (\{e_G\} \times A_{\sigma''(0)}(f^{\mathbf{v}})) \\ \subseteq \;& A_{\sigma(0)}(f) \cdots A_{\sigma(n)}(f). \end{aligned}$$

Set $W_{\mathbf{u}} = SP_{n\in\omega}A_n(f^{\mathbf{u}}) \in \tau_{\mathbf{u}}$ and $W_{\mathbf{v}} = SP_{n\in\omega}A_n(f^{\mathbf{v}}) \in \tau_{\mathbf{v}}$. Then

$$\begin{aligned} W_{\mathbf{u}} \times W_{\mathbf{v}} &= \bigcup_{n\in\omega} \bigcup_{\sigma',\sigma''\in\mathbb{S}_{n+1}} (A_{\sigma'(0)}(f^{\mathbf{u}}) \cdots A_{\sigma'(n)}(f^{\mathbf{u}})) \times (A_{\sigma''(0)}(f^{\mathbf{v}}) \cdots A_{\sigma''(n)}(f^{\mathbf{v}})) \\ &\subseteq \bigcup_{n\in\omega} \bigcup_{\sigma\in\mathbb{S}_{2n+1}} A_{\sigma(0)}(f) \cdots A_{\sigma(n)}(f) = W. \end{aligned}$$

Therefore, we can conclude that $\tau_{\mathbf{u}} \times \tau_{\mathbf{v}} = \tau_{\mathbf{d}}$. □

In the rest of this section, all groups are Abelian. By Corollary 4.3, we have the following result, which is also obtained in [16].

Corollary 4.5. *Let $\mathbf{u} = \{u_n\}_{n\in\omega}$ be a sequence in a group G. Then the following statements (a), (b), and (c) are equivalent.*

(a) The topology $\tau_{\mathbf{u}}$ on G exists;

(b) $\mathbf{u}$ is a PT-sequence in G;

(c) $\bigcap\{\sum_{k\in\omega} A_{n_k} : \{n_k\}_{k\in\omega} \subseteq \omega \text{ with } 0 \le n_0 < n_1 < \ldots\} = \{e\}$.

Moreover, if one of the statements (a), (b) or (c) holds, then the family $\{\sum_{k\in\omega} A_{n_k} : \{n_k\}_{k\in\omega} \subseteq \omega \text{ with } 0 \le n_0 < n_1 < \ldots\}$ is a base at the identity 0 in $(G, \tau_{\mathbf{u}})$.

Lemma 4.6. *Let* $\mathbf{u} = \{u_n\}_{n\in\omega}$ *be a T-sequence in G. If the sequence* $\mathbf{v} = \{v_n\}_{n\in\omega}$ *converges to 0 in* $(G, \tau_{\mathbf{u}})$*, then for some* $k, n_0 \in \omega$*,* $v_n \in A(k, 0)$ *holds for all* $n > n_0$*.*

Proof. Assuming the converse, we may assume (after passing to subsequences if required) that $v_n \notin A(n, 0)$ for all $n \in \omega$. We shall construct a neighborhood of 0 in $(G, \tau_{\mathbf{u}})$ of the form $\sum_{m\in\omega} A^*_{n_m}$ not containing any value of the sequence $\mathbf{v} = \{v_n\}_{n\in\omega}$. Let $n_0 = 0$. Since $A^*_{n_0} = A(0, 0)$ and $v_0 \notin A(0, 0)$, $\{v_0\} \cap A^*_{n_0} = \emptyset$. Suppose that $n_0, \ldots, n_k \in \mathbb{N}$ have been chosen so that

$$\{v_0, \ldots, v_k\} \cap (A^*_{n_0} + \cdots + A^*_{n_k}) = \emptyset.$$

Assume that, for every $l \in \mathbb{N}$, we have

$$\{v_0, \ldots, v_k, v_{k+1}\} \cap (A^*_{n_0} + \cdots + A^*_{n_k} + A^*_l) \neq \emptyset.$$

Then we can choose sequences $\{x_l\}_{l\in\omega}$ and $\{y_l\}_{l\in\omega}$ such that $x_l \in A^*_{n_0} + \cdots + A^*_{n_k}$, $y_l \in A^*_l$ and $x_l + y_l \in \{v_0, \ldots, v_k, v_{k+1}\}$ for all $l \in \omega$. On passing to subsequence, without loss of generality, we may assume that the sequence $\{x_l\}_{l\in\omega}$ converges to some x. Since $\{y_l\}_{l\in\omega}$ converges to 0 and $(G, \tau_{\mathbf{u}})$ is a paratopological group, the sequence $\{x_l + y_l\}_{l\in\omega}$ converges to x. Note that $A^*_{n_0} + \cdots + A^*_{n_k}$ is compact and $(G, \tau_{\mathbf{u}})$ is Hausdorff, thus $A^*_{n_0} + \cdots + A^*_{n_k}$ is closed. It follows that $x \in A^*_{n_0} + \cdots + A^*_{n_k}$. Therefore,

$$\{v_0, \ldots, v_k, v_{k+1}\} \cap (A^*_{n_1} + \cdots + A^*_{n_k}) \neq \emptyset.$$

By the choice of $n_0, \ldots, n_k \in \omega$, we have $v_{k+1} \in A^*_{n_1} + \cdots + A^*_{n_k} \subseteq A(k, 0) \subseteq A(k+1, 0)$, which is a contradiction with the assumption $v_n \notin A(n, 0)$ for all $n \in \omega$. Hence there is an $n_{k+1} \in \omega$ such that

$$\{v_0, \ldots, v_k, v_{k+1}\} \cap (A^*_{n_0} + \cdots + A^*_{n_k} + A^*_{n_{k+1}}) = \emptyset.$$

Therefore, by inductive construction, we can choose a neighborhood $\sum_{m\in\omega} A^*_{n_m}$ of 0 such that $\mathbf{v} \cap \sum_{m\in\omega} A^*_{n_m} = \emptyset$, which is a contradiction. □

Lemma 4.7. *Let* $\mathbf{u} = \{u_n\}_{n\in\omega}$ *be a T-sequence in G. Then a sequence* $\mathbf{v} = \{v_n\}_{n\in\omega}$ *converges to 0 in* $(G, \tau_{\mathbf{u}})$ *if and only if there is* $m \in \omega$ *and* $n_0 \in \omega$ *such that for every* $n \geq n_0$ *each member* $v_n \neq 0$ *can be represented in the form*

$$v_n = a^n_1 u_{k^n_1} + \cdots + a^n_{l_n} u_{k^n_{l_n}}, \tag{a}$$

where $k^n_1 < \cdots < k^n_{l_n}$*,* $k^n_1 \to \infty$*,* $a^n_k \in \mathbb{N}$ *for all* $k \in \{1, \ldots, l_n\}$ *and* $a^n_1 + \cdots + a^n_{l_n} \leq m + 1$*.*

Proof. If either $\mathbf{u}$ or $\mathbf{v}$ is trivial, the conclusion is evident. Assume that $\mathbf{u}$ and $\mathbf{v}$ are non-trivial. Since $(G, \tau_{\mathbf{u}})$ is a paratopological group, the sufficiency is clear. We will prove the necessity.

Since the subgroup $\langle \mathbf{u} \rangle$ of G is open in $\tau_{\mathbf{u}}$ and $\mathbf{v}$ converges to 0, there is $n_1 \in \omega$ such that $v_n \in \langle \mathbf{u} \rangle$ for every $n \geq n_1$. By Lemma 4.6, there is $m \in \omega$ and $n_2 \in \omega$ such that $v_n \in A(m, 0)$ for all $n > n_2$. Let $n_0 = \max\{n_1, n_2\}$. So, if $n > n_0$ and $v_n \neq 0$, then

$$v_n = a^n_1 u_{k^n_1} + \cdots + a^n_{l_n} u_{k^n_{l_n}},$$

where $k^n_1 < \cdots < k^n_{l_n}$, and $a^n_1 + \cdots + a^n_{l_n} \leq m + 1$. We can choose a representation of v_n of the form (a) with the minimal value of the sum $a^n_1 + \cdots + a^n_{l_n} \leq m + 1$. For this chosen representation of v_n, every sum of terms of the form $a^n_i u^n_{k_i}$ in (a) is non-zero. Therefore, $a^n_k \in \mathbb{N}$ for all $k \in \{1, \ldots, l_n\}$.

Let us show that $k^n_1 \to \infty$. Assuming the converse and passing to a subsequence we may suppose that $k^n_1 = k_1, a^n_1 = a_1$, and $a^n_1 u_{k^n_1} = a_1 u_{k_1} \neq 0$ for every n. So

$$v_n = a^n_1 u_{k^n_1} + \cdots + a^n_{l_n} u_{k^n_{l_n}} = a_1 u_{k_1} + w^1_n,$$

where $w^1_n = a^n_2 u_{k^n_2} + \cdots + a^n_{l_n} u_{k^n_{l_n}}$. If $k^n_2 \to \infty$, then w^1_n converges to 0. Hence $v_n = a_1 u_{k_1} + w^1_n \to a_1 u_{k_1} \neq 0$. This is impossible. Thus, there is a bounded subsequence of k^n_2. Passing to a subsequence we may suppose that $k^n_2 = k_2, a^n_2 = a_2$, and $a^n_2 u_{k^n_2} = a_2 u_{k_2}$ for every n. So

$$v_n = a_1 u_{k_1} + a_2 u_{k_2} + \cdots + a^n_{l_n} u_{k^n_{l_n}} = a_1 u_{k_1} + a_2 u_{k_2} + w^2_n,$$

where $w_n^2 = a_3^n u_{k_3^n} + \cdots + a_{l_n}^n u_{k_{l_n}^n}$. By hypothesis, $a_1 u_{k_1} + a_2 u_{k_2} \neq 0$. Continuing this process and taking into account that

$$0 < a_1 < a_1 + a_2 < \cdots \leq m + 1,$$

after at most $m + 1$ steps, we see that there is a fixed and non-zero subsequence of $\mathbf{v}$. Thus $v_n \nrightarrow 0$, which is a contradiction. Thus $k_1^n \to \infty$. $\square$

Theorem 4.8. *Let $\mathbf{u} = \{u_n\}$ and $\mathbf{v} = \{v_n\}$ be T-sequences in G. Then $\tau_{\mathbf{u}} = \tau_{\mathbf{v}}$ if and only if there are $m \in \omega$ and $n_0 \in \omega$ such that for every $n \geq n_0$ each $v_n \neq 0$ and $u_n \neq 0$ can be represented in the form*

$$v_n = a_1^n u_{k_1^n} + \cdots + a_{l_n}^n u_{k_{l_n}^n}, \tag{b}$$

$$k_1^n < \cdots < k_{l_n}^n, k_1^n \to \infty, a_k^n \in \mathbb{N} \text{ for all } k \in \{1, \ldots, l_n\}, \text{ and } a_1^n + \cdots + a_{l_n}^n \leq m + 1;$$

$$u_n = b_1^n v_{s_1^n} + \cdots + b_{q_n}^n v_{s_{q_n}^n}, \tag{c}$$

$$s_1^n < \cdots < s_{q_n}^n, s_1^n \to \infty, b_i^n \in \mathbb{N} \text{ for all } i \in \{1, \ldots, q_n\}, \text{ and } b_1^n + \cdots + b_{q_n}^n \leq m + 1.$$

Proof. If $\tau_{\mathbf{u}} = \tau_{\mathbf{v}}$, then $v_n \to 0$ in $\tau_{\mathbf{u}}$. By Lemma 4.7, $\mathbf{v}$ has representation (b) for some $m_1, n_1 \in \omega$. The same is true for the sequence $\mathbf{u}$, i.e., the sequence $\mathbf{u}$ has representation (c) for some $m_2, n_2 \in \omega$. Putting $m = \max\{m_1, m_2\}$ and $n_0 = \max\{n_1, n_2\}$ we obtain (b) and (c).

Conversely, if $v_n \neq 0$ has representation (2), since $(G, \tau_{\mathbf{u}})$ is a paratopological group, we have that $v_n \to 0$ in $\tau_{\mathbf{u}}$. Thus, $\tau_{\mathbf{u}} \subseteq \tau_{\mathbf{v}}$ by the definition of $\mathbf{v}$. Analogously, $\tau_{\mathbf{v}} \subseteq \tau_{\mathbf{u}}$. Hence $\tau_{\mathbf{u}} = \tau_{\mathbf{v}}$. $\square$

Let $\{G_i\}_{i\in I}$ be a family of groups, where I is a non-empty set of indices. The direct sum of G_i is denoted by

$$\bigoplus_{i\in I} G_i = \{(g_i)_{i\in I} \in \prod_{i\in I} G_i : g_i = 0 \text{ for almost all } i\}.$$

We denote by j_k the natural inclusion of G_k into $\bigoplus_{i\in I} G_i$, i.e.

$$j_k(g) = (g_i)_{i\in I} \in \bigoplus_{i\in I} G_i, \quad \text{where } g_i = g \text{ if } i = k \text{ and } g_i = 0 \text{ if } i \neq k.$$

Note that $\bigoplus_{i\in I} G_i$ is the coproduct of the family $\{G_i\}_{i\in I}$ in the category of all Abelian groups.

Let us denote by $\mathbb{Z}_0^\omega$ the derect sum $\bigoplus_\omega \mathbb{Z} \subseteq \mathbb{Z}^\omega$. The sequence $\mathbf{e} = \{e_n\} \subseteq \mathbb{Z}_0^\omega$, where $e_0 = (1, 0, 0, \ldots), e_1 = (0, 1, 0, \ldots), \ldots$, converges to zero in the topology induced on $\mathbb{Z}_0^\omega$ by the product topology on $(\mathbb{Z}_d)^\omega$, where $\mathbb{Z}_d$ is the groups $\mathbb{Z}$ endowed with the discrete topology. Thus $\mathbf{e}$ is a T-sequence, and then a PT-sequence.

Let $f : X \to Y$ be a continuous onto map. f is *sequence-covering* if for each sequence $\{y_n : n \in \omega\}$ in Y converging to a point in Y, there is a sequence $\{x_n : n \in \omega\}$ in X converging to a point in X such that $f(x_n) = y_n$ for all $n \in \omega$ [18].

Theorem 4.9. *Let $\mathbf{u} = \{u_n\}_{n\in\omega}$ be a T-sequence in G such that $G = \langle \mathbf{u} \rangle$. Then $(G, \tau_{\mathbf{u}})$ is a quotient group of $(\mathbb{Z}_0^\omega, \tau_{\mathbf{e}})$ under the sequence-covering homomorphism*

$$\pi((n_0, n_1, \ldots, n_m, 0, \ldots,)) = n_0 u_0 + n_1 u_1 + \ldots + n_m u_m,$$

where $m, n_0, n_1, \ldots, n_m \in \omega$.

Proof. It is clear that π is a surjective homomorphism. Since $\pi(e_n) = u_n \to 0$ in $\tau_{\mathbf{u}}$, π is continuous. By Corollary 4.2, the quotient group $(\mathbb{Z}_0^\omega; \tau_{\mathbf{e}})/\ker\pi$ is topologically isomorphic to $(G, \tau_{\mathbf{u}})$.

Let us show that π is sequence-covering. Since $G = \langle \mathbf{u} \rangle$, each number v_n can be represented in the form

$$v_n = a_1^n u_{k_1^n} + \cdots + a_{l_n}^n u_{k_{l_n}^n}.$$

Let

$$z_n = a_1^n e_{k_1^n} + \cdots + a_{l_n}^n e_{k_{l_n}^n} \text{ if } v_n \neq 0, \text{ and } z_n = 0 \text{ if } v_n = 0.$$

If $\mathbf{v} = \{v_n\}_{n\in\omega} \in S(G, \tau_{\mathbf{u}})$. By Lemma 4.7, there is $m \in \omega$ and $n_0 \in \omega$ such that for every $n \geq n_0$, the representation of $v_n \neq 0$ can be enhanced that $k_1^n < \cdots < k_{l_n}^n$, $k_1^n \to \infty$, $a_k^n \in \mathbb{N}$ for all $k \in \{1, \ldots, l_n\}$ and $a_1^n + \cdots + a_{l_n}^n \leq m + 1$. Then $z_n \to 0$ and $\pi(z_n) = v_n$. This implies π is sequence-covering. □

Two T_1-topologies τ_1 and τ_2 on a set X are called *T_1-complementary* if the intersection $\tau_1 \cap \tau_2$ is the cofinite topology and their supremum is the discrete topology on X [3, 22]. More information of this topic and resent advances can be found in [7, 16, 17]. As mentioned in Introduction, F. Lin proved that if G is an paratopological group, which is endowed with the finest paratopological group topology being determined by a T-sequence, then G does not admit a T_1-complementary Hausdorff paratopological group topology on G. Thus the following question is natural.

Question 4.10. *Is there a Hausdorff s-paratopological group G admiting a T_1-complementary Hausdorff paratopological group topology on G?*

5. The s-sum of s-paratopological groups

All groups considered in this section are assumed to be Abelian. We aim to define the s-sum of s-paratopological groups, and then give a characterization of s-paratopological groups using s-sums.

Proposition 5.1. *Assume that $G_i = (G_i, \tau_i)$ is a family of paratopological groups. For every $i \in I$ fix $U_i \in \mathcal{U}_{G_i}$ and put*

$$\bigoplus_{i\in I} U_i = \{(g_i)_{i\in I} \in \bigoplus_{i\in I} G_i : g_i \in U_i \text{ for all } i \in I\}.$$

Then the sets of the form $\bigoplus_{i\in I} U_i$, where $U_i \in \mathcal{U}_{G_i}$ for every $i \in I$, form a neighborhood basis at the unit of a paratopological group topology $\mathcal{T}_b$ on $\bigoplus_{i\in I} G_i$.

Let $\mathbf{u} = \{g_n\}_{n\in\omega}$ be an arbitrary sequence in $S(G_i, \tau_i)$. Evidently, the sequence $j_i(\mathbf{u})$ converges to the unit in $\mathcal{T}_b$. Thus, the set $\bigcup_{i\in I} j_i(S(G_i, \tau_i))$ is a PT-set of sequences in $\bigoplus_{i\in I} G_i$. If (G_i, τ_i) is an s-paratopological group for all $i \in I$, we can define the s-sum of G_i.

Definition 5.2. Let $\{(G_i, \tau_i)\}_{i\in I}$ be a non-empty family of s-paratopological groups. The group $\bigoplus_{i\in I} G_i$ endowed with the finest paratopological group topology $\mathcal{T}_s$ in which every sequence of $\bigcup_{i\in I} j_i(S(G_i, \tau_i))$ converges to zero is called the s-sum of G_i, and it is denoted by $\bigoplus_{i\in I}^{(s)} G_i$.

Proposition 5.3. *Let $\{(G_i, \tau_i)\}_{i\in I}$ be a non-empty family of s-paratopological groups. Set $S = \bigcup_{i\in I} j_i(S(G_i, \tau_i))$ and $G = \bigoplus_{i\in I}^{(s)} G_i$. The topology τ_S on G coincides with the finest paratopological group topology τ' on G for which all inclusions j_i are continuous.*

Proof. Fix $i \in I$. By construction, for every $\{u_n\}_{n\in\omega} \in S(G_i, \tau_i)$, $j_i(u_n) \to e_G$ in τ_S. By Theorem 3.1 , the inclusion j_i is continuous. Thus $\tau_S \subseteq \tau'$. Conversely, if j_i is continuous with respect to τ', then $j_i(S(G_i, \tau_i)) \subseteq S(G, \tau')$. Therefore, $S \subseteq S(G, \tau')$ and $\tau' \subseteq \tau_S$ by the definition of τ_S. □

Theorem 5.4. *Let (X, τ) be an s-paratopological group. Set $I = S(X, \tau)$. For every $\mathbf{u} \in I$, let $p_{\mathbf{u}} : (\langle \mathbf{u} \rangle, \tau_{\mathbf{u}}) \to X, p_{\mathbf{u}}(g) = g$, be the natural inclusion of $(\langle \mathbf{u} \rangle, \tau_{\mathbf{u}})$ into X. Then the natural homomorphism*

$$p : \bigoplus_{\mathbf{u}\in I}^{(s)} (\langle \mathbf{u} \rangle, \tau_{\mathbf{u}}) \to X, \quad p((x_{\mathbf{u}})) = \sum_{\mathbf{u}\in I} p_{\mathbf{u}}(x_{\mathbf{u}}) = \sum_{\mathbf{u}} x_{\mathbf{u}},$$

is a quotient sequence-covering map.

Proof. Let

$$G = \bigoplus_{\mathbf{u}\in I}^{(s)} (\langle \mathbf{u} \rangle, \tau_{\mathbf{u}}), \qquad S = \bigcup_{\mathbf{u}\in I} j_i(S(\langle \mathbf{u} \rangle, \tau_{\mathbf{u}})).$$

Since each element of X can be regarded as the first element of some sequence $\mathbf{u} \in I$, p is surjective. By construction, p is sequence-covering.

Let $\mathbf{v} = \{v_n\}_{n\in\omega} \in S$. By construction, $p(v_n) = v_n \to 0$ in τ. According to Lemma 3.1, p is continuous. Let $H = \ker p$. By Theorem 3.2, $G/H \cong (X, \tau_{p(S)})$. Since $p(S) = S(X, \tau)$, we obtain $G/H \cong (X, \tau)$. Thus p is quotient. □

Theorem 5.5. *Let (X, τ) be a paratopological group. The following statements are equivalent:*
(1) (X, τ) is an s-paratopological group;
(2) every continuous sequence-covering homomorphism from an s-paratopological group onto (X, τ) is quotient.

Proof. Let $I = S(X, \tau)$. For every $\mathbf{u} \in I$, put $X_{\mathbf{u}} = (\langle \mathbf{u} \rangle, \tau_{\mathbf{u}})$ and let $p_{\mathbf{u}} : (\langle \mathbf{u} \rangle, \tau_{\mathbf{u}}) \to X, p_{\mathbf{u}}(g) = g$, be the natural inclusion of $X_{\mathbf{u}}$ into X.

(1) ⇒ (2) Let $p : G \to X$ be a sequence-covering continuous homomorphism from an s-paratopological group (G, ν) onto X. Set $H = \ker p$. Since p is surjective, by Theorem 3.2, we have $G/H \cong (X, \tau_{p(S)})$. Note that p is a sequence-covering mapping, Proposition 3.3, $p(S(G, \nu)) = S(X, \tau)$ and $\tau = \tau_{S(X,\tau)}$. Thus $G/H \cong X$.

(2) ⇒ (1) Let $G = \bigoplus_{\mathbf{u}\in I}^{(s)} X_{\mathbf{u}}$ and

$$p : G \to X, \;\; p((x_{\mathbf{u}})) = \sum_{\mathbf{u}} p_{\mathbf{u}}(x_{\mathbf{u}}) = \sum_{\mathbf{u}} x_{\mathbf{u}}.$$

By Theorem 5.4, p is continuous and sequence-covering. By hypothesis, p is quotient. Thus $(X, \tau) \cong G/\ker p$. According to Theorem 3.2, we have that $G/\ker p \cong (X, \tau_{\pi(S)})$. Thus $\tau = \tau_{p(S)}$, and (X, τ) is an s-paratopological group. □

Proposition 5.6. *Let $\{(X_i, \nu_i)\}_{i\in I}$ and $\{(G_i, \tau_i)\}_{i\in I}$ be non-empty families of s-paratop-ological groups and let $\pi_i : G_i \to X_i$ be a quotient sequence-covering map for every $i \in I$. Set $X = \bigoplus_{i\in I}^{(s)} X_i$, $G = \bigoplus_{i\in I}^{(s)} G_i$ and $\pi : G \to X, \pi((g_i)) = (\pi_i(g_i))$. Then π is a quotient mapping.*

Proof. It is clear that π is surjective. Let

$$S_X = \bigcup_{i\in I} j_i(S(X_i, \nu_i)) \;\text{ and }\; S_G = \bigcup_{i\in I} j_i(S(G_i, \tau_i)).$$

Since π_i is sequence-covering, we have $\pi_i(S(G_i, \tau_i)) = S(X_i, \nu_i)$. Hence $\pi(S_G) = S_X$. By Lemma 3.1, π is continuous. By Theorem 3.2, $G/\ker\pi \cong (X, \tau_{\pi(S_G)})$. Since X is an s-paratopological group, $G/\ker\pi \cong X$ and π is quotient. □

For Hausdorff paratopological groups, we have the following result.

Theorem 5.7. *Let (X, τ) be a Hausdorff paratopological group. The following statements are equivalent:*
(1) (X, τ) is an s-paratopological group;
(2) (X, τ) is a quotient group of the s-sum of a nonempty family of copies of $(\mathbb{Z}_0^{\omega}, \tau_{\mathbf{e}})$.

Proof. By definition of s-sum and Theorem 3.2, it is clear that (2) implies (1). We will show that (1) implies (2).

For every $\mathbf{u} \in I = S(X, \tau)$, put $G_{\mathbf{u}} = (\mathbb{Z}_0^{\mathbb{N}}, \tau_{\mathbf{e}})$, and let $\pi_{\mathbf{u}}$ be the unique group homomorphism from $G_{\mathbf{u}}$ onto $X_{\mathbf{u}}$ defined by $\pi_{\mathbf{u}}(e_i) = u_i$ for every $i \in \omega$. Since (X, τ) is a Hausdorff paratopological group, for every $\mathbf{u} \in S(X, \tau)$, $(X, \tau_{\mathbf{u}})$ is Hausdorff. By [16, Theorem 5.3], $\mathbf{u}$ is a T-sequence. Therefore, each PT-sequence in $S(X, \tau)$ is a T-sequence. Then the result immediately follows from Theorems 4.9 and 5.4 and Proposition 5.6. □

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