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More on generalizations of topology of uniform convergence and m-topology on $C(X)$

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Abstract. This paper conglomerates our findings on the space $C(X)$ of all real valued continuous functions, under different generalizations of the topology of uniform convergence and the m-topology. The paper begins with answering all the questions which were left open in our previous paper on the classifications of Z-ideals of $C(X)$ induced by the U_I and the m_I-topologies on $C(X)$ [5]. Motivated by the definition of the m^I-topology, another generalization of the topology of uniform convergence, called U^I-topology, is introduced here. Among several other results, it is established that for a convex ideal I in $C(X)$, a necessary and sufficient condition for U^I-topology to coincide with m^I-topology on $C(X)$ is the boundedness of $X \setminus \bigcap Z[I]$ in X. As opposed to the case of the U_I-topologies (and m_I-topologies) on $C(X)$, it is proved that each U^I-topology (respectively, m^I-topology) on $C(X)$ is uniquely determined by the ideal I. In the last section, the denseness of the set of units of $C(X)$ in $C_U(X)$ (= $C(X)$ with the topology of uniform convergence) is shown to be equivalent to the strong zero dimensionality of the space X. Also, the space X turns out to be a weakly P-space if and only if the set of zero divisors (including 0) in $C(X)$ is closed in $C_U(X)$. Computing the closure of $C_{\mathscr{P}}(X)$ (=$\{f \in C(X)$: the support of $f \in \mathscr{P}\}$ where $\mathscr{P}$ is an ideal of closed sets in X) in $C_U(X)$ and $C_m(X)$ (= $C(X)$ with the m-topology), the results $cl_U C_{\mathscr{P}}(X) = C_\infty^{\mathscr{P}}(X)$ (= $\{f \in C(X) : \forall n \in \mathbb{N}, \{x \in X : |f(x)| \geq \frac{1}{n}\} \in \mathscr{P}\}$) and $cl_m C_{\mathscr{P}}(X) = \{f \in C(X) : f.g \in C_\infty^{\mathscr{P}}(X)$ for each $g \in C(X)\}$ are achieved.

1. Introduction

In the entire article X designates a completely regular Hausdorff space. As is well known $C(X)$ stands for the ring of real valued continuous functions on X. Suppose $C^*(X) = \{f \in C(X) : f$ is bounded on $X\}$. If for $f \in C(X)$ and $\epsilon > 0$ in $\mathbb{R}$, $U(f, \epsilon) = \{g \in C(X) : \sup_{x \in X} |f(x) - g(x)| < \epsilon\}$, then the family $\{U(f, \epsilon) : f \in C(X), \epsilon > 0\}$ turns out to be an open base for the so-called topology of uniform convergence or in brief the U-topology on $C(X)$. Several experts have studied U-topology on $C(X)$, from various points of view. One can look at the articles [7, 10, 12] for a glimpse of some relevant facts about this topology. A generalization of this U-topology on $C(X)$ via a kind of ideal in $C(X)$, viz a Z-ideal I in $C(X)$, is already studied only recently [5].

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Incidentally the collection $\{U_I(f,\epsilon) : f \in C(X), \epsilon > 0\}$ constitutes an open base for this generalized U-topology, named as the U_I-topology on $C(X)$. Here $U_I(f,\epsilon) = \{g \in C(X) :$ there exists $Z \in Z[I] \equiv \{Z(h) : h \in I\}$ such that $\sup_{x\in Z} |f(x) - g(x)| < \epsilon\}$, $Z(h)$ standing for the zero set of the function h. It is worth mentioning in this context that an analogous type of topology, viz the m_I-topology, on $C(X)$ is introduced and investigated in some detail in [3]. Here I is a Z-ideal in $C(X)$ and a typical basic open neighborhood of $f \in C(X)$ in this topology looks like: $m_I(f,u) = \{g \in C(X) : |f(x) - g(x)| < u(x)$ for all $x \in Z$ for some $Z \in Z[I]\}$, here $u \in C(X)$ and is strictly positive on some $Z_0 \in Z[I]$. With the special choice $I = (0)$, the m_I-topology and U_I-topology reduce respectively to the well-known m-topology and U-topology on $C(X)$ [2M, 2N [7]]. In Section 4 of the article [5], two classifications of Z-ideals in $C(X)$ induced by U_I-topologies and also by the m_I-topologies are defined. To be more specific binary relations '$\sim$' and '$\approx$' on the set $\mathcal{I}$ of all Z-ideals in $C(X)$ are introduced as follows: for $I, J \in \mathcal{I}$, $I \sim J$ if U_I-topology $= U_J$-topology and $I \approx J$ if m_I-topology $= m_J$-topology. For $I \in \mathcal{I}$, we set $[I] = \{J \in \mathcal{I} : U_I$-topology $= U_J$-topology$\}$ and $[[I]] = \{J \in \mathcal{I} : m_I$-topology $= m_J$-topology$\}$. It is established in [5, Theorem 4.1, Theorem 4.4], that each equivalence class $[I]$ has a largest member and analogously all the equivalence classes $[[I]]$ also have largest members [5, Theorem 4.13, Theorem 4.20]. It is further realized that some of these equivalence classes (in both these classifications of Z-ideals in $C(X)$) have smallest members too [5, Theorem 4.10, Theorem 4.21].

In Section 2 of the present article we prove that each equivalence class $[I]$ and $[[I]]$ has a smallest member, thereby answering the questions 4.26 and 4.27 asked in [5] affirmatively. Again it was established in [5] that if X is a P-space, then each equivalence class $[I]$ and $[[I]]$ degenerates into singleton in [5, Theorem 4.12, Theorem 4.23] and hence $\sim$ and $\approx$ are identical equivalence relations on $\mathcal{I}$. In this article we check that, regardless of whether or not X is a P-space, $\sim$ and $\approx$ are indeed identical equivalence relations on $\mathcal{I}$, the set of all Z-ideals on $C(X)$. This answers negatively the question 4.25 asked in [5].

In Section 3 of this article we introduce yet another generalization of U-topology on $C(X)$, this time via an ideal I of $C(X)$ [not necessarily a Z-ideal nor even a proper ideal] but with a slightly different technique. Essentially for $f \in C(X)$ and $\epsilon > 0$, we set $\widetilde{B}(f, I, \epsilon) = \{g \in C(X) : \sup_{x\in X} |f(x) - g(x)| < \epsilon$ and $f - g \in I\}$. Then it needs a few routine computation to show that the family $\{\widetilde{B}(f, I, \epsilon) : f \in C(X), \epsilon > 0\}$ makes an open base for some topology on $C(X)$, which we designate by the U^I-topology on $C(X)$. It is not at all hard to check that $C(X)$ with this U^I-topology is an additive topological group. The U-topology on $C(X)$ is a special case of the U^I-topology with $I = C(X)$. Let us mention at this point that an analogous kind of topology, viz the m^I-topology on $C(X)$, is initiated and studied in some details in [4]. A typical basic open neighborhood of $f \in C(X)$ for this latter topology is a set of the form $\{g \in C(X) : |f(x) - g(x)| < u(x)$ for all $x \in X$ and $f - g \in I\}$, here u is a positive unit in $C(X)$. $C(X)$ with the m^I-topology is a topological ring as is proved in [4]. For notational convenience, we let $C_{U^I}(X)$ to stand for $C(X)$ equipped with the U^I-topology. Analogously $C_{m^I}(X)$ designates $C(X)$ with the m^I-topology. In general the U^I-topology on $C(X)$ is weaker than the m^I-topology. Incidentally it is proved [vide Theorem 3.7] that if I is a convex ideal in $C(X)$ (in particular I may be a Z-ideal in $C(X)$), then U^I-topology $= m^I$-topology if and only if $X \setminus \bigcap Z[I]$ is a bounded subset of X. We observe that $I \cap C^*(X)$ is a clopen set in the space $C_{U^I}(X)$ [Theorem 3.9(2)]. We use this fact to show that $I \cap C^*(X)$ is indeed the component of 0 in $C_{U^I}(X)$ [Theorem 3.12]. We recall that a topological space Y is called homogeneous if given any two points $p, q \in Y$, there exists a homoeomorphism $\phi : Y \to Y$ such that $\phi(p) = q$. A topological group is a natural example of a homogeneous space. It follows that $C_{U^I}(X)$ is either locally compact or nowhere locally compact, indeed the latter condition holds when and only when $X \setminus \bigcap Z[I]$ is a finite set [Theorem 3.16] [Compare with Theorem 4.2 in [4]]. As in the space $C_{m^I}(X)$, ideals in $C(X)$ are never compact in $C_{U^I}(X)$ [Theorem 3.22(1)] and the ideals contained in the ring $C_\psi(X)$ of all real valued continuous functions with pseudocompact support are the only candidates for Lindelöf ideals in $C_{U^I}(X)$ [Theorem 3.22(2)]. In [5], it is seen that a whole bunch of Z-ideals I in $C(X)$, can give rise to identical U_I-topologies (respectively identical m_I-topologies). In contrast we observe in the present article that U^I-topologies on $C(X)$ (respectively m^I-topologies on $C(X)$) are uniquely determined by the ideal I in $C(X)$ [Theorem 3.1].

In Section 4 of the present article on specializing $I = C(X)$ and therefore writing $C_U(X)$ instead of $C_{U^I}(X)$, we achieve characterizations of two known classes of topological spaces X viz strongly zero-dimensional spaces and pseudocompact weakly P-spaces in terms of the behavior of two chosen subsets $U(X)$ and $D(X)$ of the ring $C(X)$, in the space $C_U(X)$ [Theorem 4.2, Theorem 4.3]. Here $U(X)$ stands for the set of all units in $C(X)$ and $D(X)$, the collection of all zero-divisors in $C(X)$, including 0. We further observe that the closure of the ideal $C_K(X)$ of all real valued continuous functions with compact support in the space $C_U(X)$ is precisely the set $\{f \in C(X) : f^*(\beta X \setminus X) = \{0\}\}$, here $f^* : \beta X \to \mathbb{R} \cup \{\infty\}$ is the well known Stone-extension of the function f. This leads to the fact that the closure of $C_K(X)$ in $C_U(X)$ is the familiar ring $C_\infty(X)$ of all functions in $C(X)$ which vanish at infinity [Remark 4.9]. We would like to point out at this moment, that the same proposition is very much there in the celebrated monograph [13, Theorem 3.17] but with the additional hypothesis that X is locally compact. We also prove that the closure of the ideal $C_\psi(X)$ of all functions with pseudocompact support in the space $C_U(X)$ equals to the set $\{f \in C(X) : f^*(\beta X \setminus \upsilon X) = \{0\}\}$ [Theorem 4.11]. This ultimately leads to the proposition that the closure of $C_\psi(X)$ in $C_U(X)$ is the ring $C_\infty^\psi(X) = \{f \in C(X) : \forall n \in \mathbb{N}, \{x \in X : |f(x)| \geq \frac{1}{n}\}$ is pseudocompact$\}$. This last ring is called the pseudocompact analogue of the ring $C_\infty(X)$ and is initiated in [1]. The closure of $C_K(X)$ is $C_\infty(X)$ and that of $C_\psi(X)$ is $C_\infty^\psi(X)$ (in the space $C_U(X)$). These two apparently distinct facts are put on a common setting in view of the following result, which we establish subsequently in this article. If $\mathscr{P}$ is an ideal of closed sets in X, in the sense that whenever $E, F \in \mathscr{P}$, then $E \cup F \in \mathscr{P}$ and $E \in \mathscr{P}$ and C, a closed set in X with $C \subset E$ implies that $C \in \mathscr{P}$, then set $C_\mathscr{P}(X) = \{f \in C(X) :$ the support of $f \in \mathscr{P}\}$ and $C_\infty^\mathscr{P}(X) = \{f \in C(X) : \forall n \in \mathbb{N}, \{x \in X : |f(x)| \geq \frac{1}{n}\} \in \mathscr{P}\}$. It is proved that the closure of $C_\mathscr{P}(X)$ in $C_U(X)$ is $C_\infty^\mathscr{P}(X)$ [Theorem 4.13(2)]. Incidentally we establish a formula for the closure of $C_\mathscr{P}(X)$ in the space $C(X)$ equipped with m-topology. In fact we prove that the closure of $C_\mathscr{P}(X)$ in the m-topology $\equiv cl_m C_\mathscr{P}(X) = \{f \in C(X) : f.g \in C_\infty^\mathscr{P}(X)$ for each $g \in C(X)\}$, Theorem 4.14(3). With the special choice $\mathscr{P} \equiv$ the ideal of all compact sets in X this formula reads $cl_m C_K(X) = \bigcap_{p \in \beta X - X} M^p$. This last result is precisely Proposition 5.6 in [4]. We conclude this article with a characterization of pseudocompact spaces via denseness of ideal $C_\mathscr{P}(X)$ in $C_\infty^\mathscr{P}(X)$ in the m-topology.

2. Answer to a few open problems concerning U_I-topologies and m_I-topologies on $C(X)$

At the very outset we need to explain a few notations. For each point $p \in \beta X$, $M^p = \{f \in C(X) : p \in cl_{\beta X} Z(f)\}$, which is a maximal ideal in $C(X)$ and $O^p = \{f \in C(X) : cl_{\beta X} Z(f)$ is a neighborhood of p in the space $\beta X\}$, a well-known Z-ideal in $C(X)$. For each subset A of βX, we prefer to write M^A instead of $\bigcap_{p \in A} M^p$. Analogously we write $O^A = \bigcap_{p \in A} O^p$. We reproduce the following results from [5], to make the paper self-contained.

Theorem 2.1. ([5, Theorem 4.1]) *If A is a closed subset of βX, then* $[M^A] = \{I \in \mathcal{I} : O^A \subseteq I \subseteq M^A\}$.

Theorem 2.2. ([5, Theorem 4.2]) *For any closed subset A of βX,* $[[M^A]] = \{I \in \mathcal{I} : O^A \subseteq I \subseteq M^A\}$.

We are going to establish a generalized version of each of the last two Theorems. We need the following subsidiary fact for that purpose. The proof is straightforward.

Theorem 2.3. *For any subset A of βX, $M^A = M^{\overline{A}}$, here $\overline{A} = cl_{\beta X} A$*

Theorem 2.1 (respectively Theorem 2.2) in conjunction with Theorem 2.3 yields the following two theorems almost immediately:

Theorem 2.4. *For any subset A of βX,* $[M^A] = \{I \in \mathcal{I} : O^{\overline{A}} \subseteq I \subseteq M^A\}$.

Theorem 2.5. *If $A \subset \beta X$, then* $[[M^A]] = \{I \in \mathcal{I} : O^{\overline{A}} \subseteq I \subseteq M^A\}$.

We want to recall at this moment that given a Z-ideal I in $C(X)$, there always exists a set of maximal ideals $\{M^p : p \in A\}$ each containing I, A being a suitable subset of βX for which we can write: $[I] = [M^A] = [[I]]$ [This is proved in Theorem 4.9 and Theorem 4.20 in [5]]. In view of this fact, we can make the following comments:

Remark 2.6. Each equivalence class $[I]$ in the quotient set $\mathcal{I}/_{\sim}$ has a largest as well as a smallest member. This answers question 4.26 raised in [5] affirmatively.

Remark 2.7. Each equivalence class $[[I]]$ in the quotient set $\mathcal{I}/_{\approx}$ has a largest as well as a smallest member [This answers question 4.27 in [5]].

Remark 2.8. For each Z-ideal I in $C(X)$, $[I] = [[I]]$. Essentially this means that $\sim$ and $\approx$ are two identical binary relations on $\mathcal{I}$ [This answers question 4.25 in [5] negatively].

3. U^I-topologies versus m^I-topologies on $C(X)$

We begin with the following simple result which states that the assignment: $I \to U^I$ is a one-one map.

Theorem 3.1. *Suppose I and J are two distinct ideals in $C(X)$. Then U^I-topology is different from U^J-topology.*

Proof. Without loss of generality, we can choose a function $g \in I \setminus J$ such that $|g(x)| < 1$ for each $x \in X$. Clearly $\widetilde{B}(g, J, 1)$ is an open set in the U^J-topology. We assert that this set is not open in the U^I-topology. If possible, let $\widetilde{B}(g, J, 1)$ be open in the U^I-topology. Then there exists $\epsilon > 0$ in $\mathbb{R}$ such that $\widetilde{B}(g, I, \epsilon) \subseteq \widetilde{B}(g, J, 1)$. Since $g + \frac{\epsilon}{2}g \in \widetilde{B}(g, I, \epsilon)$, this implies that $g + \frac{\epsilon}{2}g \in \widetilde{B}(g, J, 1)$. It follows that $g + \frac{\epsilon}{2}g - g \in J$, i.e., $\frac{\epsilon}{2}g \in J$, a contradiction to the initial choice that $g \notin J$. □

Remark 3.2. A careful modification in the above chain of arguments yields that $\widetilde{B}(g, J, 1)$, which is an open set in the U^J-topology (and therefore open in the m^J-topology) is not open in the m^I-topology. Therefore, we can say that whenever I and J are distinct ideals in $C(X)$, it is the case that m^I-topology is different from m^J-topology.

Like any homogeneous space, $C_{U^I}(X)$ (respectively $C_{m^I}(X)$) is either devoid of any isolated point or all the points of this space are isolated. The following theorem clarifies the situation.

Theorem 3.3. *The following three statements are equivalent for an ideal I in $C(X)$:*

1. *$C_{U^I}(X)$ is a discrete space.*
2. *$C_{m^I}(X)$ is a discrete space.*
3. *$I = (0)$.*

Proof. If $I = (0)$, then for each $f \in C(X)$, $\widetilde{B}(f, I, 1) = \{f\}$ and therefore each point of $C_{U^I}(X)$ (and $C_{m^I}(X)$) is isolated. This settles the implication (3) $\implies$ (1) and (3) $\implies$ (2). (1) $\implies$ (2) is trivial because m^I-topology is finer than the U^I-topology. Suppose (3) is false, i.e., $I \neq (0)$. Then the Remark 3.2 and the implication (3) $\implies$ (2) imply that m^I-topology is different from the discrete topology. Thus (2) $\implies$ (3). □

It is a standard result in the study of function spaces that $C_U(X)$ is a topological vector space if and only if X is pseudocompact [2*M*6, [7]]. The following fact is a minor improvement of this result.

Theorem 3.4. *For an ideal I of $C(X)$, $C_{U^I}(X)$ is a topological vector space if and only if $I = C(X)$ and X is pseudocompact.*

Proof. If $I = C(X)$, then $C_{U^I}(X) = C_U(X)$, which is a topological vector space if X is pseudocompact as observed above. Conversely let $C_{U^I}(X)$ be a topological vector space and $f \in C(X)$. Then there exists $\epsilon > 0$ in $\mathbb{R}$ such that $(-\epsilon, \epsilon) \times \widetilde{B}(f, I, \epsilon) \subseteq \widetilde{B}(0, I, 1)$. This implies that $\frac{\epsilon}{2}f \in I$ and hence $f \in I$. Thus $I = C(X)$. Clearly then U^I-topology on $C(X)$ reduces to the U-topology. We can therefore say that $C_U(X)$ is a topological vector space. In view of the observations made above, it follows that X is a pseudocompact space. □

The following proposition gives a set of conditions in which each implies the next.

Theorem 3.5.

1. *The U^I-topology = the m^I-topology on $C(X)$.*
2. *$C_{U^I}(X)$ is a topological ring.*
3. *$I \subset C^*(X)$.*
4. *$I \cap C^*(X) = I \cap C_\psi(X)$.*

Proof.

1. $\implies$ 2. is trivial because $C_{m^I}(X)$ is a topological ring.
2. $\implies$ 3. Suppose (2) holds but $I \not\subset C^*(X)$. Choose $f \in I$ such that $f \notin C^*(X)$. Since the product function

$$\begin{aligned} C_{U^I}(X) \times C_{U^I}(X) &\to C_{U^I}(X) \\ (g,h) &\mapsto g.h \end{aligned}$$

is continuous at the point $(0, f)$, we get an $\epsilon > 0$ such that $\widetilde{B}(0,I,\epsilon) \times \widetilde{B}(f,I,\epsilon) \subseteq \widetilde{B}(0,I,1)$. Let $g = \frac{\epsilon.f}{2(1+|f|)}$. Then $g \in \widetilde{B}(0,I,\epsilon)$, this implies that $g.f \in \widetilde{B}(0,I,1)$ and hence $g(x).f(x) < 1$ for each $x \in X$ i.e., for each $x \in X$, $\frac{\epsilon.f^2(x)}{2(1+|f(x)|)} < 1$. Now since f is an unbounded function on X, $|f(x_n)| \to \infty$ along a sequence $\{x_n\}_n$ in X. Consequently $\lim_{n\to\infty} \frac{|f(x_n)|}{1+|f(x_n)|} = \lim_{n\to\infty}[1 - \frac{1}{1+|f(x_n)|}] = 1$ and therefore there exists $k \in \mathbb{N}$ such that for all $n \geq k$, $\frac{|f(x_n)|}{1+|f(x_n)|} \geq \frac{3}{4}$. This implies that for each $n \geq k$, $\frac{\epsilon.|f(x_n)|}{2}.\frac{3}{4} \leq \frac{\epsilon.|f(x_n)|}{2}.\frac{|f(x_n)|}{1+|f(x_n)|} < 1$ and so $\{|f(x_n)| : n \geq k\}$ becomes a bounded sequence in $\mathbb{R}$. This contradicts that $|f(x_n)| \to \infty$ as $n \to \infty$. Hence $I \subset C^*(X)$.

3. $\implies$ 4. Suppose (3) holds. We need to show that $I \cap C^*(X) \subset I \cap C_\psi(X)$ (because $C_\psi(X) \subseteq C^*(X)$). Since $C_\psi(X)$ is the largest bounded ideal in $C(X)$ [Theorem 3.8, [3]]. The condition (3) implies that $I \subset C_\psi(X)$. Hence $I \cap C^*(X) = I = I \cap C_\psi(X)$. □

The statement (4) may not imply the statement (1) in Theorem 3.5. Consider the following example:

Example 3.6. Take $X = \mathbb{N}$, $I = C_K(\mathbb{N})$. Then U^I-topology on $C(\mathbb{N}) \subsetneq m^I$-topology on $C(\mathbb{N})$.

Proof of this claim: First observe that $C_K(\mathbb{N}) \subset C^*(\mathbb{N})$. Now recall the function $j \in C^*(\mathbb{N})$ given by $j(n) = \frac{1}{n}$, $n \in \mathbb{N}$. Then $\widetilde{B}(0,I,j)$ is an open set in $C(\mathbb{N})$ with m^I-topology. We assert that this set is not open in $C(\mathbb{N})$ with U^I-topology. Suppose otherwise, then there exists $\epsilon > 0$ such that $0 \in \widetilde{B}(0,I,\epsilon) \subset \widetilde{B}(0,I,j)$. Now there exists $k \in \mathbb{N}$ such that $\frac{2}{k} < \epsilon$. Let $f(n) = \begin{cases} \frac{\epsilon}{2} & \text{when } n \leq k \\ 0 & \text{otherwise} \end{cases}$, then $f \in \widetilde{B}(0,I,\epsilon)$. But $f \notin \widetilde{B}(0,I,j)$

$Y \subset X$ is called a relatively pseudocompact or bounded subset of X if for every $f \in C(X)$, $f(Y)$ is a bounded subset of $\mathbb{R}$. The previous Theorem is a special case of the more general Theorem, given below:

Theorem 3.7. *For a convex ideal I of $C(X)$, U^I-topology = m^I-topology if and only if $X \setminus \bigcap Z[I]$ is a bounded subset of X.*

Proof. First let $X \setminus \bigcap Z[I]$ be bounded and $\widetilde{B}(f,I,u)$ be an open set in m^I-topology, where $f \in C(X)$ and u is a positive unit in $C(X)$. Now $\frac{1}{u}$ is bounded on $X \setminus \bigcap Z[I]$, i.e., there exists $\lambda > 0$ such that $\frac{1}{u(x)} < \lambda$ for all $x \in X \setminus \bigcap Z[I] \implies u(x) > \frac{1}{\lambda}$ for all $x \in X \setminus \bigcap Z[I]$. We claim that $\widetilde{B}(f,I,\frac{1}{\lambda}) \subset \widetilde{B}(f,I,u)$. Consider $g \in \widetilde{B}(f,I,\frac{1}{\lambda})$. Then $|g - f| < \frac{1}{\lambda}$ and $g - f \in I$. Now for all $x \in \bigcap Z[I]$, $(g-f)(x) = 0 < u(x)$ and for all $x \in X \setminus \bigcap Z[I]$, $|g(x) - f(x)| < \frac{1}{\lambda} < u(x)$, i.e., $|g - f| < u$ on X. So $g \in \widetilde{B}(f,I,u)$. For the converse part, suppose $X \setminus \bigcap Z[I]$ is not a bounded subset of X. Then there exists a positive unit u in $C(X)$ and a C-embedded copy of $\mathbb{N} \subset X \setminus \bigcap Z[I]$ on which $u \to 0$. Clearly $\widetilde{B}(0,I,u)$ is an open set in $C_{m^I}(X)$. We claim that $\widetilde{B}(0,I,u)$ is not open in the U^I-topology. If possible, let there exist $\epsilon > 0$ such that $0 \in \widetilde{B}(0,I,\epsilon) \subset \widetilde{B}(0,I,u)$. Since $u(n) \to 0$ as $n \to \infty$ for $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $u(k) < \frac{\epsilon}{2}$. As $\mathbb{N} \subset X \setminus \bigcap Z[I]$, there exists an $f(\geq 0) \in I$ such that $f(k) > 0$. Since $\mathbb{N}$ is C-embedded in X, there exists $h(\geq 0) \in C(X)$ such that $h(k) = \frac{\epsilon}{2f(k)}$. Let $g = f.h$. Then $g \in I$ and $g(k) = \frac{\epsilon}{2}$. Set $g' = g \wedge \frac{\epsilon}{2}$. Then $g' \leq g \implies g' \in I$, as I is convex. Also $g' \leq \frac{\epsilon}{2} \implies g' \in \widetilde{B}(0,I,\epsilon)$ which further implies that $g' \in \widetilde{B}(0,I,u) \implies g' < u$. But $g'(k) = \frac{\epsilon}{2} > u(k)$, a contradiction. □

Remark 3.8. With the special choice $I = C(X)$, the above Theorem reads: The U-topology = m-topology on $C(X)$ if and only if X is pseudocompact. This is a standard result in the theory of rings of continuous function [see 2M6 and 2N [7]].

It is proved in [4], Proposition 2.2 that if I is an ideal in $C(X)$ then any ideal J containing I is clopen in $C_{m^I}(X)$ and also $C^*(X) \cap I$ is clopen in $C_{m^I}(X)$. These two facts can be deduced from the following proposition, because the m^I-topology is finer than the U^I-topology.

Theorem 3.9.

1. *If J is any additive subgroup of $(C(X), +, .)$ containing the ideal I, then J is a clopen subset of $C_{U^I}(X)$.*
2. *For any ideal I in $C(X)$, $I \cap C^*(X)$ is a clopen subset of $C_{U^I}(X)$.*

Proof. 1. Let $f \in J$. Then $f \in \widetilde{B}(f, I, 1) \subset J$, because $g \in \widetilde{B}(f, I, 1) \implies g - f \in I \subset J \implies g = f + (g - f) \in J$. Thus J becomes open in $C_{U^I}(X)$. To prove that J is also closed in this space let $f \notin J$, $f \in C(X)$. Then it is not at all hard to check that $\widetilde{B}(f, I, 1) \cap J = \emptyset$ and hence J is closed in $C_{U^I}(X)$.

2. For any $f \in I \cap C^*(X)$, it is routine to check that $f \in \widetilde{B}(f, I, 1) \subset I \cap C^*(X)$. Then $I \cap C^*(X)$ is open in $C_{U^I}(X)$. To settle the closeness of $I \cap C^*(X)$ in $C_{U^I}(X)$, we need to verify that for any $f \in C(X) \setminus (I \cap C^*(X))$, $\widetilde{B}(f, I, 1) \cap I \cap C^*(X) = \emptyset$ and that verification is also routine. □

Before proceeding further we recall for any $f \in C(X)$ the map

$$\begin{aligned} \phi_f : \mathbb{R} &\to C(X) \\ r &\mapsto r.f \end{aligned}$$

already introduced in [3], [4].

Lemma 3.10. *Let I be an ideal in $C(X)$. Then for $f \in C(X)$,*

$$\begin{aligned} \phi_f : \mathbb{R} &\to C_{U^I}(X) \\ r &\mapsto r.f \end{aligned}$$

is a continuous map if and only if $f \in I \cap C^*(X)$ [compare with an analogous fact in the m^I-topology: Lemma 3.1 in [4]].

Proof. First assume that ϕ_f is continuous, in particular at the point 0. So there exists $\delta > 0$ in $\mathbb{R}$ such that $\phi_f(-\delta, \delta) \subseteq \widetilde{B}(\phi_f(0), I, 1) = \widetilde{B}(0, I, 1)$. This implies that $\phi_f(\frac{\delta}{2}) \in \widetilde{B}(0, I, 1)$ and hence $|\frac{\delta}{2} f| < 1$ and $\frac{\delta}{2} f \in I$. Clearly then $f \in C^*(X) \cap I$. Conversely let $f \in C^*(X) \cap I$. Then $|f| < M$ on X for some $M > 0$ in $\mathbb{R}$. Choose $r \in \mathbb{R}$ and $\epsilon > 0$ arbitrarily. Then it is not at all hard to check that $\phi_f(r - \frac{\epsilon}{M}, r + \frac{\epsilon}{M}) \subseteq \widetilde{B}(\phi_f(r), I, \epsilon)$. Then ϕ_f is continuous at r. □

Corollary 3.11. *For $f \in C(X)$,*

$$\begin{aligned} \phi_f : \mathbb{R} &\to C_U(X) \\ r &\mapsto r.f \end{aligned}$$

is continuous if and only if $f \in C^*(X)$.

Theorem 3.12. *The component of 0 in $C_{U^I}(X)$ is $I \cap C^*(X)$.*

Proof. It follows from Lemma 3.10 that $I \cap C^*(X) = \bigcup_{f \in I \cap C^*(X)} \phi_f(\mathbb{R})$, a connected subset of $C_{U^I}(X)$. Since $I \cap C^*(X)$ is a clopen set in $C_{U^I}(X)$ (Theorem 3.9(2)), it is the case that $I \cap C^*(X)$ is the largest connected subset of $C_{U^I}(X)$ containing 0. Hence $I \cap C^*(X)$ is the component of 0 in $C_{U^I}(X)$. □

Corollary 3.13. *$C^*(X)$ is the component of 0 in $C_U(X)$*

To find out when does the space $C_{U^I}(X)$ become locally compact, we reproduce the Lemma 4.1(a) from the article [4]:

Lemma 3.14. *For any positive unit u in $C(X)$ and for a finite subset $\{a_1, a_2, ..., a_k\}$ of $X \setminus \bigcap Z[I]$, for each $i \in \{1, 2, ..., k\}$, there exists $t_i \in I$ such that $|t_i| < u$, $t_i(a_i) = \frac{1}{2}u(a_i)$ and $t_i(a_j) = 0$ for $j \neq i$.*

We will need the following special version of this Lemma.

Lemma 3.15. *Suppose $\epsilon > 0$ and $\{a_1, a_2, ..., a_n\}$ is a finite subset of $X \setminus \bigcap Z[I]$. Then for each $i \in \{1, 2, ..., n\}$, there exists $t_i \in I$ such that $|t_i| < \epsilon$, $t_i(a_i) = \frac{1}{2}\epsilon$ and $t_i(a_j) = 0$ for all $j \neq i$.*

Theorem 3.16. *For an ideal I in $C(X)$, the following three statements are equivalent:*

1. *$C_{U^I}(X)$ is nowhere locally compact.*
2. *$C_{m^I}(X)$ is nowhere locally compact.*
3. *$X \setminus \bigcap Z[I]$ is an infinite set.*

Proof. The equivalence of the statements (2) and (3) is precisely Theorem 4.2 in [4]. So we shall establish the equivalence of (1) and (3). The proof for this later equivalence will be a close adaption of the proof of Theorem 4.2 in [4]. However we shall make a sketch of this proof in order to make the paper self contained. First assume that $X \setminus \bigcap Z[I]$ is an infinite set. If possible, let K be a compact subset of $C_{U^I}(X)$ with non-empty interior. Then there exists $f \in C(X)$ and $\epsilon > 0$ in $\mathbb{R}$ such that $\widetilde{B}(f, I, \epsilon) \subseteq K$. The compactness of K in $C_{U^I}(X)$ implies that $K \subseteq \bigcup_{i=1}^{n} \widetilde{B}(g_i, I, \frac{\epsilon}{4})$ for a suitable finite subset $\{g_1, g_2, ..., g_n\}$ of K. Since $X \setminus \bigcap Z[I]$ is an infinite set, we can pick up $(n+1)$-many distinct members $\{a_1, a_2, ..., a_{n+1}\}$ from this set. On using Lemma 3.15, we can find out for each $i \in \{1, 2, ..., n+1\}$, a function $t_i \in I$ such that $|t_i| < \epsilon$, $t_i(a_i) = \frac{\epsilon}{2}$ and $t_i(a_j) = 0$ if $j \neq i$, $j \in \{1, 2, ..., n+1\}$. Set $k_i = f + t_i, i = 1, 2, ..., n+1$. Then for each $i = 1, 2, ..., n+1, k_i \in \widetilde{B}(f, I, \epsilon) \subset K \subseteq \bigcup_{i=1}^{n} \widetilde{B}(g_i, I, \frac{\epsilon}{4})$, so there exist distinct $p, q \in \{1, 2, ..., n+1\}$ for which k_p and k_q lie in $\widetilde{B}(g_i, I, \frac{\epsilon}{4})$ for some $i \in \{1, 2, ..., n\}$. This implies that $|k_p - k_q| < \frac{\epsilon}{2}$, while $|k_p(a_p) - k_q(a_p)| = \frac{\epsilon}{2}$, a contradiction. Thus (3) $\implies$ (1) is established. If $X \setminus \bigcap Z[I]$ is a finite set, say the set $\{b_1, b_2, ..., b_k\}$, then by proceeding analogously as in the proof of Lemma 4.1(b) in [4], we can easily show that $\mathbb{R}^k$ is homeomorphic to the subspace I of the space $C_{U^I}(X)$. From Theorem 3.9(1), we get that I is an open subspace of $C_{U^I}(X)$. Hence the space $C_{U^I}(X)$ becomes locally compact at each point on I. Consequently $C_{U^I}(X)$ is locally compact at each point on X (Mind that $C_{U^I}(X)$ is a homogeneous space). □

Corollary 3.17. *$C_U(X)$ is nowhere locally compact if and only if $C_m(X)$ is nowhere locally compact if and only if X is an infinite set.*

A sufficient condition for the nowhere local compactness of $C_{U^I}(X)$ is given as follows:

Theorem 3.18. *If $I \not\subset C^*(X)$, then $C_{U^I}(X)$ is nowhere locally compact* [*compare with an analogous fact concerning $C_{m^I}(X)$ in Corollary* 4.4 *[4]*].

Proof. It is clear that $I \not\subset C^*(X) \implies X \setminus \bigcap Z[I]$ is an infinite set. It follows from Theorem 3.16 that $C_{U^I}(X)$ is nowhere locally compact. □

The following simple example shows that the converse of the last statement is not true.

Example 3.19. Take $X = \mathbb{R}$ and $I = C_K(\mathbb{R})$. Then $I \subset C^*(X)$, but $\bigcap_{f \in C_K(\mathbb{R})} Z(f) = \emptyset$ and therefore $\mathbb{R} \setminus \bigcap Z[I] = \mathbb{R} =$ an infinite set. Hence from Theorem 3.16, $C_{U^I}(\mathbb{R})$ is nowhere locally compact, though $I \subset C^*(\mathbb{R})$.

For an essential ideal I in $C(X)$ [I is called an essential ideal in $C(X)$ if $I \neq (0)$ and every non-zero ideal in $C(X)$ cuts I non-trivially], the following fact is a simple characterization of nowhere local compactness of $C_{U^I}(X)$.

Theorem 3.20. *Let I be an essential ideal in $C(X)$. Then $C_{U^I}(X)$ is nowhere locally compact if and only if X is an infinite set.*

Proof. For the essential ideal I in $C(X)$, $\bigcap Z[I]$ is nowhere dense [Proposition 2.1, [2]] and hence $cl_X(X \setminus \bigcap Z[I]) = X$. The desired result follows on using Theorem 3.16 in a straightforward manner. □

We would like to point out at this moment that it is mentioned in [3] [the proof of the implication relation $(f) \implies (b)$ in Proposition 3.14] and also in [4] (the statement lying between Corollary 3.5 and Corollary 3.6) that whenever $C_\psi(X) \neq \{0\}$, then it is an essential ideal in $C(X)$. The following counterexample shows that there exists a non-zero $C_\psi(X)$ in $C(X)$, which is not an essential ideal in $C(X)$.

Example 3.21. Consider the following subspace of $\mathbb{R} : X = \{0\} \cup \{x \in \mathbb{R} : x$ is rational and $1 \leq x \leq 2\}$. Then X is locally compact at the point 0 and therefore $C_K(X) \neq \{0\}$, because for a space Y, $C_K(Y)$ is $\{0\}$ if and only if Y is nowhere locally compact [This follows on adapting the arguments in $4D2$ [7], more generally for a nowhere locally compact space Y instead of $\mathbb{Q}$ only]. Since X is a metrizable space, there is no difference between compact and pseudocompact subsets of X. Hence $C_\psi(X) = C_K(X) \neq \{0\}$. It is clear that if $f \in C_K(X)$, then f vanishes at each point on $X \setminus \{0\}$. Consequently $\bigcap_{f \in C_K(X)} Z(f) = [1,2] \cap \mathbb{Q}$, which being a non-empty clopen set in the space X is not nowhere dense. Hence on using Proposition 2.1 in [2], $C_K(X)$ is not an essential ideal in $C(X)$.

It is proved in Proposition 4.6 in [4] that a non-zero ideal I in $C(X)$ is never compact in $C_{m^I}(X)$ and if such an I is Lindelöf, then $I \subseteq C_\psi(X)$. These two facts can be deduced from the following proposition, because the m^I-topology is finer than the U^I-topology.

Theorem 3.22. *Let J be a non-zero ideal in $C(X)$. Then:*

1. *J is not compact in $C_{U^I}(X)$.*
2. *If J is Lindelöf in $C_{U^I}(X)$, then $J \subseteq C_\psi(X)$.*

We omit the proof of this Theorem, because this can be done on closely following the arguments for the proof of Proposition 4.6 in [4].

4. A few special properties for the spaces $C_U(X)$ and $C_m(X)$

If $U(X)$ is dense in $C_m(X)$, then it is plain that $U(X)$ is dense in $C_U(X)$, because the U-topology on $C(X)$ is weaker than the m-topology. We are going to show that the converse of this statement is true. We recall in this context that a space X is strongly zero-dimensional if given a pair of completely separated sets K and W in X, there exists a clopen set C' such that $K \subseteq C' \subseteq X \setminus W$. Equivalently X is strongly zero-dimensional if and only if given a pair of disjoint zero-sets Z and Z' in X, there exists a clopen set C in X such that $Z \subseteq C \subseteq X \setminus Z'$. The following lemma gives a sufficient condition for the strongly zero-dimensionality of X.

Lemma 4.1. *Let $U(X)$ be dense in $C_U(X)$. Then X is strongly zero-dimensional.*

Proof. Let Z_1, Z_2 be disjoint zero-sets in X. Then there exists $f \in C(X)$ such that $|f| \leq 1$, $f(Z_1) = \{-1\}$ and $f(Z_2) = \{1\}$. Since $U(X)$ is dense in X, we can find out a member $u \in \widetilde{B}(f, \frac{1}{2}) \cap U(X)$. Let $C = \{x \in X : u(x) < 0\}$. Then C is a clopen set in X, $Z_1 \subseteq C \subseteq X \setminus Z_2$. Thus X becomes strongly zero-dimensional. □

Theorem 4.2. *The following statements are equivalent for a space X.*

1. *X is strongly zero-dimensional.*

2. *$U(X)$ is dense in $C_U(X)$.*
3. *$U(X)$ is dense in $C_m(X)$.*

Proof. The equivalence of (1) and (3) is precisely the Proposition 5.1 in [4]. This combined with Lemma 4.1 finishes the proof. □

Let $U^*(X) = \{u \in C(X) : |u| > \lambda \text{ for some } \lambda > 0\}$.

Theorem 4.3. *$cl_U D(X)$($\equiv$ the closure of $D(X)$ in the space $C_U(X)$) $= C(X) \setminus U^*(X)$ [compare with the fact: $cl_m D(X) = C(X) \setminus U(X)$ in Proposition 5.2 in [4]].*

Proof. It is easy to check that $U^*(X)$ is open in $C_U(X)$ because choosing $u \in C^*(X)$, we have $|u| > \lambda$ for some $\lambda > 0$, this implies that $\widetilde{B}(u, \frac{\lambda}{2}) \subseteq U^*(X)$ (We are simply writing $\widetilde{B}(u, \frac{\lambda}{2})$ instead of $\widetilde{B}(u, C(X), \frac{\lambda}{2})$). Since $D(X) \cap U(X) = \emptyset$, in particular $D(X) \cap U^*(X) = \emptyset$, it follows therefore that $cl_U D(X) \subseteq C(X) \setminus U^*(X)$. To prove the reverse inclusion relation, let $f \in C(X) \setminus U^*(X)$ and $\epsilon > 0$ be preassigned. We need to show that $\widetilde{B}(f, \epsilon) \cap D(X) \neq \emptyset$. For that purpose define as in the proof of Proposition 5.2 in [4].

$$h(x) = \begin{cases} f(x) + \frac{\epsilon}{2} & \text{if } f(x) \leq -\frac{\epsilon}{2} \\ 0 & \text{if } |f(x)| \leq \frac{\epsilon}{2} \\ f(x) - \frac{\epsilon}{2} & \text{if } f(x) \geq \frac{\epsilon}{2} \end{cases}$$

Then $h \in C(X)$. Since $f \notin U^*(X)$, f takes values arbitrarily near to zero on X. Therefore there exists $x \in X$ for which $|f(x)| < \frac{\epsilon}{2}$. This implies that $int_X Z(h) \neq \emptyset$. Thus $h \in D(X)$ and surely $|h - f| < \epsilon$. Therefore $h \in \widetilde{B}(f, \epsilon) \cap D(X)$. □

Definition 4.4. We call a space X, a weakly P-space if whenever $f \in C(X)$ is such that f takes values arbitrarily near to zero, then f vanishes on some neighborhood of a point in X, i.e., $int_X Z(f) \neq \emptyset$.

It is clear that every weakly P-space is an almost P-space and is pseudocompact. The following proposition is a characterization of weakly P-spaces.

Theorem 4.5. *X is a weakly P-space if and only if $D(X)$ is closed in $C_U(X)$ [Compare with the Proposition 5.2 in [4]].*

Proof. Let X be a weakly P-space. This means that if $f \in C(X)$ is not a zero-divisor, then it is bounded away from zero, i.e., $f \in U^*(X)$. Thus $C(X) \setminus D(X) \subseteq U^*(X)$. The implication relation $U^*(X) \subseteq C(X) \setminus D(X)$ is trivial. Therefore $C(X) \setminus D(X) = U^*(X)$ and thus $D(X) = C(X) \setminus U^*(X)$. It follows from Theorem 4.3 that $D(X)$ is closed in $C_U(X)$. Conversely let $D(X)$ be closed in $C_U(X)$. Then this implies by Theorem 4.3 that $D(X) = C(X) \setminus U^*(X)$. Now let $f \in C(X)$ be such that f takes values arbitrarily near to zero. We need to show that $int_X Z(f) \neq \emptyset$. If possible, let $int_X Z(f) = \emptyset$. Then $f \notin D(X)$ and hence $f \in U^*(X)$, a contradiction. □

The next proposition shows that weakly P-spaces are special kind of almost P-spaces.

Theorem 4.6. *X is a weakly P-space if and only if it is pseudocompact and almost P.*

Proof. It is already settled that a weakly P-space is pseudocompact and almost P. Conversely let X be pseudocompact and almost P. Suppose $f \in C(X)$ takes values arbitrarily near to zero on X. Then f must attain the value 0 at some point on X because X is pseudocompact. Thus $Z(f) \neq \emptyset$ and hence due to the almost P property of X, we shall have $int_X Z(f) \neq \emptyset$. Therefore X becomes weakly P. □

Remark 4.7. $D(X)$ is closed in $C_U(X)$ if and only if X is a pseudocompact almost P-space.

There are enough examples of pseudocompact almost P-spaces. Indeed, if X is a locally compact realcompact space, then $\beta X \setminus X$ is a compact almost P-space [Lemma 3.1, [6]].

In what follows we compute the closure of a few related ideals in the ring $C(X)$.

Theorem 4.8. $cl_U C_K(X)$ ($\equiv$ *the closure of* $C_K(X)$ *in the space* $C_U(X)$) $= \{f \in C(X) : f^*(\beta X \setminus X) = \{0\}\}$.

Proof. Set for each $p \in \beta X$, $\widetilde{M}^p = \{f \in C(X) : f^*(p) = 0\}$. Since $f \in M^p \implies p \in cl_{\beta X} Z(f)$ (Gelfand-Kolmogoroff Theorem) $\implies f^*(p) = 0$, it follows that $M^p \subseteq \widetilde{M}^p$ for each $p \in \beta X$. Furthermore, $\widetilde{M}^p = \{f \in C(X) : |M^p(f)| =$ 0 or infinitely small in the residue class field $C(X)/M^p\}$ [Theorem 7.6(b), [7]]. It is well-known [vide [11], Lemma 2.1] that $cl_U M^p = \{f \in C(X) : |M^p(f)| = 0$ or infinitely small$\}$. Hence we get that $M^p \subseteq cl_U M^p = \widetilde{M}^p$ for each $p \in \beta X$. Therefore $C_K(X) = \bigcap_{p \in \beta X \setminus X} O^p$ [7E [7]] $\subseteq \bigcap_{p \in \beta X \setminus X} M^p \subseteq \bigcap_{p \in \beta X \setminus X} \widetilde{M}^p =$ the intersection of a family of closed sets in $C_U(X) \equiv$ a closed set in $C_U(X)$. This implies that $cl_U C_K(X) \subseteq \bigcap_{p \in \beta X \setminus X} \widetilde{M}^p = \{f \in C(X) : f^*(\beta X \setminus X) =$ 0$\}$. To prove the reverse inclusion relation, let $f \in \bigcap_{p \in \beta X \setminus X} \widetilde{M}^p$. Thus $f^*(\beta X \setminus X) = \{0\}$. Consequently then f becomes bounded on X, for if f is unbounded on X, then there exists a copy of $\mathbb{N}$, C-embedded in X for which $\lim_{n \to \infty} |f(x)| = \infty$. Surely then $cl_{\beta X}\mathbb{N} = \beta\mathbb{N}$ and so $cl_{\beta X}\mathbb{N} \setminus \upsilon X \supseteq \beta\mathbb{N} \setminus \mathbb{N}$ [We use the fact that a countable C-embedded subset of a Tychonoff space is a closed subset of it 3B3 [7]]. Choose a point $p \in \beta\mathbb{N} \setminus \mathbb{N}$, it is clear that $f^*(p) = \infty$, a contradiction. Thus $f \in C^*(X)$ and we can write $f^\beta(\beta X \setminus X) = \{0\}$, here $f^\beta : \beta X \to \mathbb{R}$ is the Stone-extension of $f \in C^*(X)$. So $\beta X \setminus X \subseteq Z_{\beta X}(f^\beta)$, the zero set of f^β in the space βX. Choose $\epsilon > 0$. We claim that $\widetilde{B}(f, \epsilon) \cap C_K(X) \neq \emptyset$ and we are done.

Proof of the claim: Define a function $h : X \to \mathbb{R}$ as follows

$$h(x) = \begin{cases} f(x) + \frac{\epsilon}{2} & \text{if } f(x) \leq -\frac{\epsilon}{2} \\ 0 & \text{if } -\frac{\epsilon}{2} \leq f(x) \leq \frac{\epsilon}{2} \\ f(x) - \frac{\epsilon}{2} & \text{if } f(x) \geq \frac{\epsilon}{2} \end{cases}$$

Then $h \in C^*(X)$ and $|h(x) - f(x)| < \epsilon$ for each $x \in X$, i.e., $h \in \widetilde{B}(f, \epsilon)$. To complete this theorem, it remains only to check that $h \in C_K(X)$. Indeed let $g = (|f| \wedge \frac{\epsilon}{2}) - \frac{\epsilon}{2}$. Then $Z(f) \subseteq X \setminus Z(g)$ and $X \setminus Z(g) \subseteq Z(h)$ and hence $g.h = 0$. Since the map

$$\begin{aligned} C^*(X) &\to C(\beta X) \\ k &\mapsto k^p \end{aligned}$$

is a lattice isomorphism, from the definition of g, we can at once write: $g^\beta = (|f|^\beta \wedge \frac{\epsilon}{2}) - \frac{\epsilon}{2}$ and $g^\beta.h^\beta = 0$. Consequently then, $\beta X \setminus Z_{\beta X}(g^\beta) \subseteq Z(h^\beta)$ and also, $Z_{\beta X}(g^\beta) \subseteq \beta X \setminus Z_{\beta X}(f^\beta)$. This shows that $Z(h^\beta)$ is a neighborhood of $\beta X \setminus X$. It follows from 7E [7] that $h \in C_K(X)$. □

Remark 4.9. It is a standard result in the theory of rings of continuous functions that the complete list of free maximal ideals in $C^*(X)$ is given by $\{M^{*p} : p \in \beta X \setminus X\}$, where $M^{*p} = \{h \in C^*(X) : h^\beta(p) = 0\}$ [Theorem 7.2, [7]]. It is also well-known that [vide 7F1, [7]], $\bigcap_{p \in \beta X \setminus X} M^{*p} = C_\infty(X)$. Hence we can ultimately write $cl_U C_K(X) = C_\infty(X)$.

Remark 4.10. We can show that for a well chosen collection of naturally existing spaces, $C_K(X)$ is not dense in $C_\infty(X)$ in the m-topology on $C(X)$. Indeed let X be a locally compact, σ-compact non compact space (say $X = \mathbb{R}^n, n \in \mathbb{N}$). Since every σ-compact space is realcompact, it follows from Theorem 8.19 in [7] that $C_K(X) = \bigcap_{p \in \beta X \setminus X} M^p$. Incidentally it is proved in Proposition 5.6 in [4] that $cl_m C_K(X)$($\equiv$ the closure of $C_K(X)$ in the space $C_m(X)$) $= \bigcap_{p \in \beta X \setminus X} M^p$. Thus $C_K(X)$ is closed in $C_m(X)$. On the other hand, it follows from 7F3 [7] that with the above mentioned condition on X, the intersection of all free maximal ideals in $C(X) \subsetneq$ the intersection of all free maximal ideals in $C^*(X)$. Therefore $C_K(X) \subsetneq C_\infty(X)$ and hence $C_K(X)$ is not dense in $C_\infty(X)$ in the space $C(X)$ in the m-topology. Towards the end of this paper, we find the closure of $C_K(X)$ in $C_m(X)$.

Theorem 4.11. *$cl_U C_\psi(X)$ (≡ the closure of $C_\psi(X)$ in $C_U(X)$) $= \bigcap_{p\in\beta X\setminus \upsilon X} \widetilde{M}^p = \{f \in C(X) : f^*(\beta X \setminus \upsilon X) = \{0\}\}$.*

Proof. We shall follow closely the technique adopted to prove Theorem 4.8. First recall the well-known fact: $C_\psi(X) = \bigcap_{p\in\beta X\setminus \upsilon X} M^p$, Theorem 3.1 [8]. It follows on adapting the chain of arguments in the first part of the proof of Theorem 4.8 that $cl_U C_\psi(X) \subseteq \bigcap_{p\in\beta X\setminus \upsilon X} \widetilde{M}^p = \{f \in C(X) : f^*(\beta X \setminus \upsilon X) = \{0\}\}$. To prove the reverse inclusion relation, choose $f \in C(X)$ such that $f^*(\beta X \setminus \upsilon X) = \{0\}$, then it is not at all hard to prove that f is bounded on X and therefore we can rewrite as in the proof of Theorem 4.8 that $cl_U C_\psi(X) \subseteq \{f \in C^*(X) : f^\beta(\beta X \setminus \upsilon X) = \{0\}\}$ and hence $\beta X \setminus \upsilon X \subseteq Z_{\beta X}(f^\beta)$. Next choosing $\epsilon > 0$ and proceeding exactly as in the proof of Theorem 4.8, thereby defining the bounded continuous function $h : X \to \mathbb{R}$ verbatim, we can easily check that $h \in \widetilde{B}(f, \epsilon)$. In the next stage we set as in the proof of Theorem 4.8, $g = (|f| \wedge \frac{\epsilon}{2}) - \frac{\epsilon}{2}$ and ultimately reach the inequality:

$$\beta X \setminus \upsilon X \subseteq Z_{\beta X}(f^\beta) \subseteq \beta X \setminus Z_{\beta X}(g^\beta) \subseteq Z_{\beta X}(h^\beta) \quad \ldots (1)$$

To complete this theorem, it remains to check that $h \in C_\psi(X)$. Since $C_\psi(X) = \bigcap_{p\in\beta X\setminus \upsilon X} M^p$, it is therefore sufficient to show that (in view of Gelfand-Kolmogoroff Theorem [7]), for each point $p \in \beta X\setminus\upsilon X$, $p \in cl_{\beta X} Z(h)$. For that purpose let U be an open neighborhood of p in βX. Then $V = \beta X \setminus Z_{\beta X}(g^\beta) \cap U$ is an open neighborhood of p in βX (we exploit the inequality (1)). Therefore $V \cap X \neq \emptyset$. But from (1) we get that $V \cap X \subseteq Z(h)$. hence $Z(h) \cap U \neq \emptyset$. Thus each open neighborhood of p in βX cuts $Z(h)$ and therefore $p \in cl_{\beta X} Z(h)$. □

For notational convenience let us write for $f \in C(X)$ and $n \in \mathbb{N}$, $A_n(f) = \{x \in X : |f(x)| \geq \frac{1}{n}\}$. Since a support, i.e., a set of the form $cl_X(X \setminus Z(k))$, $k \in C(X)$ is pseudocompact if and only if it is bounded meaning that each $h \in C(X)$ is bounded on $cl_X(X \setminus Z(k))$ [Theorem 2.1, [9]]. We rewrite: $C_\infty^\psi(X) = \{f \in C(X) : A_n(f)$ is bounded for each $n \in \mathbb{N}\}$ [see [1] in this connection]. The following result relates this ring with $C_\psi(X)$.

Theorem 4.12. *$C_\psi^\infty(X) = cl_U C_\psi(X)$.*

Proof. In view of Theorem 4.11, it amounts to showing that $C_\psi^\infty(X) = \{f \in C(X) : f^*(\beta X \setminus \upsilon X) = 0\}$. For that we make the elementary but important observation that $C_\psi^\infty(X) \subseteq C^*(X)$. First assume that $f \in C(X)$ and $f^*(\beta X \setminus \upsilon X) = \{0\}$, i.e., $f^\beta(\beta X \setminus \upsilon X) = \{0\}$. Choose $n \in \mathbb{N}$ arbitrarily, we shall show that $A_n(f)$ is bounded. For that purpose select $g \in C(X)$ at random. Now by abusing notation we write $A_n(f^\beta) = \{p \in \beta X : |f^\beta(p)| \geq \frac{1}{n}\}$. Then it is clear that $A_n(f^\beta) \subseteq \upsilon X$ and surely $A_n(f^\beta)$ is compact. It follows that for the function $g^* : \beta X \to \mathbb{R} \cup \{\infty\}$, $g^*(A_n(f^\beta))$ is compact subset of $\mathbb{R}$. In particular we can say that g is bounded on $A_n(f)$, which we precisely need. Thus it is proved that $\{f \in C(X) : f^*(\beta X \setminus \upsilon X) = \{0\}\} \subseteq C_\psi^\infty(X)$. To prove the reverse containment, let $f \in C^*(X)$ and $f^*(\beta X \setminus \upsilon X) \neq \{0\}$. Without loss of generality we can take $f \geq 0$ on X, this means that there exists $p \in \beta X\setminus\upsilon X$ and $n \in \mathbb{N}$, for which $f^\beta(p) > \frac{1}{n}$. Hence there exists an open neighborhood U of p in βX for which $f^\beta > \frac{1}{n}$ on the entire U. It follows that $p \in cl_\beta A_n(f)$. On the other hand, since $p \notin \upsilon X$, there exists $g \in C(X)$ such that $g^*(p) = \infty$. These two facts together imply that g is unbounded on $A_n(f)$. Hence $A_n(f)$ is not pseudocompact and thus $f \notin C_\infty^\psi(X)$. The theorem is completely proved. □

Theorem 4.13. *Let $\mathscr{P}$ be an ideal of closed set in X. Then*

1. *$C_\infty^{\mathscr{P}}(X)$ is a closed subset of $C_U(X)$.*
2. *$cl_U C_{\mathscr{P}}(X) = C_\infty^{\mathscr{P}}(X)$.*

Proof. 1. Let us rewrite: $C_\infty^{\mathscr{P}}(X) = \{f \in C(X) :$ for each $n \in \mathbb{N}, A_n(f) \in \mathscr{P}\}$. Suppose $f \in C(X)$ is such that $f \notin C_\infty^{\mathscr{P}}(X)$. Thus there exists $n \in \mathbb{N}$ such that $A_n(f) \notin \mathscr{P}$. We claim that $\widetilde{B}(f, \frac{1}{2n}) \cap C_\infty^{\mathscr{P}}(X) = \emptyset$ and we are done. If possible, let there exists $g \in \widetilde{B}(f, \frac{1}{2n}) \cap C_\infty^{\mathscr{P}}(X)$. Then $|g - f| < \frac{1}{2n}$ and $A_k(g) \in \mathscr{P}$ for each $k \in \mathbb{N}$. The first inequality implies that $|f| < |g| + \frac{1}{2n}$, which further implies that $A_n(f) \subseteq A_{2n}(g)$. This combined with $A_{2n}(g) \in \mathscr{P}$ yields, in view of the fact that $\mathscr{P}$ is an ideal of closed sets in X that $A_n(f) \in \mathscr{P}$, a contradiction.

2. Let $f \in C_\infty^{\mathscr{P}}(X)$ and $\epsilon > 0$ in $\mathbb{R}$. Define as in the proof of Theorem 4.8, a function $g : X \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} f(x) + \frac{\epsilon}{2} & if\ f(x) \leq -\frac{\epsilon}{2} \\ 0 & if\ -\frac{\epsilon}{2} \leq f(x) \leq \frac{\epsilon}{2} \\ f(x) - \frac{\epsilon}{2} & if\ f(x) \geq \frac{\epsilon}{2} \end{cases}$$

Then $g \in \widetilde{B}(f, \epsilon)$, we assert that $g \in C_{\mathscr{P}}(X)$ and therefore $\widetilde{B}(f, \epsilon) \cap C_{\mathscr{P}}(X) \neq \emptyset$ and we are done. Proof of the assertion: $X \setminus Z(g) \subseteq \{x \in X : |f(x)| \geq \frac{\epsilon}{2}\}$, this implies that: $cl_X(X \setminus Z(g)) \subseteq \{x \in X : |f(x)| \geq \frac{\epsilon}{2}\}$. Since $f \in C_\infty^{\mathscr{P}}(X)$, if follows that $\{x \in X : |f(x)| \geq \frac{\epsilon}{2}\} \in \mathscr{P}$ and hence $cl_X(X \setminus Z(g)) \in \mathscr{P}$. Thus $g \in C_{\mathscr{P}}(X)$. □

Set $I_{\mathscr{P}} = \{f \in C(X) : f.g \in C_\infty^{\mathscr{P}}(X) \text{ for each } g \in C(X)\}$.

Theorem 4.14. *The following results hold:*

1. *$I_{\mathscr{P}}$ is an ideal in $C(X)$ with $C_{\mathscr{P}}(X) \subset I_{\mathscr{P}} \subset C_\infty^{\mathscr{P}}(X)$.*
2. *$I_{\mathscr{P}}$ is closed in $C_m(X)$.*
3. *$cl_m C_{\mathscr{P}}(X) = I_{\mathscr{P}}$.*
4. *$I_{\mathscr{P}} = \bigcap_{p \in F_{\mathscr{P}}} M^p$, where $F_{\mathscr{P}} = \{p \in \beta X : C_{\mathscr{P}}(X) \subset M^p\}$.*

Proof. 1. Let $f, g \in I_{\mathscr{P}}$ and $h \in C(X)$. Then $f.h, g.h \in C_\infty^{\mathscr{P}}(X) \implies (f + g)h \in C_\infty^{\mathscr{P}}(X)$, because $A_n(f.h + g.h) \subset A_{2n}(f.h) \cup A_{2n}(g.h)$ for each $n \in \mathbb{N}$. Also let $f \in I_{\mathscr{P}}$ and $g \in C(X)$. Consider any $h \in C(X)$. Then $g.h \in C(X) \implies f.g.h \in C_\infty^{\mathscr{P}}(X) \implies f.g \in I_{\mathscr{P}}$. Thus $I_{\mathscr{P}}$ is an ideal in $C(X)$. Clearly $C_{\mathscr{P}}(X) \subset I_{\mathscr{P}} \subset C_\infty^{\mathscr{P}}(X)$.

2. Let $f \in C(X)$ be such that $f \notin I_{\mathscr{P}}$. Then there exists $g \in C(X)$ such that $f.g \notin C_\infty^{\mathscr{P}}(X)$. Therefore there exists $p \in \mathbb{N}$ such that $A_p(f.g) \notin \mathscr{P}$. Let $u = \frac{1}{2p(1+|g|)}$. Then u is a positive unit in $C(X)$. If possible, let $h \in \widetilde{B}(f, u) \cap I_{\mathscr{P}}$. Then $|f - h| < u$ and $h \in I_{\mathscr{P}}$. Then $h.g \in C_\infty^{\mathscr{P}}(X) \implies A_n(h.g) \in \mathscr{P}$ for all $n \in \mathbb{N}$. Now $|f - h| < u \implies |f.g - h.g| < u|g| < \frac{1}{2p} \implies |f.g| < |h.g| + \frac{1}{2p} \implies A_p(f.g) \subset A_{2p}(h.g) \implies A_p(f.g) \in \mathscr{P}$, a contradiction. Therefore $\widetilde{B}(f, u) \cap I_{\mathscr{P}} = \emptyset$ and hence I is closed in $C_m(X)$.

3. Since $I_{\mathscr{P}}$ is closed in $C_m(X)$, it follows that $cl_m C_{\mathscr{P}}(X) \subseteqq I_{\mathscr{P}}$. Let $f \in I_{\mathscr{P}}$ and u be any positive unit in $C(X)$. Define a function $g : X \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} f(x) + \frac{1}{2}u(x) & if\ f(x) \leq -\frac{1}{2}u(x) \\ 0 & if\ -\frac{1}{2}u(x) \leq f(x) \leq \frac{1}{2}u(x) \\ f(x) - \frac{1}{2}u(x) & if\ f(x) \geq \frac{1}{2}u(x) \end{cases}$$

Then $g \in \widetilde{B}(f, u)$ and $cl_X(X \setminus Z(g)) \subset \{x \in X : |f(x)| \geq \frac{1}{2}u(x)\}$. Now $\frac{1}{u} \in C(X)$ and $f \in I_{\mathscr{P}} \implies \frac{f}{u} \in C_\infty^{\mathscr{P}}(X) \implies A_n(\frac{f}{u}) \in \mathscr{P}$ for all $n \in \mathbb{N}$. It is clear that $A_2(\frac{f}{u}) = \{x \in X : |f(x)| \geq \frac{1}{2}u(x)\}$ and so $cl_X(X \setminus Z(g)) \in \mathscr{P}$, i.e., $g \in C_{\mathscr{P}}(X)$. Thus $\widetilde{B}(f, u) \cap C_{\mathscr{P}}(X) \neq \emptyset$, i.e., $f \in cl_m C_{\mathscr{P}}(X)$. So $I_{\mathscr{P}} \subseteqq cl_m C_{\mathscr{P}}(X)$.

4. We know that the closure of an ideal J of $C(X)$ in the m-topology is the intersection of all maximal ideal containing J [7Q2 [7]]. Therefore $I_{\mathscr{P}} = cl_m C_{\mathscr{P}}(X) = \bigcap_{p \in F_{\mathscr{P}}} M^p$. □

Corollary 4.15. *The closure of $C_K(X)$ in the m-topology is the ideal $\{f \in C(X) : fg \in C_\infty(X) \text{ for each } g \in C(X)\}$. When $\mathscr{P}$ is the ideal of all compact sets in X, $F_{\mathscr{P}}$ will be $\beta X - X$ and hence $I_{\mathscr{P}} = \bigcap_{p \in \beta X - X} M^p$ i.e., $cl_m C_K(X) = \bigcap_{p \in \beta X - X} M^p$.*

[This last result is achieved independently in [3] [Proposition 5.6]].

Corollary 4.16. *From Theorem* 3.1 *[8],* $C_\psi(X) = \bigcap_{p \in \beta X \setminus \upsilon X} M^p$ *and so* $C_\psi(X)$ *is closed in the m-topology. Again by Theorem 4.14(3),* $cl_m(C_\psi(X)) = \{f \in C(X) : fg \in C_\infty^\psi(X)$ *for each* $g \in C(X)\}$*. Thus* $C_\psi(X)$ *can also be written as* $\{f \in C(X) : fg \in C_\infty^\psi(X)$ *for each* $g \in C(X)\}$*, this is an alternate formula for* $C_\psi(X)$ *[Compare with the known formula:* $C_\psi(X) = \{f \in C(X) : fg \in C^*(X)$ *for each* $g \in C(X)\}$*, [Theorem* 2.1*, [9]]].*

We conclude this section by establishing a characterization of pseudocompact spaces.

Theorem 4.17. *The U-topology and the m-topology on* $C(X)$ *are equal if and only if the closures of* $C_{\mathscr{P}}(X)$ *in the respective topologies are equal for every choice of ideal* $\mathscr{P}$ *of closed sets in* X*. Therefore* X *is pseudocompact if and only if for every choice of ideal* $\mathscr{P}$ *of closed sets in* X*,* $C_{\mathscr{P}}(X)$ *is dense in* $C_\infty^{\mathscr{P}}(X)$ *in the m-topology.*

Proof. If these two topologies are unequal, then X is not pseudocompact and so there exists $f \in C^*(X)$ such that $Z(f) = \emptyset$ and $f^*(\beta X \setminus X) = \{0\}$. Consider the ideal $\mathscr{P}$ of bounded subsets of X. Then $C_{\mathscr{P}}(X) = C_\psi(X)$ and by Theorem 4.11, $cl_U C_{\mathscr{P}}(X) = \{f \in C(X) : f^*(\beta X \setminus \upsilon X) = \{0\}\}$ and $cl_m C_{\mathscr{P}}(X) = C_\psi(X)$. Clearly $f \in cl_U C_{\mathscr{P}}(X) \setminus cl_m C_{\mathscr{P}}(X)$, i.e., $cl_U C_{\mathscr{P}}(X) \neq cl_m C_{\mathscr{P}}(X)$. □

The following problem is left open:

Question 4.18. *Is the convexity condition on the ideal* I *of* $C(X)$ *in Theorem 3.7 necessary for the validity of the same theorem?*

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