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The relaxed MGHSS-like method for absolute value equations

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Abstract. Based on the matrix splitting techniques and the ideas of the GPHSS-like method, we proposed the relaxed modified generalized HSS-like method (RMGHSS-like), which is more efficient and more robust than the RPHSS-like, the MBAS, the NI and the NHSS-like methods for the absolute value equation. Furthermore, the RMGHSS-like method is the general form of the relaxed PHSS-like method. The convergence of the RMGHSS-like iterative method is proved by theoretical analysis, and the relationships between the parameters are rigorously discussed when the coefficient matrix E is a Hermitian positive definite matrix under the minimum spectral radius. Numerical experiments had been given to recognize the effectiveness of the RMGHSS-like method.

1. Introduction

Consider the absolute equations(**AVE**)

$$Eu - |u| = f, \tag{1}$$

where $E \in \mathbb{R}^{n\times n},\ u,\ f \in \mathbb{R}^n$ and $|u| = (|u_1|, |u_2|, \cdots, |u_n|)^T$. The generalized scheme of AVE below

$$Eu + F|u| = f, \tag{2}$$

where $E, F \in \mathbb{R}^{n\times n}, f \in \mathbb{R}^n$, Rohn introduced (2) for the first time [1], see for further studied [1–7]. In fact, (1) is a different shape of the weakly nonlinear equations (WNE) $Eu + F\psi(u) = f$ and (2) is a different structure of $E\phi(u) + F\psi(u) = f$, respectively [8–10]

The AVE originates from many filed of scientific computing and engineering applications, such as the linear and quadratic programming, the bimatrix games, and the contact problems. The AVE can also be simplified to a linear complementarity problem (LCP) [11, 12]. Besides, the AVE has broader purposes in applied science and technological know-how such as financial equilibrium problems[13, 14]. Therefore, building efficient algorithms and related theories for AVE has high economic value and good application prospects. For the theoretical study of AVE, there are numerous research results; see [1–3, 15–21] and the references therein.

Recently, the problem of discovering effective numerical solution algorithm of AVE has attracted plenty of interest and has been discovered in the article see[22–25]. These numerical methods for AVE can be

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considered from different perspectives. Since the correlation of the LCP and AVE [4, 7, 12], the modulus coefficient method for solving LCP [26–30] is also suitable for solving AVE. The AVE can be seen as a non-smooth equation, so that we can use non-smooth theory to solve it, such as smooth Newton's method[31], non-smooth Newton's method[32]. Based on the AVE is a WNE, Bai and Yang [33] first proposed the Picard-HSS iterative method for solving the WNE. Further, Salkuyeh [34] utilized the Picard-HSS method (Picard-HSS) to deal with the AVE, but the number of internal iteration steps are frequently independent of the problem and hard to be determined in real calculation in this method. To solve the above shortcomings, Zhu et al. [35] presented HSS-like method (HSS-like). In order to solve problems more efficiently, Jian-Jun zhang presented the relaxed nonlinear preconditioned HSS-like (RPHSS-like) method [36], which is more promising than the HSS-like, Picard-HSS method and is the general form of the HSS-like method [37]. Similarly, Zhu and Pu presented the nonlinear GPHSS-like iteration method [37] (NGPHSS-like) for solving the WNE, which is more effective and more attractive than HSS-like method through choosing appropriate iteration parameters.

In this paper, using the means of the MGHSS method [38] and the ideas of the GPHSS-like method [37], we will first split the coefficient matrix E into different parts. Then we proposed an relaxed modified GHSS-like method (RMGHSS-like) for sloving AVE by utilizing the ideas of the NGPHSS-like method. In addition, the relaxed RMGHSS-like method is the universal scheme of the PHSS-like method where the Hermitian positive definite matrix P = I with three parameters that can accelerate the convergence. Furthermore, we analysis the convergence of the RMGHSS-like method and discuss the relationship between parameters in detail when E is a Hermitian positive definite matrix.

The organization of the paper is illustrated in the following. In Section 2, we give plenty of lemmas and corresponding symbols that needed to be used in the paper. In section 3, we review the modified generalized HSS method (MGHSS), the nonlinear GPHSS-like method (NGPHSS-like) and the relaxed nonlinear PHSS-like method (RPHSS-like). In section 4, we proposed the relaxed modified generalized HSS-like method (RMGHSS-like) for solving the AVE (1). At the same time, we analysis the convergence of the RMGHSS-like method. In sections 5, under the minimum spectral radius, the relationships between parameters and the convergence properties are analyzed when coefficient matrix E is a Hermitian positive definite matrix. The results and conclusions of numerical experiments are given in sections 6 and 7, respectively.

2. Preliminaries

In this section, we give a few lemmas and corresponding symbols, which can be used in this essay.

Let the symbol I denotes the identity matrix. The spectral radius of the matrix is represented by $\rho\,(.)$.

Lemma 2.1. *[39]. For any vectors $s \in R^n$ and $t \in R^n$, the following results hold:*

(1) $\||s| - |t|\| \leq \|s - t\|$;

(2) if $0 \leq s \leq t$, *then* $\|s\| \leq \|t\|$.

Lemma 2.2. *[39]. For any matrices $E \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{n\times n}$, if* $0 \leq E \leq F$, *then* $\|E\| \leq \|F\|$.

Lemma 2.3. *For any number $s \in \mathbb{R}^n$ and $t \in \mathbb{R}^n$, it must be true that* $st \leq \left(\frac{s+t}{2}\right)^2$.

Lemma 2.4. *[38]. Assume that $E, F \in \mathbb{R}^{n\times n}$, E is positive definite and F is skew-Hermitian, then*

$$y^T\,(E + F)\,y > 0, \quad any\ y \neq 0 \in \mathbb{R}^n. \tag{3}$$

Lemma 2.5. *[40]. **(Courant-Fischer Minimax Theorem).** If E ∈ $\mathbb{R}^{n\times n}$ is symmetric, then*

$$\lambda_k(E) = \max_{dim(S)=k} \min_{0\neq y\in S} \frac{y^T E y}{y^T y}, \; for\, k = 1 : n. \tag{4}$$

Lemma 2.6. *[40]. **(Theorem).** If E and E + F are n-by-n symmetric matrices, then*

$$\lambda_k(E) + \lambda_n(F) \leq \lambda_k(E+F) \leq \lambda_k(E) + \lambda_k(F), \;\; k = 1 : n. \tag{5}$$

For a symmetric matrix E we shall use the notation $\lambda_k(E)$ to designate the kth largest eigenvalue, i.e, $\lambda_n(E) \leq \cdots \leq \lambda_2(E) \leq \lambda_1(E)$.

3. The MGHSS, GPHSS, RPHSS-like iteration methods

In this section, we will shortly introduce the MGHSS, GPHSS, RPHSS-like methods, see [36–38]. Using the matrix segmentation techniques, the coefficient matrix E will be divided into two parts[41]:

$$E = \mathscr{H} + \mathscr{S}, \; \mathscr{H} = \frac{1}{2}(E + E^*) \; and \; \mathscr{S} = \frac{1}{2}(E - E^*).$$

Further, the resulting matrix can also be quadratic:

$$\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2 \; and \; \mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2,$$

where $\mathscr{H}$, $\mathscr{H}_1$, $\mathscr{H}_2$ are Hermitian matrix and $\mathscr{S}$, $\mathscr{S}_1$, $\mathscr{S}_2$ are skew-Hermitian matrices [38].

The MGHSS iteration method. ([38]) Supposed $u^{(0)}$ is an initial vector. For k = 0, 1, 2, . until $\left\{u^{(k)}\right\}_k^\infty$ converges, computing $u^{(k+1)}$ by

$$\begin{cases} (\alpha I + \mathscr{H}_1 + \mathscr{S}_1)\, u^{(k+\frac{1}{2})} = (\alpha I - \mathscr{S}_2 - \mathscr{H}_2)\, u^{(k)} + f, \\ (\alpha I + \mathscr{S}_2 + \mathscr{H}_2)\, u^{(k+1)} = (\alpha I - \mathscr{H}_1 - \mathscr{S}_1)\, u^{(k+\frac{1}{2})} + f, \end{cases} \tag{6}$$

where α is a given non-zero positive constant.

Yang, et al. [42] proposed the more efficient GPHSS method that can improve the convergence rate by choosing the appropriate iteration parameters. The GPHSS iteration method as follows.

The GPHSS iteration method. ([42]) Supposed $u^{(0)}$ is an initial vector. For k = 0, 1, 2, . until $\left\{u^{(k)}\right\}_k^\infty$ converges, computing $u^{(k+1)}$ by

$$\begin{cases} (\alpha P + \mathscr{H})\, u^{(k+\frac{1}{2})} = (\alpha P - \mathscr{S})\, u^{(k)} + f, \\ (\beta P + \mathscr{S})\, u^{(k)} = (\beta P - \mathscr{H})\, u^{(k+\frac{1}{2})} + f, \end{cases} \tag{7}$$

where α , β , and P are the Non-negative constant, positive constant and Hermitian positive definite matrix, respectively.

By introducing a relaxation parameter and using nonlinear analysis methods, J.-J Zhang proposed the RPHSS-like method [36], which solved a common problem between the Picard-GPHSS and the Picard-HSS iteration methods. The common problem is that the number of iterative steps in their inner loop is frequently problem-independent and difficult to decided in the actual calculation process [41]. The RPHSS-like iteration method as follows.

The RPHSS-like iteration method. ([36]) Supposed $u^{(0)}$ is an initial vector. For k = 0, 1, 2, . until $\left\{u^{(k)}\right\}_k^{\infty}$ converges, computing $u^{(k+1)}$ by

$$\left\{\begin{array}{l} u^{(k+\frac{1}{2})} = u^{(k)} + \varpi\,(\alpha P + \mathscr{H})^{-1}\,[-Eu^{(k)} + |u^{(k)}| + f], \\ u^{(k+1)} = u^{(k+\frac{1}{2})} + \varpi\,(\alpha P + \mathscr{S})^{-1}\,[-Eu^{(k+\frac{1}{2})} + |u^{(k+\frac{1}{2})}| + f], \end{array}\right. \tag{8}$$

where α is a given positive constant.

In order to further speed up the rate of the convergence and improve the convergence properties of the RPHSS-lie iteration method, we can obtain the Nonlinear modified GHSS-like iterative methods for AVE according to the matrix splitting techniques and the idea of GPHSS-like iteration method [37].

4. The Relaxed MGHSS-like iteration iteration methods

In this section, we proposed a relaxed MGHSS-like method (RMGHSS-like) and rewrote the HSS-like method. Meanwhile, the convergence of the relaxed MGHSS-like method is proven.

4.1. The Relaxed MGHSS-like iteration method

At first, we can obtain the nonlinear GHSS-like methods for AVE according to the GPHSS-like iteration method [37]. The GHSS-like method as follows.

Algorithm 4.1. The GHSS-like iteration method. Supposed $u^{(0)}$ is an initial vector. For k = 0, 1, 2, . until $\left\{u^{(k)}\right\}_k^{\infty}$ converges, computing $u^{(k+1)}$ by

$$\left\{\begin{array}{l} (\alpha I + \mathscr{H})\,u^{(k+\frac{1}{2})} = (\alpha I - \mathscr{S})\,u^{(k)} + |u^{(k)}| + f, \\ (\beta I + \mathscr{S})\,u^{(k)} = (\beta I - \mathscr{H})\,u^{(k+\frac{1}{2})} + |u^{(k+\frac{1}{2})}| + f, \end{array}\right. \tag{9}$$

where α , β , and P are the Non-negative constant, positive constant and Hermitian positive definite matrix, respectively. If $\alpha = \beta$, the GHSS-like method can be simplified to the PHSS-like method. If $\alpha = \beta$, the GHSS-like method can be simplified to the HSS-like method [36].

Then, we split coefficient matrix E of the AVE into Hermitian and skew-Hermitian matrices: E = $\mathscr{H} + \mathscr{S}$, then continue to split $\mathscr{H}$ and $\mathscr{S}$ into two parts : $\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2$, $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$. Therefore, we can obtain the split form of matrix E as follows.

$$E = (\mathscr{H}_1 + \mathscr{S}_1 + \alpha I) - (\alpha I - \mathscr{S}_2 - \mathscr{H}_2) = (\beta I + \mathscr{S}_2 + \mathscr{H}_2) - (\beta I - \mathscr{H}_1 - \mathscr{S}_1). \tag{10}$$

Further, we can get

$$\left\{\begin{array}{l} (\alpha I + \mathscr{H}_1 + \mathscr{S}_1)\,u^{(k+\frac{1}{2})} = (\alpha I - \mathscr{S}_2 - \mathscr{H}_2)\,u^{(k)} + |u^{(k)}| + f, \\ (\beta I + \mathscr{S}_2 + \mathscr{H}_2)\,u^{(k)} = (\beta I - \mathscr{H}_1 - \mathscr{S}_1)\,u^{(k+\frac{1}{2})} + |u^{(k+\frac{1}{2})}| + f, \end{array}\right. \tag{11}$$

where α is a Non-negative constant, β is a positive constant. $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hermitian positive definite. $\mathscr{S}_1$ and $\mathscr{S}_2$ are skew-Hermitian positive definite. Besides, $\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2$ and $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$.

Notice that, we rewrite (11) as follows:

$$\left\{\begin{array}{l} u^{(k+\frac{1}{2})} = (\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\,[(\alpha I - \mathscr{S}_2 - \mathscr{H}_2)\,u^{(k)} + |u^{(\mathscr{H}_1)}| + f\,], \\ u^{(k+1)} = (\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}\,[(\beta I - \mathscr{H}_1 - \mathscr{S}_1)\,u^{(k+\frac{1}{2})} + |u^{(\mathscr{H}_1+\frac{1}{2})}| + f\,]. \end{array}\right. \tag{12}$$

If we add a relaxed parameter ϖ in (12), we can obtain a new iteration scheme as follows:

$$\left\{\begin{array}{l} u^{(k+\frac{1}{2})} = (1-\varpi)\,u^{(k)} + \varpi\,(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\,[(\alpha I - \mathscr{S}_2 - \mathscr{H}_2)\,u^{(k)} + |u^{(k)}| + f\,], \\ u^{(k+1)} = (1-\varpi)\,u^{(k+\frac{1}{2})} + \varpi\,(\beta P + \mathscr{S}_2 + \mathscr{H}_2)^{-1}\,[(\beta I - \mathscr{H}_1 - \mathscr{S}_1)\,u^{(k+\frac{1}{2})} + |u^{(k+\frac{1}{2})}| + f\,], \end{array}\right. \tag{13}$$

By further simplifying the calculations, we can get the iteration (13) as follows:

$$\begin{cases} u^{(k+\frac{1}{2})} = u^{(k)} + \varpi\,(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\,[-Eu^{(k)} + |u^{(k)}| + f\,], \\ u^{(k+1)} = u^{(k+\frac{1}{2})} + \varpi\,(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}\,[-Eu^{(k+\frac{1}{2})} + |u^{(k+\frac{1}{2})}| + f\,], \end{cases} \tag{14}$$

The method (14) was named the relaxed modified GHSS-like method (RGHSS-like) by us. we now present the iteration method.

Algorithm 4.2. The Relaxed MGHSS-like iteration method. Supposed $u^{(0)}$ is an initial vector. For k = 0, 1, 2, . until $\left\{u^{(k)}\right\}_k^{\infty}$ converges, computing $u^{(k+1)}$ by

$$\begin{cases} u^{(k+\frac{1}{2})} = u^{(k)} + \varpi\,(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\,[-Eu^{(k)} + |u^{(k)}| + f\,], \\ u^{(k+1)} = u^{(k+\frac{1}{2})} + \varpi\,(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}\,[-Eu^{(k+\frac{1}{2})} + |u^{(k+\frac{1}{2})}| + f\,], \end{cases} \tag{15}$$

where α is a Non-negative constant, β and ϖ are positive constants. $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{S}_1$, $\mathscr{S}_2$ are Hermitian and skew-Hermitian positive definite, respectively. Besides, $\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2$ and $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$.

In addition, we can rewrite the HSS-like method [35] as the new HSS-like method. The scheme of the new HSS-like method (NHSS-like) is (16).

$$\begin{cases} (\alpha I + \mathscr{H})\,u^{(k+\frac{1}{2})} = (\alpha I - \mathscr{S})\,u^{(k)} + |u^{(k)}| + f, \\ (\alpha I + \mathscr{S})\,u^{(k+1)} = (\alpha I - \mathscr{H})\,u^{(k+\frac{1}{2})} + |u^{(k+\frac{1}{2})}| + f, \end{cases}$$

$$\Longrightarrow \begin{cases} u^{(k+\frac{1}{2})} = (1-\varpi)\,u^{(k)} + \varpi\,(\alpha I + \mathscr{H})^{-1}\,[(\alpha I - \mathscr{S})\,u^{(k)} + |u^{(k)}| + f\,], \\ u^{(k+1)} = (1-\varpi)\,u^{(k+\frac{1}{2})} + \varpi\,(\alpha I + \mathscr{S})^{-1}\,[(\alpha I - \mathscr{H})\,u^{(k+\frac{1}{2})} + |u^{(k+\frac{1}{2})}| + f\,], \end{cases}$$

$$\Longrightarrow \begin{cases} u^{(k+\frac{1}{2})} = u^{(k)} + \varpi\,(\alpha I + \mathscr{H})^{-1}\,[-Eu^{(k)} + |u^{(k)}| + f\,], \\ u^{(k+1)} = u^{(k+\frac{1}{2})} + \varpi\,(\alpha I + \mathscr{S})^{-1}\,[-Eu^{(k+\frac{1}{2})} + |u^{(k+\frac{1}{2})}| + f\,], \end{cases}$$

$$\overset{\varpi=1}{\Longrightarrow} \begin{cases} u^{(k+\frac{1}{2})} = u^{(k)} + (\alpha I + \mathscr{H})^{-1}\,[-Eu^{(k)} + ||u^{(k)}| + f\,], \\ u^{(k+1)} = u^{(k+\frac{1}{2})} + (\alpha I + \mathscr{S})^{-1}\,[-Eu^{(k+\frac{1}{2})} + |u^{(k+\frac{1}{2})}| + f\,]. \end{cases} \tag{16}$$

Therefore, we can obtain the new HSS-like iterative method as follows.

Algorithm 4.3. The New HSS-like iteration method. Supposed $u^{(0)}$ is an initial vector. For k = 0, 1, 2, . until $\left\{u^{(k)}\right\}_k^{\infty}$ converges, computing $u^{(k+1)}$ by

$$\begin{cases} u^{(k+\frac{1}{2})} = u^{(k)} + (\alpha I + \mathscr{H})^{-1}\,[-Eu^{(k)} + |u^{(k)}| + f\,], \\ u^{(k+1)} = u^{(k+\frac{1}{2})} + (\alpha I + \mathscr{S})^{-1}\,[-Eu^{(k+\frac{1}{2})} + |u^{(k+\frac{1}{2})}| + f\,], \end{cases} \tag{17}$$

where α is a positive constant, $\mathscr{H}$ is Hermitian positive definite and $\mathscr{S}$ is skew-Herimitian positive definite.

In order to show that the RMGHSS-like method (15) is more general and applicable, we will make the following variations as follows.

Corollary 4.1. if $\alpha = \beta$, $\mathscr{S}_1 = 0$ and $\mathscr{H}_2 = 0$, then the RMGHSS-like method can be simplified to the PHSS-like method [36] where the Hermitian matrix P = I.

Corollary 4.2. if $\alpha = \beta$, $\varpi = 1$, $\mathscr{S}_1 = 0$ and $\mathscr{H}_2 = 0$, then the RMGHSS-like method can be simplified to the NHSS-like method (17).

Since the NHSS-like method is the special form of the RMGHSS-like method, the convergence of the NHSS-like is same to the RMGHSS-like method. Therefore, we only need to discuss the convergence of the RMGHSS-like method. Now, the convergence of the RMGHSS-like method is proven as follows.

4.2. The convergence of the Relaxed MGHSS-like iterative methods

Theorem 4.1. *Let the matrix $E \in \mathbb{R}^{n\times n}$ be positive definite. $\mathscr{H}$ and $\mathscr{S}$ are its Hermitian and skew-Hermitian parts, respectively. $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hermitian positive definite. $\mathscr{S}_1$ and $\mathscr{S}_2$ are skew-Hermitian positive definite. Suppose the matrix I is identity matrix and both of ϖ and β are positive constants. Besides, α is a Non-negative constant, $\mathscr{H} = \frac{1}{2}(E + E^*)$, $\mathscr{S} = \frac{1}{2}(E - E^*)$, $\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2$ and $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$.*
Define

$$\begin{gathered}
\eta_1(\alpha,\ \varpi) = 1 + \|\varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\| - \|\varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1} E\|,\\
\eta_2(\beta,\ \varpi) = 1 + \|\varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}\| - \|\varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1} E\|,\\
c_1(\alpha,\ \varpi) = \|\varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1} E\| - \|\varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\|,\\
c_2(\beta,\ \varpi) = \|\varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1} E\| - \|\varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}\|,\\
\theta(\alpha,\ \beta,\ \varpi) = \|\ \varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\ \| + \|\ \varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)\|,\\
\eta_1(\alpha,\ \varpi) = 1 - c_1(\alpha,\ \varpi),\ \ \eta_2(\beta,\ \varpi) = 1 - c_2(\beta,\ \varpi),\\
\delta = \eta_1(\alpha,\ \varpi)\,\eta_2(\beta,\ \varpi),\ \ \nu = \|E\| - 1.
\end{gathered}$$

with $\|.\|$ being an arbitrary matrix norm .

If the parameter $\varpi,\ \alpha,\ \beta$, satisfy $0 < c_1(\alpha,\varpi) + c_2(\beta,\varpi) \le \theta(\alpha,\beta,\varpi)\,\nu < 4$, then the iterative sequence $\left\{u^{(k)}\right\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ resulted from the RMGHSS-like iterative method for AVE converges to the unique solution $u^\dagger \in \mathbb{R}^n$ of the absolute value equation (1).

Proof. Suppose that the vector $u^\dagger$ is the exact solution of the AVE, we can know $u^\dagger$ satisfying fixed-point equation:

$$\begin{cases} u^\dagger = u^\dagger + \varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}[-Eu^\dagger + |u^\dagger| + f\,],\\ u^\dagger = u^\dagger + \varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}[-Eu^\dagger + |u^\dagger| + \ f\,],\end{cases} \tag{18}$$

It is easy to obtain via subtracting (15) from (18),

$$\begin{cases} u^{(k+\frac{1}{2})} - u^\dagger = [u^{(k)} - u^\dagger\,] + \varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}[[|u^{(k)}| - |u^\dagger|] - E[u^{(k)} - u^\dagger\,]],\\ u^{(k+1)} - u^\dagger = [u^{(k+\frac{1}{2})} - u^\dagger] + \varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}[[|u^{(k+\frac{1}{2})}| - |u^\dagger|] - E[u^{(k+\frac{1}{2})} - u^\dagger\,]],\end{cases}$$

Taking advantage of the properties of the norm, we can get the following inequality:

$$\begin{aligned}
\|u^{(k+\frac{1}{2})} - u^\dagger\| &= \|[u^{(k)} - u^\dagger] + \varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}[[|u^{(k)}| - |u^\dagger|] - E[u^{(k)} - u^\dagger]]\|\\
&\le [\|u^{(k)} - u^\dagger\|] + \|\varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\|\|\|u^{(k)}| - |u^\dagger\|| - \|\varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1} E\|\|u^{(k)} - u^\dagger\|\\
&\le 1 + \|\varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\| - \|\varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1} E\| = \eta_1(\alpha,\ \varpi)\,\|u^{(k)} - u^\dagger\|
\end{aligned}$$

$$\begin{aligned}
\|u^{(k+1)} - u^\dagger\| &= \|[u^{(k+\frac{1}{2})} - u^\dagger] + \varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}[[|u^{(k+\frac{1}{2})}| - |u^\dagger|] - E[u^{(k+\frac{1}{2})} - u^\dagger]]\|\\
&\le [\|u^{(k)} - u^\dagger\|] + \|\varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}\|\|\|u^{(k+\frac{1}{2})}| - |u^\dagger\|| - \|\varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1} E\|\|u^{(k+\frac{1}{2})} - u^\dagger\|\\
&\le 1 + \|\varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}\| - \|\varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1} E\| = \eta_2(\beta,\ \varpi)\,\|u^{(k+\frac{1}{2})} - u^\dagger\|
\end{aligned}$$

According to the above inequality, it is evident that

$$\|u^{(k+1)} - u^\dagger\| \le \eta_2(\beta,\ \varpi)\,\|u^{(k+\frac{1}{2})} - u^\dagger\| \le \eta_2(\beta,\ \varpi)\,\eta_1(\alpha,\ \varpi)\,\|u^{(k)} - u^\dagger\| = \delta\|u^{(k)} - u^\dagger\|$$

Therefore, when $\delta < 1$, $\lim\limits_{k\to+\infty} u^{(k)} = u^\dagger$, and the RMGHSS-like iterative method for AVE is convergent.

In order to simplify the conditions for convergence, we have further simplified the conditions to make the requirements easier to satisfy. Then, the spectral radius $\rho(M_{\alpha,\,\beta,\,\varpi})$ of the iteration matrix $\rho(M_{\alpha,\,\beta,\,\varpi})$ of the RMGHSS-like iterative method for AVE (15) satisfies the following relational formula:

$$\rho(M_{\alpha,\,\beta,\,\varpi}) \leq \eta_2\,(\beta,\ \varpi)\,\eta_1\,(\alpha,\ \varpi) = \delta = (1 - c_2\,(\beta,\ \varpi))\,(1 - c_1\,(\alpha,\ \varpi))\,. \tag{19}$$

According to Lemma 2.3, we have

$$\begin{aligned}\rho(M_{\alpha,\,\beta,\,\varpi}) &\leq \eta_2\,(\beta,\ \varpi)\,\eta_1\,(\alpha,\ \varpi) = \delta = (1 - c_2\,(\beta,\ \varpi))\,(1 - c_1\,(\alpha,\ \varpi))\\ &\leq \left(\frac{1 - c_2\,(\beta,\ \varpi) + 1 - c_1\,(\alpha,\ \varpi)}{2}\right)^2 = \left(1 - \frac{c_1\,(\alpha,\ \varpi) + c_2\,(\beta,\ \varpi)}{2}\right)^2.\end{aligned}$$

Thus, only condition $\left(1 - \frac{c_1(\alpha,\ \varpi)+c_2(\beta,\ \varpi)}{2}\right)^2 < 1$ is required to meet, and the conclusion

$$\rho(M_{\alpha,\,\beta,\,\varpi}) \leq \eta_2\,(\beta,\ \varpi)\,\eta_1\,(\alpha,\ \varpi) = \delta = (1 - c_2\,(\beta,\ \varpi))\,(1 - c_1\,(\alpha,\ \varpi))\ < 1$$

is established.

From $\left(1 - \frac{c_1(\alpha,\ \varpi)+c_2(\beta,\ \varpi)}{2}\right)^2 < 1$, we can obtain $-1 < \left(1 - \frac{c_1(\alpha,\ \varpi)+c_2(\beta,\ \varpi)}{2}\right) < 1$. So, the RMGHSS-like iterative method for AVE is convergent where $0 < (1 - c_1\,(\alpha,\ \varpi) + c_2\,(\beta,\ \varpi)) < 4$.

Further, simplify the conditions for convergence, it is not difficult to obtain

$$\begin{aligned}c_1\,(\alpha,\ \varpi) &= \|\varpi\,(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\,E\| - \|\varpi\,(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\|\\ &\leq \|\varpi\,(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\|\|E\| - \|\varpi\,(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\|\\ &\leq \|\varpi\,(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\|\,(\|E\| - 1)\,,\end{aligned}$$

$$\begin{aligned}c_2\,(\beta,\ \varpi) &= \|\varpi\,(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}\,E\| - \|\varpi\,(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}\|\\ &\leq \|\varpi\,(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}\|\|E\| - \|\varpi\,(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}\|\\ &\leq \|\varpi\,(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}\|\,(\|E\| - 1)\,.\end{aligned}$$

Hence,

$$c_1\,(\alpha,\ \varpi) + c_2\,(\beta,\ \varpi) \leq \varpi\,(\alpha,\ \beta,\ \varpi)\,\nu.$$

Now, under the condition $0 < c_1\,(\alpha,\ \varpi) + c_2\,(\beta,\varpi) \leq \theta\,(\alpha,\ \beta,\ \varpi)\,\nu < 4$, we have

$$\rho(M_{\alpha,\,\beta,\,\varpi}) \leq \eta_2\,(\beta,\ \varpi)\,\eta_1\,(\alpha,\ \varpi) = \delta = (1 - c_2\,(\beta,\ \varpi))\,(1 - c_1\,(\alpha,\ \varpi)) < 1$$

.

The above process is the process of complete proof of Theorem 3.1.

5. The relationships between parameters and the conditions of the convergence under the minimum spectral radius

In order for this part of the theorems to prove more convenient, now I will repeat some of the previous conditions below:

$\mathscr{H} = \frac{1}{2}\,(E + E^*)$ and $\mathscr{S} = \frac{1}{2}\,(E - E^*)$ are the Hermitian and skew-Hermitian parts of the matrix E, respectively. $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{S}_1$, $\mathscr{S}_2$ are Hermitian and skew-Hermitian positive definite, respectively. Suppose both of ϖ and β are positive constants. Besides,α is a Non-negative constant, $\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2$ and $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$, $E_1 = \mathscr{H}_1 + \mathscr{S}_1$ and $E_2 = \mathscr{H}_2 + \mathscr{S}_2$.

Theorem 5.1. *Let the matrix $E \in \mathbb{R}^{n\times n}$ be Hermitian positive definite and $E_1 = \mathscr{H}_1+\mathscr{S}_1$, $E_2 = \mathscr{H}_2+\mathscr{S}_2$, $E = E_1+E_2$. Let $\lambda_{\min}$ and $\lambda_{\max}$ be the smallest and largest eigenvalue of matrix E, respectively. Let $\mu_{\min}$ and $\mu_{\max}$ be the smallest and largest eigenvalue of matrix E_1 , respectively. Let $\sigma_{\min}$ and $\sigma_{\max}$ be the smallest and largest eigenvalue of matrix E_2 , respectively. Under the minimum spectral radius, the equation relationship between parameters α and β , the parameters ϖ value range, and the conditions of the RMGHSS-like iterative method as follows: with $\|.\|$ being a matrix 2-norm at the section 5.*

The equation relationship between parameters α, β is

$$\beta - \alpha = \mu_{\min} - \sigma_{\min}$$

The parameters ϖ value range is

$$\text{where } \lambda_{\min} > 1, \quad 0 < \varpi < \frac{2(\beta + \sigma_{\min})}{\lambda_{\max} - 1} \text{ or } 0 < \varpi < \frac{2(\alpha + \mu_{\min})}{\lambda_{\max} - 1}$$

The convergence conditions of the RMGHSS-like iterative method are

$$0 < \frac{\varpi(\lambda_{\max} - 1)}{\alpha + \mu_{\min}} < 2 \text{ or } 0 < \frac{\varpi(\lambda_{\max} - 1)}{\beta + \sigma_{\min}} < 2$$

Proof. Since the matrix $\mathscr{H}_1$, $\mathscr{H}_2$ are Hermitian positive definite matrix and the matrix $\mathscr{S}_1$, $\mathscr{S}_2$ are skew-Hermitian positive definite matrix, we can obtain the matrix $E_1 = \mathscr{H}_1 + \mathscr{S}_1$ and $E_2 = \mathscr{H}_2 + \mathscr{S}_2$ are positive definite matrix according to the Lemma 2.4 .

From the process of the sections 4, we can obtain

$$\rho(M_{\alpha,\beta,\varpi}) \leq \eta_2(\beta, \varpi)\,\eta_1(\alpha, \varpi) = \delta = (1 - c_2(\beta, \varpi))(1 - c_1(\alpha, \varpi)). \tag{20}$$

Thus, we only need to find the minimum upper bound of $\eta_2(\beta, \varpi)\,\eta_1(\alpha, \varpi)$, at this time the RMGHSS-like iteration method has the smallest spectral radius $\rho\left(M_{\alpha,\beta,\varpi}\right)$, and the parameters have the following relationship as follows.

$$\left(\frac{1 - c_2(\beta, \varpi) + 1 - c_1(\alpha, \varpi)}{2}\right)^2 = \left(1 - \frac{c_1(\alpha, \varpi) + c_2(\beta, \varpi)}{2}\right)^2. \tag{21}$$

If and only when $1 - c_1(\alpha, \varpi) = 1 - c_2(\beta, \varpi)$, $\eta_2(\beta, \varpi)\,\eta_1(\alpha, \varpi)$ takes the smallest upper bound, we get $c_1(\alpha, \varpi) = c_2(\beta, \varpi)$. $c_1(\alpha, \varpi) = c_2(\beta, \varpi)$ is equal to

$$\begin{aligned} &\|\varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1} E\| - \|\varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\| \\ &= \|\varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1} E\| - \|\varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}\| \end{aligned}$$

Hence, where $(\alpha I + \mathscr{H}_1 + \mathscr{S}_1) = (\beta I + \mathscr{S}_2 + \mathscr{H}_2)$, we can obtain $c_1(\alpha, \varpi) = c_2(\beta, \varpi)$. The proof is as follows. By $(\alpha I + \mathscr{H}_1 + \mathscr{S}_1) = (\beta I + \mathscr{S}_2 + \mathscr{H}_2)$, we have

$$\begin{aligned} &(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1} = (\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1} \\ &(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1} E = (\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1} E \\ &\|(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\| = \|(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1}\| \\ &\|(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1} E\| = \|(\beta I + \mathscr{S}_2 + \mathscr{H}_2)^{-1} E\| \end{aligned}$$

Therefore, $c_1(\alpha, \varpi) = c_2(\beta, \varpi)$ is proven.

From the Sufficient conditions $(\alpha I + \mathscr{H}_1 + \mathscr{S}_1) = (\beta I + \mathscr{S}_2 + \mathscr{H}_2)$ for the establishment of $c_1(\alpha,\ \varpi) = c_2(\beta,\ \varpi)$, we can obtain

$$\begin{aligned}&(\alpha I + \mathscr{H}_1 + \mathscr{S}_1) = (\beta I + \mathscr{S}_2 + \mathscr{H}_2)\\ \Leftrightarrow\ &(\mathscr{H}_1 + \mathscr{S}_1) - (\mathscr{S}_2 + \mathscr{H}_2) = (\beta - \alpha) I\\ \Leftrightarrow\ &E - 2(\mathscr{S}_2 + \mathscr{H}_2) = (\beta - \alpha) I\end{aligned}$$

Then, we have

$$\begin{cases} E_1 - E_2 = (\beta - \alpha) I \\ E_1 + E_2 = E \end{cases} \Rightarrow \begin{cases} E_1 = \frac{1}{2}(E + (\beta - \alpha) I) \\ E_2 = \frac{1}{2}(E - (\beta - \alpha) I) \end{cases} \tag{22}$$

According to the initial assumption: E is a Hermitian positive definite matrix, and the above (22), so that E_1 and E_2 are known to be Hermitian positive definite matrices.

So, by $(\alpha I + \mathscr{H}_1 + \mathscr{S}_1) = (\beta I + \mathscr{S}_2 + \mathscr{H}_2)$, then it is equal to $(\alpha I + E_1) = (\beta I + E_2)$, we have

$$\begin{aligned}&\|E_1 + \alpha I\|_2 = \|E_2 + \beta I\|_2\\ \Leftrightarrow\ &\max_{\mu_i \epsilon sp(E_1)} \left|\frac{1}{\mu_i + \alpha}\right| = \max_{\sigma_i \epsilon sp(E_2)} \left|\frac{1}{\sigma_i + \beta}\right|\\ \Leftrightarrow\ &\frac{1}{\mu_i + \alpha} = \frac{1}{\sigma_i + \beta}\\ \Leftrightarrow\ &\beta - \alpha = \mu_{\min} - \sigma_{\min}\end{aligned}$$

$sp(E)$ is the spectral set of the matrix E.

This completes the proof of the equation relationship between parameters $\alpha,\ \beta$.

From Theorem 4.1, there are some relationships between parameter and some conclusion as follows.

$$\varpi(\alpha,\ \beta,\ \varpi) = \|\ \varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\ \| + \|\ \varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)\|,\ \nu = \|E\| - 1..$$

If the parameter $\varpi,\ \ \alpha,\ \ \beta$, satisfy

$$0 < c_1(\alpha,\ \varpi) + c_2(\beta, \varpi) \le \theta(\alpha,\ \beta,\ \varpi)\nu < 4,$$

then the RMGHSS-like iterative method for AVE is convergent.

Now, we define the norm $\|.\|$ as two norms $\|.\|_2$. we have

$$\begin{aligned}\theta(\alpha,\ \beta,\ \varpi) &= \|\ \varpi(\alpha I + \mathscr{H}_1 + \mathscr{S}_1)^{-1}\ \|_2 + \|\ \varpi(\beta I + \mathscr{S}_2 + \mathscr{H}_2)\|_2\\ &= 2\varpi\|(E_1 + \alpha I)^{-1}\|_2 = 2\varpi\|(E_2 + \beta I)^{-1}\|_2\\ &= 2\varpi \max_{\mu_i \epsilon sp(E_1)} \left|\frac{1}{\mu_i + \alpha}\right| = \frac{2\varpi}{\alpha + \mu_{\min}}\\ &= 2\varpi \max_{\sigma_i \epsilon sp(E_2)} \left|\frac{1}{\sigma_i + \alpha}\right| = \frac{2\varpi}{\alpha + \sigma_{\min}}\end{aligned}$$

$\nu = \|E\|_2 - 1 = \lambda_{\max} - 1.$

By $0 < c_1(\alpha,\ \varpi) + c_2(\beta, \varpi) \le \theta(\alpha,\ \beta,\ \varpi)\nu < 4$, We can get the simplified convergence condition of the RMGHSS-like iterative method as follows.

$$0 < \frac{\varpi(\lambda_{\max} - 1)}{\alpha + \mu_{\min}} < 2\ or\ 0 < \frac{\varpi(\lambda_{\max} - 1)}{\beta + \sigma_{\min}} < 2 \tag{23}$$

This completes the proof of the convergence conditions of the RMGHSS-like iterative method.

$\beta - \alpha = \mu_{\min} - \sigma_{\min}$ is the relationship between the parameters $\alpha,\ \beta$ that meet the minimum spectral radius, and the corresponding parameters ϖ also have a relation:

(1) if $\lambda_{\max} > 1$, then $\lambda_{\max} - 1 > 0$, Thus according to (23), we have

$$0 < \varpi < \frac{2(\beta + \sigma_{\min})}{\lambda_{\max} - 1} \ \ or\ 0 < \varpi < \frac{2(\alpha + \mu_{\min})}{\lambda_{\max} - 1} \tag{24}$$

(2) if $\lambda_{\max} \leq 1$, from $\varpi > 0$, it can be seen that $\frac{\varpi(\lambda_{\max}-1)}{\beta+\sigma_{\min}} \leq 0$ and $\frac{\varpi(\lambda_{\max}-1)}{\beta+\sigma_{\min}} > 0$ contradict each other.

Therefore, we can obtain that

$$where\ \lambda_{\max} > 1,\ \ 0 < \varpi < \frac{2(\beta + \sigma_{\min})}{\lambda_{\max} - 1} \ \ or\ 0 < \varpi < \frac{2(\alpha + \mu_{\min})}{\lambda_{\max} - 1} \tag{25}$$

This completes the proof of the parameter ϖ value range.

The above processes are the proof of the Theorem 5.1.

Theorem 5.2. *Let the matrix $E \in \mathbb{R}^{n\times n}$ be Hermitian positive definite and $E_1 = \mathscr{H}_1 + R$, $E_2 = \mathscr{S}_1 + \mathscr{S}_2$, $E = E_1 + E_2$. Let $\lambda_{\min}$ and $\lambda_{\max}$ be the smallest and largest eigenvalue of matrix E, respectively. Let $\mu_{\min}$ and $\mu_{\max}$ be the smallest and largest eigenvalue of matrix E_1, respectively. Let $\sigma_{\min}$ and $\sigma_{\max}$ be the smallest and largest eigenvalue of matrix E_2, respectively. Under the minimum spectral radius, inequality relationships between parameters α and β, the parameters ϖ value range, and the conditions of the RMGHSS-like iterative method as follows:*
with $\|.\|$ being a matrix 2-norm at the section 5.

The inequality equation relationship between parameters $\alpha,\ \beta$ and the parameters ϖ value range are

(1) where $\mu_{\max} - \mu_{\min} < \sigma_{\max} - \sigma_{\min}$, we have

$$\begin{cases} \mu_{\min} + \sigma_{\min} - 2\mu_{\max} \leq \alpha - \beta \leq \mu_{\max} + \sigma_{\max} - 2\mu_{\min} \\ 0 < \varpi < \frac{4}{(\mu_{\min}+\sigma_{\max}-1)\left(\frac{1}{\alpha+\mu_{\min}} + \frac{1}{\beta+\sigma_{\min}}\right)} \end{cases}$$

(2) where $\mu_{\max} - \mu_{\min} \geq \sigma_{\max} - \sigma_{\min}$, we have

$$\begin{cases} 2\sigma_{\min} - \mu_{\max} - \sigma_{\max} \leq \alpha - \beta \leq 2\sigma_{\max} - \sigma_{\min} - \mu_{\min} \\ 0 < \varpi < \frac{4}{(\mu_{\max}+\sigma_{\min}-1)\left(\frac{1}{\alpha+\mu_{\min}} + \frac{1}{\beta+\sigma_{\min}}\right)} \end{cases}$$

The condition of the RMGHSS-like iterative method is

$$0 < \varpi(\mu_{\max} + \sigma_{\max} - 1)\left(\frac{1}{\alpha + \mu_{\min}} + \frac{1}{\beta + \sigma_{\min}}\right) < 4$$

Proof. Since the matrix $\mathscr{H}_1$, $\mathscr{H}_2$ are Hermitian positive definite matrix and the matrix $\mathscr{S}_1$, $\mathscr{S}_2$ are skew-Hermitian positive definite matrix,we can obtain the matrix $E_1 = \mathscr{H}_1 + \mathscr{S}_1$ and $E_2 = \mathscr{H}_2 + \mathscr{S}_2$ are positive definite matrix according to the Lemma 2.4. Further, Since E is a Hermitian positive definite matrix, we can know thatE_1 and E_2 are Hermitian positive definite matrices according to the above (22).

Combing the equality in (22), we get

$$\begin{cases} E = 2E_1 + (\alpha - \beta)I \\ E = 2E_2 + (\beta - \alpha)I \\ E = E_1 + E_2 \end{cases} \tag{26}$$

According to Lemma 2.6, we have

$$\begin{cases} \lambda_k(2E_1) + \lambda_1((\alpha-\beta)I) \le \lambda_k(2E_1 + (\alpha-\beta)I) \le \lambda_k(2E_1) + \lambda_n((\alpha-\beta)I) \\ \lambda_k((\alpha-\beta)I) + \lambda_1(2E_1) \le \lambda_k((\alpha-\beta)I + 2E_1) \le \lambda_k((\alpha-\beta)I) + \lambda_n(2E_1) \end{cases} \tag{27}$$

$$\begin{cases} \lambda_k(2E_2) + \lambda_1((\beta-\alpha)I) \le \lambda_k(2E_2 + (\beta-\alpha)I) \le \lambda_k(2E_2) + \lambda_n((\beta-\alpha)I) \\ \lambda_k((\beta-\alpha)I) + \lambda_1(2E_2) \le \lambda_k((\beta-\alpha)I + 2E_2) \le \lambda_k((\beta-\alpha)I) + \lambda_n(2E_2) \end{cases} \tag{28}$$

$$\begin{cases} \lambda_k(E_1) + \lambda_1(E_2) \le \lambda_k(E_1+E_2) \le \lambda_k(E_1) + \lambda_n(E_2) \\ \lambda_k(E_2) + \lambda_1(E_1) \le \lambda_k(E_2+E_1) \le \lambda_k(E_2) + \lambda_n(E_1) \end{cases} \tag{29}$$

For a Hermitian matrix E, we shall use the notation $\lambda_k(E)$ to designate the kth largest eigenvalue, i.e, $\lambda_n(E) \le \cdots \le \lambda_2(E) \le \lambda_1(E)$.

From the the inequality in (27), we get

$$\begin{cases} \lambda_{\max} \le 2\mu_{\max} + \lambda_{\max}((\alpha-\beta)I);\ \lambda_{\min} \le 2\mu_{\min} + \lambda_{\max}((\alpha-\beta)I). \\ \lambda_{\max} \le 2\mu_{\max} + \lambda_{\max}((\alpha-\beta)I);\ \lambda_{\min} \le 2\mu_{\max} + \lambda_{\min}((\alpha-\beta)I). \\ \lambda_{\max} \ge 2\mu_{\max} + \lambda_{\min}((\alpha-\beta)I);\ \lambda_{\min} \ge 2\mu_{\min} + \lambda_{\min}((\alpha-\beta)I). \\ \lambda_{\max} \ge 2\mu_{\min} + \lambda_{\max}((\alpha-\beta)I);\ \lambda_{\min} \ge 2\mu_{\min} + \lambda_{\min}((\alpha-\beta)I). \end{cases} \tag{30}$$

From the the inequality in (28) , we get

$$\begin{cases} \lambda_{\max} \le 2\sigma_{\max} + \lambda_{\max}((\beta-\alpha)I);\ \lambda_{\min} \le 2\sigma_{\min} + \lambda_{\max}((\beta-\alpha)I). \\ \lambda_{\max} \le 2\sigma_{\max} + \lambda_{\max}((\beta-\alpha)I);\ \lambda_{\min} \le 2\sigma_{\max} + \lambda_{\min}((\beta-\alpha)I). \\ \lambda_{\max} \ge 2\sigma_{\max} + \lambda_{\min}((\beta-\alpha)I);\ \lambda_{\min} \ge 2\sigma_{\min} + \lambda_{\min}((\beta-\alpha)I). \\ \lambda_{\max} \ge 2\sigma_{\min} + \lambda_{\max}((\beta-\alpha)I);\ \lambda_{\min} \ge 2\sigma_{\min} + \lambda_{\min}((\beta-\alpha)I). \end{cases} \tag{31}$$

From the the inequality in (29) , we get

$$\begin{cases} \lambda_{\max} \le \mu_{\max} + \sigma_{\max};\ \lambda_{\min} \le \mu_{\min} + \sigma_{\max}. \\ \lambda_{\max} \le \sigma_{\max} + \mu_{\max};\ \lambda_{\min} \le \sigma_{\min} + \mu_{\max}. \\ \lambda_{\max} \ge \mu_{\max} + \sigma_{\min};\ \lambda_{\min} \ge \mu_{\min} + \sigma_{\min}. \\ \lambda_{\max} \ge \sigma_{\max} + \mu_{\min};\ \lambda_{\min} \ge \sigma_{\min} + \mu_{\min}. \end{cases} \tag{32}$$

By (30), we have

$$\lambda_{\min} - 2\mu_{\max} \le \lambda_{\min}((\alpha-\beta)I) \le \lambda_{\max}((\alpha-\beta)I) \le \lambda_{\max} - 2\mu_{\min} \tag{33}$$

Combining (32) and $\lambda_{\min}((\alpha-\beta)I) = \lambda_{\max}((\alpha-\beta)I)$ further, we can get

$$\mu_{\min} + \sigma_{\min} - 2\mu_{\max} \le \alpha - \beta \le \mu_{\max} + \sigma_{\max} - 2\mu_{\min} \tag{34}$$

By (30), we have

$$\lambda_{\min} - 2\sigma_{\max} \le \lambda_{\min}((\beta-\alpha)I) \le \lambda_{\max}((\beta-\alpha)I) \le \lambda_{\max} - 2\sigma_{\min} \tag{35}$$

Combining (32) and $\lambda_{\min}((\beta-\alpha)I) = \lambda_{\max}((\beta-\alpha)I)$ further, we can get

$$2\sigma_{\min} - \sigma_{\max} - \mu_{\max} \le \alpha - \beta \le 2\sigma_{\max} - \sigma_{\min} - \mu_{\min} \tag{36}$$

Hence, we can know the inequality equation relationship between parameters α, β are as follows.

(1) where $\mu_{\max} - \mu_{\min} < \sigma_{\max} - \sigma_{\min}$, we have

$$\because \begin{cases} \mu_{\max} + \sigma_{\max} - 2\mu_{\min} < 2\sigma_{\max} - \sigma_{\min} - \mu_{\min} \\ 2\sigma_{\min} - \mu_{\max} - \sigma_{\max} < \sigma_{\min} + \mu_{\min} - 2\mu_{\max} \end{cases}$$

$$\therefore \quad \mu_{\min} + \sigma_{\min} - 2\mu_{\max} \le \alpha - \beta \le \mu_{\max} + \sigma_{\max} - 2\mu_{\min}$$

(2) where $\mu_{\max} - \mu_{\min} \ge \sigma_{\max} - \sigma_{\min}$, we have

$$\because \begin{cases} \mu_{\max} + \sigma_{\max} - 2\mu_{\min} \ge 2\sigma_{\max} - \sigma_{\min} - \mu_{\min} \\ 2\sigma_{\min} - \mu_{\max} - \sigma_{\max} \ge \sigma_{\min} + \mu_{\min} - 2\mu_{\max} \end{cases}$$

$$\therefore \quad 2\sigma_{\min} - \mu_{\max} - \sigma_{\max} \le \alpha - \beta \le 2\sigma_{\max} - \sigma_{\min} - \mu_{\min}$$

Therefore, the parameters ϖ value range are as follows.

(1) where $\mu_{\max} - \mu_{\min} < \sigma_{\max} - \sigma_{\min}$, we have

$$\because \begin{cases} \mu_{\max} + \sigma_{\min} < \sigma_{\max} + \mu_{\min} \\ 0 < w\left(\lambda_{\max} - 1\right)\left(\frac{1}{\alpha+\mu_{\min}} + \frac{1}{\beta+\sigma_{\min}}\right) < 4 \end{cases}$$

$$\because \begin{cases} \lambda_{\max} \ge \mu_{\max} + \sigma_{\min} \\ \lambda_{\max} \ge \sigma_{\max} + \mu_{\min} \end{cases}$$

$$\therefore \frac{1}{\lambda_{\max} - 1} \le \frac{1}{\mu_{\min} + \sigma_{\max} - 1}$$

Hence, we can obtain

$$0 < \varpi < \frac{4}{\left(\mu_{\min} + \sigma_{\max} - 1\right)\left(\frac{1}{\alpha+\mu_{\min}} + \frac{1}{\beta+\sigma_{\min}}\right)}, \; where \;\; \lambda_{\max} > 1.$$

(2) where $\mu_{\max} - \mu_{\min} \ge \sigma_{\max} - \sigma_{\min}$, we have

$$0 < \varpi < \frac{4}{\left(\mu_{\max} + \sigma_{\min} - 1\right)\left(\frac{1}{\alpha+\mu_{\min}} + \frac{1}{\beta+\sigma_{\min}}\right)}, \; where \;\; \lambda_{\max} > 1.$$

From the above analysis process, we can also get the convergence condition of the RMGHSS-like iterative method is

$$0 < \varpi\left(\mu_{\max} + \sigma_{\max} - 1\right)\left(\frac{1}{\alpha + \mu_{\min}} + \frac{1}{\beta + \sigma_{\min}}\right) < 4$$

This completes the proof of the Theorem 5.2 .

6. Numerical Examples

In this section, various numerical examples are given to show the efficiency of the RMGHSS-like method (15) and the NHSS-like method (17) from perspectives of the iteration steps (denoted by IT), elapsed CPU time in seconds (denoted by CPU), and ERR. We also present the following formula about ERR and RES

$$ERR \ := \|u^{\wedge} - u^{(k)}\|, \qquad RES \ := \|Eu^{(k)} - f - |u^{(k)}|\ \|,$$

here, $u^{\wedge}$ and $u^{(k)}$ represent the exact solution and the kth approximate solution to AVE (1).

Besides, we did all the experiments using a personal computer with 2.80 GHZ CPU(Inter(R) Core(TM) i7-7700HQ) and 8GB of memory using Matlab R2018b and chose the zero vector as the initial vector. For Example 6.1 and Example 6.2, if the current iterations satisfy $RES(u^k) \ \leq 10^{-7}$ or the maximum iteration number exceed 2000, the test problems are terminated. For Example 6.3 and Example 6.4, if the current iterations satisfy $RES \ \leq 10^{-5}\|b\|$ or the maximum iteration number exceed 2000, the test problems are terminated.

Then, we label the RMGHSS-like method, the new method [43], the modified block-diagonal and anti-block-diagonal splitting method [44], the relaxed nonlinear PHSS-Like method [37], the new HSS-like method as MGHSS-like, NI, MBAS, PHSS-like and NHSS-like, respectively. The parameters of these NI, MBAS, RPHSS-Like, NHSS-Like and RMGHSS-like methods as $\alpha_{opt}, \beta_{opt}, \varpi_{opt}, \gamma$, which obtained by reducing the IT as much as possible in all experiments.

Example 6.1. *([45]). We consider the AVE in (1) with*

$$E = tridiag\,(-1, 8, -1) \in \mathbb{R}^{n\times n} \ \ and \ \ f = Eu^{\wedge} - |u^{\wedge}| \in \mathbb{R}^{n\times n}$$

here $u^{\wedge} = (-1, 1, -1, 1, \cdots, -1, 1)^T \in \mathbb{R}^n$.

Through using the Example 6.1, we can obtain the computational results in Table 1. Most importantly, the IT of the RMGHSS-like method is greatly reduced compared to other four methods. Secondly, the ERR of the RMGHSS-like method is better than the NI method by four orders of magnitude, and is better than the MBAS method by one order of magnitude. The RES of the RMGHSS-like method is better than the NI and MBAS methods. The elapsed CPU time is less than PHSS−like, NHSS−like methods.

Example 6.2. *([45]). Let* $m = n^2$ *where n is a positive integer. We consider(1) with*

$$E = tridiag\,(-I_n,\ Z_n,\ -I_n) \in \mathbb{R}^{n\times n} \ \ with\ Z_n = tridiag\,(-1,\ 8,\ -1)$$

here $f = Eu^{\wedge} - |u^{\wedge}| \in \mathbb{R}^{n\times n}$ *and* $u^{\wedge} = (-1, 1, -1, 1, \cdots, -1, 1)^T \in \mathbb{R}^n$.

Through using the Example 6.2, we can obtain the experimental results in Table 2. Although the elapsed CPU times of the NI, MBAS methods is less than RMGHSS-like method, the RES of RMGHSS-like method is better than the NI method by two order of magnitude and is better than than MBAS method by one order of magnitude. Further, the ERR of RMGHSS-like method is better than the NI method by five order of magnitude and is better than the MBAS, RPHSS-like, NHSS-like methods by one order of magnitude. Mostly important, the IT is less than four other methods.

Consider the two-dimensional convection-diffusion equation [34, 35, 37]

$$\begin{cases} -(\varphi_{uu} + \varphi_{tt}) + \tau(\varphi_u + \varphi_t) + \kappa\varphi = f(u,t), & (u,t) \in \mathbb{D} \\ \varphi(u,t) = 0, & (u,t) \in \partial\mathbb{D} \end{cases}$$

where $\mathbb{D} = (0,1)\times(0,1)$, $\partial\mathbb{D}$ is the boundary of $\mathbb{D}$, the symbol τ and the symbol κ represent a constant greater than zero and a real number, respectively. The diffusion terms are discretized using the five-point finite difference method, and the convection terms are also discretized using the central difference method. The

Table 1: Numerical results for Example 6.1.

	n	1000	2000	3000	4000	5000	6000
NI	α_{opt}	0.980	0.980	0.980	0.980	0.980	0.980
	IT	7	7	7	7	7	7
	CPU	0.0135	0.0358	0.0786	0.1468	0.2491	0.3126
	RES	6.7892e−07	6.7897e−07	6.7898e−07	6.7899e−07	6.7900e−07	6.7900e−07
	ERR	3.4503e−05	4.8805e−05	5.9779e−05	6.9028e−05	7.7178e−05	8.4545e−05
MBAS	α_{opt}	0.006	0.006	0.006	0.006	0.006	0.006
	IT	11	11	11	11	11	12
	CPU	0.0206	0.0588	0.1264	0.2223	0.3429	0.5292
	RES	4.4693e−08	6.3219e−08	7.7433e−08	8.9416e−08	9.9972e−08	1.7456e−08
	ERR	7.6177e−09	1.0776e−08	1.3200e−08	1.5243e−08	1.7042e−08	2.1522e−08
RPHSS−like	α_{opt}	9.180	9.180	9.180	9.180	9.180	9.180
	ϖ_{opt}	1.10	1.10	1.10	1.10	1.10	1.10
	IT	12	12	12	12	12	12
	CPU	0.0915	0.3047	0.6476	1.2304	1.8564	3.1814
	RES	1.9816e−08	2.8060e−08	3.4381e−08	3.9708e−08	4.4401e−08	4.8643e−08
	ERR	1.9696e−09	2.7890e−09	3.4172e−09	3.9467e−09	4.4132e−09	4.8348e−09
NHSS−like	α_{opt}	7.800	7.800	7.800	7.800	7.800	7.800
	ϖ_{opt}	1.00	1.00	1.00	1.00	1.00	1.00
	IT	11	12	12	12	12	12
	CPU	0.0797	0.3230	0.7243	1.2910	2.0626	2.9731
	RES	7.8135e−08	1.4699e−08	1.8012e−08	2.0803e−08	2.3262e−08	2.5485e−08
	ERR	1.0753e−09	2.1353e−09	2.6166e−09	3.0222e−09	3.3795e−09	3.7025e−09
RMGHSS−like	α_{opt}	0.100	0.100	0.100	0.100	0.100	0.100
	β_{opt}	12.190	12.190	12.190	12.190	12.190	12.190
	ϖ_{opt}	1.10	1.10	1.10	1.10	1.10	1.10
	γ	0.65	0.65	0.65	0.65	0.65	0.65
	IT	5	5	5	5	5	5
	CPU	0.0379	0.1246	0.2878	0.4910	0.8162	1.2532
	RES	3.6989e−08	4.2395e−08	4.7713e−08	5.2617e−08	5.7146e−08	6.1362e−08
	ERR	4.4942e−09	5.1325e−09	5.8072e−09	6.4361e−09	7.0179e−09	7.5593e−09

use of symbol ϱ for equidistant steps and symbol Re for the grid Reynolds number, and has a relationship: $\varrho = 1/(m+1)$ and $Re = (\tau\varrho)/2$. Then a system of linear equations with coefficients of nth-order matrices is obtained Fu = d where $n = m^2$.

$$F = T_u \otimes I_m + I_m \otimes T_t + \kappa I_n, \tag{37}$$

where I_m and I_n are the identity matrices of order m and n, respectively. ⊗ present the Kronecker product, and T_u and T_t present the tridiagonal matrices

$$T_u = tri\mathrm{diag}\,(t_2,\ t_1,\ t_3)_{m\times m} \quad and \quad T_t = tri\mathrm{diag}\,(t_2,\ 0,\ t_3)_{m\times m}$$

with

$$t_1 = 4, \quad t_2 = -1 - Re, \quad t_3 = -1 + Re.$$

For convenience, we set $\mathscr{H}_2 = \mathscr{H} - \mathscr{H}_1 = \varrho/2$, $\mathscr{S}_1 = (1-\gamma)\mathscr{S}$ and $\mathscr{S}_2 = \gamma S$ according to [38].

Table 2: Numerical results for Example 6.2.

	n	1600	2500	3600	4900
NI	α_{opt}	0.940	0.940	0.940	0.940
	IT	9	9	9	9
	CPU	0.0337	0.0692	0.1455	0.2799
	RES	5.7693e−07	5.8249e−07	5.8620e−07	5.8885e−07
	ERR	3.9803e−05	5.0294e−05	6.0784e−05	7.1275e−05
MBAS	α_{opt}	0.006	0.006	0.006	0.006
	IT	13	14	14	14
	CPU	0.0490	0.1554	0.2276	0.3925
	RES	8.3264e−08	2.1207e−08	2.5706e−08	3.0205e−08
	ERR	1.8331e−08	4.4462e−09	5.3928e−09	6.3393e−09
RPHSS−like	α_{opt}	6.120	6.120	6.120	6.120
	ϖ_{opt}	1.00	1.00	1.00	1.00
	IT	16	16	16	16
	CPU	0.2850	0.6565	1.4041	2.4823
	RES	1.9816e−08	2.8060e−08	3.4381e−08	3.9708e−08
	ERR	1.9696e−09	2.7890e−09	3.4172e−09	3.9467e−09
NHSS−like	α_{opt}	6.142	6.142	6.142	6.142
	ϖ_{opt}	1.00	1.00	1.00	1.00
	IT	16	16	16	16
	CPU	0.2638	0.6548	1.3437	2.5905
	RES	4.8647e−08	5.5069e−08	6.0869e−08	6.6212e−08
	ERR	4.3679e−09	5.0371e−09	5.6681e−09	6.2723e−09
RMGHSS−like	α_{opt}	0.633	0.633	0.633	0.633
	β_{opt}	12.210	12.210	12.210	12.210
	ϖ_{opt}	1.20	1.20	1.20	1.20
	γ	0.65	0.65	0.65	0.65
	IT	7	8	8	8
	CPU	0.1289	0.3238	0.6721	2.2753
	RES	8.9156e−08	7.0492e−09	7.7679e−09	8.4351e−09
	ERR	9.9991e−09	6.6449e−10	7.4922e−10	8.3128e−10

Example 6.3. *In this example, we first set the matrix $E = F$ where F is defined by (37) and let $\tau = 0$, $\kappa = 0$. Computational results with different values of m (m = 10, 20, 40, 80) are given in Table 3.*

It is handy to discover the matrix E in AVE (1) is symmetric positive definite.The experimental results for Examples 6.3 are presented in Table 3. In Table 3, we can obtain that the NI method does no longer resolve the problem. The RES of the RMGHSS-like is comparable to the RES of the MBAS, RPHSS−like and NHSS-like. In addition, we can find the number of iteration steps of the proposed MGHSS-like is greatly reduced compared to other three methods. The elapsed CPU time is less than PHSS−like, NHSS−like methods. we remark here that the RMGHSS-like iteration method is more efficient than the RPHSS−like iteration method by choosing suitable choices of $\varpi, \gamma, \alpha, \beta$.

Example 6.4. *In this example, we set the matrix $E = F + 0.5(L\text{-}L^T)$ where F is defined by (37). Computational results with different values of n (n = 100, 400, 9000, 1600, 2500), different values of τ (τ = 1, 10, 100) and κ = -1 are given in Table 4, Table 5 and Table 6, respectively.*

From Tables 4, 5 and 6, we get that the ERR of the RMGHSS-like method is smaller than that of the

Table 3: Numerical results for Example 6.3.

	m	10	20	40	80
NI	α_{opt}	–	–	–	–
	IT	–	–	–	–
	CPU	–	–	–	–
	RES	–	–	–	–
MBAS	α_{opt}	0.400	0.400	0.400	0.400
	IT	29	27	25	23
	CPU	0.0020	0.0017	0.0835	1.1302
	RES	8.5303e−06	8.3904e−06	8.0517e−06	8.6230e−06
RPHSS−like	α_{opt}	1.800	1.800	1.800	1.800
	ϖ_{opt}	1.20	1.20	1.20	1.20
	IT	18	16	15	13
	CPU	0.0010	0.0188	0.2278	3.7599
	RES	8.4831e−06	8.4027e−06	6.6096e−06	8.4504e−06
NHSS−like	α_{opt}	1.800	1.800	1.800	1.800
	ϖ_{opt}	1.00	1.00	1.00	1.00
	IT	26	20	18	17
	CPU	0.8769	0.0453	1.3437	2.5905
	RES	8.4831e−06	8.4027e−06	6.6096e−06	8.4504e−06
RMGHSS−like	α_{opt}	1.640	1.640	1.640	1.640
	β_{opt}	2.000	2.000	2.000	2.000
	ϖ_{opt}	1.20	1.20	1.20	1.20
	γ	0.40	0.40	0.40	0.40
	IT	13	12	11	10
	CPU	0.0005	0.0110	0.2076	2.8502
	RES	8.4716e−06	8.6833e−06	8.6540e−06	8.9804e−06

rest of the methods, which is smaller than that of the rest of the methods. From Tables 5 and Tables 6, we find that the proposed method has achieved a qualitative leap and reduced the number of iterative steps to within five steps, and the ERR and the RES of the proposed RMGHSS-like method have also been greatly reduced, which is unprecedented. The CPU time of the MBAS is less than the RMGHSS-like method , but the iteration steps of the RMGHSS-like method is Greatly less than the MBAS method. Therefore, we can infer that the RMGHSS-like method is generally more efficent than other four methods.

Table 4: Computational results for Example 6.4 with different values of n and κ = -1 and τ = 1.

	n	100	400	900	1600	2500
NI	α_{opt}	0.600	0.600	0.600	0.600	0.600
	IT	22	24	24	25	25
	CPU	0.0087	0.1056	0.7266	3.3418	10.5789
	ERR	2.9207e−04	5.6496e−04	1.0792e−03	1.0580e−03	1.4039e−03
	RES	9.2354e−06	7.5608e−06	9.2031e−06	6.6512e−06	6.9966e−06
MBAS	α_{opt}	0.200	0.200	0.200	0.200	0.200
	IT	16	18	18	18	18
	CPU	0.0007	0.0009	0.0229	0.0577	0.1741
	ERR	2.5242e−04	4.5506e−04	9.4177e−04	1.4334e−03	1.9250e−03
	RES	7.8344e−06	5.7642e−06	7.5006e−06	8.3818e−06	8.9100e−06
RPHSS−like	α_{opt}	4.600	4.600	4.600	4.600	4.600
	ϖ_{opt}	1.20	1.20	1.20	1.20	1.20
	IT	14	17	18	18	19
	CPU	0.0005	0.0148	0.1064	0.3105	0.8088
	ERR	1.9408−04	9.0802e−04	1.5130e−03	2.5143e−03	2.1991e−03
	RES	4.9558e−06	8.3513e−06	8.2858e−06	9.9737e−06	6.8761e−06
NHSS−like	α_{opt}	2.900	2.900	2.900	2.900	2.900
	ϖ_{opt}	1.00	1.00	1.00	1.00	1.00
	IT	13	15	16	16	16
	CPU	0.0006	0.0190	0.0798	0.2789	0.6649
	ERR	1.2914e−04	8.7241e−04	1.3192e−03	2.1893e−03	3.0562e−03
	RES	4.7199e−06	8.1330e−06	7.2904e−06	8.7578e−06	9.6287e−06
RMGHSS−like	α_{opt}	1.960	1.960	1.960	1.960	1.960
	β_{opt}	3.500	3.500	3.500	3.500	3.500
	ϖ_{opt}	1.20	1.20	1.20	1.20	1.20
	γ	0.40	0.40	0.40	0.40	0.40
	IT	8	10	11	11	11
	CPU	0.0003	0.0086	0.0540	0.1629	0.4734
	ERR	5.3861e−05	6.9996e−04	9.2827e−04	1.6533e−04	2.3730e−04
	RES	4.0810e−06	9.8772e−06	7.0297e−06	8.8508e−06	9.9257e−06

Table 5: Computational results for Example 6.4 with different values of n and κ = -1 and τ = 10.

	n	100	400	900	1600	2500
NI	α_{opt}	0.500	0.500	0.500	0.500	0.500
	IT	15	15	15	15	15
	CPU	0.0032	0.0674	0.4630	2.0905	6.4030
	ERR	1.2149e−04	2.4836e−04	3.7523e−04	5.0210e−04	6.2897e−04
	RES	9.2684e−06	9.3743e−06	9.4103e−06	9.4284e−06	9.43e93−06
MBAS	α_{opt}	0.700	0.700	0.700	0.700	0.700
	IT	13	13	13	13	13
	CPU	0.0002	0.0009	0.0154	0.0377	0.1119
	ERR	7.3282e−05	1.5035e−04	2.2742e−04	3.0449e−04	3.8156e−04
	RES	5.6581e−06	5.7487e−06	5.7792e−06	5.7945e−06	5.8037e−06
RPHSS−like	α_{opt}	3.430	3.430	3.430	3.430	3.430
	ϖ_{opt}	1.20	1.20	1.20	1.20	1.20
	IT	15	16	16	17	17
	CPU	0.0005	0.0154	0.0814	0.3146	0.7282
	ERR	8.9377−05	2.0445e−04	3.6908e−04	2.7192e−04	3.5571e−04
	RES	8.6365e−06	8.6094e−06	9.8717e−06	5.3638e−06	5.5600e−06
NHSS−like	α_{opt}	11.800	11.800	11.800	11.800	11.800
	ϖ_{opt}	1.00	1.00	1.00	1.00	1.00
	IT	6	6	6	6	6
	CPU	0.0003	0.0086	0.0342	0.0986	0.2556
	ERR	4.5508e−05	6.8051e−05	8.5089e−05	9.9571e−05	1.1250e−04
	RES	5.6578e−06	4.2264e−06	3.5153e−06	3.0779e−06	2.7755e−06
RMGHSS−like	α_{opt}	4.070	4.070	4.070	4.070	4.070
	β_{opt}	17.188	17.188	17.188	17.188	17.188
	ϖ_{opt}	1.20	1.20	1.20	1.20	1.20
	γ	0.40	0.40	0.40	0.40	0.40
	IT	3	3	3	3	3
	CPU	0.0002	0.0046	0.0258	0.0549	0.1381
	ERR	6.4733e−07	1.0711e−06	1.5820e−06	2.1050e−06	2.6323e−06
	RES	6.5092e−08	5.2860e−08	5.1842e−08	5.1646e−08	5.1615e−08

Table 6: Computational results for Example 6.4 with different values of n and κ = -1 and τ = 100.

	n	100	400	900	1600	2500
NI	α_{opt}	1.500	1.500	1.500	1.500	1.500
	IT	12	12	12	12	12
	CPU	0.0024	0.0500	0.3984	1.6593	5.2037
	ERR	5.3281e−05	1.0680e−04	1.6032e−04	2.1384e−04	2.6735e−04
	RES	5.1886e−06	5.1909e−06	5.1917e−06	5.1921e−06	5.1923e−06
MBAS	α_{opt}	0.900	0.900	0.900	0.900	0.900
	IT	11	11	11	11	11
	CPU	0.0002	0.0009	0.0118	0.0450	0.0975
	ERR	5.8419e−05	1.1707e−04	1.7571e−04	2.3436e−04	2.9301e−04
	RES	5.6296e−06	5.6296e−06	5.6296e−06	5.6296e−06	5.6296e−06
RPHSS−like	α_{opt}	15.700	15.700	15.700	15.700	15.700
	ϖ_{opt}	1.20	1.20	1.20	1.20	1.20
	IT	14	14	14	14	14
	CPU	0.0006	0.0137	0.0741	0.2597	0.5973
	ERR	7.6042e−05	1.3499e−04	1.9272e−04	2.5014e−04	3.0744e−04
	RES	7.7135e−06	6.8019e−06	6.4540e−06	6.2716e−06	6.1593e−06
NHSS−like	α_{opt}	97.400	97.400	97.400	97.400	97.400
	ϖ_{opt}	1.00	1.00	1.00	1.00	1.00
	IT	4	4	4	4	4
	CPU	0.0002	0.0047	0.0295	0.0747	0.1669
	ERR	1.4172e−05	2.7122e−05	3.9949e−05	5.2741e−05	6.5520e−05
	RES	1.4277e−06	1.3634e−06	1.3376e−06	1.3238e−06	1.3152e−06
RMGHSS−like	α_{opt}	21.100	21.100	21.100	21.100	21.100
	β_{opt}	146.240	146.240	146.240	146.240	146.240
	ϖ_{opt}	1.20	1.20	1.20	1.20	1.20
	γ	0.40	0.40	0.40	0.40	0.40
	IT	2	2	2	2	2
	CPU	0.0001	0.0034	0.0145	0.0411	0.1229
	ERR	6.4733e−07	1.0711e−06	1.5820e−06	2.1050e−06	2.6323e−06
	RES	6.5092e−08	5.2860e−08	5.1842e−08	5.1646e−08	5.1615e−08

7. Conclusions

In this paper, we have introduced the RMGHSS-like method for the AVE. We prove the convergence of the method and analyse the relationship between the parameters and the convergence properties in detail when matrix E is a Hermitian positive definite matrix under the minimum spectral radius. Numerical experiments had been given to apprehend the effectiveness of the RMGHSS-like method. However, the values of the optimal parameters of this iteration method need to be considered.

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