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Normal family of holomorphic mappings and a generalization of Lappan's five-valued theorem to holomorphic mapping in several variables

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Abstract. In this paper, we prove some results in normal family of holomorphic mappings intersecting with moving hypersurfaces. Our results strongly extend the Montel's normal criterion in several variables and some previous works. Moreover, we also give a generalization of Lappan's five-valued theorem to φ-normal mapping in several complex variables. In particular, we correct a result due to Hahn [Complex Variables, 6, 109-121,1986] about Lappan-type theorem for holomorphic curve in several variables.

1. Normal family of meromorphic mappings

1.1. Introduction and main results

P. Montel [14] first defined the notion of normal family in one complex variable. By definition, a family $\mathscr{F}$ of meromorphic functions defined on a domain $\mathfrak{D} \subseteq \mathbb{C}$ is said to be normal on $\mathfrak{D}$ if every sequence functions of $\mathscr{F}$ has a subsequence which converges uniformly on every compact subset of $\mathfrak{D}$ with respect to the spherical metric to a meromorphic function or identically ∞ on $\mathfrak{D}$. Perhaps the most celebrated criteria for normality in one complex variable is the following Montel-type theorems related to Picard's theorem of value distribution theory.

Theorem A.[12] *Let $\mathscr{F}$ be a family of meromorphic functions on a domain $\mathfrak{D} \subseteq \mathbb{C}$. Suppose that there exist three distinct points w_1, w_2 and w_3 on the Riemann sphere such that $f(z) - w_i$ $(i = 1, 2, 3)$ has no zero on $\mathfrak{D}$ for each*

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$f \in F$. Then $\mathscr{F}$ is a normal family on $\mathfrak{D}$.

Theorem B.[13] *Let $\mathscr{F}$ be a family of meromorphic functions on a domain $\mathfrak{D} \subseteq \mathbb{C}$. Suppose that there exist distinct points $w_1, w_2, \dots, w_q$ $(q \geq 3)$ on the Riemann sphere such that $f(z) - w_i$ has no zero with multiplicities less than m_i $(i = 1, 2, \dots, q)$ on $\mathfrak{D}$ for each $f \in \mathscr{F}$, where m_i $(i = 1, 2, \dots, q)$ are q fixed positive integers and may be ∞, with $\sum_{j=1}^{q} \dfrac{1}{m_j} < q - 2$. Then $\mathscr{F}$ is a normal family on $\mathfrak{D}$.*

Rutishauser [16] and Stoll [17] first introduced the concept of normal families of meromorphic mappings of several complex variables. Later, Wu [23] initiated the study of normal family for holomorphic mappings. Afterwards, Fujimoto [6] introduced the notion of a meromorphically normal family into the $N-$dimensional complex projective space $\mathbb{P}^N(\mathbb{C})$ and obtained some results about normal family of meromorphic mappings. In the following, we will introduce some basic symbols and notations.

Let $f : \mathbb{D} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic mapping with $\mathbb{D}$ a domain in $\mathbb{C}^m$ and U a nonempty connected open subset of $\mathbb{D}$. If there exists a holomorphic mapping $\tilde{f} = (f_0, \dots, f_n) : U \to \mathbb{C}^{n+1}$ such that $\mathbb{P}(\tilde{f}(z)) \equiv f(z)$ for $z \in U$, where $\mathbb{P} : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{P}^n(\mathbb{C})$ is the standard quotient mapping, then $\tilde{f}$ is called a representation of f on U. Moreover, the representation $\tilde{f}$ of f on U is said to be reduced if $f_0, \dots, f_n$ have no common zeros. Of course, for any point $y \in \mathbb{D}$, one can always find a reduced representation of f near y.

Let $\mathcal{F}$ be a family of holomorphic mappings from $\mathbb{D}$ to $\mathbb{P}^n(\mathbb{C})$. The family $\mathcal{F}$ is said to be normal on $\mathbb{D}$ if any sequence in $\mathcal{F}$ contains a subsequence which converges uniformly to a holomorphic mapping from $\mathbb{D}$ into $\mathbb{P}^n(\mathbb{C})$ on compact subsets of $\mathbb{D}$. Further, we say that $\mathcal{F}$ is normal at a point a in $\mathbb{D}$ if $\mathcal{F}$ is normal on some neighborhood of a in $\mathbb{D}$. By using the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$, we see that a sequence $\left\{f_p\right\}_{p=1}^{\infty}$ in $\mathcal{F}$ converges uniformly to a holomorphic mapping $f : \mathbb{D} \longrightarrow \mathbb{P}^n(\mathbb{C})$ on compact subsets of $\mathbb{D}$ if and only if, for any $a \in \mathbb{D}$, each f_p has a reduced representation $\tilde{f_p} = \left(f_{p0}, f_{p1}, \dots, f_{pn}\right)$ on some fixed neighborhood U of a in $\mathbb{D}$ such that $\left\{f_{pi}\right\}_{p=1}^{\infty}$ converges uniformly on compact subsets of U to a holomorphic function f_i(for $i = 0, \dots, n$) on U with the property that $\tilde{f} = (f_0, f_1, \dots, f_n)$ is a reduced representation of f on U.

In 2002, by using the technique of Stoll [17] on the normality criteria for families of non-negative divisors on a domain $\mathbb{D}$ in $\mathbb{C}^n$, Tu [20] obtained some meromorphic normality criteria for families of meromorphic mappings of several complex variables into $\mathbb{P}^N(\mathbb{C})$ and generalized the above results of Fujimoto [6]. For more related results, we refer the readers to [19, 21, 22, 24].

In 2015, Dethloff-Thai-Trang [5] generalized the result of Tu-Li [21] to the case of holomorphic mappings intersecting a class of hypersurfaces in weak general position. Before state the results of Dethloff-Thai-Trang [5], we need some definitions as follows.

Let $H_{\mathbb{D}}$ be the ring of holomorphic functions on $\mathbb{D}$ and let $H_{\mathbb{D}}[x_0, \dots, x_n]$ be the set of homogeneous polynomials of variables $\mathbf{x} = (x_0, \dots, x_n)$ over $H_{\mathbb{D}}$. Take $Q \in H_{\mathbb{D}}[x_0, \dots, x_n] \setminus \{0\}$ with $\deg(Q) = d$ and write

$$Q(\mathbf{x}) = Q(x_0, \dots, x_n) = \sum_{k=0}^{n_d} a_k \mathbf{x}^{I_k} = \sum_{k=0}^{n_d} a_k x_0^{i_{k0}} \dots x_n^{i_{kn}},$$

where $a_k \in H_{\mathbb{D}}$, $I_k = (i_{k0}, \dots, i_{kn})$ with $|I_k| = i_{k0} + \cdots + i_{kn} = d$ for $k = 0, \dots, n_d$ and $n_d = \binom{n+d}{n} - 1$. Thus each $z \in \mathbb{D}$ corresponds to an element Q_z in $\mathbb{C}[x_0, \dots, x_n]$ defined by

$$Q_z(\mathbf{x}) = \sum_{k=0}^{n_d} a_k(z) \mathbf{x}^{I_k},$$

and hence a hypersurface

$$D_z := \{\mathbf{x} \in \mathbb{C}^{n+1} : Q_z(\mathbf{x}) = 0\}$$

is associated to z or Q_z. The correspondence D from $\mathbb{D}$ into hypersurfaces is called a *moving hypersurface* in $\mathbb{P}^n(\mathbb{C})$ defined by Q. Let $\mathbf{a} = (a_0, \dots, a_{n_d})$ be the vector function associated to D (or Q). In this paper, we only consider homogeneous polynomials Q over $H_{\mathbb{D}}[x_0, \ldots, x_n]$ such that coefficients of Q have no common zeros, so that the hypersurface D_z is well defined for each $z \in \mathbb{D}$.

Let $P_0, \ldots, P_n$ be $n+1$ moving homogeneous polynomials of common degree in $H_{\mathbb{D}}[x_0, \ldots, x_n]$. Denote by $S(\{P_i\}_{i=0}^n)$ the set of all homogeneous not identically zero polynomials $Q = \sum_{i=0}^n b_i P_i, b_i \in H_{\mathbb{D}}$. Let $T_0, \ldots, T_n$ be moving hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ with common degree, where T_i is defined by the (not identically zero) polynomial P_i $(0 \leq i \leq N)$. Denote by $\widetilde{S}(\{T_i\}_{i=0}^n)$ the set of all moving hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ which are defined by $Q \in S(\{P_i\}_{i=0}^n)$.

Let $\{Q_j = \sum_{i=0}^n b_{ij} P_i\}_{j=1}^q$ be q $(q \geq n+1)$ homogeneous polynomials in $S(\{P_i\}_{i=0}^n)$. We say that $\{Q_j\}_{j=1}^q$ are located pointwise in general position in $S(\{P_i\}_{i=0}^n)$ if

$$\det(b_{ij_k})_{0 \leq k, i \leq n} \neq 0$$

for all $1 \leq j_0 < \cdots < j_n \leq q$ and all $z \in \mathbb{D}$.

Definition 1.1. Let D_j be moving hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j which is defined by polynomial homogeneous $Q_j \in H_{\mathbb{D}}[x_0, \ldots, x_n]$, $j = 1, \ldots, q$. We say that moving hypersurfaces $\{D_j\}_{j=1}^q$ $(q \geq n+1)$ in $\mathbb{P}^n(\mathbb{C})$ are located in (weakly) general position if there exists $z \in \mathbb{D}$ such that, for any $1 \leq j_0 < \cdots < j_n \leq q$, the system of equations

$$\begin{cases} Q_{j_i}(z)(x_0, \ldots, x_n) = 0 \\ \qquad 0 \leq i \leq n \end{cases}$$

has only the trivial solution $\mathbf{x} = (0, \ldots, 0)$ in $\mathbb{C}^{n+1}$.

This is equivalent to

$$D(Q_1, \ldots, Q_q)(z) = \prod_{1 \leq j_0 < \cdots < j_n \leq q} \inf_{\|\mathbf{x}\|=1} (|Q_{j_0}(z)(\mathbf{x})|^2 + \cdots + |Q_{j_n}(z)(\mathbf{x})|^2) > 0,$$

where $\mathbf{x} = (x_0, \ldots, x_n)$, $Q_j(z)(\mathbf{x}) = \sum_{|I|=d_j} a_{jI}(z)\mathbf{x}^I$ and $\|\mathbf{x}\| = (\sum_{j=0}^n |x_j|^2)^{1/2}$.

Let f be a meromorphic mapping from a domain $\mathbb{D}$ in $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. We denote D_f by the hypersurface in $\mathbb{P}^n(\mathbb{C})$ depending on f, and Q_f by the homogeneous polynomial depending on f, where Q_f is homogeneous polynomial defining by D_f. In fact, Dethloff-Thai-Trang[5] proved the results as follows:

Theorem C. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain $\mathbb{D}$ in $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $q \geq 2n+1$ be a positive integer. Let $m_1, \ldots, m_q$ be positive integers or ∞ such that*

$$\sum_{j=1}^q \frac{1}{m_j} < \frac{q-n-1}{n}.$$

Suppose that for each $f \in F$, there exist $n+1$ moving hypersurfaces $T_{0,f}, \ldots, T_{n,f}$ in $\mathbb{P}^n(\mathbb{C})$ of common degree and that there exist q moving hypersurfaces $D_{1,f}, \ldots, D_{q,f}$ in $\widetilde{S}(\{T_{i,f}\}_{i=0}^n)$ such that the following conditions are satisfied:

(i) For each $0 \leq i \leq n$, the coefficients of the homogeneous polynomials $P_{i,f}$ which define the $T_{i,f}$ are bounded above uniformly for all f in F on compact subsets of $\mathbb{D}$ for all $1 \leq j \leq q$, the coefficients $b_{ij}(f)$ of the linear combinations of the $P_{i,f}, i = 0, \ldots, n$, which define the homogeneous polynomials $Q_{j,f}$ are bounded above uniformly for all f in $\mathcal{F}$ on compact subsets of $\mathbb{D}$, where $Q_{j,f} \in S(\{P_{i,f}\}_{i=0}^n)$ is a homogeneous polynomial defined the $D_{j,f}$, and for any fixed $z \in \mathbb{D}$,

$$\inf_{f \in F} D(Q_{1,f}, \ldots, Q_{q,f})(z) > 0.$$

(ii) Assume that f intersects $D_{j,f}$ with multiplicity at least m_j for each $1 \leq j \leq q$, for all $f \in \mathcal{F}$.

Then $\mathcal{F}$ is a holomorphically normal family on $\mathbb{D}$.

Now, we will state our first result which is an extension of above Theorem C.

Theorem 1.2. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain $\mathbb{D}$ in $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $q \geq 2n+1$ be a positive integer. Let $m_1, \ldots, m_q$ be positive integers or ∞ such that*

$$\sum_{j=1}^{q} \frac{1}{m_j} < \frac{q-n-1}{n}.$$

Suppose that for each $f \in \mathcal{F}$, there exist $n+1$ moving hypersurfaces $T_{0,f}, \ldots, T_{n,f}$ in $\mathbb{P}^n(\mathbb{C})$ of common degree and that there exist q moving hypersurfaces $D_{1,f}, \ldots, D_{q,f}$ in $\widetilde{S}(\{T_{i,f}\}_{i=0}^n)$ such that the following conditions are satisfied on each compact subset $K \subset \mathbb{D}$:
(i) For each $0 \leq i \leq n$, the coefficients of the homogeneous polynomials $P_{i,f}$ which define the $T_{i,f}$ are bounded above uniformly for all f in $\mathcal{F}$ on compact subset K of $\mathbb{D}$ for all $1 \leq j \leq q$, the coefficients $b_{ij}(f)$ of the linear combinations of the $P_{i,f}, i = 0, \ldots, n$, which define the homogeneous polynomials $Q_{j,f}$ are bounded above uniformly for all f in $\mathcal{F}$ on compact subset K of $\mathbb{D}$, where $Q_{j,f} \in S(\{P_{i,f}\}_{i=0}^n)$ is a homogeneous polynomial defined the $D_{j,f}$, and for any fixed $z \in \mathbb{D}$,

$$\inf_{f \in F} D(Q_{1,f}, \ldots, Q_{q,f})(z) > 0.$$

(ii) Let $\tilde{f} = (f_0, \ldots, f_n)$ be a reduced representation of $f \in \mathcal{F}$ on $\mathbb{D}$ with $f_i(z) \neq 0$ for some i and z, and when $m_j \geq 2$

$$\sup_{1 \leq |\alpha| \leq m_j - 1, z \in f^{-1}(D_{j,f}) \cap K} \left| \frac{\partial^\alpha (Q_{j,f}(\tilde{f_i}))}{\partial z^\alpha}(z) \right| < \infty$$

hold for all $j \in \{1, \ldots, q\}$.

Then $\mathcal{F}$ is a holomorphically normal family on $\mathbb{D}$.

Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of arbitrary hypersurfaces and Q_j be the homogeneous polynomial in $\mathbb{C}[x_0, \ldots, x_n]$ of degree d_j defining D_j for $j = 1, \ldots, q$. Let $m_{\mathcal{D}}$ be the least common multiple of the d_j for $j = 1, \ldots, q$ and denote

$$n_{\mathcal{D}} = \binom{n + m_{\mathcal{D}}}{n} - 1.$$

For $j = 1, \ldots, q$, we set $Q_j^* = Q_j^{m_{\mathcal{D}}/d_j}$ and let $\mathbf{a}_j^*$ be the vector associated with Q_j^*.

We recall the *lexicographic ordering* on the $(n+1)$-tuples of natural numbers: let $J = (j_0, \ldots, j_n), I = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$, $J < I$ if and only if for some $b \in \{0, \ldots, n\}$ we have $j_l = i_l$ for $l < b$ and $j_b < i_b$. With the $(n+1)$-tuples $I = (i_0, \ldots, i_n)$ of non-negative integers, we denote $|I| := \sum_j i_j$.

Let $(x_0 : \cdots : x_n)$ be a homogeneous coordinates in $\mathbb{P}^n(\mathbb{C})$ and let $\{I_0, \ldots, I_{n_{\mathcal{D}}}\}$ be a set of $(n+1)$-tuples such that $|I_j| = m_{\mathcal{D}},\ j = 0, \ldots, n_{\mathcal{D}}$, and $I_i < I_j$ for $i < j \in \{0, \ldots, n_{\mathcal{D}}\}$. We denote $\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$, and write $\mathbf{x}^I$ as $x_0^{i_0} \ldots x_n^{i_n}$ where $I = (i_0, \ldots, i_n) \in \{I_0, \ldots, I_{n_{\mathcal{D}}}\}$. Then we may order the set of monomials of degree $m_{\mathcal{D}}$ as $\{\mathbf{x}^{I_0}, \ldots, \mathbf{x}^{I_{n_{\mathcal{D}}}}\}$.

Denote

$$\varrho_{m_{\mathcal{D}}} : \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$$

be the *Veronese embedding of degree $m_{\mathcal{D}}$*. Let $(w_0 : \cdots : w_{n_{\mathcal{D}}})$ be the homogeneous coordinates in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$. Then $\varrho_{m_{\mathcal{D}}}$ is given by

$$\varrho_{m_{\mathcal{D}}}(x) = (w_0(\mathbf{x}) : \cdots : w_{n_{\mathcal{D}}}(\mathbf{x})), \ \text{ where } \ w_j(x) = \mathbf{x}^{I_j},\ j = 0, \ldots, n_{\mathcal{D}}.$$

For any hypersurface $D_j \in \{D_1, \ldots, D_q\}$ and $\mathbf{a}_j = (a_{j0}, \ldots, a_{jn_{\mathcal{D}}})$ be the vector associated with Q_j^*, we set

$$L_j = Q_j^* = a_{j0} w_0 + \cdots + a_{jn_{\mathcal{D}}} w_{n_{\mathcal{D}}}.$$

Then L_j is a linear form in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$. Let H_j^* be a hyperplane in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$, which is defined by L_j, we say that the hyperplane H_j^* associated with Q_j^* (or D_j). Hence for the collection of hypersurfaces $\{D_1, \ldots, D_q\}$ in $\mathbb{P}^n(\mathbb{C})$, we have the collection $\mathcal{H}^* = \{H_1^*, \ldots, H_q^*\}$ of associated hyperplanes in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$.

Let $q > n_{\mathcal{D}}$ be a positive integer. We say that collection $\mathcal{D}$ is in general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$ if $\{H_1^*, \ldots, H_q^*\}$ is in general position in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$. Hence, the collection $\mathcal{D}$ is in *general position for Veronese embedding* if for any distinct $i_0, \ldots, i_{n_{\mathcal{D}}} \in \{1, \ldots, q\}$, the vectors $\mathbf{a}_{i_0}^*, \ldots, \mathbf{a}_{i_{n_{\mathcal{D}}}}^*$ have rank $n_{\mathcal{D}}$. We can see that the concept of hyperplanes are in general position for Veronese embedding is equivalent to the hyperplanes are in general position usual. For hypersurfaces, general position for Veronese embedding implies $n_{\mathcal{D}}$-subgeneral position.

Let $\mathcal{D} = \{D_1, \ldots, D_q\}$ be a collection of arbitrary moving hypersurfaces and Q_j be the homogeneous polynomial in $H_{\mathbb{D}}[x_0, \ldots, x_n]$ of degree d_j defining D_j for $j = 1, \ldots, q$. Let $m_{\mathcal{D}}$ be the least common multiple of the d_j for $j = 1, \ldots, q$ and denote

$$n_{\mathcal{D}} = \binom{n + m_{\mathcal{D}}}{n} - 1.$$

For $j = 1, \ldots, q$, we set $Q_j^* = Q_j^{m_{\mathcal{D}}/d_j}$ and let $\mathbf{a}_j^*$ be the function vector associated with Q_j^*. We set

$$L_j = Q_j^* = a_{j0}w_0 + \cdots + a_{jn_{\mathcal{D}}}w_{n_{\mathcal{D}}}.$$

Then L_j is a moving linear form in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$. Let H_j^* be a moving hyperplane in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$, which is defined by L_j (or Q_j^*), we say that the hyperplane H_j^* associated with Q_j^* (or D_j). Hence for the collection of moving hypersurfaces $\{D_1, \ldots, D_q\}$ in $\mathbb{P}^n(\mathbb{C})$, we have the collection $\mathcal{H}^* = \{H_1^*, \ldots, H_q^*\}$ of associated moving hyperplanes in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$.

Definition 1.3. Let $\{D_j\}_{j=1}^q$ be moving hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ which are defined by homogeneous $Q_j \in H_{\mathbb{D}}[x_0, \ldots, x_n]$ of degree d_j ($q \geq n_{\mathcal{D}} + 1$), $j = 1, \ldots, q$. We say that moving hypersurfaces $\{D_j\}_{j=1}^q$ are located in (weakly) general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$ if there exists $z \in \mathbb{D}$ such that, the collection $\mathcal{H}^*(z) = \{H_1^*(z), \ldots, H_q^*(z)\}$ of associated hyperplanes in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$ at z are in general position in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$.

Remark 1.4. *We see that hypersurfaces $\{D_j\}_{j=1}^q$ are located in (weakly) general position for Veronese embedding if there exists $z \in \mathbb{D}$ such that*

$$D(Q_1^*, \ldots, Q_q^*)(z) > 0$$

in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$.

Based on above notations and definitions, we generalize the Montel's normal criterion in the higher dimensional case which due to Tu [19] as follows:

Theorem 1.5. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain $\mathbb{D}$ in $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $q \geq 2n_{\mathcal{D}} + 1$ be a positive integer. Let $m_1, \ldots, m_q$ be positive integers or ∞ such that*

$$\sum_{j=1}^{q} \frac{d_j}{m_j} < \frac{(q - n_{\mathcal{D}} - 1)m_{\mathcal{D}}}{n_{\mathcal{D}}}.$$

Suppose that for each $f \in \mathcal{F}$, there exist q moving hypersurfaces $D_{1,f}, \ldots, D_{q,f}$ which are defined by homogeneous polynomials $Q_{1,f}, \ldots, Q_{q,f} \in H_{\mathbb{D}}[x_0, \ldots, x_n]$ with degree $d_1, \ldots, d_q$ respectively such that the following conditions are satisfied on each compact subset $K \subset \mathbb{D}$:
(i) For each $1 \leq j \leq q$, the coefficients of the homogeneous polynomials $Q_{j,f}$ are bounded above uniformly for all f in $\mathcal{F}$ on compact subset K of $\mathbb{D}$. Set $m_{\mathcal{D}} = lcm(d_1, \ldots, d_q)$, $n_{\mathcal{D}} = \binom{n+m_{\mathcal{D}}}{n} - 1$, and $Q_{j,f}^* = Q_{j,f}^{m_{\mathcal{D}}/d_j}$. Suppose that $Q_{j,f}^* = \sum_{i=0}^{n_{\mathcal{D}}} c_{ij}(f)\omega_i$, ($\omega_i = \mathbf{x}^{I_i}$) in $H_{\mathbb{D}}[\omega_0, \ldots, \omega_{n_{\mathcal{D}}}]$ is moving linear form defines the associated hyperplane $H_{j,f}^*$ of $D_{j,f}$ ($j = 1, \ldots, q$) and for any $\{f_p\} \subset \mathcal{F}$, and for any fixed $z \in \mathbb{D}$,

$$\inf_{p \in \mathbb{N}} D(Q_{1,f_p}^*, \ldots, Q_{q,f_p}^*)(z) > 0.$$

(*ii*) *Assume that for each* $f \in \mathcal{F}$, *and* $\tilde{f} = (f_0, \dots, f_n)$ *is a reduced representation of* $f \in \mathcal{F}$ *on* $\mathbb{D}$ *with* $f_i(z) \neq 0$ *for some* i *and* z, *and when* $m_j \geq 2$

$$\sup_{1\leq|\alpha|\leq m_j-1, z\in f^{-1}(D_{j,f})\cap K} \left|\frac{\partial^{\alpha}(Q_{j,f}(\tilde{f}_i))}{\partial z^{\alpha}}(z)\right| < \infty$$

hold for all $j \in \{1, \dots, q\}$ *and all* $f \in \mathcal{F}$.

Then $\mathcal{F}$ *is a holomorphically normal family on* $\mathbb{D}$.

From Theorem 1.5, we get the result as follows:

Corollary 1.6. *Let* $\mathcal{F}$ *be a family of holomorphic mappings of a domain* $\mathbb{D}$ *in* $\mathbb{C}^m$ *into* $\mathbb{P}^n(\mathbb{C})$. *Let* $q \geq 2n_{\mathcal{D}} + 1$ *be a positive integer. Let* $m_1, \dots, m_q$ *be positive integers or* ∞ *such that*

$$\sum_{j=1}^{q} \frac{d_j}{m_j} < \frac{(q - n_{\mathcal{D}} - 1)m_{\mathcal{D}}}{n_{\mathcal{D}}}.$$

Suppose that for each $f \in \mathcal{F}$, *there exist* q *moving hypersurfaces* $D_{1,f}, \dots, D_{q,f}$ *which are defined by homogeneous polynomials* $Q_{1,f}, \dots, Q_{q,f} \in H_{\mathbb{D}}[x_0, \dots, x_n]$ *with degree* $d_1, \dots, d_q$ *respectively such that the following conditions are satisfied on each compact subset* $K \subset \mathbb{D}$:

(*i*) *For each* $1 \leq j \leq q$, *the coefficients of the homogeneous polynomials* $Q_{j,f}$ *are bounded above uniformly for all* f *in* $\mathcal{F}$ *on compact subset* $K \subset \mathbb{D}$. *Set* $m_{\mathcal{D}} = lcm(d_1, \dots, d_q)$, $n_{\mathcal{D}} = \binom{n+m_{\mathcal{D}}}{n} - 1$, *and* $Q^*_{j,f} = Q^{m_{\mathcal{D}}/d_j}_{j,f}$. *Suppose that* $Q^*_{j,f} = \sum_{i=0}^{n_{\mathcal{D}}} c_{ij}(f)\omega_i$, $(\omega_i = \mathbf{x}^{I_i})$ *in* $H_{\mathbb{D}}[\omega_0, \dots, \omega_{n_{\mathcal{D}}}]$ *is moving linear form defines the associated hyperplane* $H^*_{j,f}$ *of* $D_{j,f}$ $(j = 1, \dots, q)$ *and for any* $\{f_p\} \subset \mathcal{F}$, *and for any fixed* $z \in \mathbb{D}$,

$$\inf_{p\in\mathbb{N}} D(Q^*_{1,f_p}, \dots, Q^*_{q,f_p})(z) > 0.$$

(*ii*) *Assume that for each* $f \in \mathcal{F}$, f *intersects* $D_{j,f}$ *with multiplicity at least* m_j *for each* $1 \leq j \leq q$.

Then $\mathcal{F}$ *is a holomorphically normal family on* $\mathbb{D}$.

Finally, we consider the case that moving hypersurfaces which are in weakly general position in $\mathbb{P}^n(\mathbb{C})$ and prove the results as follows:

Theorem 1.7. *Let* $\mathcal{F}$ *be a family of holomorphic mappings of a domain* $\mathbb{D}$ *in* $\mathbb{C}^m$ *into* $\mathbb{P}^n(\mathbb{C})$. *Let* $q \geq nn_{\mathcal{D}} + 2n + 1$ *be a positive integer. Let* $m_1, \dots, m_q$ *be positive integers or* ∞ *such that*

$$\sum_{j=1}^{q} \frac{1}{m_j} < \frac{q - n - (n-1)(n_{\mathcal{D}} + 1)}{n_{\mathcal{D}}(n_{\mathcal{D}} + 2)}.$$

Suppose that for each $f \in \mathcal{F}$, *there exist* q *moving hypersurfaces* $D_{1,f}, \dots, D_{q,f}$ *which are defined by homogeneous polynomials* $Q_{1,f}, \dots, Q_{q,f} \in H_{\mathbb{D}}[x_0, \dots, x_n]$ *with degree* $d_1, \dots, d_q$ *respectively, such that the following conditions are satisfied on each compact subset* $K \subset \mathbb{D}$:

(*i*) *For each* $1 \leq j \leq q$, *the coefficients of the homogeneous polynomials* $Q_{j,f}$ *are bounded above uniformly for all* f *in* $\mathcal{F}$ *on compact subset* K *of* $\mathbb{D}$, *and for any* $\{f_p\} \subset \mathcal{F}$, *and for any fixed* $z \in \mathbb{D}$,

$$\inf_{p\in\mathbb{N}} D(Q_{1,f_p}, \dots, Q_{q,f_p})(z) > 0.$$

(*ii*) *Assume that for each* $f \in \mathcal{F}$, *and* $\tilde{f} = (f_0, \dots, f_n)$ *is a reduced representation of* $f \in \mathcal{F}$ *on* $\mathbb{D}$ *with* $f_i(z) \neq 0$ *for some* i *and* z, *and when* $m_j \geq 2$

$$\sup_{1\leq|\alpha|\leq m_j-1, z\in f^{-1}(D_{j,f})\cap K} \left|\frac{\partial^{\alpha}(Q_{j,f}(\tilde{f}_i))}{\partial z^{\alpha}}(z)\right| < \infty$$

hold for all $j \in \{1, \dots, q\}$ *and all* $f \in \mathcal{F}$.

Then $\mathcal{F}$ *is a holomorphically normal family on* $\mathbb{D}$.

From Theorem 1.7, we get the result as follows:

Corollary 1.8. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain $\mathbb{D}$ in $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $q \geq nn_{\mathcal{D}} + 2n + 1$ be a positive integer. Let $m_1, \ldots, m_q$ be positive integers or ∞ such that*

$$\sum_{j=1}^{q} \frac{1}{m_j} < \frac{q - n - (n-1)(n_{\mathcal{D}} + 1)}{n_{\mathcal{D}}(n_{\mathcal{D}} + 2)}.$$

Suppose that for each $f \in \mathcal{F}$, there exist q moving hypersurfaces $D_{1,f}, \ldots, D_{q,f}$ which are defined by homogeneous polynomials $Q_{1,f}, \ldots, Q_{q,f} \in H_{\mathbb{D}}[x_0, \ldots, x_n]$ with degree $d_1, \ldots, d_q$ respectively, such that the following conditions are satisfied on each compact subset $K \subset \mathbb{D}$:
(i) For each $1 \leq j \leq q$, the coefficients of the homogeneous polynomials $Q_{j,f}$ are bounded above uniformly for all f in $\mathcal{F}$ on compact subset K of $\mathbb{D}$, and for any $\{f_p\} \subset \mathcal{F}$, and for any fixed $z \in \mathbb{D}$,

$$\inf_{p \in \mathbb{N}} D(Q_{1,f_p}, \ldots, Q_{q,f_p})(z) > 0.$$

(ii) Assume that f intersects $D_{j,f}$ with multiplicity at least m_j for each $1 \leq j \leq q$, for all $f \in \mathcal{F}$.
Then $\mathcal{F}$ is a holomorphically normal family on $\mathbb{D}$.

1.2. Some lemmas

In order to prove our results, we need some lemmas as follows:

Lemma 1.9. *[4] Let F be a family of holomorphic mappings of a domain $\mathbb{D}$ in $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Then the family F is not normal on $\mathbb{D}$ if and only if there exist a compact subset $K_0 \subset \mathbb{D}$ and sequences $\{f_p\} \subset F$, $\{y_p\} \subset K_0$, $\{r_p\} \subset \mathbb{R}$ with $r_p > 0$ and $r_p \to 0^+$, and $\{u_p\} \subset \mathbb{C}^m$ which are unit vectors such that*

$$g_p(\xi) := f_p(y_p + r_p u_p \xi),$$

where $\xi \in \mathbb{C}$ such that $y_p + r_p u_p \xi \in \mathbb{D}$, converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic map g of $\mathbb{C}$ to $\mathbb{P}^n(\mathbb{C})$.

Definition 1.10. Let $\Omega \subset \mathbb{C}^m$ be a hyperbolic domain and let M be a complete complex Hermittian manifold with metric ds^2_M. A holomorphic mapping $f(z)$ from Ω into M is said to be a normal holomorphic mapping from Ω into M if and only if there exists a positive constant c such that for all $z \in \Omega$ and all $\xi \in T_z(\Omega)$,

$$|ds^2_M(f(z), df(z)(\xi))| \leq cK_{\Omega}(z, \xi),$$

where $df(z)$ is the mapping from $T_z(\Omega)$ into $T_{f(z)}(M)$ induced by f and K_Ω denote the infinitesimal Kobayashi metric on Ω.

Lemma 1.11. *[5] Let natural numbers n and $q \geq n+1$ be fixed. Let D_{kp} $(1 \leq k \leq q, p \geq 1)$ be moving hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ such that the following conditions are satisfied:*
(i) For each $1 \leq k \leq q, p \geq 1$, the coefficients of the homogeneous polynomials Q_{kp} which define the D_{kp} are bounded above uniformly for all $p \geq 1$ on compact subsets of $\mathbb{D}$.
(ii) There exists $z_0 \in \mathbb{D}$ such that

$$\inf_{p \in \mathbb{N}} \{D(Q_{1p}, \ldots, Q_{qp})(z_0)\} > \delta > 0$$

Then, we have the following:
(a) There exists a subsequence $\{j_p\} \subset \mathbb{N}$ such that for $1 \leq k \leq q$, Q_{kj_p} converge uniformly on compact subsets of $\mathbb{D}$ to not identically zero homogeneous polynomials Q_k (meaning that the Q_{kj_p} and Q_k are homogeneous polynomials in $H_{\mathbb{D}}[x_0, \ldots, x_N]$ of the same degree, and all their coefficients converge uniformly on compact subsets of $\mathbb{D}$). Moreover, we have that $D(Q_1, \ldots, Q_q)(z_0) > \delta > 0$, the hypersurfaces $Q_1(z_0), \ldots, Q_q(z_0)$ are located in general position, and the moving hypersurfaces $Q_1(z), \ldots, Q_q(z)$ are located in (weakly) general position.
(b) There exist a subsequence $\{j_p\} \subset \mathbb{N}$ and $r = r(\delta) > 0$ such that

$$\inf_{p \in \mathbb{N}} D(Q_{1j_p}, \ldots, Q_{qj_p})(z) > \frac{\delta}{4}, \forall z \in B(z_0, r).$$

The following lemma is Picard type theorem in Nevanlinna-Cartan theory due to Nochka:

Lemma 1.12. *[11] Suppose that $q \geq 2n+1$ hyperplanes $H_1, \ldots, H_q$ are given in general position in $\mathbb{P}^n(\mathbb{C})$ and that q positive integers (may be ∞) $m_1, \ldots, m_q$ are given such that*

$$\sum_{j=1}^{q} \frac{1}{m_j} < \frac{q-n-1}{n}.$$

Then there does not exist a nonconstant holomorphic mapping $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ such that f intersects H_j with multiplicity at least m_j $(1 \leq j \leq q)$.

Lemma 1.13. *[5] Let $f = (f_0 : \cdots : f_n) : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic mapping, and let $\{P_i\}_{i=0}^{n+1}$ be $n+1$ homogeneous polynomials in general position of common degree in $\mathbb{C}[x_0, \ldots, x_n]$. Assume that*

$$F = (F_0 : \cdots : F_n) : \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^n(\mathbb{C})$$

is the mapping defined by

$$F_i(\mathbf{x}) = P_i(\mathbf{x}) \ (0 \leq i \leq n).$$

Then, $F(f)$ is a constant map if and only if f is a constant map.

Lemma 1.14. *[8] Let f be a meromorphic nonconstant mappings from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let D_i $(i = 1, \ldots, q), q \geq nn_{\mathcal{D}} + n + 1$ be slowly (with respective f) moving hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in weakly general position which is defined by homogeneous polynomials $Q_i \in H_{\mathbb{D}}[x_0, \ldots, x_n]$ with $\deg Q_i = d_i$ $(i = 1, \ldots, q)$. Set $m_{\mathcal{D}} = lcm(d_1, \ldots, d_q)$ and $n_{\mathcal{D}} = \binom{n+m_{\mathcal{D}}}{n} - 1$. Let $m_1, \ldots, m_q$ be positive integers or ∞ such that*

$$\sum_{j=1}^{q} \frac{1}{m_j} < \frac{q-(n-1)(n_{\mathcal{D}}+1)}{n_{\mathcal{D}}(n_{\mathcal{D}}+2)}.$$

Assume that $Q_j(\tilde{f}) \not\equiv 0$ $(1 \leq j \leq q)$ and f intersects D_j with multiplicity at least m_j for each $1 \leq j \leq q$, then f must be constant mapping.

1.3. *Proofs of Theorems*

Proof. [Proof of Theorem 1.5] Without loss of generality, we may assume that $\mathbb{D}$ is the unit disc. Suppose that F is not normal on $\mathbb{D}$. Then, by Lemma 1.9, there exists a subsequence denoted by $\{f_p\} \subset F$ and $y_0 \in K_0$, $\{y_p\}_{p=1}^{\infty} \subset K_0$ with $y_p \to y_0$, $\{r_p\} \subset (0, +\infty)$ with $r_p \to 0^+$, and $\{u_p\} \subset \mathbb{C}^m$, which are unit vectors, such that $g_p(\xi) := f_p(y_p + r_p u_p \xi)$ converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic map g of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$. Let

$$\varrho_{m_{\mathcal{D}}} : \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$$

be the *Veronese embedding of degree $m_{\mathcal{D}}$* which is given by

$$\varrho_{m_{\mathcal{D}}}(x) = (w_0(\mathbf{x}) : \cdots : w_{n_{\mathcal{D}}}(\mathbf{x})), \text{ where } w_j(\mathbf{x}) = \mathbf{x}^{I_j},\ j = 0, \ldots, n_{\mathcal{D}}.$$

For $w = (w_0 : \cdots : w_{n_{\mathcal{D}}})$, we denote $\mathbf{w} = (w_0, \ldots, w_{n_{\mathcal{D}}})$. We see that $G_p := \rho_{m_{\mathcal{D}}}(g_p)$ converges uniformly on compact subsets of $\mathbb{C}$ to $G := \rho_{m_{\mathcal{D}}}(g)$. By Lemma 1.11, $Q^*_{jp} = Q_{jp}^{m_{\mathcal{D}}/d_j} := Q_{j,f_p}^{m_{\mathcal{D}}/d_j}$ converge uniformaly on compact subset of $\mathbb{D}$ to $Q^*_j, 1 \leq j \leq q$, and

$$D(Q^*_1, \ldots, Q^*_q)(z) > \delta(z) > 0$$

for any fixed $z \in \mathbb{D}$. In particular, the moving hyperplanes $H^*_1, \ldots, H^*_q$ defining by moving linear forms $Q^*_1, \ldots, Q^*_q$ (respectively) are located in pointwise general position in $\mathbb{P}^{n_{\mathcal{D}}}(\mathbb{C})$.

We see that $y_p + r_p u_p \xi$ belong to a closed disc $K = \left\{ z \in \mathbb{C}^m : \|z\| \leq \frac{1+\|y_0\|}{2} \right\}$ of $\mathbb{D}$ for all ξ in a subset compact of $\mathbb{C}$ if p is large enough. Thus for $j \in \{1, ..., q\}$, there exist reduced representations $\tilde{f}_p = (f_{0p}, ..., f_{np}), \tilde{g} = (g_0, \ldots, g_n)$ of f_p, g respectively such that as $p \to \infty$, $\tilde{g}_p(\xi) = \tilde{f}_p(y_p + r_p u_p \xi) \to \tilde{g}(\xi)$ and hence

$$Q_{jp, y_p + r_p u_p \xi}(\tilde{g}_p(\xi)) \to Q_{j,y_0}(\tilde{g}(\xi)) \tag{3.1}$$

uniformly on compact subsets of $\mathbb{C}$.

Claim 1. $g(\mathbb{C})$ intersects with D_{j,y_0} with multiplicity at least m_j.

Indeed, for $\xi_0 \in \mathbb{C}$ with $Q_{j,y_0}(\tilde{g}(\xi_0)) = 0$, there exist a disc $B(\xi_0, r_0)$ in $\mathbb{C}$ and an index $i \in \{0, \ldots, n\}$ satisfying $g(B(\xi_0, r_0)) \subset \{[x_0 : ... : x_n] \in \mathbb{P}^n(\mathbb{C}) : x_i \neq 0\}$. Since $g_i(\xi) \neq 0$ on $B(\xi_0, r_0)$, Hurwitz's theorem shows that $g_{ip}(\xi) = f_{ip}(y_p + r_p u_p \xi) \neq 0$ on $B(\xi_0, r_0)$ when p is large enough. Thus $\tilde{g}_{ip}(\xi) = \left(\frac{g_{0p}}{g_{ip}}, \ldots, \frac{g_{np}}{g_{ip}} \right) \to \tilde{g}_i(\xi)$, and hence $Q_{jp, y_p + r_p u_p \xi}(\tilde{g}_{ip}(\xi)) \to Q_{j,y_0}(\tilde{g}_i(\xi))$ converge uniformly on $B(\xi_0, r_0)$, which further implies that

$$(Q_{jp, y_p + r_p u_p \xi}(\tilde{g}_{ip}(\xi)))^{(k)} \to (Q_{j,y_0}(\tilde{g}_i(\xi)))^{(k)} \tag{3.2}$$

uniformly on $B(\xi_0, r_0)$ for all $k \geq 1$.

By the condition (ii) in Theorem 1.5, when $m_j \geq 2$,

$$\sup_{1 \leq |\alpha| \leq m_j - 1, z \in f_p^{-1}(D_{jp}) \cap K} \left| \frac{\partial^\alpha (Q_{jp}(\tilde{f}_{ip}))}{\partial z^\alpha}(z) \right| < \infty$$

hold for all $j \in \{1, \ldots, q\}$ and all $p \geq 1$, where $f_{ip}(z) \neq 0$ and D_{jp} is the moving hypersurface associated to Q_{jp}. Since $Q_{j,y_0}(\tilde{g}(\xi_0)) = 0$, then Hurwitz's theorem shows that there exits $\xi_p \to \xi_0$ such that $Q_{jp, y_p + r_p u_p \xi_p}(\tilde{g}_p(\xi_p)) = 0$, which implies $y_p + r_p u_p \xi_p \in f_p^{-1}(D_{jp})$. Hence there exists $M > 0$ such that when $m_j \geq 2$ $(1 \leq j \leq q)$ and p is large enough

$$\left| \frac{\partial^\alpha (Q_{jp}(\tilde{f}_{ip}))}{\partial z^\alpha}(y_p + r_p u_p \xi_p) \right| \leq M$$

for all $\alpha = (\alpha_1, \ldots, \alpha_m) : 1 \leq |\alpha| \leq m_j - 1$ and some $i \in \{0, \ldots, n\}$, which means

$$\left| (Q_{jp, y_p + r_p u_p \xi}(\tilde{g}_{ip}(\xi)))^{(|\alpha|)} \right|_{\xi = \xi_p} = \left| \sum_\alpha c_\alpha r_p^{|\alpha|} \frac{\partial^\alpha (Q_{jp}(\tilde{f}_{ip}))}{\partial z_1^{\alpha_1} \ldots \partial z_m^{\alpha_m}}(y_p + r_p u_p \xi_p) \right| \leq C r_p^{|\alpha|} M, \tag{3.3}$$

where C, c_α are suitable constants. Combine (3.2) and (3.3), we get

$$(Q_{j,y_0}(\tilde{g}_i(\xi)))^{(|\alpha|)}|_{\xi = \xi_0} = \lim_{p \to \infty} (Q_{jp, y_p + r_p u_p \xi}(\tilde{g}_{ip}(\xi)))^{(|\alpha|)}_{\xi = \xi_p} = 0$$

for $1 \leq |\alpha| \leq m_j - 1$. Hence ξ_0 is a zero of $Q_{j,y_0}(\tilde{g})$ with multiplicity at least m_j. Therefore, Claim 1 is proved. We know that $G_p = \rho_{m_\mathcal{D}}(g_p)$ converges uniformly on compact subsets of $B(\xi_0, r_0)$ to $G = \rho_{m_\mathcal{D}}(g)$. Denote $\tilde{G}$ by a reduced representaion of G. Note that $Q_{j,y_0}^{m_\mathcal{D}/d_j}(\tilde{g}(\xi)) = Q_j^*(y_0)(\tilde{G}(\xi))$, then G intersects $H_j^*(y_0)$ with multiplicity at least $\frac{m_\mathcal{D}}{d_j} m_j$ for each $1 \leq j \leq q$, where $H_j^*(y_0)$ is a associated hyperplane with linear form $Q_j^*(y_0)$ in $\mathbb{P}^{n_\mathcal{D}}(\mathbb{C})$. Since $H_1^*(y_0), \ldots, H_q^*(y_0)$ are in general position in $\mathbb{P}^{n_\mathcal{D}}(\mathbb{C})$. From assumption,

$$\sum_{j=1}^q \frac{d_j}{m_j} < \frac{(q - n_\mathcal{D} - 1) m_\mathcal{D}}{n_\mathcal{D}}$$

and apply Lemma 1.12, we get G is a constant holomorphic map. Then g is a constant holomorphic map from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$. This is a contradiction. Therefore, $\mathcal{F}$ is holomorphically normal family on $\mathbb{D}$. □

Proof. [Proof Theorem 1.2] The proof of Theorem 1.2 is similarly to Theorem 1.5 by using Lemma 1.9 and Lemma 1.13. We omit it at here. □

Proof. [Proof of Theorem 1.7] Suppose that F is not normal on $\mathbb{D}$. Then, by Lemma 1.9, there exists a subsequence denoted by $\{f_p\} \subset F$ and $y_0 \in K_0$, $\{y_p\}_{p=1}^{\infty} \subset K_0$ with $y_p \to y_0$, $\{r_p\} \subset (0, +\infty)$ with $r_p \to 0^+$, and $\{u_p\} \subset \mathbb{C}^m$, which are unit vectors, such that $g_p(\xi) := f_p(y_p + r_p u_p \xi)$ converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic map g of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$. By Lemma 1.11, $Q_{jp} := Q_{j,f_p}$ converge uniformaly on compact subset of $\mathbb{D}$ to Q_j, $1 \le j \le q$, and

$$D(Q_1, \ldots, Q_q)(z) > \delta(z) > 0$$

for any fixed $z \in \mathbb{D}$. In particular, the moving hypersurfaces $D_1, \ldots, D_q$ defining by moving homogeneous polynomial $Q_1, \ldots, Q_q$ (respectively) are located in pointwise general position in $\mathbb{P}^n(\mathbb{C})$. We recall that with writing both variables $z \in \mathbb{D}$ and $x \in \mathbb{P}^n(\mathbb{C})$ we have

$$Q_{jp}(z)(\mathbf{x}) \to Q_j(z)(\mathbf{x})$$

uniformly on compact subsets in the variable $z \in \mathbb{D}$.

By arguments as the proof of Theorem 1.5, the map g intersects $D_j(y_0)$ with multiplicity at least m_j. Now, we show that g is a constant mapping. Since $Q_1(y_0), \ldots, Q_q(y_0)$ are in general position in $\mathbb{P}^n(\mathbb{C})$, then there are at most n hypersurfaces in the family $D_1(y_0), \ldots, D_q(y_0)$ containing image of g. We denote $\mathcal{I} = \{i \in \{1, \ldots, q\} : Q_i(y_0)(\tilde{g}) \equiv 0\}$. Therefore, we must have $0 \le |\mathcal{I}| \le n$. Note that $q \ge n n_{\mathcal{D}} + 2n + 1$, then $q - |\mathcal{I}| \ge n n_{\mathcal{D}} + n + 1$. We see that g intersects with $\{D_j(y_0)\}_{j \notin \mathcal{I}}$ with multiplicity at least m_j, and

$$\begin{aligned} \sum_{j \notin \mathcal{I}} \frac{1}{m_j} \le \sum_{j=1}^{q} \frac{1}{m_j} &< \frac{q - n - (n-1)(n_{\mathcal{D}} + 1)}{n_{\mathcal{D}}(n_{\mathcal{D}} + 2)} \\ &\le \frac{q - |\mathcal{I}| - (n-1)(n_{\mathcal{D}} + 1)}{n_{\mathcal{D}}(n_{\mathcal{D}} + 2)}. \end{aligned}$$

Apply Lemma 1.14 for map g and hypersurfaces $\{D_j(y_0)\}_{j \notin I}$, we get g is a constant map. This is a contradiction. Therefore, F is holomorphically normal family on $\mathbb{D}$. □

2. Lappan's five-valued theorem to holomorphic mapping in several variables

In this section, we give a version of Lappan's five-valued theorem to holomorphic mapping in several variables.

2.1. *Introduction and main results*

Marty's theorem is an important normal criterion related closely to study of normal functions, which states that $\mathscr{F}$ is normal on $\mathfrak{D}$ if and only if for each compact subset $K \subset \mathfrak{D}$, there exists a constant $M(K)$ such that

$$f^{\#}(z) := \frac{|f'(z)|}{1 + |f(z)|^2} \le M(K)$$

for all $z \in K$ and all $f \in \mathscr{F}$. For example, Lehto-Virtanen [10] showed that a meromorphic function f defined on the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of $\mathbb{C}$ is normal if and only if $\sup_{z \in \mathbb{U}}(1 - |z|^2) f^{\#}(z) < \infty$. Recall that a meromorphic function f defined on $\mathbb{U}$ is said to be *normal* if the family $\{f \circ \tau : \tau \in \mathcal{T}\}$ is normal in $\mathbb{U}$, where $\mathcal{T}$ is the set of all conformal self-mappings of $\mathbb{U}$. The following five-valued theorem of Lappan [9] strongly answers a question posed by Pommerenke [15]: if $M > 0$ is given, does there exists a finite set E such that a meromorphic function f on $\mathbb{U}$ is normal under the following condition

$$\sup_{z\in f^{-1}(E)} (1-|z|^2)f^{\#}(z) \le M? \tag{3.4}$$

Theorem D. *A meromorphic function f on $\mathbb{U}$ is normal if there exist a positive number M and a set E consisting of five distinct elements in $\mathbb{P}^1$ such that the condition* (3.4) *holds.*

Hahn [7] generalized Lappan's five-valued theorem to several complex variables. However, the key Lemma 2 in [7] has a gap. In this paper, we give a version of Lappan's five-valued theorem to φ-normal mapping in several complex variables.

Let $\mathbb{U}^m$ be the unit ball in $\mathbb{C}^m$ centered at 0 and let $\mathcal{H}(\mathbb{U}^m)$ be the set of holomorphic mappings from $\mathbb{U}^m$ into $\mathbb{P}^n(\mathbb{C})$. When $n = 1$, $\mathcal{H}(\mathbb{U}^m)$ just is the set $\mathcal{M}(\mathbb{U}^m)$ of meromorphic functions on $\mathbb{U}^m$. Let $\varphi : [0,1) \to \mathbb{R}^+$ be an increasing function satisfying the following properties

$$\varphi(r)(1-r) \ge 1 \quad \text{for all} \quad r \in [0,1) \tag{3.5}$$

and

$$\mathcal{R}_a(z) := \frac{\varphi(\|a + z/\varphi(\|a\|)\|)}{\varphi(\|a\|)} \to 1 \quad \text{as} \quad \|a\| \to 1^- \tag{3.6}$$

uniformly on compact subsets of $\mathbb{U}^m$, where $\|a\|$ is the Euclidean norm of $a \in \mathbb{C}^m$. Then an element $f \in \mathcal{H}(\mathbb{U}^m)$ is called *φ-normal* and hence belongs to *class $\mathcal{N}^\varphi$* if

$$\|f\|_{\mathcal{N}^\varphi} := \sup_{\|\xi\|=1, z\in\mathbb{U}^m} \frac{E_{\mathbb{P}^n(\mathbb{C})}(f(z), df(z)(\xi))}{\varphi(\|z\|)} < \infty, \tag{3.7}$$

in which $ds^2_{\mathbb{P}^n(\mathbb{C})}$ is the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$.

From (3.6), there exists $M(K) > 0$ such that

$$\sup_{a\in\mathbb{U}^m} \mathcal{R}_a(z) \le M(K) \tag{3.8}$$

for each compact set $K \subset \mathbb{U}^m$. In particular, when $m = n = 1$, if we choose $\varphi(r) = \frac{1}{1-r}$, we obtain again the definition of normal meromorphic function due to Lehto-Virtanen [10].

Let $\mathcal{F} \subset \mathcal{H}(\mathbb{U}^m)$ be a family of holomorphic mapping from $\mathbb{U}^m$ into $\mathbb{P}^n(\mathbb{C})$. The family $\mathcal{F}$ is called uniformly φ-normal if

$$\sup_{f\in\mathcal{F}} \|f\|_{\mathcal{N}^\varphi} < +\infty. \tag{3.9}$$

We also generalize Lappan's five-valued theorem to φ-normal mapping in several complex variables as follows:

Theorem 2.1. *Let $\mathcal{F}$ be a family of holomorphic mappings from $\mathbb{U}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $D_1, \ldots, D_q$ be q $(\ge 2n_{\mathcal{D}} + 1)$ hypersurfaces which are located in general position for Veronese embedding in $\mathbb{P}^n(\mathbb{C})$, and $Q_j \in \mathbb{C}[x_0, \ldots, x_n]$, $j = 1, \ldots, q$ are homogeneous polynomial with degree d_j defined D_j, $j = 1, \ldots, q$. For each $f \in \mathcal{F}$, assume that $\tilde{f} = (f_0, \ldots, f_n)$ is a reduced representation of f on $\mathbb{U}^m$ with $f_i(z) \ne 0$ for some i and z, and when $m_j \ge 2$*

$$\sup_{1\le|\alpha|\le m_j-1, z\in f^{-1}(D_j)} \frac{\left|\dfrac{\partial^\alpha(Q_j(\tilde{f_i}))}{\partial z^\alpha}(z)\right|}{\varphi^{|\alpha|}(\|z\|)} < \infty$$

hold for all $j \in \{1, \ldots, q\}$ *and all* $f \in \mathcal{F}$, *where* $m_1, \ldots, m_q$ *are positive integers and may be* ∞, *with* $\sum_{j=1}^{q} \frac{d_j}{m_j} < \frac{(q - n_{\mathcal{D}} - 1)m_{\mathcal{D}}}{n_{\mathcal{D}}}$, $n_{\mathcal{D}} = \binom{n+m_{\mathcal{D}}}{n} - 1$, *and* $m_{\mathcal{D}}$ *is the least common multiple of the* $d_1, ..., d_q$. *Then* $\mathcal{F}$ *is a uniformly* φ*-normal family.*

In Theorem 2.1, when $\mathcal{F} = \{f\}$, we can get two results as follows:

Corollary 2.2. *Let* f *be a holomorphic mapping from* $\mathbb{U}^m$ *into* $\mathbb{P}^n(\mathbb{C})$. *Let* $D_1, \ldots, D_q$ *be* q $(\geq 2n_{\mathcal{D}} + 1)$ *hypersurfaces which are located in general position for Veronese embedding in* $\mathbb{P}^n(\mathbb{C})$, *and* $Q_j \in \mathbb{C}[x_0, \ldots, x_n]$, $j = 1, \ldots, q$ *are homogeneous polynomial with degree* d_j *defined* D_j, $j = 1, \ldots, q$. *Let* $\tilde{f} = (f_0, \ldots, f_n)$ *be a reduced representation of* f *on* $\mathbb{U}^m$ *with* $f_i(z) \neq 0$ *for some* i *and* z, *and when* $m_j \geq 2$

$$\sup_{1\leq|\alpha|\leq m_j-1, z\in f^{-1}(D_j)} \frac{\left|\frac{\partial^{\alpha}(Q_j(\tilde{f_i}))}{\partial z^{\alpha}}(z)\right|}{\varphi^{|\alpha|}(\|z\|)} < \infty$$

hold for all $j \in \{1, \ldots, q\}$, *where* $m_1, \ldots, m_q$ *are positive integers and may be* ∞, *with* $\sum_{j=1}^{q} \frac{d_j}{m_j} < \frac{(q - n_{\mathcal{D}} - 1)m_{\mathcal{D}}}{n_{\mathcal{D}}}$, $n_{\mathcal{D}} = \binom{n+m_{\mathcal{D}}}{n} - 1$, *and* $m_{\mathcal{D}}$ *is the least common multiple of the* $d_1, ..., d_q$. *Then* f *is a* φ*-normal on* $\mathbb{U}^m$.

Corollary 2.3. *Take positive integers* $q \geq 2n + 1$ *and* m_j *(or* ∞*) with* $\sum_{j=1}^{q} \frac{1}{m_j} < \frac{q - n - 1}{n}$. *An element* $f \in \mathcal{H}(\mathbb{U}^m)$ *is* φ*-normal if there are linear independency of polynomials* $L_j \in \mathbb{C}[x_0, \ldots, x_n]$ $(1 \leq j \leq q)$ *such that if* $\tilde{f} = (f_0, ..., f_n)$ *is a reduced representation of* f *on* $\mathbb{U}^m$ *with* $f_i(z) \neq 0$ *for some* i *and* z, *then when* $m_j \geq 2$,

$$\sup_{1\leq|\alpha|\leq m_j-1, z\in f^{-1}(H_j)} \frac{|\frac{\partial^{\alpha}(L_j(\tilde{f_i}))}{\partial z^{\alpha}}(z)|}{\varphi^{|\alpha|}(\|z\|)} < \infty$$

hold for all $j \in \{1, \ldots, q\}$, *where* H_j *is the hyperplane associated to* L_j.

Theorem 2.4. *Let* $\mathcal{F}$ *be a family of holomorphic mappings from* $\mathbb{U}^m$ *into* $\mathbb{P}^n(\mathbb{C})$. *Take positive integers* $q \geq nn_{\mathcal{D}}+2n+1$, d_j *and* m_j *(or* ∞*) such that* $\sum_{j=1}^{q} \frac{1}{m_j} < \frac{q - n - (n-1)(n_{\mathcal{D}} + 1)}{n_{\mathcal{D}}(n_{\mathcal{D}} + 2)}$, *where* $n_{\mathcal{D}} = \binom{n+m_{\mathcal{D}}}{n} - 1$, *and* $m_{\mathcal{D}}$ *is the least common multiple of the* $d_1, ..., d_q$. *Let* $Q_j \in \mathbb{C}[x_0, \ldots, x_n]$ *be homogeneous polynomials of degree* d_j $(1 \leq j \leq q)$ *in general position such that if* $\tilde{f} = (f_0, ..., f_n)$ *is a reduced representation of each* $f \in \mathcal{F}$ *on* $\mathbb{U}^m$ *with* $f_i(z) \neq 0$ *for some* i *and* z, *then when* $m_j \geq 2$,

$$\sup_{1\leq|\alpha|\leq m_j-1, z\in f^{-1}(D_j)} \frac{|\frac{\partial^{\alpha}(Q_j(\tilde{f_i}))}{\partial z^{\alpha}}(z)|}{\varphi^{|\alpha|}(\|z\|)} < \infty$$

hold for all $j \in \{1, \ldots, q\}$ *for all* $f \in \mathcal{F}$, *where* D_j *is the hypersurface associated to* Q_j. *Then* $\mathcal{F}$ *is a uniformly* φ*-normal family.*

In Theorem 2.4, set $\mathcal{F} = \{f\}$, we get the following result:

Corollary 2.5. *Take positive integers* $q \geq nn_{\mathcal{D}}+2n+1$, d_j *and* m_j *(or* ∞*) such that* $\sum_{j=1}^{q} \frac{1}{m_j} < \frac{q - n - (n-1)(n_{\mathcal{D}} + 1)}{n_{\mathcal{D}}(n_{\mathcal{D}} + 2)}$, *where* $n_{\mathcal{D}} = \binom{n+m_{\mathcal{D}}}{n} - 1$, *and* $m_{\mathcal{D}}$ *is the least common multiple of the* $d_1, ..., d_q$. *An element* $f \in \mathcal{H}(\mathbb{U}^m)$ *is* φ*-normal* *if there exist homogeneous polynomials $Q_j \in \mathbb{C}[x_0, \ldots, x_n]$ of degree d_j $(1 \le j \le q)$ in general position such that if $\tilde{f} = (f_0, ..., f_n)$ is a reduced representation of f on $\mathbb{U}^m$ with $f_i(z) \neq 0$ for some i and z, then when $m_j \geq 2$,*

$$\sup_{1\le|\alpha|\le m_j-1, z\in f^{-1}(D_j)} \frac{|\dfrac{\partial^\alpha(Q_j(\tilde{f_i}))}{\partial z^\alpha}(z)|}{\varphi^{|\alpha|}(\|z\|)} < \infty$$

hold for all $j \in \{1, \ldots, q\}$, where D_j is the hypersurface associated to Q_j.

2.2. Proofs of Theorems 2.1 and 2.4

First, we need the following lemma:

Lemma 2.6. *[18] Let Ω be a domain in $\mathbb{C}^m$ and let M be a complex space. Let $\mathcal{F}$ be a family in the set $Hol(\Omega, M)$ of holomorphic mappings from Ω into M. If the family $\mathcal{F}$ is normal, then for each norm E_M on $T(M)$ and for each compact subset K of Ω, there is a constant $C_K > 0$ such that*

$$E_M(f(z), df(z)(\xi)) \le C_K\|\xi\| \tag{3.10}$$

for every $z \in K$, $\xi \in \mathbb{C}^m \setminus \{0\}$, $f \in \mathcal{F}$. Conversely, if M is a complete Hermitian complex space and if the family $\mathcal{F}$ is not compactly divergent such that (3.10) *holds, then $\mathcal{F}$ is normal.*

We see that the inequality (3.10) is equivalent to

$$E_M(f(z), df(z)(\xi)) \le C_K, \|\xi\| = 1. \tag{3.11}$$

Lemma 2.7. *The family $\mathcal{F} \in \mathcal{H}(\mathbb{U}^m)$ is not uniformly φ-normal if and only if there exist a sequence $\{f_p\} \subset \mathcal{F}$, a subset compact $K_0 \subset \mathbb{U}^m$, three sequences $\{z_p^*\} \subset \mathbb{U}^m$, $\{z_p\} \subset \mathbb{U}^m$, $\{\rho_p\} \subset \mathbb{R}^+$ satisfying*

(i) $\{z_p^*\} \subset K_0$;

(ii) $w_p = z_p + \dfrac{z_p^*}{\varphi(\|z_p\|)} \in \mathbb{U}^m$ *for p large sufficiently;*

(iii) $\rho_p \to 0^+$ *such that the sequence $g_p(\xi) = f_p(w_p + \frac{\rho_p}{\varphi(\|z_p\|)}\xi_p\xi)$ converges uniformly on compact subsets of $\mathbb{C}$ to a non-constant holomorphic mapping $g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$, where $\xi_p \in \mathbb{C}^m$ with $\|\xi_p\| = 1$.*

Proof. Suppose that $\mathcal{F}$ is not uniformly φ-normal, then (3.9) is not true, so that there exists sequences $\{z_p\}, \{\varsigma_p\}$ in $\mathbb{U}^m$ with $\|\varsigma_p\| = 1$ such that

$$\lim_{p\to\infty} \frac{E_{\mathbb{P}^n(\mathbb{C})}(f_p(z_p), df_p(z_p)(\varsigma_p))}{\varphi(\|z_p\|)} = +\infty. \tag{3.12}$$

Set $\mathcal{G} = \left\{ g_{z_p}(\zeta) = f_p\left(z_p + \dfrac{\zeta}{\varphi(\|z_p\|)}\right)\right\}$. Then (3.12) implies

$$\lim_{p\to\infty} E_{\mathbb{P}^n(\mathbb{C})}(g_{z_p}(0), dg_{z_p}(0)(\varsigma_p)) = \lim_{p\to\infty} \frac{E_{\mathbb{P}^n(\mathbb{C})}(f_p(z_p), df_p(z_p)(\varsigma_p))}{\varphi(\|z_p\|)} = +\infty,$$

which implies $\mathcal{G}$ is not normal on some neighborhood of 0 by Lemma 2.6. By the definition and Lemma 1.9, we can deduce that there exist a compact subset $K_0 \subset \mathbb{U}^m$ and sequences $\{z_{t_p}^*\} \subset K_0$, $\{g_{z_{t_p}}\} \subset \mathcal{G}$, $\{\rho_{t_p}\} \subset \mathbb{R}^+$, $\{\xi_{t_p}\} \subset \mathbb{C}^m$ with $\|\xi_{t_p}\| = 1$ such that $\rho_{t_p} \to 0^+$, and $g_p(\xi) = g_{z_{t_p}}(z_{t_p}^* + \rho_{t_p}\xi_{t_p}\xi) = f_p\left(z_{t_p} + \dfrac{z_{t_p}^* + \rho_{t_p}\xi_{t_p}\xi}{\varphi(\|z_{t_p}\|)}\right)$ converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic mapping $g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$. Up to a subsequence, we can suppose that $t_p = p$. Setting $w_p = z_p + \dfrac{z_p^*}{\varphi(\|z_p\|)}$, then we obtain the sequence $g_p(\xi) = f_p(w_p + \rho_p \dfrac{\xi_p \xi}{\varphi(\|z_p\|)})$ converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic mapping $g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$.

Conversely assume, to the contrary, that $\mathcal{F}$ is uniformly φ-normal under the conditions (i) to (iii) in Lemma 2.7. Then by (3.9), there exists a constant $M > 0$ satisfying

$$\frac{E_{\mathbb{P}^n(\mathbb{C})}(f(a), df(a)(\eta))}{\varphi(\|a\|)} \leq M \tag{3.13}$$

for $a \in \mathbb{U}^m$, $\eta \in \mathbb{C}^m$ with $\|\eta\| = 1$ and for all $f \in \mathcal{F}$. Note that

$$\begin{aligned} E_{\mathbb{P}^n(\mathbb{C})}(g_p(\xi), dg_p(\xi)(1)) &= \rho_p \frac{E_{\mathbb{P}^n(\mathbb{C})}(f_p(u_p), df_p(u_p)(\xi_p))}{\varphi(\|z_p\|)} \\ &= \rho_p \frac{\varphi(\|u_p\|)}{\varphi(\|z_p\|)} \frac{E_{\mathbb{P}^n(\mathbb{C})}(f_p(u_p), df_p(u_p)(\xi_p))}{\varphi(\|u_p\|)}, \end{aligned} \tag{3.14}$$

where

$$u_p = u_p(\xi) := w_p + \frac{\rho_p}{\varphi(\|z_p\|)} \xi_p \xi = z_p + \frac{z_p^* + \rho_p \xi_p \xi}{\varphi(\|z_p\|)}.$$

Take a compact subset K of $\mathbb{C}$. When $\xi \in K$, then $z_p^* + \rho_p \xi_p \xi$ belong to a compact subset of $\mathbb{U}^m$ if p is large enough. Hence by (3.8), (3.13) and (3.14), we get

$$E_{\mathbb{P}^n(\mathbb{C})}(g(\xi), dg(\xi)(1)) = \lim_{p\to\infty} E_{\mathbb{P}^n(\mathbb{C})}(g_p(\xi), dg_p(\xi)(1)) = 0$$

for $\xi \in K$. Therefore, g is constant. It is a contradiction since g is nonconstant by the assumption. This completes the proof of Lemma 2.7. □

Proof. [Proofs of Theorem 2.1 and Theorem 2.4] We now give the proofs of Theorem 2.1 and Theorem 2.4 by contradictions. Assume that $\mathcal{F}$ is not uniformly φ-normal. By Lemma 2.7, there exist a compact subset $K_0 \subset \mathbb{U}^m$, and four sequences $\{z_p^*\} \subset K_0$, $\{z_p\} \subset \mathbb{U}^m$ with $\{\rho_p\} \subset \mathbb{R}^+$ with $\rho_p \to 0^+$, $\{\xi_p\} \subset \mathbb{C}^m$ with $\|\xi_p\| = 1$ such that $w_p = z_p + \frac{z_p^*}{\varphi(\|z_p\|)} \in \mathbb{U}^m$ when p is large sufficiently, and such that the sequence $g_p(\xi) = f_p(u_p(\xi))$ converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic mapping $g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$, where

$$u_p = u_p(\xi) := w_p + \frac{\rho_p}{\varphi(\|z_p\|)} \xi_p \xi.$$

Therefore, there exists a reduced representation $\tilde{g} = (g_0, \ldots, g_n) : \mathbb{C} \longrightarrow \mathbb{C}^{n+1}$ such that for each $j \in \{1, \ldots, q\}$, as $p \to \infty$,

$$\tilde{g}_p(\xi) = (g_{0p}(\xi), \ldots, g_{np}(\xi)) \to \tilde{g}(\xi),\ Q_j(\tilde{g}_p(\xi)) \to Q_j(\tilde{g}(\xi)) \tag{3.15}$$

uniformly on each compact subset of $\mathbb{C}$, where $\tilde{g}_p(\xi) = \tilde{f}_p(u_p(\xi))$. We claim that $g(\mathbb{C})$ intersects with D_j with multiplicity at least $m_j (j = 1, \ldots, q)$.

In fact, if $Q_j(\tilde{g}(\xi_0)) = 0$ for $\xi_0 \in \mathbb{C}$, then there exist a disc $B(\xi_0, r_0)$ in $\mathbb{C}$ and an index $i \in \{0, \ldots, n\}$ satisfying

$$g(B(\xi_0, r_0)) \subset \{[x_0 : ... : x_n] \in \mathbb{P}^n(\mathbb{C}) : x_i \neq 0\}$$

and $g_i(\xi) \neq 0$ on $B(\xi_0, r_0)$. By Hurwitz's theorem, we may suppose that $g_{ip}(\xi) \neq 0$ on $B(\xi_0, r_0)$ when p is large enough, and hence $\dfrac{g_{kp}}{g_{ip}}(\xi) \to \dfrac{g_k}{g_i}(\xi)$ converges uniformly on $B(\xi_0, r_0)$ for all $k \neq i$, which further implies that

$Q_j(\tilde{g}_{i,p}(\xi))$ converges uniformly to $Q_j(\tilde{g}_i(\xi))$ on $B(\xi_0, r_0)$, where $\tilde{g}_{i,p} = (\frac{g_{0p}}{g_{ip}}, \ldots, \frac{g_{np}}{g_{ip}})$ as usual and similarly for $\tilde{g}_i$. Thus we get

$$(Q_j(\tilde{g}_{i,p}(\xi)))^{(k)} \to (Q_j(\tilde{g}_i(\xi)))^{(k)} \tag{3.16}$$

uniformly on $B(\xi_0, r_0)$ for all $k \in \mathbb{N}$. Since $Q_j(\tilde{g}(\xi_0)) = 0$, Hurwitz's theorem means that there exits $\eta_p \to \xi_0$ such that $Q_j(\tilde{f_p}(u_p(\eta_p))) = 0$, which implies $u_p(\eta_p) \in f_p^{-1}(D_j)$.

By the assumption, when $m_j \geq 2$,

$$\sup_{1 \leq |\alpha| \leq m_j - 1, z \in f_p^{-1}(D_j)} \frac{\left|\frac{\partial^\alpha (Q_j(\tilde{f}_{i,p}))}{\partial z^\alpha}(z)\right|}{\varphi^{|\alpha|}(\|z\|)} < \infty$$

hold for all $j \in \{1, \ldots, q\}$ and all $p \geq 1$, where $f_{ip}(z) \neq 0$ and D_j is the hypersurface associated to Q_j. Since $Q_j(\tilde{g}(\xi_0)) = 0$, then Hurwitz's theorem shows that there exits $\nu_p \to \xi_0$ such that $Q_j(\tilde{g}_p(\nu_p)) = 0$, which implies $u_p(\nu_p) \in f_p^{-1}(D_j)$. Hence there exists $M > 0$ such that when $m_j \geq 2$ $(1 \leq j \leq q)$ and p is large enough

$$\left|\frac{\partial^\alpha (Q_j(\tilde{f}_{i,p}))}{\partial z^\alpha}(u_p(\nu_p))\right| \leq M \varphi^{|\alpha|}(\|u_p(\nu_p)\|)$$

for all $\alpha = (\alpha_1, \ldots, \alpha_m) : 1 \leq |\alpha| \leq m_j - 1$ and some $i \in \{0, \ldots, n\}$, which means

$$\begin{aligned} \left|(Q_j(\tilde{g}_{i,p}(\xi)))^{(|\alpha|)}\right|_{\xi = \nu_p} &= \left|\sum_\alpha c_\alpha \frac{r_p^{|\alpha|}}{\varphi^{|\alpha|}(\|z_p\|)} \frac{\partial^\alpha (Q_j(\tilde{f}_{i,p}))}{\partial z_1^{\alpha_1} \ldots \partial z_m^{\alpha_m}}(w_p + \frac{\rho_p}{\varphi(\|z_p\|)} \xi_p \nu_p)\right| \\ &\leq C r_p^{|\alpha|} M \left(\frac{\varphi(\|u_p(\nu_p)\|)}{\varphi(\|z_p\|)}\right)^{|\alpha|}, \end{aligned} \tag{3.17}$$

where C, c_α are suitable constants. Combine (3.16), (3.17) and (3.8), we get

$$(Q_j(\tilde{g}_i(\xi)))^{(|\alpha|)}|_{\xi = \xi_0} = \lim_{p \to \infty} (Q_j(\tilde{g}_{i,p}(\xi)))^{(|\alpha|)}_{\xi = \nu_p} = 0$$

for $1 \leq |\alpha| \leq m_j - 1$. Hence ξ_0 is a zero of $Q_j(\tilde{g})$ with multiplicity at least m_j. Therefore, the above claim is proved.

By the assumption of Theorem 2.1, we know that $G_p = \rho_{m_\mathcal{D}}(g_p)$ converges uniformly on compact subsets of $B(\xi_0, r_0)$ to $G = \rho_{m_\mathcal{D}}(g)$. Denote $\tilde{G}$ by a reduced representaion of G. Note that $Q_j^{m_\mathcal{D}/d_j}(\tilde{g}(\xi)) = Q_j^*(\tilde{G}(\xi))$, then G intersects H_j^* with multiplicity at least $\frac{m_\mathcal{D}}{d_j} m_j$ for each $1 \leq j \leq q$, where H_j^* is a associated hyperplane with linear form Q_j^* in $\mathbb{P}^{n_\mathcal{D}}(\mathbb{C})$. Since $H_1^*, \ldots, H_q^*$ are in general position in $\mathbb{P}^{n_\mathcal{D}}(\mathbb{C})$. From assumption,

$$\sum_{j=1}^q \frac{d_j}{m_j} < \frac{(q - n_\mathcal{D} - 1) m_\mathcal{D}}{n_\mathcal{D}}$$

and apply Lemma 1.12, we get G is a constant holomorphic map. Then g is a constant holomorphic map from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$. This is a contradiction. Therefore, $\mathcal{F}$ is a uniformly φ-normal mapping on $\mathbb{U}^m$. Thus Theorem 2.1 is proved.

Now we continue to prove Theorem 2.4. Since $D_1, \ldots, D_q$ are in general position in $\mathbb{P}^n(\mathbb{C})$, then there are at most n hypersurfaces in the family $D_1, \ldots, D_q$ containing the image of g. Set $\mathcal{I} = \{i \in \{1, \ldots, q\} : Q_i(\tilde{g}) \equiv 0\}$. We must have $0 \leq |\mathcal{I}| \leq n$. Note that $q \geq n n_\mathcal{D} + 2n + 1$, and hence $q - |\mathcal{I}| \geq n n_\mathcal{D} + n + 1$. Similar arguments as that in the proof of Theorem 2.1, we have that $g(\mathbb{C})$ intersects with each of $\{D_j\}_{j\notin \mathcal{I}}$ with multiplicity at least m_j, and

$$\sum_{j\notin\mathcal{I}} \frac{1}{m_j} \le \sum_{j=1}^{q} \frac{1}{m_j} < \frac{q-n-(n-1)(n_{\mathcal{D}}+1)}{n_{\mathcal{D}}(n_{\mathcal{D}}+2)} \le \frac{q-|\mathcal{I}|-(n-1)(n_{\mathcal{D}}+1)}{n_{\mathcal{D}}(n_{\mathcal{D}}+2)},$$

apply Lemma 1.14 to g and $\{D_j\}_{j\notin\mathcal{I}}$, it follows that g is constant. This is a contradiction. Therefore, $\mathcal{F}$ is a uniformly φ-normal mapping on $\mathbb{U}^m$. Thus Theorem 2.4 is proved. □

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