Filomat 37:26 (2023), 8829–8844
https://doi.org/10.2298/FIL2326829B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Exponential stability of delayed neutral impulsive stochastic integro-differential systems perturbed by fractional Brownian motion and Poisson jumps

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Abstract. In this manuscript, we investigate the existence, uniqueness, and exponential stability of a delayed neutral impulsive stochastic integro-differential equation driven by fractional Brownian motion in a separable Hilbert space and Poisson jumps. The results are obtained, using the theory of resolvent operators, stochastic analysis, and a fixed-point technique. Lastly, an example is provided to show the validity of the obtained results.

1. Introduction

The theory of functional differential equations have been widely applied to modeling real phenomena of quite different natures (such as physics, biology, mechanics, medicine, control theory, information theory, chemistry, economic, etc). A defining property of functional differential equations is that the evolution rate of the processes described by these equations depends on the past history. Moreover, impulsive functional differential equations have been extensively developed to describe evolutionary processes in which the values of the parameters undergo short-term rapid changes, whose duration is negligible in contrast with the length of an entire evolution. Besides, Random perturbation are common in actual systems, which might be caused by unexpected environmental changes, stochastic failures, repairs of components, and so on. These random influences can result in a variety of complicated dynamic performances. As a result, it is important to include the stochastic impact in describing such systems. Such systems are frequently modelled by taking into account the theory of stochastic differential equations.

As an important and practical self-similar stochastic process, fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$ is gaussian, has stationary increments, and exhibits long range dependence for $1/2 < H < 1$. It has been extensively used to model dynamic random phenomena in diverse fields such as physics, finance, biology, etc. Driven by fBm, stochastic differential equations has gained a lot of attention. An existence and asymptotic behavior of solutions result to stochastic delay evolution equations perturbed by a fBm have been studied by Caraballo et al. [8]. Boufoussi et al. [6] have proved the existence, uniqueness, and exponential stability of mild solution for neutral stochastic differential equations with

2020 Mathematics Subject Classification. Primary 35B35, 35R10, 60G22, 60H15, 60J76;
Keywords. Neutral stochastic functional integro-differential equations, exponential stability, fractional Brownian motion, Resolvent operator, Poisson jumps.
Received: 13 December 2022; Revised: 26 April 2023; Accepted: 10 May 2023
Communicated by Miljana Jovanović
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finite delay driven by fBm. Recently, Benkabdi et al. [4] have established some new criteria ensuring the controllability of a class of retarded time-dependent neutral stochastic integro-differential equations driven by fBm. Very recently, Dhayal et al. [12] have derived a set of sufficient conditions guaranteeing the existence, uniqueness, and stability of mild solutions and controllability result for a class of non-instantaneous impulsive stochastic differential equations driven by mixed fBm. For more results on the topic, we refer to [5, 11, 19, 24] and references therein.

In reality, however, where the path continuity assumption does not appear suitable for the model (for instance, sudden price changes due to market crushes, epidemics, earthquake, etc), stochastic processes with jumps should be included in modeling such systems. Generally, these jump models are based on the Poisson random measure. Such systems have right-continuous sample paths with left limits. The study of stochastic differential equations with jumps has recently attracted more attention. Taniguchi et al.[23] have established the existence and asymptotic behaviour of mild solution to stochastic evolution equations with infinite delays driven by Poisson jumps. Lakhel et al.[17] have proved the existence, uniqueness and asymptotic behavior of mild solutions for a class of neutral functional stochastic differential equations with Poisson jumps. Very recently, the exponential behavior of a class of neutral impulsive stochastic integro-differential equations driven by Poisson jumps and Rosenblatt process have been studied by kasinathan et al. [14]. For more results, we refer to [1–3, 5, 7, 18, 22] and references therein.

The objective of this paper is to investigate the existence and exponential stability of the unique solution of the following system

$$
\begin{aligned}
& d\Big[x(t) - \Gamma_1(t, x_t, \int_0^t \mu_1(t, s, x_s)ds)\Big] \\
& \quad = \Big[A\Big[x(t) - \Gamma_1(t, x_t, \int_0^t \mu_1(t, s, x_s)ds)\Big] + \Gamma_2(t, x_t, \int_0^t \mu_2(t, s, x_s)ds)\Big]dt \\
& \quad + \Big[\int_0^t \overline{B}(t-s)\Big[x(s) - \Gamma_1(s, x_s, \int_0^s \mu_1(s, r, x_r)dr)\Big]ds\Big]dt \\
& \quad + \int_{\mathcal{U}} f(t, x_t, \eta)\widetilde{N}(dt, d\eta) + v(t)dB^H(t), \qquad t \in J := [0, T], \quad t \neq t_k, \\
& \quad \Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, ..., m, \ m \in \mathbb{N} \\
& \quad x_0 = \varphi \ \in \ Ca
\end{aligned} \tag{1}
$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators, $(S(t))_{t\geq 0}$, in a Hilbert space X; B^H is a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ on a real and separable Hilbert space Y; $\overline{B}(t)$ is a closed linear operator on X with Domain $D(\overline{B}(t)) \supset D(A)$. For any continuous function x and $t \in J$, $x_t : (-\tau, 0] \to X$, $x_t(r) = x(t+r)$, and $\Gamma_1, \Gamma_2 : J \times Ca \times X \to X$, $\mu_1, \mu_2 : D \times Ca \to X$, $v : [0, \infty) \to \mathcal{L}_2^0(Y, X)$, and $f : J \times Ca \times \mathcal{U} \to X$ are appropriate functions and will be specified later, where $\mathcal{L}_2^0(Y, X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X, $D = \{(t, s) \in I \times I : s < t\}$, and $Ca = \{\varphi \ : \ [-\tau, 0] \longrightarrow X, \varphi$ is càdlàg everywhere except a finit number of points τ_i at which $\varphi(\tau_i^-), \varphi(\tau_i^+)$ exist $\}$. For $\varphi \in Ca$, $\|\varphi\|_{Ca} = \sup_{-\tau\leq s\leq 0}(\mathbb{E}\|\varphi(s)\|^2)^{\frac{1}{2}} < +\infty$. Moreover, the fixed moments of time t_k satisfy $0 < t_1 < t_2 < ... < t_m < T$. $\Delta x(t_k)$ represents the leap in the state x at time t_k with $I(.) : X \longrightarrow X$ determining the size of the leap.

Although there are numerous results concerning the solvability and stability of stochastic differential equations, to our knowledge, no research has considered the existence, uniqueness, and exponential stability of delayed impulsive neutral stochastic integro-differential equations driven by fBm and Poisson jumps. To illuminate this uncharted area, we will attempt to research this issue for the first time in this work.

The outline of this paper is as follows: Some preliminary results and notations are provided in section 2. Section 3 demonstrates the existence, uniqueness, and stability of system (1) by employing a fixed point strategy. Section 4 gives an illustrative example of our result. Lastly, section 5 concludes the paper.

2. Preliminaries

For more information on the subjects covered in this section, we refer to [13, 16, 20].

Let $(\mathcal{U}, \mathfrak{B}, \nu(du))$ be a σ-finite measurable space, given a Poisson point process $(q(t))_{t>0}$ which is defined on a complete probabilty space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathcal{U}$ and with characteristic measure ν (see [16]). $N(dt, du)$ denote the counting random measure associated to $q(\cdot)$,i.e $N(t, \Lambda) := N([0, t), \Lambda) = \sum_{s\in(0,t]} 1_\Lambda(q(s))$ such that $E(N(t, \Lambda)) = t\nu(\Lambda)$ for $\Lambda \in \mathfrak{B}$. Define $\widetilde{N}(dt, du) := N(dt, du) - dt\nu(du)$, the Poisson martingale measure generated by $q(t)$.

2.1. *Fractional Brownian motion.*

Let given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A standard fractional Brownian motion (fBm) $\{\beta^H(t), t \in \mathbb{R}\}$ with Hurst parameter $H \in (0, 1)$ is a zero mean Gaussian process with continuous sample paths such that

$$R_H(t, s) = \mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}\Big(t^{2H} + s^{2H} - |t - s|^{2H}\Big), \qquad s, t \in \mathbb{R}. \tag{2}$$

Remark 2.1. *In the case $H > \frac{1}{2}$, it follows from [20] that the second partial derivative of the covariance function*

$$\frac{\partial R_H}{\partial t \partial s} = \alpha_H |t - s|^{2H-2},$$

where $\alpha_H = H(2H - 1)$, is integrable, and we can write

$$R_H(t, s) = \alpha_H \int_0^t \int_0^s |u - v|^{2H-2} du dv. \tag{3}$$

Moreover, β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t, s) d\beta(s), \tag{4}$$

where $\beta = \{\beta(t) : \ t \in [0, T]\}$ is a Wiener process and kernel $K_H(t, s)$ is the kernel given by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{g(2-2H, H-\frac{1}{2})}}$ and $g(\cdot, \cdot)$ denotes the Beta function. We take $K_H(t, s) = 0$ if $t \leq s$.

We will denote by $\mathcal{H}$ the reproducing kernel Hilbert space of the fBm. Precisely, $\mathcal{H}$ is the closure of set of indicator functions $\{1_{[0;t]} : t \in [0, T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

The mapping $1_{[0,t]} \to \beta^H(t)$ can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos and we will denote by $\beta^H(\varphi)$ the image of φ by the previous isometry.

Recall that for $\psi, \varphi \in \mathcal{H}$, the scalar product in $\mathcal{H}$ is given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}} = H(2H - 1) \int_0^T \int_0^T \psi(s)\varphi(t)|t - s|^{2H-2} ds dt.$$

Consider the operator K_H^* from $\mathcal{H}$ to $L^2([0, T])$ defined by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K_H}{\partial r}(r, s) dr.$$

The proof of the fact that K_H^* is an isometry between $\mathcal{H}$ and $L^2([0,T])$ can be found in [20]. Moreover, for any $\varphi \in \mathcal{H}$, we have

$$\beta^H(\varphi) = \int_0^T (K_H^*\varphi)(t)d\beta(t).$$

It follows from [20] that the elements of $\mathcal{H}$ are distributions of negative order. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions ψ such that

$$\|\psi\|^2_{|\mathcal{H}|} := \alpha_H \int_0^T \int_0^T |\psi(s)||\psi(t)||s-t|^{2H-2}dsdt < \infty.$$

Note that $|\mathcal{H}|$ is Banach space with the norm $\|.\|_{|\mathcal{H}|}$ and we have the following result.

Lemma 2.2 (Nualart [20]). *Let $H > \frac{1}{2}$, the following inclusions hold*

$$\mathbb{L}^2([0,T]) \subseteq \mathbb{L}^{1/H}([0,T]) \subseteq |\mathcal{H}| \subseteq \mathcal{H};$$

and for any $\varphi \in \mathbb{L}^2([0,T])$,

$$\|\psi\|^2_{|\mathcal{H}|} \leq 2HT^{2H-1} \int_0^T |\psi(s)|^2 ds.$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y,X)$ be the space of bounded linear operator from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X,Y and $\mathcal{L}(Y,X)$. Let $Q \in \mathcal{L}(Y,Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$. where $\lambda_n \geq 0$ $(n = 1,2...)$ are non-negative real numbers and $\{e_n\}$ $(n = 1,2...)$ is a complete orthonormal basis in Y. We define the infinite dimensional fBm on Y with covariance Q as

$$B^H(t) = B^H_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta^H_n(t),$$

where β^H_n are real, independent fBm's. This process is Gaussian, it starts from 0, has zero mean and covariance:

$$E\langle B^H(t), x\rangle\langle B^H(s), y\rangle = R_H(s,t)\langle Q(x), y\rangle \text{ for all } x, y \in Y \text{ and } t, s \in [0,T].$$

In order to define Weiner integrals with respect to the Q-fBm, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y,X)$ of all Q-Hilbert-Schmidt operators $\psi : Y \to X$. We recall that $\psi \in \mathcal{L}(Y,X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|^2_{\mathcal{L}_2^0} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty,$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $\langle\varphi,\psi\rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty}\langle\varphi e_n, \psi e_n\rangle$ is a separable Hilbert space.

Let $\psi(s)$; $s \in [0,T]$ be a function with values in $\mathcal{L}_2^0(Y,X)$, such that

$$\sum_{n=1}^{\infty} \|K^*\psi Q^{\frac{1}{2}} e_n\|^2_{L^2([0,T])} < \infty.$$

The Weiner integral of ψ with respect to B^H is defined by

$$\int_0^t \psi(s)dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\psi(s)e_n d\beta^H_n(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}(K_H^*(\phi e_n)(s)d\beta_n(s). \tag{5}$$

We conclude by stating the following lemma which is critical in the proof of our result, see for example [6].

Lemma 2.3. *If $\psi : [0,T] \to \mathcal{L}_2^0(Y,X)$ satisfies $\int_0^T \|\psi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$, then* (5) *is well-defined as an X-valued random variable and*

$$\mathbb{E}\|\int_0^t \psi(s)dB^H(s)\|^2 \leq 2Ht^{2H-1}\int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

2.2. Resolvent operator.

For better comprehension of the subject we shall introduce some definitions, hypothesis and results (see [13, 15]). Throughout this work we assume that X is a Banach space, A and $\overline{B}(t)$ are closed linear operators on X, and Y denotes the Banach space $D(A)$ equipped with the graph norm defined by

$$\|y\|_Y = \|Ay\| + \|y\|, \qquad \text{for } y \in Y.$$

We consider the following abstract integro-differential problem

$$\begin{cases} \dfrac{du(t)}{dt} & = Au(t) + \int_0^t \overline{B}(t-s)u(s)\,ds, \\ u(0) & = x \in X. \end{cases} \tag{6}$$

Definition 2.4. A one-parameter family of bounded linear operators $(R(t))_{t\geq 0}$ on X is called a resolvent operator of (6) if the following conditions are satisfied.

(a) Function $R(\cdot) : [0,\infty) \to \mathcal{L}(X)$ is strongly continuous and $R(0)x = x$ for all $x \in X$.

(b) For $x \in D(A)$, $R(\cdot)x \in C([0,\infty),[D(A)]) \cap C^1([0,\infty),X)$, and

$$\frac{dR(t)x}{dt} = AR(t)x + \int_0^t \overline{B}(t-s)R(s)xds, \tag{7}$$

$$\frac{dR(t)x}{dt} = R(t)Ax + \int_0^t R(t-s)\overline{B}(s)xds, \tag{8}$$

for every $t \geq 0$,

(c) There exists some constants $M > 0, \delta$ such that $\|R(t)\| \leq Me^{\delta t}$ for every $t \geq 0$.

Definition 2.5. *[13, Theorem 4.1]* A resolvent operator $(R(t))_{t\geq 0}$ of (6) is called exponentially stable if there exists positive constants M, β such that $\|R(t)\| \leq Me^{-\beta t}$.

Resolvent operators are crucial in studying integro-differential equations. So it is important to know when the linear system (6) has a resolvent operator. In this work, we assume that the following two assumptions are satisfied.

($\mathcal{A}$.1) A is the infinitesimal generator of a strongly continuous semigroup on X.

($\mathcal{A}$.2) For all $t \geq 0$, $\overline{B}(t)$ is a closed linear operator from $D(A)$ to X, and $\overline{B}(t) \in \mathcal{L}(Y,X)$. For any $y \in Y$, the map $t \longrightarrow \overline{B}(t)y$ is bounded, differentiable and the derivative $B'(t)y$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Theorem 2.6. *[13, Theorem 3.7] Assume that* ($\mathcal{A}$.1) *and* ($\mathcal{A}$.2) *hold. Then there exists a unique resolvent operator of the Cauchy problem (6).*

In the remaining of this section we discuss the existence of solutions to

$$\begin{cases} \dfrac{du(t)}{dt} & = Au(t) + \int_0^t \overline{B}(t-s)u(s)\,ds + f(t), \quad t \geq 0, \\ u(0) & = u_0 \in X. \end{cases} \tag{9}$$

where $f : [0, +\infty) \longrightarrow X$ is a continuous function. First, we give the definition of strict solution.

Definition 2.7. *A function $u : [0, +\infty) \to X$, is called a strict solution of* (9) *on* $[0, +\infty)$ *if* $u \in C([0, +\infty), [D(A)]) \cap C^1([0, +\infty), X)$*, and u satisfies* (9) *on* $[0, +\infty)$.

Theorem 2.8 ([13, Theorem 2.5]). *Let $u_0 \in X$. Assume that $f \in C([0, +\infty), X)$ and $u(\cdot)$ is a strict solution of* (9) *on* $[0, +\infty)$*. Then*

$$u(t) = R(t)u_0 + \int_0^t R(t-s)f(s)\,ds, \quad t \in [0, +\infty). \tag{10}$$

Motivated by (10), we introduce the concept of mild solution.

Definition 2.9. A function $u \in C([0, +\infty), X)$ is called a mild solution of (9) if

$$u(t) = R(t)u_0 + \int_0^t R(t-s)f(s)\,ds, \quad t \in [0, +\infty).$$

3. Main result

In this section, we prove the existence, uniqueness, and stability of system (1). To obtain our result, we use the theory of resolvent operator and a fixed point theorem. First, we introduce the concept of mild solution for equation (1).

Definition 3.1. *A stochastic process $x(\cdot) : [-\tau, T] \longrightarrow X$ is a mild solution of* (1) *if*

1. *$x(\cdot)$ has càdlàg path on $[0, T] - \{t_1, t_2, ..., t_m\}$, and $\int_0^T \|x(t)\|^2 dt < \infty$ almost surely, and $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $k = 1, ..., m$;*
2. *$x(t) = \varphi(t)$ on $[-\tau, 0]$;*
3. *for every $0 \leq t \leq T$, the process $x(t)$ satisfies*

$$\begin{aligned} x(t) \quad &= R(t)\Big[\varphi(0) - \Gamma_1(0, \varphi, 0)\Big] + \Gamma_1(t, x_t, \int_0^t \mu_1(t, s, x_s)ds) \\ &+ \int_0^t R(t-s)\Gamma_2(s, x_s, \int_0^s \mu_2(s, \tau, x_\tau)d\tau)ds + \int_0^t R(t-s)v(s)dB^H(s) \\ &+ \int_0^t \int_{\mathcal{U}} R(t-s)g(s, x_s, \eta)\widetilde{N}(dt, d\eta) \\ &+ \sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-)), \quad \mathbb{P} - a.s. \end{aligned} \tag{11}$$

To establish our main result, we introduce the following assumptions.

($\mathcal{H}$.1) The resolvent operator $(R(t))_{t\geq 0}$ given by ($\mathcal{A}$.1) and ($\mathcal{A}$.2) satisfies the following condition:

$$\|R(t)\| \leq Me^{-\lambda t} \text{ for } t \geq 0, \text{ where } M \geq 1 \text{ and } \lambda > 0.$$

($\mathcal{H}$.2) The mapping $\Gamma_i : J \times Ca \times X \to X$, $i = 1, 2$, satisfies the following conditions.

(i) The function Γ_1 is continuous in the quadratic mean sense, i.e for $x_1, x_2 \in Ca$, $y_1, y_2 \in X$

$$\lim_{t \longrightarrow s} \mathbb{E}\|\Gamma_1(t, x_1, y_1) - \Gamma_1(s, x_2, y_2)\|^2 = 0.$$

(ii) There exist positive constants $\gamma_i, i = 1, 2$ such that for $t \in J, x_1, x_2 \in Ca,\ y_1, y_2 \in X$

$$\mathbb{E}\|\Gamma_i(t, x_1, y_1) - \Gamma_i(t, x_2, y_2)\|^2 \leq \gamma_i[\|x_1 - x_2\|^2_{Ca} + \mathbb{E}\|y_1 - y_2\|^2],$$

and $\Gamma_i(t, 0, 0) = 0$.

($\mathcal{H}$.3) The functions $\mu_i : D \times Ca \to X,\ i = 1, 2$ satisfies the following condition. There exists a constant $\delta_i > 0, i = 1, 2$, such that for $x_1, x_2 \in Ca$ we have

$$\mathbb{E}\|\int_0^t [\mu_i(t, s, x_1) - \mu_i(t, s, x_2)]ds\|^2 \leq \delta_i\|x_1 - x_2\|^2_{Ca}, \quad t \in J,$$

and $\mu_i(t, s, 0) = 0$, for $(t, s) \in D$.

($\mathcal{H}$.4) There exist a positive constant $\bar{k} > 0$ such that, for all $t \in J$ and $x, y \in Ca$

$$\mathbb{E}\int_{\mathcal{U}} \|f(t, x, \eta) - f(t, y, \eta)\|^2 \nu(d\eta) \leq \bar{k}\|x - y\|^2_{Ca},$$

and $f(t, 0, \eta) = 0$, for $\eta \in \mathcal{U}$.

($\mathcal{H}$.5) The function $v : [0, \infty) \to \mathcal{L}^0_2(Y, X)$ satisfies

$$\int_0^T e^{2\lambda s}\|v(s)\|_{\mathcal{L}^0_2(Y,X)} ds < \infty, \forall T > 0.$$

($\mathcal{H}$.6) The impulses functions $I_k,\ k = 1, 2, ..., m$, satisfy the following conditions. There exist positive constants M_k, such that $\mathbb{E}\|I_k(y_1) - I_k(y_2)\|^2 \leq M_k\mathbb{E}\|y_1 - y_2\|^2$ for all $y_1, y_2 \in X$, and $I_k(0) = 0$.

Next, we prove the paper's main result.

Theorem 3.2. *Suppose the hypotheses $\mathcal{H}.1 - \mathcal{H}.6$ are satisfied. Moreover, the initial value φ satisfies*

$$\mathbb{E}\|\varphi(t)\|^2 \leq M_0\|\varphi\|^2_{Ca}e^{-bt}, \quad -\tau \leq t \leq 0,$$

for some $M_0 > 0$ and $b > 0$, and

$$4\Big(\gamma_1(1 + \delta_1) + M^2\gamma_2(1 + \delta_2)\lambda^{-2} + M^2\bar{k}(2\lambda)^{-1} + mM^2\Big(\sum_{i=1}^m M_k\Big)\Big) < 1. \tag{12}$$

Then, there exist a unique mild solution of system (1). *moreover, it exponentially decays to zero in mean square, i.e, there exist a pair of positive constants $\alpha < \lambda$ and $M^* = M^*(\varphi, \alpha)$ such that*

$$\mathbb{E}\|x(t)\|^2 \leq M^*e^{-\alpha t}$$

*P*roof. For a fixed $T > 0$, let Ca_T be the space of càdlàg processes $y(s)$ on $[0, T]-\{t_1, t_2, ..., t_m\}$ such that $y(t_i^-)$ and $y(t_i^+)$ exist, and $y(s) = \varphi(s)$, $s \in [-\tau, 0]$, equipped with the supremum norm $\|y\|_{Ca_T} = \sup_{\theta\in[-\tau,T]}\Big(\mathbb{E}\|y(\theta)\|^2\Big)^{1/2}$. Assume further that there exist some constants $b > 0$ and $M^* = M^*(\varphi, b) > 0$ such that

$$\mathbb{E}\|x(t)\|^2 \leq M^*e^{-bt}. \tag{13}$$

It is routine to check that Ca_T is a Banach space endowed with the norm $\|.\|_{Ca_T}$. Define the operator Π on Ca_T by

$$\Pi(x)(t) = \begin{cases} \varphi(t), \quad \text{if } t \in [-\tau, 0], \\ R(t)\Big[\varphi(0) - \Gamma_1(0, \varphi, 0)\Big] + \Gamma_1(t, x_t, \int_0^t \mu_1(t, s, x_s)ds) \\ + \int_0^t R(t-s)\Gamma_2(s, x_s, \int_0^s \mu_2(s, \tau, x_\tau)d\tau)ds + \int_0^t R(t-s)v(s)dB^H(s) \\ + \int_0^t \int_{\mathcal{U}} R(t-s)f(s, x_s, \eta)\widetilde{N}(dt, d\eta) \\ + \sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-)), \quad \text{if } t \in J. \end{cases}$$

Thus it is clear that proving the existence of a unique mild solution to equation (1), which exponentially decays to zero in mean square, is equivalent to finding a unique fixed point of the operator Π. We will provide the proof in the steps that follow.

Step 1: We verify that $\Pi(x)(t)$ is a càdlàg process. Let $t \in J$ and h be sufficiently small, then for all $x \in Ca_T$, we have

$$\begin{aligned} E\|\Pi(x)(t+h) - \Pi(x)(t)\|^2 &\leq 6\mathbb{E}\|[R(t+h) - R(t)]\Big(\varphi(0) - \Gamma_1(0, \varphi, 0)\Big)\|^2 \\ &+ 6\mathbb{E}\|\Gamma_1(t+h, x_{t+h}, \int_0^{t+h} \mu_1(t+h, s, x_s)ds) \\ &- \Gamma_1(t, x_t, \int_0^t \mu_1(t, s, x_s)ds)\|^2 \\ &+ 6\mathbb{E}\| \int_0^{t+h} R(t+h-s)\Gamma_2(s, x_s, \int_0^s \mu_2(s, \eta, x_\eta)d\eta)ds \\ &- \int_0^t R(t-s)\Gamma_2(s, x_s, \int_0^s \mu_2(s, \eta, x_\eta)d\eta)ds\|^2 \\ &+ 6\mathbb{E}\| \int_0^{t+h} R(t+h-s)v(s)dB^H(s) \\ &- \int_0^t R(t-s)v(s)dB^H(s)\|^2 \\ &+ 6\mathbb{E}\| \int_0^{t+h} R(t+h-s) \int_{\mathcal{U}} f(s, x_s, \eta)\widetilde{N}(dt, d\eta) \\ &- \int_0^t R(t-s) \int_{\mathcal{U}} f(s, x_s, \eta)\widetilde{N}(dt, d\eta)\|^2 \\ &+ 6\mathbb{E}\|\sum_{0<t_k<t+h} R(t+h-t_k)I_k(x(t_k^-)) \\ &- \sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-))\|^2 \\ &= 6\sum_{i=1}^{6} \mathbb{E}\|F_i(h)\|^2. \end{aligned}$$

By Definition 2.4, we have

$$\lim_{h\to 0}(R(t+h) - R(t))\Big(\varphi(0) - \Gamma_1(0, \varphi, 0)\Big) = 0.$$

And from condition $\mathcal{H}.1$, we get

$$\mathbb{E}\|(R(t+h) - R(t))\Big(\varphi(0) - \Gamma_1(0, \varphi, 0)\Big)\| \leq Me^{-\lambda t}(e^{-\lambda h} + 1)\mathbb{E}\|\Big(\varphi(0) - \Gamma_1(0, \varphi, 0)\Big)\|.$$

Then using the Lebesgue dominated theorem, we obtain

$$\lim_{h\to 0} \mathbb{E}\|F_1(h)\|^2 = 0.$$

From condition (i) in $\mathcal{H}.2$, we conclude that

$$\lim_{h\to 0}\mathbb{E}\|F_2(h)\|^2 = 0.$$

For the third terme $F_3(h)$, we have

$$\begin{aligned}\|F_3(h)\| &\leq \|\int_0^t [R(t+h-s)-R(t-s)]\Gamma_2(s,x_s,\int_0^s \mu_2(s,\tau,x_\tau)d\tau)ds\| \\ &+\|\int_t^{t+h} R(t+h-s)\Gamma_2(s,x_s,\int_0^s \mu_2(s,\tau,x_\tau)d\tau)ds\| \\ &= \|F_{31}(h)\| + \|F_{32}(h)\|.\end{aligned}$$

By using Holder's inequality, we get

$$\mathbb{E}\|F_{31}(h)\|^2 \leq t\int_0^t \mathbb{E}\|[R(t+h-s)-R(t-s)]\Gamma_2(s,x_s,\int_0^s \mu_2(s,\tau,x_\tau)d\tau)\|^2 ds.$$

By using conditions $\mathcal{H}.1$–$\mathcal{H}.3$, one has that

$$\begin{aligned}&\mathbb{E}\|[R(t+h-s)-R(t-s)]\Gamma_2(s,x_s,\int_0^s \mu_2(s,\tau,x_\tau)d\tau)\|^2 \\ &\leq 2M^2e^{-2\lambda(t-s)}(e^{-2\lambda h}+1)\gamma_2(1+\delta_2)\sup_{-\tau\leq s\leq T}\mathbb{E}\|x(s)\|^2.\end{aligned}$$

Then by using Definition 2.4 and by applying the Lebesgue dominated theorem, we obtain

$$\lim_{h\to 0}\mathbb{E}\|F_{31}(h)\|^2 = 0.$$

Through the use of $\mathcal{H}.1$–$\mathcal{H}.2$ along with Holder's inequality, we get

$$E\|F_{32}(h)\|^2 \quad \leq M^2(2\lambda)^{-1}(1-e^{-2\lambda h})\gamma_2(1+\delta_2)\sup_{-\tau\leq s\leq T}E\|x(s)\|^2,$$

therefore

$$\lim_{h\to 0}\mathbb{E}\|F_3(h)\|^2 = 0.$$

Moreover, we have

$$\begin{aligned}\|F_4(h)\| \leq &\left\|\int_0^t (R(t+h-s)-R(t-s))v(s)dB_H(s)\right\| \\ &+\left\|\int_t^{t+h} R(t+h-s)v(s)dB_H(s)\right\| \\ &\leq \|F_{41}(h)\| + \|F_{42}(h)\|.\end{aligned}$$

From Lemma 2.3, we get that

$$\mathbb{E}\,\|F_{41}(h)\|^2 \leq 2Ht^{2H-1}\int_0^t \|(R(t+h-s)-R(t-s))v(s)\|^2_{\mathcal{L}_2}ds.$$

Since $R(t)$ is strongly continuous and by using condition $\mathcal{H}.1$ we conclude, by the dominated convergence theorem that,

$$\lim_{h\to 0}\mathbb{E}\,\|F_{41}(h)\|^2 = 0.$$

Again by virtue of Lemma 2.3 , we obtain that

$$\begin{aligned}\mathbb{E}\,\|F_{42}(h)\|^2 &\leq 2Hh^{2H-1}\int_t^{t+h} M^2e^{-2\lambda(t+h-s)}\|v(s)\|^2_{\mathcal{L}_2}ds \\ &\leq 2M^2Hh^{2H-1}\int_t^{t+h} e^{2\lambda s}\|v(s)\|^2_{\mathcal{L}_2}ds \quad \longrightarrow 0 \text{ as } h\to 0.\end{aligned}$$

By assumption $\mathcal{H}.1$, we get

$$\begin{aligned}
\mathbb{E}\|F_5(h)\|^2 &\leq 2\mathbb{E}\left\|\int_0^t [R(t+h-s)-R(t-s)]\int_{\mathcal{U}} f(s,x_s,\eta)\widetilde{N}(ds,d\eta)\right\|^2 \\
&+ 2\mathbb{E}\left\|\int_t^{t+h} R(t+h-s)\int_{\mathcal{U}} f(s,x_s,\eta)\widetilde{N}(ds,d\eta)\right\|^2 \\
&\leq 4M^2[e^{-2\lambda h}+1]\int_0^t\int_{\mathcal{U}} e^{-2\lambda(t-s)}\mathbb{E}\|f(s,x_s,\eta)\|^2\nu(d\eta)ds \\
&+ 2M^2e^{-2\lambda h}\int_t^{t+h}\int_{\mathcal{U}} e^{-2\lambda(t-s)}\mathbb{E}\|f(s,x_s,\eta)\|^2\nu(d\eta)ds,
\end{aligned} \tag{14}$$

from assumption $\mathcal{H}.5$, we have

$$\begin{aligned}
\int_0^t\int_{\mathcal{U}} e^{-2\lambda(t-s)}\mathbb{E}\|f(s,x_s,\eta)\|^2\nu(d\eta)ds &\leq \int_0^t e^{-2\lambda(t-s)}\bar{k}\|x_s\|^2_{Ca}ds \\
&\leq \bar{k}\sup_{-\tau\leq s\leq T}\mathbb{E}\|x(s)\|^2(2\lambda)^{-1}(1-e^{-2\lambda t}) \\
&:= K.
\end{aligned} \tag{15}$$

Using the Lebesgue dominated theorem along with the continuous property of the resolvent operator and the inequalities (14) and (15), we obtain that

$$\lim_{h\to 0}\mathbb{E}\|F_5(h)\|^2 = 0.$$

Finally, we have

$$\begin{aligned}
\mathbb{E}\|F_6(h)\|^2 \leq\ & 2\mathbb{E}\left\|\sum\nolimits_{0<t_i<t}(R(t+h-t_i)-R(t-t_i))I_i(x(t_i^-))\right\|^2 \\
& +2\mathbb{E}\left\|\sum\nolimits_{t\leq t_i<t+h} R(t+h-t_i)I_i(x(t_i^-))\right\|^2.
\end{aligned}$$

From condition $\mathcal{H}.1$ and $\mathcal{H}.6$, we have

$$\mathbb{E}\left\|(R(t+h-t_i)-R(t-t_i))I_i(x(t_i^-))\right\|^2 \leq M^2[e^{-2\lambda h}-1]e^{-2\lambda(t-s)}M_k\sup_{-\tau\leq s\leq T}\mathbb{E}\|x(s)\|^2,$$

and

$$\mathbb{E}\left\|R(t+h-t_i)I_i(x(t_i^-))\right\|^2 \leq M^2e^{-2\lambda(t+h-s)}M_k\sup_{-\tau\leq s\leq T}\mathbb{E}\|x(s)\|^2.$$

So, we obtain that

$$\mathbb{E}\|F_6(h)\|^2 \to 0 \text{ as } h\to 0.$$

The above arguments show that $\lim_{h\to 0}\mathbb{E}\|\Pi(x)(t+h)-\Pi(x)(t)\|^2 = 0$. Therefore, we conclude that $t\longmapsto\Pi(x)(t)$ is càdlàg on $J-\{t_1,t_2,...,t_m\}$ in the $\mathbb{L}^2$-sense.

Step 2: In this step, we show that $\Pi(Ca_T)\subset Ca_T$. Let $x\in Ca_T$ then there exist some constants $b>0$ and

$M^* = M^*(\varphi, b) > 0$ such that for all $t \in J$, $\mathbb{E}\|x(t)\|^2 \leq M^* e^{-bt}$, and we have

$$\begin{aligned} E\|\Pi(x)(t)\|^2 \leq\ & 6\mathbb{E}\|R(t)\Big(\varphi(0) - \Gamma_1(0, \varphi, 0)\Big)\|^2 + 6\mathbb{E}\|\Gamma_1(t, x_t, \int_0^t \mu_1(t, s, x_s)ds)\|^2 \\ & +6\mathbb{E}\| \int_0^t R(t-s)\Gamma_2(s, x_s, \int_0^s \mu_2(s, \eta, x_\eta)d\eta)ds\|^2 \\ & +6\mathbb{E}\| \int_0^t R(t-s)v(s)dB^H(s)\|^2 \\ & +6\mathbb{E}\| \int_0^t R(t-s) \int_{\mathcal{U}} f(s, x_s, \eta)\widetilde{N}(dt, d\eta)\|^2 \\ & +6\mathbb{E}\|\textstyle\sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-))\|^2 \\ =\ & 6\textstyle\sum_{i=1}^6 H_i(t). \end{aligned}$$

First, we begin with estimating $x_t \in Ca_T$, for $t \in J$.

$$\|x_t\|^2_{Ca} \leq M_0\|\varphi\|^2_{Ca}e^{-bt} + M^* e^{-bt} \leq [M_0\|\varphi\|^2_{Ca} + M^*]e^{-bt} = \overline{M}e^{-bt},$$

where $\overline{M} = M_0\|\varphi\|^2_{Ca} + M^*$.
Without loss of generality, we can suppose that $b < \lambda$. From condition $\mathcal{H}.1$, we have

$$H_1(t) \leq M^2\mathbb{E}\|\varphi(0) - \Gamma_1(0, \varphi, 0)\|^2 e^{-\lambda t} \leq N_1 e^{-bt},$$

where $N_1 = M^2\mathbb{E}\|\varphi(0) - \Gamma_1(0, \varphi, 0)\|^2$.
By condition $\mathcal{H}.2$ and $\mathcal{H}.3$, we get that

$$\begin{aligned} H_2(t) &\leq \gamma_1[\|x_t\|^2_{Ca} + \mathbb{E}\| \int_0^t \mu(t, s, x_s)ds\|^2] \\ &\leq \gamma_1(1 + \delta_1)\|x_t\|^2_{Ca} \\ &\leq \gamma_1(1 + \delta_1)\overline{M}e^{-bt} \\ &\leq N_2 e^{-bt}, \end{aligned}$$

where $N_2 = \gamma_1(1 + \delta_1)\overline{M}$.
By using Holder's inequality together with conditions $\mathcal{H}.1$-$\mathcal{H}.3$, we get

$$\begin{aligned} H_3(t) &\leq \mathbb{E}\Big[\int_0^t Me^{-\lambda(t-s)}\|\Gamma_2(s, x_s, \int_0^s \mu_2(s, \eta, x_\eta)d\eta)\|ds\Big]^2 \\ &\leq M^2\Big(\int_0^t e^{-\lambda(t-s)}ds\Big)\Big(\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|\Gamma_2(s, x_s, \int_0^s \mu_2(s, \eta, x_\eta)d\eta)\|^2 ds\Big) \\ &\leq M^2\lambda^{-1}(1 - e^{-\lambda t})\gamma_2(1 + \delta_2) \int_0^t e^{-\lambda(t-s)}\|x_s\|^2_{Ca}ds \\ &\leq M^2\lambda^{-1}\gamma_2(1 + \delta_2)(1 - e^{-\lambda t}) \int_0^t e^{-\lambda(t-s)}\overline{M}e^{-bs}ds \\ &\leq M^2\overline{M}\lambda^{-1}\gamma_2(1 + \delta_2)e^{-\lambda t} \int_0^t e^{(\lambda-b)s}ds \\ &\leq M^2\overline{M}\lambda^{-1}(\lambda - b)^{-1}\gamma_2(1 + \delta_2)e^{-\lambda t}(e^{(\lambda-b)t} - 1) \\ &\leq N_3 e^{-bt}, \end{aligned}$$

where $N_3 = M^2\overline{M}\lambda^{-1}(\lambda - b)^{-1}\gamma_2(1 + \delta_2)$.
Using Lemma 2.3, we obtain

$$\begin{aligned}
H_4(t) &\leq 2Ht^{2H-1}\int_0^t M^2 e^{-2\lambda(t-s)}\|v(s)\|^2_{\mathcal{L}_2}ds\\
&\leq 2HM^2t^{2H-1}e^{-2\lambda t}\int_0^t e^{2\lambda s}\|v(s)\|^2_{\mathcal{L}_2}ds\\
&\leq 2HM^2(t^{2H-1}e^{-\lambda t})\Big(\int_0^t e^{2\lambda s}\|v(s)\|^2_{\mathcal{L}_2}ds\Big)e^{-\lambda t},
\end{aligned}$$

using hypothesis $\mathcal{H}.5$ and since $\sup_{t>0} t^{2H-1}e^{-\lambda t} < \infty$, we get that

$$H_4(t) \leq N_4 e^{-bt},$$

where $N_4 = 2HM^2t^{2H-1}e^{-\lambda t}\int_0^t e^{2\lambda s}\|v(s)\|^2_{\mathcal{L}_2}ds$.
By assumptions $\mathcal{H}.1$ and $\mathcal{H}.4$, we get

$$\begin{aligned}
H_5(t) &\leq \int_0^t M^2e^{-2\lambda(t-s)}\int_{\mathcal{U}}\mathbb{E}\|f(s,x_s,\eta)\|^2\nu(d\eta)ds\\
&\leq M^2e^{-2\lambda t}\overline{k}\int_0^t e^{2\lambda s}\|x_s\|^2_{Ca}ds\\
&\leq \overline{k}M^2\overline{M}e^{-2\lambda t}\int_0^t e^{2\lambda s}e^{-bs}ds\\
&\leq \overline{k}M^2\overline{M}e^{-2\lambda t}(2\lambda - b)^{-1}(e^{(2\lambda-b)t} - 1)\\
&\leq \overline{k}M^2\overline{M}(2\lambda - b)^{-1}e^{-bt}\\
&\leq N_5e^{-bt},
\end{aligned}$$

where $N_5 = \overline{k}M^2\overline{M}(2\lambda - b)^{-1}$. By assumptions $\mathcal{H}.1$ and $\mathcal{H}.6$, we get

$$\begin{aligned}
H_6(t) &\leq m\mathbb{E}\sum_{0<t_k<t}\mathbb{E}\|R(t-t_k)I_k(x(t_k^-))\|^2\\
&\leq m\sum_{0<t_k<t}M^2e^{-2\lambda(t-t_k)}M_k\mathbb{E}\|x(t_k^-)\|^2\\
&\leq mM^2M^*e^{-2\lambda t}\sum_{0<t_k<t}M_ke^{2\lambda t_k}e^{-bt_k}\\
&\leq mM^2M^*e^{-2\lambda t}\Big(\sum_{0<t_k<t}M_k\Big)e^{(2\lambda-b)t}\\
&\leq mM^2M^*\Big(\sum_{0<t_k<t}M_k\Big)e^{-bt}\\
&\leq N_6e^{-bt},
\end{aligned}$$

where $N_6 = mM^2M^*(\sum_{0<t_k<t}M_k)$. Therefore, we obtain

$$\mathbb{E}\|\Pi(x)(t)\|^2 \leq Ne^{-bt}$$

for some constants $b > 0$ and $N = 6\sum_{i=1}^6 N_i > 0$. That is $\Pi(x) \in Ca_T$, and hence, we conclude that $\Pi(Ca_T) \subset Ca_T$.

Step 3: Finally, we show that Π is a contraction. Let $y, z \in Ca_T$, and $t \in J$

$$\begin{aligned}\mathbb{E}\|\Pi(y)(t)-\Pi(z)(t)\|^2 \leq\ & 4\mathbb{E}\|\Gamma_1(t, y_t, \int_0^t \mu_1(t,s,y_s)ds) - \Gamma_1(t, z_t, \int_0^t \mu_1(t,s,z_s)ds)\|^2 \\ & +4\mathbb{E}\| \int_0^t R(t-s)[\Gamma_2(s, y_s, \int_0^s \mu_2(s,\eta,y_\eta)d\eta) \\ & -\Gamma_2(s, z_s, \int_0^s \mu_2(s,\eta,z_\eta)d\eta)]ds\|^2 \\ & +4\mathbb{E}\| \int_0^t R(t-s) \int_{\mathcal{U}}[f(s,y_s,\eta) - f(s,z_s,\eta)]\widetilde{N}(dt,d\eta)\|^2 \\ & +4\mathbb{E}\|\textstyle\sum_{0<t_k<t} R(t-t_k)[I_k(y(t_k^-)) - I_k(z(t_k^-))]\|^2.\end{aligned}$$

From $(\mathcal{H}.1)-(\mathcal{H}.6)$ combined with Holder's inequality, we obtain

$$\begin{aligned}\mathbb{E}\|\Pi(y)(t)-\Pi(z)(t)\|^2 & \leq 4\gamma_1(1+\delta_1)\sup_{-\tau\leq s\leq T}\mathbb{E}\|y(s)-z(s)\|^2 \\ & +4M^2\gamma_2(1+\delta_2)\lambda^{-2}\sup_{-\tau\leq s\leq T}\mathbb{E}\|y(s)-z(s)\|^2 \\ & +4M^2\overline{k}(2\lambda)^{-1}\sup_{-\tau\leq s\leq T}\mathbb{E}\|y(s)-z(s)\|^2 \\ & +4mM^2\Big(\textstyle\sum_{i=1}^m M_k\Big)\sup_{-\tau\leq s\leq T}\mathbb{E}\|y(s)-z(s)\|^2.\end{aligned}$$

Thus

$$\|\Pi(y)-\Pi(z)\|_{Ca_T} \leq \widehat{k}\|y-z\|_{Ca_T},$$

where

$$\widehat{k} = 4\Big(\gamma_1(1+\delta_1) + M^2\gamma_2(1+\delta_2)\lambda^{-2} + M^2\overline{k}(2\lambda)^{-1} + mM^2\Big(\sum_{i=1}^m M_k\Big)\Big) < 1.$$

Therefore, Π is a contractive mapping. Hence, Π has a unique fixed point $x \in Ca_T$, which is a unique mild solution to equation (1). Furthermore, this solution is exponentially stable in mean square. This complete the proof.

□

4. Exponential stability in the case of time-varying delay

Consider now the following impulsive neutral stochastic integro-differential equation driven by fractional Brownian motion with time-varying delay and Poisson jumps

$$\begin{cases} d\Big[x(t)-\Gamma_1\Big(t, x(t-r_1(t)), \displaystyle\int_0^t \mu_1(t,s,x(s-r_1(s)))ds\Big)\Big] = \\ A\Big[x(t)-\Gamma_1\Big(t, x(t-r_1(t)), \displaystyle\int_0^t \mu_1(t,s,x(s-r_1(s))\Big)ds)\Big]dt \\ +\Big[\Gamma_2\Big(t, x(t-r_2(t)), \displaystyle\int_0^t \mu_2(t,s,x(s-r_2(s)))ds\Big)\Big]dt \\ +\Big[\displaystyle\int_0^t \overline{B}(t-s)\Big[x(s)-\Gamma_1\Big(s, x(s-r_1(s)), \int_0^s \mu_1(s,\delta,x(\delta-r_1(\delta)))d\delta\Big)\Big]ds\Big]dt \\ +\displaystyle\int_{\mathcal{U}} f(t, x(t-r_3(t)), \eta)\widetilde{N}(dt,d\eta) + v(t)dB^H(t), \quad t\in J := [0,T], \quad t\neq t_k, \\ \Delta x|_{t=t_k} = x(t_k^+)-x(t_k^-) = I_k(x(t_k^-)),\ k=1,\dots,m,\ m\in\mathbb{N}, \\ x(t) = \varphi(t) \ \in\ Ca \quad \text{for a.e } t\in[-\tau,0], \end{cases} \tag{16}$$

where $r_1, r_2, r_3 : [0, \infty) \longrightarrow [0, \tau]$ are continuous functions. The conditions $\mathcal{H}.2 - \mathcal{H}.4$ are replaced with the following ones:

($\mathcal{H}.2'$) The mapping $\Gamma_i : J \times X \times X \to X,\ i = 1, 2$, satisfies the following conditions.

(i) The function Γ_1 is continuous in the quadratic mean sense, i.e for $x_1, x_2, y_1, y_2 \in X$

$$\lim_{t \longrightarrow s} \mathbb{E}\|\Gamma_1(t, x_1, y_1) - \Gamma_1(s, x_2, y_2)\|^2 = 0.$$

(ii) There exist positive constants $\gamma_i, i = 1, 2$ such that for $t \in J, x_1, x_2, y_1, y_2 \in X$

$$\mathbb{E}\|\Gamma_i(t, x_1, y_1) - \Gamma_i(t, x_2, y_2)\|^2 \le \gamma_i[\mathbb{E}\|x_1 - x_2\|^2 + \mathbb{E}\|y_1 - y_2\|^2],$$

and $\Gamma_i(t, 0, 0) = 0$.

($\mathcal{H}.3'$) The functions $\mu_i : D \times X \to X,\ i = 1, 2$ satisfies the following condition. There exists a constant $\delta_i > 0, i = 1, 2$, such that for $x_1, x_2 \in X$ we have

$$\mathbb{E}\|\int_0^t [\mu_i(t, s, x_1) - \mu_i(t, s, x_2)]ds\|^2 \le \delta_i \mathbb{E}\|x_1 - x_2\|^2, \quad t \in J,$$

and $\mu_i(t, s, 0) = 0$, for $(t, s) \in D$.

($\mathcal{H}.4'$) There exist a positive constant $\bar{k} > 0$ such that, for all $t \in J$ and $x, y \in X$

$$\mathbb{E}\int_{\mathcal{U}} \|f(t, x, \eta) - f(t, y, \eta)\|^2 \nu(d\eta) \le \bar{k}\mathbb{E}\|x - y\|^2,$$

and $f(t, 0, \eta) = 0$, for $\eta \in \mathcal{U}$.

Using the same arguments as in theorem 3.2 and the fact that

$$\mathbb{E}\|x(t - r_i(t))\|^2 \le (M_0\|\varphi\|^2_{Ca} + M^*)e^{-b(t - r_i(t))} \le (M_0\|\varphi\|^2_{Ca} + M^*)e^{-bt}e^{b\tau} \le \overline{M}e^{-bt},$$

where $\overline{M} = (M_0\|\varphi\|^2_{Ca} + M^*)e^{b\tau}$, we can easily obtain the following result.

Theorem 4.1. *Assume the assumptions $\mathcal{H}.1, \mathcal{H}.2' - \mathcal{H}.4'$, and $\mathcal{H}.5 - \mathcal{H}.6$ hold, then the unique mild solution to the system* (16) *exists and is exponentially stable in mean square provide that the inequality 12 is satisfied.*

5. example

As an application for our theoretical result, let us consider the stochastic system:

$$\begin{cases} \frac{\partial}{\partial t}[u(t, \xi) - F_1(t, u(t - b_1, \xi), \int_0^t f_1(t, s, u(s - b_1, \xi))ds)] \\ = \frac{\partial^2}{\partial^2 \xi}[u(t, \xi) - F_1(t, u(t - b_1, \xi), \int_0^t f_1(t, s, u(s - b_1, \xi))ds)] + F_2(t, u(t - b_2, \xi), \int_0^t f_2(t, s, u(s - b_2, \xi))ds) \\ + \int_0^t a(t - s)\frac{\partial^2}{\partial^2 \xi}[u(t, \xi) - F_1(t, u(t - b_1, \xi), \int_0^t f_1(t, s, u(s - b_1, \xi))ds)]ds \\ +\sigma(t)\frac{dB^H(t)}{dt} + \int_{\mathcal{U}} h(t, u(t - b_3, \xi), \eta)\widetilde{N}(dt, d\eta), \quad t \in J = [0, T],\ 0 \le \xi \le \pi, \\ u(t, 0) = u(t, \pi) = 0, \qquad 0 \le t \le T, \\ u(s, \xi) = \varphi(s, \xi), \ -\tau < s \le 0, \quad 0 \le \xi \le \pi, \end{cases} \tag{17}$$

where $0 \le b_i \le \tau < \infty, i = 1,2,3$, B_H is a fractional Brownian motion, $F_1, F_2 : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, a: $J \longrightarrow \mathbb{R}$ are continuous functions and $\varphi : [-\tau, 0] \times [0, \pi] \longrightarrow \mathbb{R}$ is a given continuous function such that $\varphi(s,.) \in L^2([0,\pi])$ is measurable and satisfies $\mathbb{E}\|\varphi(t)\|^2 \le M_0\|\varphi\|^2_{Ca}e^{-bt}, -\tau \le t \le 0$.

Let $X = L^2([0,\pi])$. Define the operator $A : D(A) \subset X \longrightarrow X$ given by $A = \frac{\partial^2}{\partial^2\xi}$ with domain $D(A) = H^2([0,\pi]) \cap H_0^1([0,\pi])$, then we get

$$Au = \sum_{n=1}^{\infty} n^2 < u, e_n >_X e_n, \quad u \in D(A),$$

where $e_n := \sqrt{\frac{2}{\pi}} \sin nu$, $n = 1,2,....$ is an orthogonal set of eigenvector of $-A$. It is well known that A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S(t)\}_{t\ge0}$ in X, thus $(\mathcal{A}.1)$ is true. Furthermore, $\{S(t)\}_{t\ge0}$ is given by (see [21])

$$S(t)u = \sum_{n=1}^{\infty} e^{-n^2t} < u, e_n > e_n,$$

for $u \in X$ and $t \ge 0$, that satisfies $\|S(t)\| \le e^{-\pi^2 t}$ for every $t \ge 0$.

Let $B : D(A) \subset X \longrightarrow X$ be the operator given by $B(t)u = a(t)Au$, for $t \ge 0$ and $u \in D(A)$. If a is bounded and C^1 function such that a' is bounded and uniformly continuous, then $(\mathcal{A}.1)$ and $(\mathcal{A}.2)$ are satisfied and hence, by Theorem 2.6, Equation (17) has a resolvent operator $(R(t))_{t\ge0}$ on X.

Let $q(t)_{t>0}$ be a Poisson point process with a σ-finite measure $\nu(d\eta)$. Denote by $N(dt, d\eta)$ the Poisson counting measure, which is induced by $q(\cdot)$, then $\widetilde{N}(dt, d\eta) = N(dt, d\eta) - dt\nu(d\eta)$ is the compensating martingale measure. Let $Q : Y := L^2([0,\pi], \mathbb{R}) \longrightarrow Y$, we choose a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$, set $Qe_n = \lambda_n e_n$, and assume that $tr(Q) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty$. Define the fBm in Y by $B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} b_n^H(t) e_n$, where $H \in (\frac{1}{2}, 1)$ and $\{b_n^H\}_{n\in\mathbb{N}}$ is a sequence of one-dimensional fBm mutually independent. Let us assume the function $\sigma : [0, +\infty) \to \mathcal{L}_2^0(L^2([0,\pi]), L^2([0,\pi]))$ satisfies $\int_0^T e^{2\lambda s}\|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty, \ \ \forall T > 0$.

Then the condition $(\mathcal{H}.5)$ is satisfied. Suppose further that there exist positive constants $k_i, \widehat{k_i}, k_3$, $i = 1,2$, and M_j, $j = 1,2,...,m$, such that

$$F_i(t,0,0) = f_i(t,s,0) = h(t,0,\eta) = I_j(0) = 0, \tag{18}$$

$$\mathbb{E}\|F_i(t,u_1,v_1) - F_i(t,u_2,v_2)\|^2 \le k_i[\mathbb{E}\|u_1 - u_2\|^2 + \mathbb{E}\|v_1 - v_2\|^2], \tag{19}$$

$$\mathbb{E}\| \int_0^t [f_i(t,s,u_1) - f_i(t,s,u_2)]ds\|^2 \le \widehat{k_i}\mathbb{E}\|u_1 - u_2\|^2, \tag{20}$$

$$\int_{\mathcal{U}} \mathbb{E}\|h(t,u_1,\eta) - h(t,u_2,\eta)\|^2 \nu(d\eta) \le k_3\mathbb{E}\|u_1 - u_2\|^2 \tag{21}$$

$$\mathbb{E}\|I_j(u_1) - I_j(u_2)\|^2 \le M_j\mathbb{E}\|u_1 - u_2\|^2, \tag{22}$$

for $t \in [0,T]$, $u_i, v_i \in X$, $i \in \{1,2\}$. For $(t,\psi) \in J \times Ca$, where $\psi(s)(\xi) = \psi(s,\xi)$, $(s,\xi) \in [-\tau,0] \times [0,\pi]$, we put $y(t)(\xi) = y(t,\xi)$. Define the functions $\Gamma_i : J \times Ca \times X \to X, i = 1,2$, $f : J \times Ca \times \mathcal{U} \to X$ and $v : [0,+\infty) \to \mathcal{L}_2^0(L^2([0,\pi]), L^2([0,\pi]))$ as follow

$$\begin{aligned}
\Gamma_i(t,\psi, \int_0^t \mu_i(t,s,\psi)ds)(\xi) &= F_i(t,\psi(-b_i,\xi), \int_0^t f_i(t,s,\psi(-b_i,\xi))ds), \ i = 1,2 \\
f(t,\psi,\eta)(\xi) &= h(t,\psi(-b_3,\xi),\eta), \\
v(t) &= \sigma(t).
\end{aligned}$$

Thus, we can rewrite system (17) in the abstract form of system (1). Moreover, if we assume that

$$\|R(t)\| \le Me^{-\lambda t} \text{ for } t \ge 0, \text{ where } M \ge 1 \text{ and } \lambda > 0. \tag{23}$$

It follows from assumptions (4.2), (4.3), (4.4), (4.5), (4.6) and (4.7) that the functions Γ_1, Γ_2, f, v satisfy all assumptions on theorem 3.2. Hence, system (17) has a unique mild solution, which is exponentially stable provided that

$$4\Big(k_1(1 + \widehat{k_1}) + M^2k_2(1 + \widehat{k_2})\lambda^{-2} + M^2k_3(2\lambda)^{-1} + mM^2\Big(\sum_{j=1}^{m} M_j\Big)\Big) < 1.$$

6. Conclusion

In the paper sufficient conditions for the existence, uniqueness, and exponential stability of a class of delayed neutral stochastic integro-differential systems driven by a fBm in a separable Hilbert space and Poisson jumps with instantaneous impulses have been established. In the proof resolvent operator, stochastic analysis, and a fixed-point strategy have been used. Finally, an example is provided to illustrate the applicability of the obtained result. Here, the result was obtained for delayed neutral stochastic integro-differential systems with instantaneous impulses. Recently, there have been an increasing interest in studying dynamic systems with non-instantaneous impulses, see for example [9, 10] and references therein. So as a future work, one can consider solvability and exponential stability of system (1) with non-instantaneous impulses.

Acknowledgements
The authors would like to thank the referee for their valuable comments and suggestions.

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