Filomat 37:26 (2023), 8791–8811
https://doi.org/10.2298/FIL2326791C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Conditional Wiener integral associated with Gaussian processes and applications

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Abstract. Let $C_0[0,T]$ denote the one-parameter Wiener space and let $C_0'[0,T]$ be the Cameron–Martin space in $C_0[0,T]$. Given a function k in $C_0'[0,T]$, define a stochastic process $\mathcal{Z}_k : C_0[0,T] \times [0,T] \to \mathbb{R}$ by $\mathcal{Z}_k(x,t) = \int_0^t Dk(s)dx(s)$, where $Dk \equiv \frac{d}{dt}k$. Let a random vector $X_{\mathcal{G},k} : C_0[0,T] \to \mathbb{R}^n$ be given by

$$X_{\mathcal{G},k}(x) = ((g_1, \mathcal{Z}_k(x,\cdot))^\sim, \ldots, (g_n, \mathcal{Z}_k(x,\cdot))^\sim),$$

where $\mathcal{G} = \{g_1, \ldots, g_n\}$ is an orthonormal set with respect to the weighted inner product induced by the function k on the space $C_0'[0,T]$, and $(g, \mathcal{Z}_k(x,\cdot))^\sim$ denotes the Paley–Wiener–Zygmund stochastic integral. In this paper, using the reproducing kernel property of the Cameron–Martin space, we establish a very general evaluation formula for expressing conditional generalized Wiener integrals, $E\big(F(\mathcal{Z}_k(x,\cdot))\big|X_{\mathcal{G},k}(x) = \vec{\eta}\big)$, associated with the Gaussian processes $\mathcal{Z}_k$. As an application, we establish a translation theorem for the conditional Wiener integral and then use it to obtain various conditional Wiener integration formulas on $C_0[0,T]$.

1. Introduction

Let $C_0[0,T]$ denote the one-parameter Wiener space; that is, the space of all real-valued continuous functions x on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let $\mathfrak{m}$ denote the Wiener measure. Then, as is well known, $(C_0[0,T], \mathcal{M}, \mathfrak{m})$ is a complete probability measure space. The coordinate process $W \equiv \{W_t\}_{t\in[0,T]}$ given by $W_t(x) \equiv W(x,t) = x(t)$ on $C_0[0,T] \times [0,T]$ forms a standard Brownian motion. Thus the Wiener measure $\mathfrak{m}$ is the Gaussian measure on $C_0[0,T]$ with mean zero and covariance function $r(s,t) = \min\{s,t\}$. Throughout this paper, we denote the Wiener integral of a Wiener integrable functional F by

$$E[F] \equiv E_x[F(x)] = \int_{C_0[0,T]} F(x)d\mathfrak{m}(x).$$

We start this paper with the definition of the conditional Wiener integral [9, 10, 20–22].

2020 *Mathematics Subject Classification*. Primary 46B09, 46G12, 60G15; Secondary 28C20, 60J65.
Keywords. Wiener space, Cameron–Martin space, reproducing kernel property, evaluation formula, conditional generalized Wiener integral, Gaussian process.
Received: 25 December 2022; Accepted: 02 June 2023
Communicated by Dragan S. Djordjević
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Definition 1.1. *Let $X : C_0[0,T] \to \mathbb{R}^n$ be a Wiener measurable function whose probability distribution $\mathfrak{m}_X$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$, and let F be a complex-valued Wiener integrable functional on $C_0[0,T]$. Then, by the definition of conditional expectation, the conditional Wiener integral of F given X, denoted by $E(F|X = \vec{\eta})$, is a Lebesgue measurable and $\mathfrak{m}_X$-integrable function of $\vec{\eta}$, unique up to null sets in $\mathbb{R}^n$, satisfying the equation*

$$\int_{X^{-1}(B)} F(x)d\mathfrak{m}(x) = \int_B E(F|X = \vec{\eta})d\mathfrak{m}_X(\vec{\eta})$$

for all Borel sets B in $\mathbb{R}^n$.

Using the Fourier transform, Yeh [21] derived an inversion formula for a conditional expectation on probability spaces. Since then, in [22, 23], Yeh applied the inversion formula to the conditional Wiener integral in order to derive very useful formulas to analyze the Kac–Feynman integral equation and the conditional Cameron–Martin translation theorem. See [2, 3, 8] for further work involving Yeh's inversion formula. The evaluation formulas for the conditional Wiener integrals in [3, 8, 22, 23] were performed on the functionals of standard Brownian motion paths in the Wiener space $C_0[0,T]$.

In [16], Park and Skoug pointed out that Yeh's inversion formula for evaluating the conditional Wiener integral is very complicated and difficult to use. But, in [16], Park and Skoug established a simple formula for expressing the conditional Wiener integrals, $E(F|X_\tau = \vec{\eta})$, in terms of ordinary Wiener integrals, with the conditioning function $X_\tau : C_0[0,T] \to \mathbb{R}^n$ given by

$$X_\tau(x) = (x(t_1), \dots, x(t_n)), \tag{1.1}$$

where $\tau = \{t_1, \dots, t_n\}$ is a partition of $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_n \le T$.

In [17], Park and Skoug improved this simple formula from the conditional Wiener integral to the conditional 'generalized' Wiener integral. The generalized Wiener integral [10, 19] was defined by the Wiener integral

$$E_x[F(z_h(x,\cdot))] = \int_{C_0[0,T]} F(z_h(x,\cdot))d\mathfrak{m}(x),$$

where $z_h : C_0[0,T] \times [0,T] \to \mathbb{R}$ is the stochastic process given by

$$z_h(x,t) = \int_0^t h(s)\tilde{d}x(s), \tag{1.2}$$

with $h \in L_2[0,T]$, and where $\int_0^t h(s)\tilde{d}x(s)$ denotes the Paley–Wiener–Zygmund stochastic integral [14, 15]. Given a partition $\tau = \{t_1, \dots, t_n\}$ of $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_n = T$, let $X_{h,\tau} : C_0[0,T] \to \mathbb{R}^n$ be defined by

$$X_{h,\tau}(x) = (z_h(x,t_1), \dots, z_h(x,t_n)). \tag{1.3}$$

In this setting, the evaluation formula obtained by Park and Skoug in [17] was given by

$$\begin{aligned} E(F(z_h(x,\cdot))|X_{h,\tau}(x) = \vec{\eta}) &= E_x\Big[F(z_h(x,\cdot) - [z_h(x,\cdot)](\cdot) + [\vec{\eta}](\cdot))\Big] \\ &\equiv \int_{C_0[0,T]} F(z_h(x,\cdot) - [z_h(x,\cdot)](\cdot) + [\vec{\eta}](\cdot))d\mathfrak{m}(x) \end{aligned} \tag{1.4}$$

where

$$[z_h(x,\cdot)](t) = z_h(x,t_{j-1}) + \frac{\beta_h(t) - \beta_h(t_{j-1})}{\beta_h(t_j) - \beta_h(t_{j-1})}(z_h(x,t_j) - z_h(x,(t_{j-1}))), \qquad t_{j-1} \le t \le t_j,\ j \in \{1, \dots, n\} \tag{1.5}$$

and

$$[\vec{\eta}](t) = \eta_{j-1} + \frac{\beta_h(t) - \beta_h(t_{j-1})}{\beta_h(t_j) - \beta_h(t_{j-1})}(\eta_j - \eta_{j-1}), \qquad t_{j-1} \le t \le t_j,\ j \in \{1, \dots, n\},$$

where $\eta_0 = 0$ and $\beta_h(t) = \int_0^t h^2(s)ds$. One can see that equation (1.3) with $h \equiv 1$ reduces equation (1.1).

We emphasized that the evaluation formulas studied in [16, 17] expressed the conditional Wiener integrals in terms of ordinary (i.e., nonconditional) Wiener integrals. But, as seen equation (1.3), the conditioning functions used in [17], as well as in [16], are depended upon the values of $z_h(x, \cdot)$ at the set τ of finitely many time moments in $(0, T]$.

The aim of this paper is to establish a more general evaluation formula for the conditional generalized Wiener integral with much more general conditioning functions. The conditioning function suggested in this paper is generally not rely on the time moments; see equation (3.2) below. Nevertheless, the result established in this paper includes the results in [16–18], as special cases. As an application, we establish a translation theorem for the conditional Wiener integral and then use it to obtain various conditional Wiener integration formulas on $C_0[0, T]$.

2. Gaussian processes

In order to establish our evaluation formula involving the Gaussian processes, we follow the exposition of [4–7].

For each $v \in L_2[0, T]$ and $x \in C_0[0, T]$, we let $\langle v, x\rangle = \int_0^T v(t)\tilde{d}x(t)$ denote the Paley–Wiener–Zygmund stochastic integral [5, 12, 14, 15]. It is well known that for each $v \in L_2[0, T]$, the Paley–Wiener–Zygmund stochastic integral $\langle v, x\rangle$ exists for $\mathfrak{m}$-a.e. $x \in C_0[0, T]$ and it is a Gaussian random variable with mean zero and variance $\|v\|_2^2$, where $\|\cdot\|_2$ denotes the $L_2[0, T]$-norm. It is also known that for $v_1, v_2 \in L_2[0, T]$,

$$E_x[\langle v_1, x\rangle\langle v_2, x\rangle] \equiv \int_{C_0[0,T]} \langle v_1, x\rangle\langle v_2, x\rangle d\mathfrak{m}(x) = (v_1, v_2)_2 \tag{2.1}$$

where $(\cdot, \cdot)_2$ denotes the L_2-inner product. Furthermore, if $v \in L_2[0, T]$ is of bounded variation on $[0, T]$, then the Paley–Wiener–Zygmund stochastic integral $\langle v, x\rangle$ equals the Riemann–Stieltjes integral $\int_0^T v(t)dx(t)$.

Throughout the rest of this paper, we consider the Cameron–Martin space

$$C_0'[0, T] = \left\{w : w(t) = \int_0^t v(s)ds \text{ for some } v \in L_2[0, T]\right\}$$

to define the Gaussian processes used in this paper. For $w \in C_0'[0, T]$, with $w(t) = \int_0^t v(s)ds$ for $t \in [0, T]$, let $D : C_0'[0, T] \to L_2[0, T]$ be defined by the formula

$$Dw(t) = \frac{d}{dt}w(t) = v(t). \tag{2.2}$$

Then, as is well-known, the space $C_0' \equiv C_0'[0, T]$ with inner product

$$(w_1, w_2)_{C_0'} = \int_0^T Dw_1(t)Dw_2(t)dt \tag{2.3}$$

is a separable Hilbert space. Note that the two separable Hilbert spaces $L_2[0, T]$ and $C_0'[0, T]$ are isometric under the linear operator given by equation (2.2). The inverse operator of D is given by

$$(D^{-1}v)(t) = \int_0^t v(s)ds$$

for $t \in [0,T]$.

Given a function $w \in C_0'[0,T]$ and $x \in C_0[0,T]$, we let $(w,x)^\sim = \langle Dw, x\rangle$ for notational convenience. Then it follows that for any w and x in $C_0'[0,T]$, $(w,x)^\sim = (w,x)_{C_0'}$ and equation (2.1) can be rewritten as follows: for $w_1, w_2 \in C_0'[0,T]$,

$$E_x[(w_1,x)^\sim (w_2,x)^\sim] = (w_1,w_2)_{C_0'}. \tag{2.4}$$

For each $t \in [0,T]$, let

$$\nu_t(s) = (D^{-1}I_{[0,t]})(s) = \int_0^s I_{[0,t]}(\tau)d\tau = \begin{cases} s, & 0 \le s \le t \\ t, & t < s \le T \end{cases} \tag{2.5}$$

where I_A denotes the indicator function of A. Then ν_T is the identity function on the interval $[0,T]$. Let $C_0^*[0,T]$ be the set of functions k in $C_0'[0,T]$ such that Dk is continuous except for a finite number of finite jump discontinuities and is of bounded variation on $[0,T]$. For any $w \in C_0'[0,T]$ and $k \in C_0^*[0,T]$, let the operation $\odot$ between $C_0'[0,T]$ and $C_0^*[0,T]$ be defined by

$$w \odot k = D^{-1}(DwDk), \text{ i.e., } D(w \odot k) = DwDk, \tag{2.6}$$

where $DwDk$ denotes the pointwise multiplication of the functions Dw and Dk. In this case, $(C_0^*[0,T], \odot)$ forms a commutative algebra with the identity ν_T. We note that given any functions $w \in C_0'[0,T]$ and g_1, g_2 and g_3 in $C_0^*[0,T]$,

$$(w \odot g_1 \odot g_2, g_3)_{C_0'} = (w \odot g_{\pi(1)}, g_{\pi(2)} \odot g_{\pi(3)})_{C_0'} \tag{2.7}$$

for any permutation π of the set $\{1,2,3\}$.

Given any $k \in C_0'[0,T]$ with $Dk = h$ and with

$$\|k\|_{C_0'} \equiv [(k,k)_{C_0'}]^{1/2} = \|h\|_2 > 0,$$

let $\mathcal{Z}_k$ be the stochastic process on $C_0[0,T] \times [0,T]$ given by

$$\mathcal{Z}_k(x,t) = \int_0^t Dk(s)\tilde{d}x(s) = \int_0^t h(s)\tilde{d}x(s) = (k \odot \nu_t, x)^\sim. \tag{2.8}$$

Of course if $k(t) = \nu_T(t)$ on $[0,T]$, then $\mathcal{Z}_{\nu_T}(x,t) = x(t)$ is an ordinary Wiener process.

Remark 2.1. *By a simple examination, one can see that given a function k in $C_0'[0,T]$ with $Dk = h$ and $\|k\|_{C_0'} = \|h\|_2 \neq 0$, the Gaussian process $\mathcal{Z}_k$ given by* (2.8) *is essentially equal to the process z_h given by* (1.2). *For further work with the process z_h, see [4–7, 10, 17, 19].*

Using equations (2.8), (2.6), (2.5), and (2.3), we obtain the following two identities: given any functions k and w in $C_0'[0,T]$,

$$\mathcal{Z}_k(w,t) = (k \odot \nu_t, w)^\sim = \int_0^t Dk(s)dw(s) = \int_0^t Dk(s)Dw(s)ds = (k \odot w)(t) \tag{2.9}$$

and

$$\mathcal{Z}_k(w,t) = (w \odot k)(t) = \int_0^T D(k \odot w)(s)D\nu_t(s)ds = (w \odot k, \nu_t)_{C_0'} \tag{2.10}$$

for each $t \in [0,T]$.

Remark 2.2. *In view of equation* (2.10) *with k replaced with ν_T, we assert that the family of functions $\{\nu_t : 0 \le t \le T\}$ from $C_0'[0,T]$ has the reproducing property $w(t) = (w, \nu_t)_{C_0'}$ for all $w \in C_0'[0,T]$. Note that $\nu_t(s) = \min\{s,t\}$, the covariance function of the standard Brownian motion W illustrated in Section 1. We also note that for each $(x,t) \in C_0[0,T] \times [0,T]$,*

$$W(x,t) = x(t) = \int_0^T I_{[0,t]}(\tau)dx(\tau) = (\nu_t, x)^{\sim}.$$

Throughout this paper, we let

$$\mathrm{Supp}_{C_0'}[0,T] = \{k \in C_0'[0,T] : Dk \neq 0 \ m_L\text{-a.e on } [0,T]\}$$

and

$$\mathrm{Supp}_{C_0^*}[0,T] = \{k \in C_0^*[0,T] : Dk \neq 0 \ m_L\text{-a.e on } [0,T]\},$$

where m_L denotes the Lebesgue measure on $[0,T]$. Then one can see that $\mathrm{Supp}_{C_0^*}[0,T] \subset \mathrm{Supp}_{C_0'}[0,T]$.

Given a function k in $\mathrm{Supp}_{C_0'}[0,T]$ with $Dk = h$, let

$$\beta_k(t) = \int_0^t [Dk(u)]^2 du = \int_0^t [h(u)]^2 du. \tag{2.11}$$

It is easy to see that $\mathcal{Z}_k$ is a Gaussian process with mean zero and covariance function

$$E_x[\mathcal{Z}_k(x,s)\mathcal{Z}_k(x,t)] = \int_0^{\min\{s,t\}} [Dk(u)]^2 du = \beta_k(\min\{s,t\}).$$

In addition, by [24, Theorem 21.1], $\mathcal{Z}_k(\cdot,t)$ is stochastically continuous in t on $[0,T]$ and for any $k_1, k_2 \in \mathrm{Supp}_{C_0'}[0,T]$,

$$E_x[\mathcal{Z}_{k_1}(x,s)\mathcal{Z}_{k_2}(x,t)] = \int_0^{\min\{s,t\}} Dk_1(u)Dk_2(u)du.$$

It is also easy to check that for $w \in C_0'[0,T]$ and $k \in \mathrm{Supp}_{C_0^*}[0,T]$,

$$(w, \mathcal{Z}_k(x,\cdot))^{\sim} = (w \odot k, x)^{\sim} \tag{2.12}$$

for $\mathfrak{m}$-a.e. $x \in C_0[0,T]$. Thus it follows that $E_x[(w, \mathcal{Z}_k(x,\cdot))^{\sim}] = 0$. Furthermore, for any function k in $\mathrm{Supp}_{C_0^*}[0,T]$, $\mathcal{Z}_k$ is a continuous process. For our evaluation formula for the conditional generalized Wiener integral, we thus require k to be basically in $\mathrm{Supp}_{C_0^*}[0,T]$ rather than simply in $C_0'[0,T]$.

3. Evaluation formula

In this section, we establish an evaluation formula for expressing conditional generalized Wiener integrals in terms of nonconditional generalized Wiener integrals for a very general conditioning function $X_{\mathcal{G},k}(x)$ given by (3.2) below.

Given a function k in $\mathrm{Supp}_{C_0^*}[0,T]$, we consider the weighted inner product $(\cdot,\cdot)_{C_0'}^{k\odot k}$ given by

$$(w_1, w_2)_{C_0'}^{k\odot k} = \int_0^T Dw_1(t)Dw_2(t)Dk(t)Dk(t)dt = (w_1 \odot k, w_2 \odot k)_{C_0'}. \tag{3.1}$$

Given a positive integer n, let $\mathcal{G} = \{g_1, \dots, g_n\}$ be an orthonormal set of functions in $(C_0'[0,T], \|\cdot\|_{C_0'}^{k\odot k})$ where $\|\cdot\|_{C_0'}^{k\odot k} = \sqrt{(\cdot,\cdot)_{C_0'}^{k\odot k}}$. These orthonormal sets with respect to the weighted inner product given by (3.1) will be called a $k \odot k$-orthonormal set in $C_0'[0,T]$.

Remark 3.1. *Every $\nu_T \odot \nu_T$-orthonormal set $\mathcal{G}$ of functions in $C_0'[0,T]$ is an orthonormal set in $(C_0'[0,T], \|\cdot\|_{C_0'})$.*

In order to establish an evaluation formula for conditional generalized Wiener integral, we will always condition by the function $X_{\mathcal{G},k} : C_0[0,T] \to \mathbb{R}^n$ defined by

$$X_{\mathcal{G},k}(x) = \Big((g_1, \mathcal{Z}_k(x,\cdot))^\sim, \dots, (g_n, \mathcal{Z}_k(x,\cdot))^\sim\Big), \tag{3.2}$$

where $\mathcal{G} = \{g_1, \dots, g_n\}$ is a $k \odot k$-orthonormal set in $C_0'[0,T]$. Using (2.12) and (2.9) with w replaced with g_j, $j \in \{1, \dots, n\}$, it follows that

$$X_{\mathcal{G},k}(x) = \Big((g_1 \odot k, x)^\sim, \dots, (g_n \odot k, x)^\sim\Big) = \Big((\mathcal{Z}_k(g_1,\cdot), x)^\sim, \dots, (\mathcal{Z}_k(g_n,\cdot), x)^\sim\Big)$$

for $\mathfrak{m}$-a.e. $x \in C_0[0,T]$.

Remark 3.2. *Since $\{g_1, \dots, g_n\}$ is a $k \odot k$-orthonormal set in $C_0[0,T]$, it follows that for $j, l \in \{1, \cdots, n\}$,*

$$E_x[(g_j, \mathcal{Z}_k(x,\cdot))^\sim (g_l, \mathcal{Z}_k(x,\cdot))^\sim] = \delta_{jl}$$

where δ_{jl} denotes the Kronecker delta. Thus $(g_j, \mathcal{Z}_k(x,\cdot))^\sim$'s are (stochastically) independent.

Given a function k in $\mathrm{Supp}_{C_0^*}[0,T]$, and a $k \odot k$-orthonormal set $\mathcal{G} = \{g_1, \dots, g_n\}$ of functions in $C_0'[0,T]$, we define a map $[\cdot]_{\mathcal{G},k} : \mathbb{R}^n \to C_0[0,T]$ by

$$[\vec{\eta}]_{\mathcal{G},k} = \sum_{j=1}^n \eta_j g_j \odot k \odot k = \sum_{j=1}^n \eta_j \mathcal{Z}_k(g_j \odot k, \cdot) \tag{3.3}$$

for $\vec{\eta} = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$ and we write

$$[x]_{\mathcal{G},k} \equiv [X_{\mathcal{G},k}(x)] = \sum_{j=1}^n (g_j \odot k, x)^\sim g_j \odot k \odot k = \sum_{j=1}^n (g_j \odot k, x)^\sim \mathcal{Z}_k(g_j \odot k, \cdot) \tag{3.4}$$

for $x \in C_0[0,T]$.

Lemma 3.3. *Let k be a function in $\mathrm{Supp}_{C_0^*}[0,T]$, and let $\mathcal{G} = \{g_1, \dots, g_n\}$ be a $k \odot k$-orthonormal set of functions in $C_0'[0,T]$. Then, the processes $\{\mathcal{Z}_k(x,\cdot) - [x]_{\mathcal{G},k} : t \in [0,T]\}$ and $\{[x]_{\mathcal{G},k} : t \in [0,T]\}$ are (stochastically) independent.*

Proof. Since the processes are Gaussian with mean zero, it suffices to show that for every $s, t \in [0,T]$,

$$E_x\Big[\Big(\mathcal{Z}_k(x,s) - [x]_{\mathcal{G},k}(s)\Big)[x]_{\mathcal{G},k}(t)\Big] = 0.$$

Using equations (2.8) and (3.4), we observe that for any $s, t \in [0,T]$,

$$\begin{aligned}
&\Big(\mathcal{Z}_k(x,s) - [x]_{\mathcal{G},k}(s)\Big)[x]_{\mathcal{G},k}(t) \\
&= \Big((k \odot \nu_s, x)^\sim - \sum_{j=1}^n (g_j \odot k, x)^\sim (g_j \odot k \odot k)(s)\Big)\Big(\sum_{j=1}^n (g_j \odot k, x)^\sim (g_j \odot k \odot k)(t)\Big) \\
&= (k \odot \nu_s, x)^\sim \sum_{j=1}^n (g_j \odot k, x)^\sim (g_j \odot k \odot k)(t) - \sum_{j=1}^n \sum_{l=1}^n (g_j \odot k, x)^\sim (g_l \odot k, x)^\sim (g_j \odot k \odot k)(s)(g_l \odot k \odot k)(t).
\end{aligned}$$

Then, using equations (2.4), (2.6), (2.5), and (3.1), it follows that

$$
\begin{aligned}
&E_x\Big[\Big(\mathcal{Z}_k(x,s)-[x]_{\mathcal{G},k}(s)\Big)[x]_{\mathcal{G},k}(t)\Big]\\
&=\sum_{j=1}^{n}E_x\Big[(k\odot\nu_s,x)^{\sim}(g_j\odot k,x)^{\sim}\Big](g_j\odot k\odot k)(t)-\sum_{j=1}^{n}\sum_{l=1}^{n}E_x\Big[(g_j\odot k,x)^{\sim}(g_l\odot k,x)^{\sim}\Big](g_j\odot k\odot k)(s)(g_l\odot k\odot k)(t)\\
&=\sum_{j=1}^{n}\Big[\int_0^T D(k\odot\nu_s)(\tau)D(g_j\odot k)(\tau)d\tau\Big](g_j\odot k\odot k)(t)\\
&\quad-\sum_{j=1}^{n}\sum_{l=1}^{n}\Big[\int_0^T D(g_j\odot k)(\tau)D(g_l\odot k)(\tau)d\tau\Big](g_j\odot k\odot k)(s)(g_l\odot k\odot k)(t)\\
&=\sum_{j=1}^{n}\Big[\int_0^T Dk(\tau)D\nu_s(\tau)Dg_j(\tau)Dk(\tau)d\tau\Big](g_j\odot k\odot k)(t)\\
&\quad-\sum_{j=1}^{n}\sum_{l=1}^{n}\Big[\int_0^T Dg_j(\tau)Dk(\tau)Dg_l(\tau)Dk(\tau)d\tau\Big](g_j\odot k\odot k)(s)(g_l\odot k\odot k)(t)\\
&=\sum_{j=1}^{n}\Big[\int_0^s Dg_j(\tau)Dk(\tau)Dk(\tau)d\tau\Big](g_j\odot k\odot k)(t)-\sum_{j=1}^{n}\sum_{l=1}^{n}(g_j,g_l)_{C_0'}^{k\odot k}(g_j\odot k\odot k)(s)(g_l\odot k\odot k)(t)\\
&=\sum_{j=1}^{n}(g_j\odot k\odot k)(s)(g_j\odot k\odot k)(t)-\sum_{j=1}^{n}\sum_{l=1}^{n}\delta_{jl}(g_j\odot k\odot k)(s)(g_l\odot k\odot k)(t)\\
&=0,
\end{aligned}
$$

where δ_{jl} denotes the Kronecker delta. Hence the theorem is proved. □

Throughout the remainder of this section, we assume that $F(\mathcal{Z}_k(x,\cdot))$ is Wiener integrable with respect to x on $C_0[0,T]$.

Lemma 3.4. *Let k and $\mathcal{G}$ be as in Lemma 3.3. Let $F(\mathcal{Z}_k(x,\cdot))$ be a functional in $L_1(C_0[0,T],\mathcal{M},\mathfrak{m})$, and let $X_{\mathcal{G},k}$ be given by* (3.2). *Then it follows that*

$$\int_{C_0[0,T]}F(\mathcal{Z}_k(x,\cdot))d\mathfrak{m}(x)=\int_{\mathbb{R}^n}E_x\Big[F\Big(\mathcal{Z}_k(x,\cdot)-[x]_{\mathcal{G},k}(\cdot)+[\vec{\eta}]_{\mathcal{G},k}(\cdot)\Big)\Big]d(\mathfrak{m}\circ X_{\mathcal{G},k}^{-1})(\vec{\eta}). \tag{3.5}$$

Proof. Let $\mathfrak{Y}$ and $\mathfrak{Z}$ be the Wiener measurable map from $C_0[0,T]$ to $C_0[0,T]$ given by

$$\mathfrak{Y}(x)=\mathcal{Z}_k(x,\cdot)-[x]_{\mathcal{G},k}(\cdot)$$

and

$$\mathfrak{Z}(x)=[x]_{\mathcal{G},k}(\cdot)\equiv[X_{\mathcal{G},k}(x)]=([\cdot]_{\mathcal{G},k}\circ X_{\mathcal{G},k})(x),$$

respectively. Let $S_1=\mathfrak{Y}(C_0[0,T])$ and $S_2=\mathfrak{Z}(C_0[0,T])$. Since $\mathfrak{Y}$ and $\mathfrak{Z}$ are independent processes by Lemma 3.3, it follows that

$$
\begin{aligned}
\int_{C_0[0,T]}F(\mathcal{Z}_k(x,\cdot))d\mathfrak{m}(x)&=\int_{C_0[0,T]}F\Big(\mathcal{Z}_k(x,\cdot)-[x]_{\mathcal{G},k}(\cdot)+[x]_{\mathcal{G},k}(\cdot)\Big)d\mathfrak{m}(x)\\
&=\int_{S_1\times S_2}F\Big(y(\cdot)+z(\cdot)\Big)d\Big(\mathfrak{m}\circ\mathfrak{Y}^{-1}\times\mathfrak{m}\circ\mathfrak{Z}^{-1}\Big)(y,z)\\
&=\int_{S_2}\int_{S_1}F\Big(y(\cdot)+z(\cdot)\Big)d\Big(\mathfrak{m}\circ\mathfrak{Y}^{-1}\Big)(y)d\Big(\mathfrak{m}\circ\mathfrak{Z}^{-1}\Big)(z)
\end{aligned} \tag{3.6}
$$

by the change of variable theorem [11, p. 163]. Applying the change of variable theorem again, and using (3.3) and (3.4), it follows that

$$\begin{aligned}
&\int_{S_2}\int_{S_1} F\Big(y(\cdot)+z(\cdot)\Big)d\Big(\mathfrak{m}\circ\mathfrak{Y}^{-1}\Big)(y)d\Big(\mathfrak{m}\circ\mathfrak{Z}^{-1}\Big)(z)\\
&=\int_{S_2}\int_{S_1} F\Big(y(\cdot)+z(\cdot)\Big)d\Big(\mathfrak{m}\circ\mathfrak{Y}^{-1}\Big)(y)d\Big(\mathfrak{m}\circ[X_{\mathcal{G},k}(\cdot)]^{-1}\Big)(z)\\
&=\int_{S_2}\int_{S_1} F\Big(y(\cdot)+z(\cdot)\Big)d\Big(\mathfrak{m}\circ\mathfrak{Y}^{-1}\Big)(y)d\Big(\mathfrak{m}\circ X_{\mathcal{G},k}^{-1}\circ[\cdot]_{\mathcal{G},k}^{-1}\Big)(z)\\
&=\int_{\mathbb{R}^n}\int_{S_1} F\Big(y(\cdot)+[\vec{\eta}]_{\mathcal{G},k}(\cdot)\Big)d\Big(\mathfrak{m}\circ\mathfrak{Y}^{-1}\Big)(y)d\Big(\mathfrak{m}\circ X_{\mathcal{G},k}^{-1}\Big)(\vec{\eta})\\
&=\int_{\mathbb{R}^n}\int_{C_0[0,T]} F\Big(\mathcal{Z}_k(x,\cdot)-[x]_{\mathcal{G},k}(\cdot)+[\vec{\eta}]_{\mathcal{G},k}(\cdot)\Big)d\mathfrak{m}(x)d\Big(\mathfrak{m}\circ X_{\mathcal{G},k}^{-1}\Big)(\vec{\eta})\\
&=\int_{\mathbb{R}^n} E_x\Big[F\Big(\mathcal{Z}_k(x,\cdot)-[x]_{\mathcal{G},k}(\cdot)+[\vec{\eta}]_{\mathcal{G},k}(\cdot)\Big)\Big]d\Big(\mathfrak{m}\circ X_{\mathcal{G},k}^{-1}\Big)(\vec{\eta}).
\end{aligned}$$

This, together with equation (3.6), completes the proof. □

The next lemma is crucial in the proof of our main assertion for the conditional generalized Wiener integral.

Lemma 3.5. *Let k, $\mathcal{G}$, F, and $X_{\mathcal{G},k}$ be as in Lemma 3.4. Then*

$$\int_{X_{\mathcal{G},k}^{-1}(B)} F(\mathcal{Z}_k(x,\cdot))d\mathfrak{m}(x)=\int_B E_x\Big[F\Big(\mathcal{Z}_k(x,\cdot)-[x]_{\mathcal{G},k}(\cdot)+[\vec{\eta}]_{\mathcal{G},k}(\cdot)\Big)\Big]d(\mathfrak{m}\circ X_{\mathcal{G},k}^{-1})(\vec{\eta}) \tag{3.7}$$

for every $B\in\mathcal{B}(\mathbb{R}^n)$.

Proof. Given a $k\odot k$-orthonormal set $\mathcal{G}=\{g_1,\cdots,g_n\}$, let

$$Y_{\mathcal{G}}(x)=((g_1,x)^\sim,\dots,(g_n,x)^\sim). \tag{3.8}$$

Then we first see that $X_{\mathcal{G},k}(x)=Y(\mathcal{Z}_k(x,\cdot))$ in view of (3.2), and that for $j,l\in\{1,\dots,n\}$,

$$(g_j,g_l\odot k\odot k)^\sim=(g_j,g_l\odot k\odot k)_{C_0'}=(g_j,g_l)_{C_0'}^{k\odot k}=\delta_{jl}, \tag{3.9}$$

since $\mathcal{G}=\{g_1,\dots,g_n\}$ is a $k\odot k$-orthonormal set. Next, using equations (3.8), (2.12), (3.4), (3.9), and (3.3), it follows that

$$\begin{aligned}
Y_{\mathcal{G}}\Big(\mathcal{Z}_k(x,\cdot)-[x]_{\mathcal{G},k}(\cdot)\Big)&=Y_{\mathcal{G}}\Big(\mathcal{Z}_k(x,\cdot)\Big)-Y_{\mathcal{G}}\Big([x]_{\mathcal{G},k}(\cdot)\Big)\\
&=((g_1\odot k,x)^\sim,\dots,(g_n\odot k,x)^\sim)-((g_1,[x]_{\mathcal{G},k}(\cdot))^\sim,\dots,(g_n,[x]_{\mathcal{G},k}(\cdot))^\sim)\\
&=0
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
Y_{\mathcal{G}}([\vec{\eta}]_{\mathcal{G},k}(\cdot))&=\bigg(\Big(g_1,\sum_{j=1}^n\eta_j g_j\odot k\odot k\Big)^\sim,\dots,\Big(g_n,\sum_{j=1}^n\eta_j g_j\odot k\odot k\Big)^\sim\bigg)\\
&=\bigg(\sum_{j=1}^n\eta_j(g_1,g_j\odot k\odot k)^\sim,\dots,\sum_{j=1}^n\eta_j(g_n,g_j\odot k\odot k)^\sim\bigg)\\
&=(\eta_1,\dots,\eta_n)
\end{aligned} \tag{3.11}$$

for all $\vec{\eta} = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$.

Finally, applying equations (3.5), (3.10), and (3.11), it follows that given any Borel set B in $\mathcal{B}(\mathbb{R}^n)$,

$$
\begin{aligned}
&\int_{X_{\mathcal{G},k}^{-1}(B)} F(\mathcal{Z}_k(x,\cdot))d\mathfrak{m}(x)\\
&= \int_{C_0[0,T]} I_{X_{\mathcal{G},k}^{-1}(B)}(x)F(\mathcal{Z}_k(x,\cdot))d\mathfrak{m}(x)\\
&= \int_{C_0[0,T]} (I_B \circ X_{\mathcal{G},k})(x)F(\mathcal{Z}_k(x,\cdot))d\mathfrak{m}(x)\\
&= \int_{C_0[0,T]} I_B(X_{\mathcal{G},k}(x))F(\mathcal{Z}_k(x,\cdot))d\mathfrak{m}(x)\\
&= \int_{C_0[0,T]} I_B(Y_{\mathcal{G}}(\mathcal{Z}_k(x,\cdot)))F(\mathcal{Z}_k(x,\cdot))d\mathfrak{m}(x)\\
&= \int_{\mathbb{R}^n} E_x\Big[\Big((I_B \circ Y_{\mathcal{G}}) \cdot F\Big)\Big(\mathcal{Z}_k(x,\cdot) - [x]_{\mathcal{G},k}(\cdot) + [\vec{\eta}]_{\mathcal{G},k}(\cdot)\Big)\Big]d(\mathfrak{m} \circ X_{\mathcal{G},k}^{-1})(\vec{\eta})\\
&= \int_{\mathbb{R}^n} E_x\Big[I_B\Big(Y_{\mathcal{G}}\Big(\mathcal{Z}_k(x,\cdot) - [x]_{\mathcal{G},k}(\cdot) + [\vec{\eta}]_{\mathcal{G},k}(\cdot)\Big)\Big)F\Big(\mathcal{Z}_k(x,\cdot) - [x]_{\mathcal{G},k}(\cdot) + [\vec{\eta}]_{\mathcal{G},k}(\cdot)\Big)\Big]d(\mathfrak{m} \circ X_{\mathcal{G},k}^{-1})(\vec{\eta})\\
&= \int_{\mathbb{R}^n} I_B(\vec{\eta})E_x\Big[F\Big(\mathcal{Z}_k(x,\cdot) - [x]_{\mathcal{G},k}(\cdot) + [\vec{\eta}]_{\mathcal{G},k}(\cdot)\Big)\Big]d(\mathfrak{m} \circ X_{\mathcal{G},k}^{-1})(\vec{\eta})\\
&= \int_{B} E_x\Big[F\Big(\mathcal{Z}_k(x,\cdot) - [x]_{\mathcal{G},k}(\cdot) + [\vec{\eta}]_{\mathcal{G},k}(\cdot)\Big)\Big]d(\mathfrak{m} \circ X_{\mathcal{G},k}^{-1})(\vec{\eta})
\end{aligned}
$$

as desired. □

Remark 3.6. *Let $F(\mathcal{Z}_k(x,\cdot))$ be in $L_1(C_0[0,T], \mathcal{M}, \mathfrak{m})$. Then the conditional generalized Wiener integral of F given $X_{\mathcal{G},k}$, denoted by*

$$E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|X_{\mathcal{G},k}(x) = \vec{\eta}\Big),$$

is a Lebesgue measurable function of $\vec{\eta}$, unique up to null sets in $\mathbb{R}^n$, satisfying the equation

$$\int_{X_{\mathcal{G},k}^{-1}(B)} F(\mathcal{Z}_k(x,\cdot))d\mathfrak{m}(x) = \int_B E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|X_{\mathcal{G},k}(x) = \vec{\eta}\Big)d(\mathfrak{m} \circ X_{\mathcal{G},k}^{-1})(\vec{\eta}) \tag{3.12}$$

for all Borel sets B in $\mathbb{R}^n$.

The main assertion of this paper now follows readily from Remark 3.6 and Lemma 3.5 above.

Theorem 3.7. *Let k, $\mathcal{G}$, F, and $X_{\mathcal{G},k}$ be as in Lemma 3.4. Then it follows that*

$$
\begin{aligned}
E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|X_{\mathcal{G},k}(x) = \vec{\eta}\Big) &= E_x\Big[F\Big(\mathcal{Z}_k(x,\cdot) - [x]_{\mathcal{G},k}(\cdot) + [\vec{\eta}]_{\mathcal{G},k}(\cdot)\Big)\Big]\\
&\equiv E_x\Big[F\Big(\mathcal{Z}_k(x,\cdot) - \sum_{j=1}^{n}(g_j \odot k, x)^{\sim} g_j \odot k \odot k + \sum_{j=1}^{n} \eta_j g_j \odot k \odot k\Big)\Big]
\end{aligned}
\tag{3.13}
$$

for a.e. $\vec{\eta} \in \mathbb{R}^n$.

Proof. By equations (3.12) and (3.7), it follows that for any $B \in \mathcal{B}(\mathbb{R}^n)$,

$$
\begin{aligned}
\int_B E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|X_{\mathcal{G},k}(x) = \vec{\eta}\Big)d(\mathfrak{m} \circ X_{\mathcal{G},k}^{-1})(\vec{\eta}) &= \int_{X_{\mathcal{G},k}^{-1}(B)} F(\mathcal{Z}_k(x,\cdot))d\mathfrak{m}(x)\\
&= \int_B E_x\Big[F\Big(\mathcal{Z}_k(x,\cdot) - [X_{\mathcal{G},k}](\cdot) + [\vec{\eta}]_{\mathcal{G},k}(\cdot)\Big)\Big]d(\mathfrak{m} \circ X_{\mathcal{G},k}^{-1})(\vec{\eta})
\end{aligned}
$$

for every $B \in \mathcal{B}(\mathbb{R}^n)$. This yields the desired result. □

Remark 3.8. *Note that equation* (3.13) *is indeed a very simple formula equating the conditional generalized Wiener integral* $E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|X_{\mathcal{G},k}(x)=\vec{\eta}\Big)$ *and the nonconditional generalized Wiener integral* $E[F(\mathcal{Z}_k(x,\cdot)-[x]_{\mathcal{G},k}(\cdot)+[\vec{\eta}]_{\mathcal{G},k}(\cdot))]$.

Remark 3.9. *Using the linearity of the Paley–Wiener–Zygmund stochastic integral as a random variable on* $C_0[0,T]$, *equation* (3.13) *can be rewritten as*

$$\begin{aligned} E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|X_{\mathcal{G},k}(x)=\vec{\eta}\Big) &= E_x\Big[F\Big(\mathcal{Z}_k(x,\cdot)-\sum_{j=1}^n (g_j\odot k,x)^\sim \mathcal{Z}_k(g_j\odot k,\cdot)+\sum_{j=1}^n \eta_j\mathcal{Z}_k(g_j\odot k,\cdot)\Big)\Big] \\ &= E_x\Big[F\Big(\mathcal{Z}_k\Big(x-\sum_{j=1}^n (g_j\odot k,x)^\sim g_j\odot k+\sum_{j=1}^n \eta_j g_j\odot k,\cdot\Big)\Big)\Big] \end{aligned} \tag{3.14}$$

for a.e. $\vec{\eta}\in\mathbb{R}^n$.

4. Corollaries

Let ν_T be given by (2.5) with t replaced with T and let m_L denote the Lebesgue measure on $[0,T]$. Given a positive integer n, let $\tau=\{t_1,\ldots,t_n\}$ be a partition of $[0,T]$ with $0=t_0<t_1<\cdots<t_{n-1}<t_n=T$. For each $j=1,\ldots,n$, let

$$p_j(t)=\int_0^t \frac{1}{\sqrt{t_j-t_{j-1}}} I_{(t_{j-1},t_j]}(s)ds. \tag{4.1}$$

Then $\mathcal{P}_\tau=\{p_1,\ldots,p_n\}$ is a $\nu_T\odot\nu_T$-orthonormal (i.e., orthonormal with respect to the Lebesgue measure m_L on $[0,T]$) set of functions in $C_0'[0,T]$. Then it follows that for each $t\in[0,T]$,

$$\begin{aligned} \sum_{j=1}^n (p_j\odot\nu_T,x)^\sim p_j\odot\nu_T(t) &= \sum_{j=1}^n (p_j,x)^\sim p_j(t) \\ &= \sum_{j=1}^n \int_0^T \frac{I_{(t_{j-1},t_j]}(s)}{\sqrt{t_j-t_{j-1}}}\tilde{d}x(s)\int_0^t \frac{I_{(t_{j-1},t_j]}(s)}{\sqrt{t_j-t_{j-1}}}ds \\ &= \sum_{j=1}^n \frac{x(t_j)-x(t_{j-1})}{t_j-t_{j-1}} m_L([0,t]\cap(t_{j-1},t_j]) \\ &= \sum_{j=1}^n \Big\{x(t_{j-1})+\frac{t-t_{j-1}}{t_j-t_{j-1}}(x(t_j)-x(t_{j-1}))\Big\} I_{(t_{j-1},t_j]}(t) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \sum_{j=1}^n \frac{\eta_j-\eta_{j-1}}{\sqrt{t_j-t_{j-1}}} p_j\odot\nu_T(t) &= \sum_{j=1}^n \frac{\eta_j-\eta_{j-1}}{\sqrt{t_j-t_{j-1}}} p_j(t) \\ &= \sum_{j=1}^n \frac{\eta_j-\eta_{j-1}}{\sqrt{t_j-t_{j-1}}}\int_0^t \frac{I_{(t_{j-1},t_j]}(s)}{\sqrt{t_j-t_{j-1}}}ds \\ &= \sum_{j=1}^n \frac{\eta_j-\eta_{j-1}}{t_j-t_{j-1}} m_L([0,t]\cap(t_{j-1},t_j]) \\ &= \sum_{j=1}^n \Big\{\eta_{j-1}+\frac{t-t_{j-1}}{t_j-t_{j-1}}(\eta_j-\eta_{j-1})\Big\} I_{(t_{j-1},t_j]}(t) \end{aligned} \tag{4.3}$$

where $\eta_0 = 0$.

Next let $X_{\mathcal{P}_\tau,\nu_T} : C_0'[0,T] \to \mathbb{R}^n$ be defined by

$$\begin{aligned} X_{\mathcal{P}_\tau,\nu_T}(x) &= ((p_1, \mathcal{Z}_{\nu_T}(x,\cdot))^\sim, \dots, (p_n, \mathcal{Z}_{\nu_T}(x,\cdot))^\sim) \\ &= ((p_1 \odot \nu_T, x)^\sim, \dots, (p_n \odot \nu_T, x)^\sim) \\ &= ((p_1, x)^\sim, \dots, (p_n, x)^\sim). \end{aligned} \tag{4.4}$$

Then, by (4.2), it follows that

$$[x]_{\mathcal{P}_\tau,\nu_T} \equiv [X_{\mathcal{P}_\tau,\nu_T}] = \sum_{j=1}^n (p_j \odot \nu_T, x)^\sim p_j \odot \nu_T \odot \nu_T = \sum_{j=1}^n (p_j, x)^\sim p_j. \tag{4.5}$$

Corollary 4.1 ([16]). *Let $F \in L_1(C_0[0,T], \mathcal{M}, \mathfrak{m})$ and let $X_{\mathcal{P}_\tau,\nu_T}$ be given by* (4.4)*. Then it follows that*

$$E\Big(F(x(\cdot))\Big|x(t_j) = \eta_j,\ j = 1,\dots,n\Big) = E_x\Big[F\Big(x(\cdot) - [x]_\tau(\cdot) + [\vec{\eta}]_\tau(\cdot)\Big)\Big] \tag{4.6}$$

for a.e. $\vec{\eta} \in \mathbb{R}^n$, where $[x]_\tau \equiv [x]_{\mathcal{P}_\tau,\nu_T}$ is given by the last expression of (4.2) *and $[\vec{\eta}]_\tau \equiv [\vec{\eta}]_{\mathcal{P}_\tau,\nu_T}$ is given by the last expression of* (4.3).

Proof. Using (3.13) with $\mathcal{G}$, k and $X_{\mathcal{G},k}$ replaced with $\mathcal{P}_\tau$, ν_T and $X_{\mathcal{P}_\tau,\nu_T}$, respectively, and (4.5), it follows that for a.e. $\vec{\eta} \in \mathbb{R}^n$,

$$\begin{aligned} E\Big(F(x(\cdot))\Big|x(t_j) = \eta_j,\ j = 1,\dots,n\Big) &= E\Big(F(x(\cdot))\Big|(p_j, x)^\sim = \frac{\eta_j - \eta_{j-1}}{\sqrt{t_j - t_{j-1}}},\ j = 1,\dots,n\Big) \\ &= E\Big(F(\mathcal{Z}_{\nu_T}(x,\cdot))\Big|X_{\mathcal{P}_\tau,\nu_T}(x) = \Big(\frac{\eta_1 - \eta_0}{\sqrt{t_1 - t_0}}, \dots, \frac{\eta_n - \eta_{n-1}}{\sqrt{t_n - t_{n-1}}}\Big)\Big) \\ &= E_x\Big[F\Big(\mathcal{Z}_{\nu_T}(x,\cdot) - \sum_{j=1}^n (p_j, x)^\sim p_j + \sum_{j=1}^n \frac{\eta_j - \eta_{j-1}}{\sqrt{t_j - t_{j-1}}} p_j\Big)\Big], \end{aligned} \tag{4.7}$$

where $\eta_0 = 0$. Equation (4.7) together with (4.2) and (4.3) yields equation (4.6) as desired. □

Given a function k in $\mathrm{Supp}_{C_0^*}[0,T]$ with $Dk = h$ and a partition $\tau = \{t_1, \dots, t_n\}$ of $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, let β_k be given by (2.11) and let

$$r_{k,j}(t) = \int_0^t \frac{1}{\sqrt{\beta_k(t_j) - \beta_k(t_{j-1})}} I_{(t_{j-1},t_j]}(s)ds \tag{4.8}$$

for each $j = 1,\dots,n$. Then $\mathcal{R}_{\tau,k} = \{r_{k,1}, \dots, r_{k,n}\}$ is a $k \odot k$-orthonormal set of functions in $C_0'[0,T]$, and it follows that for each $t \in [0,T]$,

$$\begin{aligned} \sum_{j=1}^n (r_{k,j} \odot k, x)^\sim (r_{k,j} \odot k \odot k)(t) &= \sum_{j=1}^n \int_0^T \frac{I_{(t_{j-1},t_j]}(s)h(s)}{\sqrt{\beta_k(t_j) - \beta_k(t_{j-1})}} \tilde{d}x(s) \int_0^t \frac{I_{(t_{j-1},t_j]}(s)h^2(s)}{\sqrt{\beta_k(t_j) - \beta_k(t_{j-1})}} ds \\ &= \sum_{j=1}^n \frac{\mathcal{Z}_k(x,t_j) - \mathcal{Z}_k(x,t_{j-1})}{\beta_k(t_j) - \beta_k(t_{j-1})} \int_{[0,t]\cap(t_{j-1},t_j]} h^2(s)ds \\ &= \sum_{j=1}^n \Big\{\mathcal{Z}_k(x,t_{j-1}) + \frac{\beta_k(t) - \beta_k(t_{j-1})}{\beta_k(t_j) - \beta_k(t_{j-1})}(\mathcal{Z}_k(x,t_j) - \mathcal{Z}_k(x,t_{j-1}))\Big\} I_{(t_{j-1},t_j]}(t) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\sum_{j=1}^{n}\frac{\eta_j-\eta_{j-1}}{\sqrt{\beta_k(t_j)-\beta_k(t_{j-1})}}(r_{k,j}\odot k\odot k)(t) &= \sum_{j=1}^{n}\frac{\eta_j-\eta_{j-1}}{\sqrt{\beta_k(t_j)-\beta_k(t_{j-1})}}\int_0^t\frac{I_{(t_{j-1},t_j]}(s)h^2(s)}{\sqrt{\beta_k(t_j)-\beta_k(t_{j-1})}}ds\\ &= \sum_{j=1}^{n}\frac{\eta_j-\eta_{j-1}}{\beta_k(t_j)-\beta_k(t_{j-1})}\int_{[0,t]\cap(t_{j-1},t_j]}h^2(s)ds\\ &= \sum_{j=1}^{n}\Big\{\eta_{j-1}+\frac{\beta_k(t)-\beta_k(t_{j-1})}{\beta_k(t_j)-\beta_k(t_{j-1})}(\eta_j-\eta_{j-1})\Big\}I_{(t_{j-1},t_j]}(t)\end{aligned} \tag{4.10}$$

where $\eta_0 = 0$.

Let $X_{\mathcal{R}_{\tau,k},k} : C_0'[0,T] \to \mathbb{R}^n$ be defined by

$$\begin{aligned}X_{\mathcal{R}_{\tau,k},k}(x) &= ((r_{k,1},\mathcal{Z}_k(x,\cdot))^\sim,\dots,(r_{k,n},\mathcal{Z}_k(x,\cdot))^\sim)\\ &= ((r_{k,1}\odot k,x)^\sim,\dots,(r_{k,n}\odot k,x)^\sim).\end{aligned} \tag{4.11}$$

Then, by (4.9), it follows that for each $t \in [0,T]$,

$$[x]_{\mathcal{R}_{\tau,k},k}(t) = [X_{\mathcal{R}_{\tau,k},k}(x)](t) = \sum_{j=1}^{n}(r_{k,j}\odot k,x)^\sim(r_{k,j}\odot k\odot k)(t) \tag{4.12}$$

is given by the right-hand side of (1.5).

Corollary 4.2 ([17]). *Given a function k in $\mathrm{Supp}_{C_0^*}[0,T]$, let $F(\mathcal{Z}_k(x,\cdot)) \in L_1(C_0[0,T],\mathcal{M},\mathfrak{m})$ and let $X_{\mathcal{R}_{\tau,k},k}$ be given by* (4.11). *Then it follows that*

$$E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|\mathcal{Z}_k(x,t_j)=\eta_j,\ j=1,\dots,n\Big) = E_x\Big[F\Big(\mathcal{Z}_k(x,\cdot)-[x]_{\tau,k}(\cdot)+[\vec{\eta}]_{\tau,k}(\cdot)\Big)\Big] \tag{4.13}$$

for a.e. $\vec{\eta} \in \mathbb{R}^n$, where $[x]_{\tau,k} \equiv [x]_{\mathcal{R}_{\tau,k},k}$ is given by the last expression of (4.9) *and $[\vec{\eta}]_{\tau,k} \equiv [\vec{\eta}]_{\mathcal{R}_{\tau,k},k}$ is given by the last expression of* (4.10).

Proof. Using (3.13) with $\mathcal{G}$ and $X_{\mathcal{G},k}$ replaced with $\mathcal{R}_{\tau,k}$ and $X_{\mathcal{R}_{\tau,k},k}$, respectively, and (4.12), it follows that for a.e. $\vec{\eta} \in \mathbb{R}^n$,

$$\begin{aligned}&E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|\mathcal{Z}_k(x,t_j)=\eta_j,\ j=1,\dots,n\Big)\\ &= E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|(r_{k,j}\odot k,x)^\sim=\frac{\eta_j-\eta_{j-1}}{\sqrt{\beta_k(t_j)-\beta_k(t_{j-1})}},\ j=1,\dots,n\Big)\\ &= E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|X_{\mathcal{R}_{\tau,k},k}(x)=\Big(\frac{\eta_1-\eta_0}{\sqrt{\beta_k(t_1)-\beta_k(t_0)}},\dots,\frac{\eta_n-\eta_{n-1}}{\sqrt{\beta_k(t_n)-\beta_k(t_{n-1})}}\Big)\Big)\\ &= E_x\Big[F\Big(\mathcal{Z}_k(x,\cdot)-\sum_{j=1}^{n}(r_{k,j}\odot k,x)^\sim r_{k,j}\odot k\odot k+\sum_{j=1}^{n}\frac{\eta_j-\eta_{j-1}}{\sqrt{\beta_k(t_j)-\beta_k(t_{j-1})}}r_{k,j}\odot k\odot k\Big)\Big],\end{aligned} \tag{4.14}$$

where $\eta_0 = 0$. Equation (4.14) together with (4.9) and (4.10) yields equation (4.13). □

Remark 4.3. *In view of Remark 2.1, equations* (1.4) *and* (4.13) *are essentially same structure.*

Choosing $k = \nu_T$ in (3.13), we also have the following corollary.

Corollary 4.4 ([18]). *Let $F \in L_1(C_0[0,T],\mathcal{M},\mathfrak{m})$ and let $X_{\mathcal{G},\nu_T}$ be given by* (3.2) *with k replaced with ν_T. Then it follows that*

$$E\Big(F(x)\Big|(g_j,x)^\sim=\eta_j,\ j=1,\dots,n\Big) = E_x\Big[F\Big(x(\cdot)-\sum_{j=1}^{n}(g_j,x)^\sim g_j(\cdot)+\sum_{j=1}^{n}\eta_j g_j(\cdot)\Big)\Big]$$

for a.e. $\vec{\eta} \in \mathbb{R}^n$.

5. Examples

In this section we present interesting examples to which equation (3.13) can be applied.

Let $S : C_0'[0,T] \to C_0'[0,T]$ be a bounded linear operator. In Example 5.1 below, we use equation (3.13) to obtain the conditional Wiener integration formula

$$E\Big((Sw, \mathcal{Z}_k(x,\cdot))^\sim \Big| X_{\mathcal{G},k}(x) = \vec{\eta}\Big) = \sum_{j=1}^{n} \eta_j (Sw, g_j)_{C_0'}^{k\odot k} \tag{5.1}$$

where $(\cdot,\cdot)_{C_0'}^{k\odot k}$ is the weighted inner product given by (3.1). In particular, we have

$$E\Big((w, x)^\sim \Big| X_{\mathcal{G},k}(x) = \vec{\eta}\Big) = \sum_{j=1}^{n} \eta_j (w, g_j)_{C_0'}^{k\odot k}. \tag{5.2}$$

Example 5.1. *Let $S : C_0'[0,T] \to C_0'[0,T]$ be a bounded linear operator. Given a function k in $\mathrm{Supp}_{C_0^*}[0,T]$, and a $k\odot k$-orthonormal set $\mathcal{G} = \{g_1, \dots, g_n\}$ of functions in $C_0'[0,T]$, let $X_{\mathcal{G},k}$ be given by equation* (3.2)*, and let $w \in C_0'[0,T]$. Then using equations* (3.13)*,* (2.12)*,* (2.7)*, and* (3.1)*, it follows that for a.e. $\vec{\eta} \in \mathbb{R}^n$,*

$$\begin{aligned}
&E\Big((Sw, \mathcal{Z}_k(x,\cdot))^\sim \Big| X_{\mathcal{G},k}(x) = \vec{\eta}\Big)\\
&= E_x\Big[\Big(Sw, \mathcal{Z}_k(x,\cdot) - \sum_{j=1}^{n} (g_j \odot k, x)^\sim g_j \odot k \odot k + \sum_{j=1}^{n} \eta_j g_j \odot k \odot k\Big)\Big]\\
&= E_x\Big[((Sw)\odot k, x)^\sim\Big] - \sum_{j=1}^{n} (Sw, g_j \odot k \odot k)^\sim E_x\Big[(g_j \odot k, x)^\sim\Big] + \sum_{j=1}^{n} \eta_j (Sw, g_j \odot k \odot k)_{C_0'}\\
&= \sum_{j=1}^{n} \eta_j ((Sw)\odot k, g_j \odot k)_{C_0'}\\
&= \sum_{j=1}^{n} \eta_j (Sw, g_j)_{C_0'}^{k\odot k}.
\end{aligned} \tag{5.3}$$

Hence equation (5.1) *is established. Equation* (5.2) *follows from equation* (5.3) *by letting S be the identity operator on $C_0'[0,T]$.*

Let $S_1 : C_0'[0,T] \to C_0'[0,T]$ be the linear operator defined by

$$S_1 w(t) = t w(T) - \int_0^t w(s)ds = \int_0^t [w(T) - w(s)]ds. \tag{5.4}$$

Then, we see that

$$(S_1 w, x)^\sim = \int_0^T x(t)dw(t)$$

and the adjoint operator S_1^* of S_1 is given by

$$S_1^* w(t) = \int_0^t w(s)ds. \tag{5.5}$$

Example 5.2. *Let S_1 be the linear operator on $C_0'[0,T]$ given by* (5.4). *Given a function k in $\mathrm{Supp}_{C_0^*}[0,T]$ and a $k\odot k$-orthonormal set $\mathcal{G}=\{g_1,\ldots,g_n\}$ of functions in $C_0'[0,T]$, let $X_{\mathcal{G},k}$ be given by equation* (3.2)*, and let $w\in C_0'[0,T]$. Then, using* (5.1) *with S replaced with S_1,* (3.1), (2.7), *and* (5.5), *it follows that for a.e. $\vec{\eta}\in\mathbb{R}^n$,*

$$
\begin{aligned}
E\Big(\int_0^T \mathcal{Z}_k(x,\cdot)dw(t)\Big|X_{\mathcal{G},k}(x)=\vec{\eta}\Big) &= E\Big((S_1w,\mathcal{Z}_k(x,\cdot))^{\sim}\Big|X_{\mathcal{G},k}(x)=\vec{\eta}\Big)\\
&=\sum_{j=1}^n \eta_j (S_1 w, g_j)_{C_0'}^{k\odot k}\\
&=\sum_{j=1}^n \eta_j (w, S_1^*(g_j\odot k\odot k))_{C_0'}\\
&=\sum_{j=1}^n \eta_j \int_0^T (g_j\odot k\odot k)(t)dw(t).
\end{aligned}
\tag{5.6}
$$

In particular, equation (5.6) *with $X_{\mathcal{G},k}$, w and k replaced with $X_{\mathcal{G},\nu_T}$, ν_T and ν_T, respectively, yields the formula*

$$
\begin{aligned}
E\Big(\int_0^T x(t)dt\Big|(g_j,x)^{\sim}=\eta_j,\ j=1,\ldots,n\Big) &= E\Big(\int_0^T \mathcal{Z}_{\nu_T}(x,\cdot)d\nu_T(t)\Big|(g_j,x)^{\sim}=\eta_j,\ j=1,\ldots,n\Big)\\
&=\sum_{j=1}^n \eta_j \int_0^T g_j(t)dt.
\end{aligned}
\tag{5.7}
$$

Example 5.3. *Given a partition $\{t_1,\ldots,t_n\}$ of $[0,T]$ with $0=t_0<t_1<\cdots<t_{n-1}<t_n=T$, consider the orthonormal set $\mathcal{P}_\tau=\{p_1,\ldots,p_n\}$ of functions in $C_0'[0,T]$ where p_j is given by* (4.1) *above, for each $j\in\{1,\ldots,n\}$. Applying equations* (5.7) *with g_j ($j\in\{1,\ldots,n\}$) replaced with p_j ($j\in\{1,\ldots,n\}$), and* (4.3), *it follows that for a.e. $\vec{\eta}\in\mathbb{R}^n$,*

$$
\begin{aligned}
E\Big(\int_0^T x(t)dt\Big|x(t_j)=\eta_j,\ j=1,\ldots,n\Big) &= E\Big(\int_0^T \mathcal{Z}_{\nu_T}(x,\cdot)d\nu_T(t)\Big|(p_j,x)^{\sim}=\frac{\eta_j-\eta_{j-1}}{\sqrt{t_j-t_{j-1}}},\ j=1,\ldots,n\Big)\\
&=\sum_{j=1}^n \frac{\eta_j-\eta_{j-1}}{\sqrt{t_j-t_{j-1}}}\int_0^T p_j(t)dt\\
&=\sum_{j=1}^n \int_{t_{j-1}}^{t_j}\Big(\eta_{j-1}+\frac{\eta_j-\eta_{j-1}}{t_j-t_{j-1}}(t-t_{j-1})\Big)dt\\
&=\frac{1}{2}\sum_{j=1}^n(\eta_j+\eta_{j-1})(t_j-t_{j-1})
\end{aligned}
$$

where $\eta_0=0$. This result agrees with the results in [3, 16].

In our next example, we use the following Wiener integration formula which follows quite easily from the change of variable theorem:

$$
E\Big[\exp\{\lambda(w,x)^{\sim}\}\Big]=\exp\Big\{\frac{\lambda^2}{2}\|w\|_{C_0'}^2\Big\}
\tag{5.8}
$$

holds for all $\lambda\in\mathbb{C}$ and $w\in C_0'[0,T]$.

Example 5.4. *Let S, k, $\mathcal{G}$, $X_{\mathcal{G},k}$, and w be as in Example 5.1 above. Then, using equations* (3.13), (3.1), (2.7), (2.12) *and* (5.8), *it follows that*

$$\begin{aligned}
&E\Big(\exp\{\lambda(Sw,\mathcal{Z}_k(x,\cdot))^{\sim}\}\Big|X_{\mathcal{G},k}(x)=\vec{\eta}\Big)\\
&=E_x\Big[\exp\Big\{\lambda\Big(Sw,\mathcal{Z}_k(x,\cdot)-\sum_{j=1}^{n}(g_j\odot k,x)^{\sim}g_j\odot k\odot k+\sum_{j=1}^{n}\eta_j g_j\odot k\odot k\Big)^{\sim}\Big\}\Big]\\
&=\exp\Big\{\lambda\sum_{j=1}^{n}\eta_j(Sw,g_j)_{C_0'}^{k\odot k}\Big\}E_x\Big[\exp\Big\{\lambda\Big(\Big[(Sw)\odot k-\sum_{j=1}^{n}(Sw,g_j)_{C_0'}^{k\odot k}g_j\odot k\Big],x\Big)^{\sim}\Big\}\Big]\\
&=\exp\Big\{\lambda\sum_{j=1}^{n}\eta_j(Sw,g_j)_{C_0'}^{k\odot k}+\frac{\lambda^2}{2}\Big[(Sw,Sw)_{C_0'}^{k\odot k}-\sum_{j=1}^{n}\Big[(Sw,g_j)_{C_0'}^{k\odot k}\Big]^2\Big]\Big\}
\end{aligned}\tag{5.9}$$

for all $\lambda\in\mathbb{C}$ and for a.e. $\vec{\eta}=(\eta_1,\ldots,\eta_n)\in\mathbb{R}^n$.

The formulas and results in this paper are more complicated than the corresponding formulas and results in [16–18]. However, by appropriate choices of k, $\mathcal{G}$, and F in Theorem 3.7, and calculations as done as in Examples 5.2 and 5.3, the expected results on with standard Brownian motion processes are immediate corollaries of the results in this paper.

6. Translation result for conditional generalized Wiener integral

In this section, we establish a translation theorem for the conditional Wiener integral and then use it to obtain conditional Wiener integration formulas for certain Wiener integrable functionals F on $C_0[0,T]$.

We start this section by stating the following well-known translation theorem [1, 13], using the notation of this article.

Theorem 6.1 (Translation Theorem). *Let $F\in L_1(C_0[0,T],\mathcal{M},\mathfrak{m})$ and let $x_0\in C_0'[0,T]$. Then*

$$E_x[F(x)]=E_x\Big[F(x+x_0)J(x_0,x)\Big]\tag{6.1}$$

where

$$J(x_0,x)=\exp\Big\{-\frac{1}{2}\|x_0\|_{C_0'}^2-(x_0,x)^{\sim}\Big\}.\tag{6.2}$$

Using equations (3.1), and (2.12) with w replaced with θ, equation (6.2) with x_0 replaced with $\theta\odot k$ can be rewritten as equation (6.3) below.

Lemma 6.2. *Let k be a function in $\mathrm{Supp}_{C_0^*}[0,T]$. Then it follows that for each function θ in $C_0'[0,T]$,*

$$J(\theta\odot k,x)=\exp\Big\{-\frac{1}{2}\Big[\|\theta\|_{C_0'}^{k\odot k}\Big]^2-(\theta,\mathcal{Z}_k(x,\cdot))^{\sim}\Big\},\tag{6.3}$$

where $\|\cdot\|_{C_0'}^{k\odot k}=[(\cdot,\cdot)^{k\odot k}]^{1/2}$.

Lemma 6.3. *Let k be a function in $\mathrm{Supp}_{C_0^*}[0,T]$ and let $\mathcal{G}=\{g_1,\ldots,g_n\}$ be a $k\odot k$-orthonormal set of functions in $C_0'[0,T]$. Then it follows that for each function θ in $C_0'[0,T]$,*

$$E_x\Big[\exp\Big\{-\sum_{j=1}^{n}(g_j\odot k,x)^{\sim}\Big(\theta,\mathcal{Z}_k(g_j\odot k,\cdot)\Big)^{\sim}\Big\}\Big]=\exp\Big\{\frac{1}{2}\sum_{j=1}^{n}\Big[(\theta,g_j)_{C_0'}^{k\odot k}\Big]^2\Big\}.\tag{6.4}$$

Proof. Applying the fact that $(g_j \odot k, x)^{\sim}$'s, $j \in \{1, \dots, n\}$, are independent Gaussian random variables, and using equations (2.12) and (5.8), one can derive equation (6.4) above, immediately. □

The following theorem is a translation theorem for conditional generalized Wiener integrals associated with Gaussian paths on Wiener space $C_0[0,T]$.

Theorem 6.4. *Let k, $\mathcal{G}$, F and $X_{\mathcal{G},k}$ be as in Lemma 3.4, and let θ be a function in $C_0'[0,T]$. Then it follows that for a.e. $\vec{\eta} \in \mathbb{R}^n$,*

$$\begin{aligned} &E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|X_{\mathcal{G},k}(x)=\vec{\eta}\Big) \\ &= E\Big(F\Big(\mathcal{Z}_k(x,\cdot)+\mathcal{Z}_k(\theta\odot k,\cdot)\Big)J(\theta\odot k,x)\Big|X_{\mathcal{G},k}(x)=\vec{\eta}-(\vec{g},\theta)_{C_0'}^{k\odot k}\Big)\exp\Big\{\sum_{j=1}^{n}\eta_j(\theta,g_j)_{C_0'}^{k\odot k}-\frac{1}{2}\sum_{j=1}^{n}\Big[(\theta,g_j)_{C_0'}^{k\odot k}\Big]^2\Big\} \\ &= E\Big(F\Big(\mathcal{Z}_k(x,\cdot)+\mathcal{Z}_k(\theta\odot k,\cdot)\Big)J(\theta\odot k,x)\Big|X_{\mathcal{G},k}(x+\theta\odot k)=\vec{\eta}\Big)\exp\Big\{\sum_{j=1}^{n}\eta_j(\theta,g_j)_{C_0'}^{k\odot k}-\frac{1}{2}\sum_{j=1}^{n}\Big[(\theta,g_j)_{C_0'}^{k\odot k}\Big]^2\Big\} \end{aligned} \tag{6.5}$$

where $J(\theta \odot k, x)$ is given by (6.3) and

$$(\vec{g},w)_{C_0'}^{k\odot k} = ((g_1,w)_{C_0'}^{k\odot k},\dots,(g_n,w)_{C_0'}^{k\odot k}). \tag{6.6}$$

Proof. Given a function k in $\mathrm{Supp}_{C_0^*}[0,T]$, the conditioning function $X_{\mathcal{G},k}$ given by (3.2), and $\vec{\eta} \in \mathbb{R}^n$, let

$$F_{k,X_{\mathcal{G},k},\vec{\eta}}(x) \equiv F\Big(\mathcal{Z}_k\Big(x-\sum_{j=1}^{n}(g_j\odot k,x)^{\sim}g_j\odot k+\sum_{j=1}^{n}\eta_j g_j\odot k,\cdot\Big)\Big),$$

$$\mathfrak{S}_{k,X_{\mathcal{G},k}}(x,\cdot) \equiv \mathcal{Z}_k(x,\cdot)-\sum_{j=1}^{n}(g_j\odot k,x)^{\sim}\mathcal{Z}_k(g_j\odot k,\cdot), \tag{6.7}$$

and

$$\mathfrak{T}_{k,X_{\mathcal{G},k}}(x,\cdot) \equiv \sum_{j=1}^{n}(g_j\odot k,x)^{\sim}\mathcal{Z}_k(g_j\odot k,\cdot). \tag{6.8}$$

Using (3.14), (6.1) with F and x_0 replaced with $F_{k,X_{\mathcal{G},k},\vec{\eta}}$ and $\theta \odot k$, respectively, and (6.7), it follows that

$$\begin{aligned} E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|X_{\mathcal{G},k}(x)=\vec{\eta}\Big) &= E_x\Big[F\Big(\mathcal{Z}_k\Big(x-\sum_{j=1}^{n}(g_j\odot k,x)^{\sim}g_j\odot k+\sum_{j=1}^{n}\eta_j g_j\odot k,\cdot\Big)\Big)\Big] \\ &= E_x\Big[F_{k,X_{\mathcal{G},k},\vec{\eta}}(x)\Big] \\ &= E_x\Big[F_{k,X_{\mathcal{G},k},\vec{\eta}}(x+\theta\odot k)J(\theta\odot k,x)\Big] \\ &= E_x\Big[F\Big(\mathcal{Z}_k\Big(x+\theta\odot k-\sum_{j=1}^{n}(g_j\odot k,x+\theta\odot k)^{\sim}g_j\odot k+\sum_{j=1}^{n}\eta_j g_j\odot k,\cdot\Big)\Big)J(\theta\odot k,x)\Big] \\ &= E_x\Big[F\Big(\mathfrak{S}_{k,X_{\mathcal{G},k}}(x,\cdot)+\mathcal{Z}_k(\theta\odot k,\cdot)+\sum_{j=1}^{n}\Big(\eta_j-(g_j,\theta)_{C_0'}^{k\odot k}\Big)\mathcal{Z}_k(g_j\odot k,\cdot)\Big)J(\theta\odot k,x)\Big]. \end{aligned} \tag{6.9}$$

Next, using equations (6.3), (6.7), and (6.8), we rewrite $J(\theta \odot k, x)$ in the form

$$\begin{aligned} J(\theta\odot k,x) &= \exp\Big\{-\frac{1}{2}\Big[\|\theta\|_{C_0'}^{k\odot k}\Big]^2-\Big(\theta,\mathcal{Z}_k\Big(x-\sum_{j=1}^{n}(g_j\odot k,x)^{\sim}g_j\odot k\Big)\Big)^{\sim}-\Big(\theta,\mathcal{Z}_k\Big(\sum_{j=1}^{n}(g_j\odot k,x)^{\sim}g_j\odot k\Big)\Big)^{\sim}\Big\} \\ &= \exp\Big\{-\frac{1}{2}\Big[\|\theta\|_{C_0'}^{k\odot k}\Big]^2-\Big(\theta,\mathfrak{S}_{k,X_{\mathcal{G},k}}(x,\cdot)\Big)^{\sim}-\Big(\theta,\mathfrak{T}_{k,X_{\mathcal{G},k}}(x,\cdot)\Big)^{\sim}\Big\}. \end{aligned}$$

Using this, equation (6.9) can be rewritten by

$$\begin{aligned} &E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|X_{\mathcal{G},k}(x)=\vec{\eta}\Big)\\ &=E_x\Big[F\Big(\mathfrak{S}_{k,X_{\mathcal{G},k}}(x,\cdot)+\mathcal{Z}_k(\theta\odot k,\cdot)+\sum_{j=1}^{n}\Big(\eta_j-(g_j,\theta)_{C_0'}^{k\odot k}\Big)\mathcal{Z}_k(g_j\odot k,\cdot)\Big)\\ &\qquad\times\exp\Big\{-\frac{1}{2}\Big[\|\theta\|_{C_0'}^{k\odot k}\Big]^2-\Big(\theta,\mathfrak{S}_{k,X_{\mathcal{G},k}}(x,\cdot)\Big)^{\sim}\Big\}\exp\Big\{-\Big(\theta,\mathfrak{T}_{k,X_{\mathcal{G},k}}(x,\cdot)\Big)^{\sim}\Big\}\Big]. \end{aligned} \tag{6.10}$$

Since

$$\Big\{\mathfrak{S}_{k,X_{\mathcal{G},k}}(x,t):t\in[0,T]\Big\}\equiv\Big\{\mathcal{Z}_k(x,t)-[x]_{\mathcal{G},k}(t):t\in[0,T]\Big\}$$

and

$$\Big\{\mathfrak{T}_{k,X_{\mathcal{G},k}}(x,t):t\in[0,T]\Big\}\equiv\Big\{[x]_{\mathcal{G},k}(t):t\in[0,T]\Big\}$$

are independent processes by Lemma 3.3, we see that the random variables appeared in the last expression of (6.10),

$$F\Big(\mathfrak{S}_{k,X_{\mathcal{G},k}}(x,\cdot)+\mathcal{Z}_k(\theta\odot k,\cdot)+\sum_{j=1}^{n}\Big(\eta_j-(g_j,\theta)_{C_0'}^{k\odot k}\Big)\mathcal{Z}_k(g_j\odot k,\cdot)\Big)\exp\Big\{-\frac{1}{2}\Big[\|\theta\|_{C_0'}^{k\odot k}\Big]^2-\Big(\theta,\mathfrak{S}_{k,X_{\mathcal{G},k}}(x,\cdot)\Big)^{\sim}\Big\}$$

and

$$\exp\Big\{-\Big(\theta,\mathfrak{T}_{k,X_{\mathcal{G},k}}(x,\cdot)\Big)^{\sim}\Big\}$$

are independent. Using this fact and applying equations (6.7), (6.8), (6.4), (6.3), and (3.14) with $F(\mathcal{Z}_k(x,\cdot))$ replaced with $F(\mathcal{Z}_k(x,\cdot))J(\theta\odot k,x)$, we next have that

$$\begin{aligned} &E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|X_{\mathcal{G},k}(x)=\vec{\eta}\Big)\\ &=E_x\Big[F\Big(\mathfrak{S}_{k,X_{\mathcal{G},k}}(x,\cdot)+\mathcal{Z}_k(\theta\odot k,\cdot)+\sum_{j=1}^{n}\Big(\eta_j-(g_j,\theta)_{C_0'}^{k\odot k}\Big)\mathcal{Z}_k(g_j\odot k,\cdot)\Big)\exp\Big\{-\frac{1}{2}\Big[\|\theta\|_{C_0'}^{k\odot k}\Big]^2-\Big(\theta,\mathfrak{S}_{k,X_{\mathcal{G},k}}(x,\cdot)\Big)^{\sim}\Big\}\Big]\\ &\quad\times E_x\Big[\exp\Big\{-\Big(\theta,\mathfrak{T}_{k,X_{\mathcal{G},k}}(x,\cdot)\Big)^{\sim}\Big\}\Big]\\ &=E_x\Big[F\Big(\mathfrak{S}_{k,X_{\mathcal{G},k}}(x,\cdot)+\sum_{j=1}^{n}\Big(\eta_j-(g_j,\theta)_{C_0'}^{k\odot k}\Big)\mathcal{Z}_k(g_j\odot k,\cdot)+\mathcal{Z}_k(\theta\odot k,\cdot)\Big)\\ &\qquad\times\exp\Big\{-\frac{1}{2}\Big[\|\theta\|_{C_0'}^{k\odot k}\Big]^2-\Big(\theta,\mathfrak{S}_{k,X_{\mathcal{G},k}}(x,\cdot)\Big)^{\sim}-\sum_{j=1}^{n}\Big(\eta_j-(g_j,\theta)_{C_0'}^{k\odot k}\Big)\Big(\theta,\mathcal{Z}_k(g_j\odot k,\cdot)\Big)_{C_0'}\Big\}\Big]\\ &\qquad\times\exp\Big\{\sum_{j=1}^{n}\eta_j\Big(\theta,\mathcal{Z}_k(g_j\odot k,\cdot)\Big)_{C_0'}-\sum_{j=1}^{n}(g_j,\theta)_{C_0'}^{k\odot k}\Big(\theta,\mathcal{Z}_k(g_j\odot k,\cdot)\Big)_{C_0'}\Big\}\\ &\qquad\times\exp\Big\{\frac{1}{2}\sum_{j=1}^{n}\Big[(\theta,g_j)_{C_0'}^{k\odot k}\Big]^2\Big\}\\ &=E_x\Big[F\Big(\Big\{\mathcal{Z}_k(x,\cdot)-\sum_{j=1}^{n}(g_j\odot k,x)^{\sim}\mathcal{Z}_k(g_j\odot k,\cdot)+\sum_{j=1}^{n}\Big(\eta_j-(g_j,\theta)_{C_0'}^{k\odot k}\Big)\mathcal{Z}_k(g_j\odot k,\cdot)\Big\}+\mathcal{Z}_k(\theta\odot k,\cdot)\Big)\\ &\qquad\times\exp\Big\{-\frac{1}{2}\Big[\|\theta\|_{C_0'}^{k\odot k}\Big]^2-\Big(\theta,\mathcal{Z}_k(x,\cdot)-\sum_{j=1}^{n}(g_j\odot k,x)^{\sim}\mathcal{Z}_k(g_j\odot k,\cdot)+\sum_{j=1}^{n}\Big(\eta_j-(g_j,\theta)_{C_0'}^{k\odot k}\Big)\mathcal{Z}_k(g_j\odot k,\cdot)\Big)^{\sim}\Big\}\Big]\\ &\quad\times\exp\Big\{\sum_{j=1}^{n}\eta_j(\theta\odot k,g_j\odot k)_{C_0'}-\sum_{j=1}^{n}(\theta,g_j)_{C_0'}^{k\odot k}(\theta\odot k,g_j\odot k)_{C_0'}+\frac{1}{2}\sum_{j=1}^{n}\Big[(\theta,g_j)_{C_0'}^{k\odot k}\Big]^2\Big\} \end{aligned}$$

$$= E\Big(F\Big(\mathcal{Z}_k(x,\cdot)+\mathcal{Z}_k(\theta\odot k,\cdot)\Big)J(\theta\odot k,x)\Big|X_{\mathcal{G},k}(x)=\eta_j-(g_j,\theta)_{C_0'}^{k\odot k}\Big)\exp\Big\{\sum_{j=1}^n \eta_j(\theta,g_j)_{C_0'}^{k\odot k}-\frac{1}{2}\sum_{j=1}^n\Big[(\theta,g_j)_{C_0'}^{k\odot k}\Big]^2\Big\},$$

which completes the proof of the theorem. □

Next we use Theorem 6.4 to obtain various conditional generalized Wiener integration formulas.

Example 6.5. *Let k, $\mathcal{G}$, and $X_{\mathcal{G},k}$ be as in Example 5.1 above. Also for $\vec{\eta}\in\mathbb{R}^n$, let $\vec{\eta}=\vec{\xi}+(\vec{g},w)_{C_0'}^{k\odot k}$ where $(\vec{g},w)_{C_0'}^{k\odot k}$ is given by* (6.6) *above. Then, using equation* (6.5) *with $F(x)\equiv 1$ on $C_0[0,T]$, we obtain that*

$$\begin{aligned}
1 &= E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|X_{\mathcal{G},k}(x)=\vec{\xi}+(\vec{g},w)_{C_0'}^{k\odot k}\Big)\\
&= E\Big(J(\theta\odot k,x)\Big|X_{\mathcal{G},k}(x)=\vec{\xi}\Big)\exp\Big\{\sum_{j=1}^n\Big(\xi_j+(\theta,g_j)_{C_0'}^{k\odot k}\Big)(\theta,g_j)_{C_0'}^{k\odot k}-\frac{1}{2}\sum_{j=1}^n\Big[(\theta,g_j)_{C_0'}^{k\odot k}\Big]^2\Big\}\\
&= E\Big(\exp\Big\{-\frac{1}{2}\Big[\|\theta\|_{C_0'}^{k\odot k}\Big]^2-(\theta,\mathcal{Z}_k(x,\cdot))^{\sim}\Big\}\Big|X_{\mathcal{G},k}(x)=\vec{\xi}\Big)\exp\Big\{\sum_{j=1}^n\Big(\xi_j+(\theta,g_j)_{C_0'}^{k\odot k}\Big)(\theta,g_j)_{C_0'}^{k\odot k}-\frac{1}{2}\sum_{j=1}^n\Big[(\theta,g_j)_{C_0'}^{k\odot k}\Big]^2\Big\}\\
&= E\Big(\exp\Big\{-(\theta,\mathcal{Z}_k(x,\cdot))^{\sim}\Big\}\Big|X_{\mathcal{G},k}(x)=\vec{\xi}\Big)\exp\Big\{-\frac{1}{2}\Big[\|\theta\|_{C_0'}^{k\odot k}\Big]^2+\sum_{j=1}^n\Big(\xi_j+(\theta,g_j)_{C_0'}^{k\odot k}\Big)(\theta,g_j)_{C_0'}^{k\odot k}-\frac{1}{2}\sum_{j=1}^n\Big[(\theta,g_j)_{C_0'}^{k\odot k}\Big]^2\Big\}.
\end{aligned} \tag{6.11}$$

Using equation (6.11)*, we immediately obtain the conditional generalized Wiener integration formula*

$$E\Big(\exp\Big\{-(\theta,\mathcal{Z}_k(x,\cdot))^{\sim}\Big\}\Big|X_{\mathcal{G},k}(x)=\vec{\xi}\Big)=\exp\Big\{\frac{1}{2}\Big[\|\theta\|_{C_0'}^{k\odot k}\Big]^2-\sum_{j=1}^n\xi_j(\theta,g_j)_{C_0'}^{k\odot k}-\frac{1}{2}\sum_{j=1}^n\Big[(\theta,g_j)_{C_0'}^{k\odot k}\Big]^2\Big\}. \tag{6.12}$$

Example 6.6. *Let S, k, $\mathcal{G}$, $X_{\mathcal{G},k}$, and w be as in Example 5.1 above. Then replacing θ and $\vec{\xi}=(\xi_1,\dots,\xi_n)$ with $-\lambda Sw$ and $\vec{\eta}=(\eta_1,\dots,\eta_n)$, respectively, in equation* (6.12) *above yields the conditional generalized Wiener integration formula*

$$\begin{aligned}
E\Big(\exp\Big\{(\lambda Sw,\mathcal{Z}_k(x,\cdot))^{\sim}\Big\}\Big|X_{\mathcal{G},k}(x)=\vec{\xi}\Big) &= \exp\Big\{\frac{\lambda^2}{2}\Big[\|\lambda Sw\|_{C_0'}^{k\odot k}\Big]^2+\lambda\sum_{j=1}^n\xi_j(\lambda Sw,g_j)_{C_0'}^{k\odot k}-\frac{\lambda^2}{2}\sum_{j=1}^n\Big[(\lambda Sw,g_j)_{C_0'}^{k\odot k}\Big]^2\Big\}\\
&= \exp\Big\{\lambda\sum_{j=1}^n\xi_j(\lambda Sw,g_j)_{C_0'}^{k\odot k}+\frac{\lambda^2}{2}\Big(\Big[\|\lambda Sw\|_{C_0'}^{k\odot k}\Big]^2-\sum_{j=1}^n\Big[(\lambda Sw,g_j)_{C_0'}^{k\odot k}\Big]^2\Big)\Big\}.
\end{aligned} \tag{6.13}$$

Remark 6.7. *It can be shown, using analytic continuation, that equation* (6.13) *holds for all $\lambda\in\mathbb{C}$; this gives an alternate proof of the conditional generalized Wiener integration formula given by* (5.9) *above.*

We finish this paper with various corollaries of Theorem 6.4. Each conditional generalized Wiener integration formula provided in the following two corollaries agrees with the main results in [16, 18].

Corollary 6.8 ([16]). *Let $F\in L_1(C_0[0,T],\mathcal{M},\mathfrak{m})$, and given a partition $\tau=\{t_1,\dots,t_n\}$ of $[0,T]$ with $0=t_0<t_1<\cdots<t_{n-1}<t_n=T$, let $X_{\mathcal{P}_\tau,\nu_T}$ be given by* (4.4)*. Then it follows that for any function θ in $C_0'[0,T]$ and for a.e. $\vec{\eta}\in\mathbb{R}^n$,*

$$\begin{aligned}
E\Big(F(x(\cdot))\Big|x(t_j)=\eta_j,\ j=1,\dots,n\Big) &= E_x\Big(F\Big(x(\cdot)+\theta(\cdot)\Big)J(\theta,x)\Big|x(t_j)=\eta_j-\theta(t_j),\ j=1,\dots,n\Big)\\
&\quad\times\exp\Big\{\sum_{j=1}^n\frac{(\eta_j-\eta_{j-1})(\theta(t_j)-\theta(t_{j-1}))}{(t_j-t_{j-1})}-\frac{1}{2}\sum_{j=1}^n\frac{(\theta(t_j)-\theta(t_{j-1}))^2}{(t_j-t_{j-1})}\Big\}
\end{aligned}$$

where $J(\theta,x)$ is given by (6.2) *with x_0 replaced with θ, and $\eta_0=0$.*

Proof. In (6.5), choose $k = \nu_T$, and replace $X_{\mathcal{G},k}$ with $X_{\mathcal{P}_\tau,\nu_T}$ given by (4.4). Then, using (4.1), it follows that for each function θ in $C_0'[0,T]$,

$$
\begin{aligned}
&E\Big(F(x(\cdot))\Big|x(t_j) = \eta_j,\ j = 1,\dots,n\Big)\\
&= E\Big(F(x(\cdot))\Big|X_{\mathcal{P}_\tau,\nu_T}(x) = \Big(\frac{\eta_1-\eta_0}{\sqrt{t_1-t_0}},\dots,\frac{\eta_n-\eta_{n-1}}{\sqrt{t_n-t_{n-1}}}\Big)\Big)\\
&= E\Big(F(x(\cdot)+\theta(\cdot))J(\theta,x)\Big|X_{\mathcal{P}_\tau,\nu_T}(x) = \Big(\frac{\eta_1-\eta_0}{\sqrt{t_1-t_0}} - (p_j,\theta)_{C_0'},\dots,\frac{\eta_n-\eta_{n-1}}{\sqrt{t_n-t_{n-1}}} - (p_n,\theta)_{C_0'}\Big)\Big)\\
&\quad\times\exp\Big\{\sum_{j=1}^n \frac{\eta_j-\eta_{j-1}}{\sqrt{t_j-t_{j-1}}}(\theta,p_j)_{C_0'} - \frac12\sum_{j=1}^n(\theta,p_j)^2_{C_0'}\Big\}\\
&= E\Big(F(x(\cdot)+\theta(\cdot))J(\theta,x)\Big|X_{\mathcal{P}_\tau,\nu_T}(x) = \Big(\frac{(\eta_1-\theta(t_1))-(\eta_0-\theta(t_0))}{\sqrt{t_1-t_0}},\dots,\frac{(\eta_n-\theta(t_n))-(\eta_{n-1}-\theta(t_{n-1}))}{\sqrt{t_n-t_{n-1}}}\Big)\Big)\\
&\quad\times\exp\Big\{\sum_{j=1}^n \frac{(\eta_j-\eta_{j-1})(\theta(t_j)-\theta(t_{j-1}))}{t_j-t_{j-1}} - \frac12\sum_{j=1}^n\frac{(\theta(t_j)-\theta(t_{j-1}))^2}{t_j-t_{j-1}}\Big\}\\
&= E_x\Big(F(x(\cdot)+\theta(\cdot))J(\theta,x)\Big|x(t_j) = \eta_j-\theta(t_j),\ j = 1,\dots,n\Big)\\
&\quad\times\exp\Big\{\sum_{j=1}^n \frac{(\eta_j-\eta_{j-1})(\theta(t_j)-\theta(t_{j-1}))}{(t_j-t_{j-1})} - \frac12\sum_{j=1}^n\frac{(\theta(t_j)-\theta(t_{j-1}))^2}{(t_j-t_{j-1})}\Big\},
\end{aligned}
$$

for a.e. $\vec{\eta}\in\mathbb{R}^n$, as desired. □

Corollary 6.9 ([18]). *Let $F\in L_1(C_0[0,T],\mathcal{M},\mathfrak{m})$, and given an orthonormal set $\mathcal{G} = \{g_1,\dots,g_n\}$ of functions in $C_0'[0,T]$, let $X_{\mathcal{G},\nu_T}$ be given by (3.2) with k replaced with ν_T. Then it follows that for each $\theta\in C_0'[0,T]$,*

$$
\begin{aligned}
&E\Big(F(x)\Big|(g_j,x)^{\sim} = \eta_j,\ j = 1,\dots,n\Big)\\
&= E\Big(F(x+\theta)J(\theta,x)\Big|(g_j,x+\theta)^{\sim} = \eta_j,\ j = 1,\dots,n\Big)\exp\Big\{\sum_{j=1}^n\eta_j\big(\theta,g_j\big)_{C_0'} - \frac12\sum_{j=1}^n\big(\theta,g_j\big)^2_{C_0'}\Big\}
\end{aligned}
\tag{6.14}
$$

for a.e. $\vec{\eta}\in\mathbb{R}^n$, where $J(\theta,x)$ is given by (6.2) with x_0 replaced with θ.

Proof. Any orthonormal set of functions in $C_0'[0,T]$ is a $\nu_T\odot\nu_T$-orthonormal set in $C_0'[0,T]$. Choosing $k=\nu_T$, equation (6.5) yields equation (6.14). □

The following conditional translation theorem also can be derived by the main results in [17].

Corollary 6.10. *Given a function k in $\mathrm{Supp}_{C_0'}[0,T]$, let $F(\mathcal{Z}_k(x,\cdot))\in L_1(C_0[0,T],\mathcal{M},\mathfrak{m})$, and given a partition $\tau = \{t_1,\dots,t_n\}$ of $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, let $X_{\mathcal{R}_{\tau,k},k}$ be given by (4.11). Then it follows that for each function θ in $C_0'[0,T]$ and for a.e. $\vec{\eta}\in\mathbb{R}^n$,*

$$
\begin{aligned}
&E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|\mathcal{Z}_k(x,t_j) = \eta_j,\ j = 1,\dots,n\Big)\\
&= E\Big(F\big(\mathcal{Z}_k(x,\cdot)+\mathcal{Z}_k(\theta\odot k,\cdot)\big)J(\theta\odot k,x)\Big|\mathcal{Z}_k(x,t_j) = \eta_j - \mathcal{Z}_k(\theta\odot k,t_j),\ j = 1,\dots,n\Big)\\
&\quad\times\exp\Big\{\sum_{j=1}^n\frac{(\eta_j-\eta_{j-1})(\mathcal{Z}_k(\theta\odot k,t_n)-\mathcal{Z}_k(\theta\odot k,t_{n-1}))}{\beta_k(t_j)-\beta_k(t_{j-1})} - \frac12\sum_{j=1}^n\frac{(\mathcal{Z}_k(\theta\odot k,t_n)-\mathcal{Z}_k(\theta\odot k,t_{n-1}))^2}{\beta_k(t_j)-\beta_k(t_{j-1})}\Big\}
\end{aligned}
$$

for a.e. $\vec{\eta}\in\mathbb{R}^n$, where $J(\theta\odot k,x)$ is given by (6.3) and $\eta_0 = 0$.

Proof. First, we note that $\mathcal{R}_{\tau,k} = \{r_{k,1}, \ldots, r_{k,n}\}$ is a $k \odot k$-orthonormal set in $C_0'[0,T]$ for any function k in $\mathrm{Supp}_{C_0^*}[0,T]$, where $r_{k,j}$, $j \in \{1, \ldots, n\}$, is given by (4.8). In (6.5), next, replace $X_{\mathcal{G},k}$ with $X_{\mathcal{R}_{\tau,k},k}$ given by (4.11). Then, using (4.8), it follows that for each function θ in $C_0'[0,T]$,

$$\begin{aligned}
&E\Big(F(\mathcal{Z}_k(x,\cdot))\Big|\mathcal{Z}_k(x,t_j) = \eta_j,\ j = 1, \ldots, n\Big) \\
&= E\bigg(F(\mathcal{Z}_k(x,\cdot))\bigg|X_{\mathcal{R}_{\tau,k},k}(x) = \Big(\frac{\eta_1 - \eta_0}{\sqrt{\beta_k(t_1) - \beta_k(t_0)}}, \ldots, \frac{\eta_n - \eta_{n-1}}{\sqrt{\beta_k(t_n) - \beta_k(t_{n-1})}}\Big)\bigg) \\
&= E\bigg(F\Big(\mathcal{Z}_k(x,\cdot) + \mathcal{Z}_k(\theta \odot k,\cdot)\Big)J(\theta \odot k, x) \\
&\qquad \bigg|X_{\mathcal{R}_{\tau,k},k}(x) = \Big(\frac{\eta_1 - \eta_0}{\sqrt{\beta_k(t_1) - \beta_k(t_0)}} - (r_{k,1},\theta)_{C_0'}^{k\odot k}, \ldots, \frac{\eta_n - \eta_{n-1}}{\sqrt{\beta_k(t_n) - \beta_k(t_{n-1})}} - (r_{k,n},\theta)_{C_0'}^{k\odot k}\Big)\bigg) \\
&\quad \times \exp\bigg\{\sum_{j=1}^n \frac{\eta_j - \eta_{j-1}}{\sqrt{\beta_k(t_j) - \beta_k(t_{j-1})}}(\theta, r_{k,j})_{C_0'}^{k\odot k} - \frac{1}{2}\sum_{j=1}^n \Big[(\theta, r_{k,j})_{C_0'}^{k\odot k}\Big]^2\bigg\} \\
&= E\bigg(F\Big(\mathcal{Z}_k(x,\cdot) + \mathcal{Z}_k(\theta \odot k,\cdot)\Big)J(\theta \odot k, x) \\
&\qquad \bigg|X_{\mathcal{R}_{\tau,k},k}(x) = \Big(\frac{(\eta_1 - \mathcal{Z}_k(\theta \odot k, t_1)) - (\eta_0 - \mathcal{Z}_k(\theta \odot k, t_0))}{\sqrt{\beta_k(t_1) - \beta_k(t_0)}}, \ldots, \frac{(\eta_n - \mathcal{Z}_k(\theta \odot k, t_n)) - (\eta_{n-1} - \mathcal{Z}_k(\theta \odot k, t_{n-1}))}{\sqrt{\beta_k(t_n) - \beta_k(t_{n-1})}}\Big)\bigg) \\
&\quad \times \exp\bigg\{\sum_{j=1}^n \frac{(\eta_j - \eta_{j-1})(\mathcal{Z}_k(\theta \odot k, t_n) - \mathcal{Z}_k(\theta \odot k, t_{n-1}))}{\beta_k(t_j) - \beta_k(t_{j-1})} - \frac{1}{2}\sum_{j=1}^n \frac{(\mathcal{Z}_k(\theta \odot k, t_n) - \mathcal{Z}_k(\theta \odot k, t_{n-1}))^2}{\beta_k(t_j) - \beta_k(t_{j-1})}\bigg\} \\
&= E\bigg(F\Big(\mathcal{Z}_k(x,\cdot) + \mathcal{Z}_k(\theta \odot k,\cdot)\Big)J(\theta \odot k, x)\bigg|\mathcal{Z}_k(x,t_j) = \eta_j - \mathcal{Z}_k(\theta \odot k, t_j),\ j = 1, \ldots, n\bigg) \\
&\quad \times \exp\bigg\{\sum_{j=1}^n \frac{(\eta_j - \eta_{j-1})(\mathcal{Z}_k(\theta \odot k, t_n) - \mathcal{Z}_k(\theta \odot k, t_{n-1}))}{\beta_k(t_j) - \beta_k(t_{j-1})} - \frac{1}{2}\sum_{j=1}^n \frac{(\mathcal{Z}_k(\theta \odot k, t_n) - \mathcal{Z}_k(\theta \odot k, t_{n-1}))^2}{\beta_k(t_j) - \beta_k(t_{j-1})}\bigg\}
\end{aligned}$$

for a.e. $\vec{\eta} \in \mathbb{R}^n$. □

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