



Ψ DOs with radial symbols and spaces of type G

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Abstract. We investigate various G type spaces on \mathbb{R}_+^d and their relations with the Gelfand-Shilov S type spaces on \mathbb{R}^{2d} through the mapping $w : \mathbb{R}^{2d} \rightarrow \mathbb{R}_+^d$, $w(x, \xi) = (2x_1^2 + 2\xi_1^2, \dots, 2x_d^2 + 2\xi_d^2)$. Sufficient conditions for the hypoellipticity of symbols originating from the coordinate radial symbols in G type spaces are also given. Two open problems explained in the introduction are posed.

1. Introduction

When a solution of a certain partial differential equation of global type is smooth but not analytic, we look for a space where we can describe its decay for $|x| \rightarrow \infty$ and its regularity in \mathbb{R}^d . For these needs, Gelfand-Shilov type spaces [11] i.e. S type spaces are significantly useful. Moreover, they are used as a framework for the time frequency analysis of pseudo-differential operators with symbols being of global type. Recall that Gelfand and Shilov introduced such spaces in order to find solutions of certain parabolic initial-value problems. There is a vast literature dealing with pseudo-differential calculus in Gelfand-Shilov setting, see e.g. monographs [6, 19, 25] as well as articles [1, 2, 4, 22–24, 26] and references therein. However, there are only a few papers related to similar type of spaces defined over $\mathbb{R}_+^d = (0, \infty)^d$. Here, we mention [8, 9], where Duran introduced G type spaces and papers [13–15]. In one dimensional case, the space of tempered distributions supported by $(0, \infty)$ is determined by the corresponding space of rapidly decreasing functions in $(0, \infty)$ (cf. [10, 29]). It should be underlined that the transfer of the space of tempered distributions over \mathbb{R}^d to the one with the domain \mathbb{R}_+^d is not a trivial task, see [12].

In [16], we considered G type spaces $G_\alpha^a(\mathbb{R}_+^d)$ and their duals for $\alpha \geq 1$ and gave their full topological characterization. Furthermore, we analyzed the Weyl pseudo-differential operators on the Gelfand-Shilov spaces

$$\tilde{a}^W(x, D)f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} \tilde{a}\left(\frac{x+y}{2}, \xi\right) f(y) dy d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d), \quad (1)$$

with symbols $\tilde{a}(x, \xi)$ such that $\tilde{a}(x, \xi) = a(r_1, \dots, r_d)$, $r_i = 2x_i^2 + 2\xi_i^2$, $i = 1, \dots, d$, where $a(r)$ belongs to a dual space of a G type space. Symbol $\tilde{a}(x, \xi)$ is called a symbol with coordinate radial arguments. Essential

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relations between G type spaces and Gelfand-Shilov type spaces are determined through the mapping

$$\mathbb{R}^d \ni (x_1, \dots, x_n) \mapsto v(x) = r = (r_1, \dots, r_d) = (x_1^2, \dots, x_n^2) \in \mathbb{R}_+^d. \tag{2}$$

and

$$(x, \xi) = (x_1, \xi_1, \dots, x_d, \xi_d) \mapsto w(x, \xi) = (2x_1^2 + 2\xi_1^2, \dots, 2x_d^2 + 2\xi_d^2) = (r_1, \dots, r_d). \tag{3}$$

Wong [28, Theorem 24.5] used the Laguerre expansions of symbols with coordinate radial arguments which are tempered functions on \mathbb{R}^2 in order to obtain a sufficient condition for the boundedness of the Weyl pseudo-differential operators in the $L^2(\mathbb{R})$ setting. Especially, this expansion is useful for the analysis of the global hypoellipticity. This was well analyzed in [16] through the Laguerre expansion of symbols with radial arguments belonging to the projective limit space

$$\mathcal{R}^{[s];p}(\mathbb{R}_+) := \lim_{h \rightarrow 0} L_{\exp(-h|\cdot|^{1/s})}^p(\mathbb{R}_+^d),$$

and the Hermite expansions in $\mathcal{S}'_{\alpha/2}(\mathbb{R}^{2d}), \alpha \geq 1$. This space is a subspace of $G'_\alpha(\mathbb{R}_+^d)$. (In [16] we also considered the Beurling case.) In this paper we show by an example that we can still have symbols, with the Laguerre expansions as in [16] which are not elements of the spaces introduced there.

Except in the case explained above, we are not able to characterize pseudo differential operators which originate from symbols in $G'_\alpha(\mathbb{R}_+^d)$ even in the case when such symbols are compactly supported in \mathbb{R}_+^d . This is an open problem. More precisely, the characterization of $\int_{\mathbb{R}_+^d} e^{-ix^2\tau} \phi(x) dx, \phi \in G'_\alpha(\mathbb{R}_+^d)$ is not obtained.

Our results relates subspaces of $G'_\alpha(\mathbb{R}_+^d)$, the corresponding subspaces of $\mathcal{S}'_{\alpha/2}(\mathbb{R}^{2d}), \alpha \geq 1$ and their duals. We introduce $G_{0,\alpha}(\mathbb{R}_+^d)$ which is a subspace of $G'_\alpha(\mathbb{R}_+^d)$. From the point of view of microlocal analyses, it seems that $G_{0,\alpha}(\mathbb{R}_+^d)$ offers a new field of investigations concerning the behaviour of functions at the corner point zero. For example, let $[0, \infty] \ni t \mapsto f(t) = \phi(t)e^{-1/t} (f(0) = 0)$, where $\phi \in G_5^5(\mathbb{R}_+)$. Then, $f \in G_{0,6}^6(\mathbb{R}_+) \subset G_6^6(\mathbb{R}_+)$ and $f \in G_5^5(\mathbb{R}_+)$. We know that $G_5^5(\mathbb{R}_+) \neq G_{0,6}^6(\mathbb{R}_+)$ but we do not know whether $f \in G_{0,5}^5(\mathbb{R}_+)$. This is also an open problem.

Assuming $\alpha > 2$, we introduce $G_{c,\alpha}(\mathbb{R}_+^d)$ as a completion of $\mathcal{D}'_{\alpha/2}(\mathbb{R}_+^d)$ in the topology of $G'_\alpha(\mathbb{R}_+^d)$ and analyse the corresponding dual pairing; $G_{c,\alpha}(\mathbb{R}_+^d)$ is a subspace of $G'_\alpha(\mathbb{R}_+^d)$, but it is not dense in $G_{0,\alpha}(\mathbb{R}_+^d)$.

We finish the paper with a result on global hypoellipticity as a consequence of the fact that a symbol is a pull-back of an $a \in G'_{0,\alpha}(\mathbb{R}_+^d)$ with additional assumptions on symbols.

2. Notations and preliminaries

Sets of positive integers, non-negative integers, integers, real and complex numbers are denoted by $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}$ and \mathbb{C} , respectively. Let $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$. We use the notation: $x^n := \prod_{j=1}^d x_j^{n_j}; \langle x \rangle = (1 + |x|^2)^{1/2}$, where $|x| := (x_1^2 + \dots + x_d^2)^{1/2}; D^n := D_x^n := D_1^{n_1} \dots D_d^{n_d}$, where $D_j^{n_j} := (-i\partial/\partial x_j)^{n_j}$ and $j = 1, \dots, d$. Given $p, q \in \mathbb{N}^d$, we set $p! := p_1! \dots p_d!; |p| := p_1 + \dots + p_d; a^p = a^{|p|}, a > 0, q \leq p$ means $q_i \leq p_i$ for $i = 1, \dots, d; p - q$ is the multi-index $(p_1 - q_1, \dots, p_d - q_d)$, whenever $q \leq p$.

Let $\alpha \geq 1$ and $a > 1$. The sequence space $s^{\alpha,a}$ is defined as a space of all complex sequences $\{a_n\}_{n \in \mathbb{N}_0^d}$ for which $\|\{a_n\}_{n \in \mathbb{N}_0^d}\|_{s^{\alpha,a}} = \sup_{n \in \mathbb{N}_0^d} |a_n| a^{|n|/\alpha} < \infty$. We define $s^\alpha = \lim_{a \rightarrow 1^+} s^{\alpha,a}$. The strong dual $(s^\alpha)'$ of s^α is the space of

all complex valued sequences $\{b_n\}_{n \in \mathbb{N}_0^d}$ such that, for each $a > 1, \|\{b_n\}_{n \in \mathbb{N}_0^d}\|_{(s^\alpha)',a} = \sum_{n \in \mathbb{N}_0^d} |b_n| a^{-|n|/\alpha} < \infty$. Its topology is generated by the system of seminorms $\|\cdot\|_{(s^\alpha)',a}$. We also point out that s^α is a nuclear (DFS) space (therefore, (DFN) space) while $(s^\alpha)'$ is an (FN) space (see [27] for these notions and consequences).

Remark concerning the notation. *Contrary to the usual notation, we use notation $\mathcal{S}'_{\alpha/2}(\mathbb{R}^d), \alpha \geq 1, \mathcal{E}^{\alpha/2}(\mathbb{R}^d), \alpha \geq 0$, and for $\alpha > 2, \mathcal{D}'^{\alpha/2}(\mathbb{R}^d)$, since we will compare these spaces with the G -type spaces given below which will have symbol α in the definition. Thus, all the spaces over \mathbb{R}^d or \mathbb{R}^{2d} will have super- (and sub-) exponent $\alpha/2$ while those ones which are defined over \mathbb{R}_+^d will be signed with α .*

Let $\alpha \geq 0, h > 0$. Recall [17],

$$\mathcal{E}^{\alpha/2}(\mathbb{R}_+^d) = \lim_{\leftarrow} \lim_{h \rightarrow \infty} \mathcal{E}_{K,h}^{\alpha/2}, \text{ where } \phi \in \mathcal{E}_{K,h}^{\alpha/2} \text{ if } \|\phi\|_{K,h} = \sup_{x \in K, q \in \mathbb{N}_0^d} \frac{|\phi^{(q)}(x)|}{h^q q!^{\alpha/2}} < \infty,$$

while, for $\alpha > 2$, the Roumieu type space $\mathcal{D}^{\alpha/2}(\mathbb{R}_+^d)$ of smooth functions equals

$$\mathcal{D}^{\alpha/2}(\mathbb{R}_+^d) = \lim_{\leftarrow} \lim_{h \rightarrow \infty} \mathcal{D}_{K,h}^{\alpha/2}.$$

$\mathcal{D}_{K,h}^{\alpha/2}$ is defined by the seminorms $\|\phi\|_{K,h}$ given above, but ϕ is supported by K ($\text{supp } \phi \subset\subset K$).

Let $\alpha \geq 1$. We recall the definition of Gelfand-Shilov type spaces [11] (see also [3, 5, 18, 21]). Let $A > 0$. Then $\mathcal{S}_{\alpha/2,A}^{\alpha/2}(\mathbb{R}^d)$ is the Banach space of all $\varphi \in C^\infty(\mathbb{R}^d)$ with the norm

$$\|\varphi\|_{\mathcal{S}_{\alpha/2,A}^{\alpha/2}}^{\alpha/2} = \sup_{p,q \in \mathbb{N}_0^d} \frac{\|x^p \partial^q \varphi(x)\|_{L^2(\mathbb{R}^d)}}{(A^{p+q} p!^{\alpha/2} q!^{\alpha/2})} < \infty. \tag{4}$$

The Gelfand-Shilov space $\mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$ is the inductive limit of $\mathcal{S}_{\alpha/2,A}^{\alpha/2}(\mathbb{R}^d)$ i.e. $\mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d) = \lim_{A \rightarrow \infty} \mathcal{S}_{\alpha/2,A}^{\alpha/2}(\mathbb{R}^d)$.

The corresponding dual space of $\mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$ is the space of ultradistributions of Roumieu type $\mathcal{S}'_{\alpha/2}(\mathbb{R}^d) = (\mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d))' = \lim_{A \rightarrow 0} (\mathcal{S}_{\alpha/2,A}^{\alpha/2}(\mathbb{R}^d))'$ (cf. [17]).

Denote by $h_n, n \in \mathbb{N}^d$ the Hermite functions which form an orthonormal basis of $L^2(\mathbb{R}^d)$. Then $\iota : \mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d) \rightarrow s^\alpha$, defined by $\iota(f) = \{\langle f, h_n \rangle\}_{n \in \mathbb{N}_0^d}$ is a continuous linear bijection between $\mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$ and s^α , cf. [18], for example. Furthermore, for each $f \in \mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$, $\sum_{n \in \mathbb{N}_0^d} \langle f, h_n \rangle h_n$ converges absolutely to f in $\mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$.

Let $\alpha \geq 1$. We recall the definition of our basic space from [13]. For $A > 0$, we denote by $G_{\alpha,A}^{\alpha,A}(\mathbb{R}_+^d)$ the space of all $\psi \in C^\infty(\mathbb{R}_+^d)$ for which the norms

$$\|\psi\|_{G_{\alpha,A,j,\alpha}}^\alpha = \sup_{p,q \in \mathbb{N}_0^d} \frac{\|t^{(p+q)/2} \partial^q \psi(t)\|_{L^2(\mathbb{R}_+^d)}}{A^{p+q} p!^{\alpha/2} q!^{\alpha/2}} + \sup_{\substack{|q| \leq j \\ |p| \leq j}} \sup_{t \in \mathbb{R}_+^d} |t^p \partial^q \psi(t)|, \quad j \in \mathbb{N}_0. \tag{5}$$

are finite. Then, the G type space $G_\alpha^\alpha(\mathbb{R}_+^d)$ is the inductive limit of the Fréchet spaces $G_{\alpha,A}^{\alpha,A}(\mathbb{R}_+^d)$ i.e. $G_\alpha^\alpha(\mathbb{R}_+^d) = \lim_{A \rightarrow \infty} G_{\alpha,A}^{\alpha,A}(\mathbb{R}_+^d)$. The corresponding dual space of $G_\alpha^\alpha(\mathbb{R}_+^d)$ is the spaces of ultradistributions of Roumieu type $G_\alpha^\alpha(\mathbb{R}_+^d)' = (G_\alpha^\alpha(\mathbb{R}_+^d))' = \lim_{A \rightarrow 0} (G_{\alpha,A}^{\alpha,A}(\mathbb{R}_+^d))'$.

Using the Laguerre orthonormal basis of $L^2(\mathbb{R}_+^d)$, by $\ell_n, n \in \mathbb{N}_0^d$ Laguerre functions are denoted, it is proved in [13, Theorem 6.1], that the mapping $\iota : G_\alpha^\alpha(\mathbb{R}_+^d) \rightarrow s^\alpha$, $\iota(f) = \{\langle f, \ell_n \rangle\}_{n \in \mathbb{N}_0^d}$ is a topological isomorphism between $G_\alpha^\alpha(\mathbb{R}_+^d)$ and s^α . Hence, the G type space inherits topological properties of the sequence space s^α . Furthermore, for each $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$, $\sum_{n \in \mathbb{N}_0^d} \langle f, \ell_n \rangle \ell_n = f$ in $G_\alpha^\alpha(\mathbb{R}_+^d)$, see [8, 13].

Let $\alpha > 2$. We recall the definitions from [24] (with $\rho = 1, \sigma = s$ in [24], and $s = \alpha/2$ here). In the Roumieu case, for any fixed $h > 0, m > 0$, by $\Gamma_{\alpha/2}^{\alpha/2,\infty}(\mathbb{R}^{2d}; h; m)$ is denoted the space of all functions $a(x, \xi) \in C^\infty(\mathbb{R}^{2d})$ such that

$$\sup_{q,p \in \mathbb{N}_0^d} \sup_{(x,\xi) \in \mathbb{R}^{2d}} \frac{|D_\xi^q D_x^p a(x, \xi)| \langle (x, \xi) \rangle^{p+q} e^{-m(|x|^{2/\alpha} + |\xi|^{2/\alpha})}}{h^{p+q} p!^{\alpha/2} q!^{\alpha/2}} < \infty.$$

Then

$$\Gamma_{\alpha/2}^{\alpha/2,\infty}(\mathbb{R}^{2d}) = \lim_{h \rightarrow \infty} \lim_{m \rightarrow 0} \Gamma_{\alpha/2}^{\alpha/2,\infty}(\mathbb{R}^{2d}; h; m).$$

Let $a \in \Gamma_{\alpha/2}^{\alpha/2,\infty}(\mathbb{R}^{2d})$. It is said that a symbol a is $\Gamma_{\alpha/2}^{\alpha/2,\infty}$ -hypoelliptic if

(i) There exist $R > 0$ and $C, m > 0$ such that

$$|a(x, \xi)| \geq Ce^{-m(|x|^{2/\alpha} + |\xi|^{2/\alpha})}, \quad (x, \xi) \in Q_R^c = \{(x, \xi) : \langle x \rangle \geq R \text{ or } \langle \xi \rangle \geq R\}. \tag{6}$$

(ii) There exists $R > 0$ such that for every $h > 0$ there exists $C > 0$ such that

$$|D_\xi^q D_x^p a(x, \xi)| \leq \frac{h^{p+q} |a(x, \xi)| p!^{\alpha/2} q!^{\alpha/2}}{\langle (x, \xi) \rangle^{p+q}}, \quad p, q \in \mathbb{N}^d, (x, \xi) \in Q_R^c. \tag{7}$$

3. Relations between G-type spaces

Let $\alpha \geq 1$. The subspace $\mathcal{S}_{\alpha/2, \text{even}}^{\alpha/2}(\mathbb{R}^d)$ of $\mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$ consists of all even functions in $\mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$: $\psi(x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_d) = \psi(x), x \in \mathbb{R}^d, j = 1, \dots, d$. A topological isomorphism between $G_\alpha^\alpha(\mathbb{R}_+^d)$ and $\mathcal{S}_{\alpha/2, \text{even}}^{\alpha/2}(\mathbb{R}^d)$,

$$G_\alpha^\alpha(\mathbb{R}_+^d) = \mathcal{S}_{\alpha/2, \text{even}}^{\alpha/2}(\mathbb{R}^d), \tag{8}$$

is established in [15] through the push forward $G_\alpha^\alpha(\mathbb{R}_+^d) \rightarrow \mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$ by the mapping v given in (2). In relation to $G_\alpha^\alpha(\mathbb{R}_+^d)$, the elements of the next space have essentially different behaviour at zero.

Let $A > 0$ and $\alpha \geq 1$; $G_{0, \alpha, A}^{\alpha, A}$ is defined as an F -space of smooth functions ϕ on \mathbb{R}_+^d such that

$$\|\phi\|_{G_{0, \alpha, A, j, \alpha}}^\alpha = \sup_{\substack{p, q \in \mathbb{N}_0^d \\ x \in \mathbb{R}_+^d}} \frac{\|\langle t \rangle^{(p+q)/2} \partial^q \phi(t)\|_{L^2(\mathbb{R}_+^d)}}{A^{p+q} p!^{\alpha/2} q!^{\alpha/2}} + \sup_{\substack{|q| \leq j \\ t \in \mathbb{R}_+^d \\ |p| \leq j}} \sup |t^p \partial^q \psi(t)|, \quad j \in \mathbb{N}_0.$$

$G_{0, \alpha}^\alpha$ is the inductive limit of the spaces $G_{0, \alpha, A}^{\alpha, A}$. It is a (DFN) space; $G_\alpha^\alpha(\mathbb{R}_+^d) \subset \mathcal{E}^{\alpha/2}(\mathbb{R}_+^d)$.

Furthermore, for $\alpha > 2$, we define $G_{c, \alpha}^\alpha(\mathbb{R}_+^d)$ as the closure of $\mathcal{D}^{\alpha/2}(\mathbb{R}_+^d)$ in $G_\alpha^\alpha(\mathbb{R}_+^d)$. Clearly, $\mathcal{D}^{\alpha/2}(\mathbb{R}_+^d)$ is a dense subspace of $G_{c, \alpha}^\alpha(\mathbb{R}_+^d)$. This gives embedding $G_{c, \alpha}^\alpha(\mathbb{R}_+^d) \hookrightarrow \mathcal{D}'^{\alpha/2}(\mathbb{R}_+^d)$. Moreover, $G_{c, \alpha}^\alpha(\mathbb{R}_+^d)$ is continuously embedded in $G_\alpha^\alpha(\mathbb{R}_+^d)$ but not densely embedded. See Example 3.3 below and the comment therein.

If $F \in G_\alpha^\alpha(\mathbb{R}_+^d)$, then its restriction on $G_{c, \alpha}^\alpha(\mathbb{R}_+^d)$ belongs to $G_{c, \alpha}^\alpha(\mathbb{R}_+^d)$. Space $G_{c, \alpha}^\alpha(\mathbb{R}_+^d)$ has a good approximation property:

Proposition 3.1. *Let $\alpha > 2$ and $u \in G_{c, \alpha}^\alpha(\mathbb{R}_+^d)$. Then there is a sequence $(u_j)_j$ in $G_{c, \alpha}^\alpha(\mathbb{R}_+^d)$, such that $u_j \rightarrow u$ in $G_{c, \alpha}^\alpha(\mathbb{R}_+^d)$.*

In particular, if u is compactly supported, then $(u_j)_j$ is a sequence in $\mathcal{D}'^{\alpha/2}(\mathbb{R}_+^d)$ which converges to u in $\mathcal{D}'^{\alpha/2}(\mathbb{R}_+^d)$.

Proof. By (8), the properties of $\mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$ are transferred to $G_\alpha^\alpha(\mathbb{R}_+^d)$ through the mapping v . If $u \in L^1(\mathbb{R}_+^d)$ and $\theta \in G_{c, \alpha}^\alpha(\mathbb{R}_+^d)$, then their dual pairing is given by $\langle u, \theta \rangle_{\mathbb{R}_+^d} = \int_{\mathbb{R}_+^d} u(\rho) \theta(\rho) d\rho_1 \dots d\rho_d$. Let $\kappa \in G_\alpha^\alpha(\mathbb{R}_+^d)$ have the properties:

$$\kappa(r) = 1, r \in B(\mathbf{0}, 1) \cap \mathbb{R}_+^d, \kappa(r) = 0, r \in CB(\mathbf{0}, 2),$$

where $CB(\mathbf{0}, 2)$ is a complement of the ball $B(\mathbf{0}, 2)$ with center at zero and radius equals 2. Put $\kappa_j(r) = \kappa(r/j), r \in \mathbb{R}_+^d, j \in \mathbb{N}$, (cut-off function). Further on, let $\phi(r) \in G_\alpha^\alpha(\mathbb{R}_+^d)$ so that it is supported by the ball $B(\mathbf{1}, 1)$ so that its integral over \mathbb{R}_+^d equals one ($\mathbf{1} = (1, \dots, 1)$). Put $\phi_j = j^d \phi(jr), r \in \mathbb{R}_+^d, j \in \mathbb{N}$ (this is a delta sequence). Let $u \in G_{c, \alpha}^\alpha(\mathbb{R}_+^d)$. Define $u_j(r) = \kappa_j(u * \phi_j(r)), r \in \mathbb{R}_+^d, j \in \mathbb{N}$. This is a sequence in $G_{c, \alpha}^\alpha(\mathbb{R}_+^d)$ which converges to u in $G_{c, \alpha}^\alpha(\mathbb{R}_+^d)$ since $\langle u_j(r), \theta(r) \rangle_{\mathbb{R}_+^d} = \langle u(r), \Phi_j(r) \rangle_{\mathbb{R}_+^d}, j \in \mathbb{N}$, where

$$\Phi_j(r) = \int_{\mathbb{R}_+^d} j^d \phi(j(r - \rho)) (\kappa(\rho/j) \theta(\rho)) d\rho_1 \dots d\rho_d \text{ converges to } \theta(r) \text{ in } G_{c, \alpha}^\alpha(\mathbb{R}_+^d), j \rightarrow \infty.$$

The particular part of lemma is clear. □

We recall the Faà di Bruno formula for the partial derivatives of the composition of functions [7, Corollary 2.10]:

$$\frac{\partial^q}{\partial x^q} \sigma(g(x)) = \sum_{r=0}^q \frac{q!}{k_1! \dots k_q!} \sigma^{(r)}(g(x)) \prod_{i=1}^q \left(\frac{g^{(i)}(x)}{i!} \right)^{k_i}, \quad q \in \mathbb{N}_0 \tag{9}$$

where $r = k_1 + \dots + k_q$ and the sum is over all nonnegative integers k_1, \dots, k_q for which $k_1 + 2k_2 + \dots + qk_q = q$.

Example 3.2. Let $\phi \in G_{0,5}^5(\mathbb{R}_+)$ so that $\phi(t) = 1, t \in [0, 1]$. Our aim in this example is to show that

$$f(t) = e^{-1/t} \phi(t) \in G_{0,6}^6(\mathbb{R}_+) \cap G_5^5(\mathbb{R}_+). \tag{10}$$

We use Faà di Bruno formula (9), in the one dimensional case. Since $\left(\frac{(t^{-1})^{(i)}}{i!}\right)^{k_i} = (-1)^{ik_i} t^{-(i+1)k_i}$, it follows that

$$\prod_{i=1}^q \left(\frac{(t^{-1})^{(i)}}{i!}\right)^{k_i} = (-1)^q t^{-q-r}.$$

With this, we have

$$(e^{-1/t})^{(q)} = q! \sum_{r=0}^q \frac{e^{-1/t}}{k_1! \dots k_q!} (-1)^q t^{-q-r}.$$

By Stirling’s formula, and the fact that $e^{-1/t} t^{-q-r}$ takes its maximum at $t = 1/(q+r)$, we have

$$\max_{t \in [0,1]} \frac{e^{-1/t}}{t^{q+r}} = e^{-q-r} (q+r)^{q+r} \sim \frac{e^{-q-r} (q+r)! e^{q+r}}{\sqrt{2\pi(q+r)}} \leq \frac{2^{q+r} q! r!}{\sqrt{2\pi(q+r)}},$$

we have,

$$(e^{-1/t})^{(q)} \leq q!^3 \sum_{r=0}^q \frac{2^{q+r}}{k_1! \dots k_q! \sqrt{2\pi(q+r)}} \leq c^q q!^3, \quad t \in [0, 1].$$

By the similar calculation

$$|t^{q/2} (e^{-1/t})^{(q)}| \leq q! \sum_{r=0}^q \frac{(q/2+r)^{q/2+r} e^{-q/2-r}}{k_1! \dots k_q! \sqrt{2\pi(q/2+r)}} \leq q! \sum_{r=1}^q \frac{(q/2+2q/2)! 2^{q/2+r}}{k_1! \dots k_q! \sqrt{2\pi(q/2+r)}} \leq c_2^q q!^{5/2},$$

where we have used that $q! \leq 2^q (q/2)!^2 \leq 2^q q!$. Now, one can find that for $\alpha = 6$ there exists $h > 0$ such that

$$\sup_{t \in [0,1]} \frac{|t^{(p+q)/2} (e^{-1/t})^{(q)}|}{h^{p+q} p!^3 q!^3} < \infty. \tag{11}$$

and that for $\alpha = 5$, there exists $h > 0$ such that

$$\sup_{t \in [0,1]} \frac{|t^{(p+q)/2} (e^{-1/t})^{(q)}|}{h^{p+q} p!^{5/2} q!^{5/2}} < \infty. \tag{12}$$

Thus, by (11), $e^{-1/t} \phi(t) \in G_{0,6}^6(\mathbb{R}_+)$. On the other hand, by (12), we have $e^{-1/t} \phi(t) \in G_5^5(\mathbb{R}_+)$. As we noted, we are not able to prove that $f \in G_{0,5}^5(\mathbb{R}_+)$.

Example 3.3. The space $G_{c,\alpha}^\alpha$, $\alpha > 2$, is not dense in $G_{0,\alpha}^\alpha$, $\alpha > 2$. Let $\phi \in G_{0,\alpha}^\alpha(\mathbb{R}_+)$ so that $\phi(t) = 1, t \in [0, 1]$. There does not exist a sequence $(\psi_n)_n$ in $\mathcal{D}^{\alpha/2}(\mathbb{R})$ such that $\psi_n \rightarrow \phi, n \rightarrow \infty$, in $G_{0,\alpha}^\alpha$ because $\sup_{t \in [0,1]} |\phi(t) - \psi_n(t)| = 1, n \in \mathbb{N}$.

The proof of the next proposition is based on the multivariate Faà di Bruno formula (9). First, we formulate the one-dimensional case with $g(x) = 2x^2$. Note that i in (9) can take only values 1 and 2. This further implies $k_1 + k_2 = k$ and $k_1 + 2k_2 = p$, that is, $k_1 = 2k - p, k_2 = p - k$ and $k \geq [p/2]$. This, with $j = p - k$, gives

$$\frac{\partial^p}{\partial x^p} \sigma(2x^2) = \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} \frac{p!}{j!(p-2j)!} \sigma^{(p-j)}(2x^2)(4x)^{p-2j} 2^j, \tag{13}$$

where $[x]$ is the largest integer $\leq x$. With the help of multi-indices, the identity (13) in the $2d$ -dimensional case becomes

$$\begin{aligned} \frac{\partial_{x,\xi}^{p,q}}{\partial x^p \partial \xi^q} \sigma(2x^2 + 2\xi^2) &= \frac{\partial_{x,\xi}^{p,q}}{\partial x^p \partial \xi^q} \sigma(2x_1^2 + 2\xi_1^2, \dots, 2x_d^2 + 2\xi_d^2) \\ &= p!q! \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{q}{2} \rfloor} \frac{\sigma^{(p+q-j-i)}(2x^2 + 2\xi^2)}{j!i!(p-2j)!(q-2i)!} (4x)^{p-2j} 2^j (4\xi)^{q-2i} 2^i, \quad x, \xi \in \mathbb{R}^d, \end{aligned} \tag{14}$$

where we use notation

$$\sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{q}{2} \rfloor} = \sum_{j_1=0}^{\lfloor \frac{p_1}{2} \rfloor} \sum_{i_1=0}^{\lfloor \frac{q_1}{2} \rfloor} \dots \sum_{j_d=0}^{\lfloor \frac{p_d}{2} \rfloor} \sum_{i_d=0}^{\lfloor \frac{q_d}{2} \rfloor}.$$

Proposition 3.4. Let $\alpha \geq 2$ and $\psi \in G_{0,\alpha}^\alpha(\mathbb{R}_+^d)$ and let $\varphi(x, \xi) = w^* \psi(x, \xi) = \psi(2x^2 + 2\xi^2) = \psi(2x_1^2 + 2\xi_1^2, \dots, 2x_d^2 + 2\xi_d^2)$, (cf. (3)). Then $\varphi \in S_{\alpha/2}^{\alpha/2}(\mathbb{R}^{2d})$ and the mapping

$$w^* : G_\alpha^\alpha(\mathbb{R}_+^d) \rightarrow S_{\alpha/2}^{\alpha/2}(\mathbb{R}^{2d}), \quad \psi \mapsto w^* \psi = \psi \circ w = \varphi.$$

is continuous.

Proof. We use the change of variables $t_i = 2x_i^2 + 2\xi_i^2, i = 1, \dots, d$ and the inequalities $p!q! \leq (p+q)! \leq 2^{p+q} p!q!, p, q \in \mathbb{N}_0^d (0! = 1)$,

$$|\langle x_1 \rangle^{p_1} \langle \xi_1 \rangle^{q_1} \dots \langle x_d \rangle^{p_d} \langle \xi_d \rangle^{q_d}| \leq (1 + 2x_1^2 + 2\xi_1^2)^{p_1/2+q_1/2} \dots (1 + 2x_d^2 + 2\xi_d^2)^{p_d/2+q_d/2},$$

$$|x^{q_1-2j} \xi^{q_2-2i}| \leq \langle x \rangle^{q_1-2j} \langle \xi \rangle^{q_2-2i} \leq 2^{q_1+q_2-2j-2i} \langle t \rangle^{\frac{q_1+q_2-2j-2i}{2}}, \quad \langle x \rangle^{m_1} \langle \xi \rangle^{m_2} \leq 2^{m_1+m_2} \langle t \rangle^{(m_1+m_2)/2}.$$

It is enough to show that there exist $A > 0$ and $C > 0$ such that for given $h > 0$ (which depends on ψ),

$$\sup_{\substack{p,q \in \mathbb{N}_0^d \\ x,\xi \in \mathbb{R}^d}} \frac{|\partial_x^{q_1} \partial_\xi^{q_2} \varphi(x, \xi)| (1 + |x|^2)^{m_1/2} (1 + |\xi|^2)^{m_2/2}}{A^{q_1+q_2+m_1+m_2} (q_1 + q_2)!^{\alpha/2} (m_1)!^{\alpha/2} (m_2)!^{\alpha/2}} \leq C \sup_{\substack{p,q \in \mathbb{N}_0^d \\ t \in \mathbb{R}_+^d}} \frac{|\langle t \rangle^{(p+q)/2} \partial^q \psi(t)|}{h^{p+q} p!^{\alpha/2} q!^{\alpha/2}}.$$

It should be noticed that in the successive calculation we obtain the expression $(|x|^2 + |\xi|^2)^{-j-i}$ which can not be controlled by other factors. Therefore, the assumption that $\psi \in G_{0,\alpha}^\alpha(\mathbb{R}_+)$ is needed.

Let $c_{A,m_1,m_2}^{q_1,q_2} = A^{q_1+q_2+m_1+m_2} (q_1 + q_2)!^{\alpha/2} (m_1)!^{\alpha/2} (m_2)!^{\alpha/2}$. Using (14) and the change of variable $t = 2|x|^2 + 2|\xi|^2$ (also written as $t = 2x^2 + 2\xi^2$), we conclude, with the use of suitable constants

$$\|\varphi\|_{S_{A,\alpha/2}}^{\alpha/2} = \sup_{\substack{q_1,q_2,m_1,m_2 \in \mathbb{N}_0^d \\ x,\xi \in \mathbb{R}^d}} \frac{|\partial_x^{q_1} \partial_\xi^{q_2} \varphi(x, \xi)| \langle x \rangle^{m_1} \langle \xi \rangle^{m_2}}{c_{A,m_1,m_2}^{q_1,q_2}}$$

$$\begin{aligned} &\leq \sup_{\substack{q_1, q_2, m_1, m_2 \in \mathbb{N}_0^d \\ t \in \mathbb{R}_+^d}} \sum_{j=0}^{\lfloor \frac{q_1}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{q_2}{2} \rfloor} \frac{q_1! q_2! |\psi^{(q_1+q_2-i)}(t)|}{c_{A, m_1, m_2}^{q_1, q_2} j! i! (q_1 - 2j)! (q_2 - 2i)!} \langle t \rangle^{\frac{q_1+q_2-2i-2j+m_1+m_2}{2}} 2^{2(q_1+q_2)-3(i+j)+m_1+m_2} \\ &\leq \left(\frac{C_1}{A}\right)^{q_1+q_2+m_1+m_2} \sup_{\substack{q_1, q_2, m_1, m_2 \in \mathbb{N}_0^d \\ t \in \mathbb{R}_+^d}} \sum_{j=0}^{\lfloor \frac{q_1}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{q_2}{2} \rfloor} \frac{|\psi^{(q_1+q_2-j-i)}(t)| \langle t \rangle^{\frac{q_1+q_2-j-i}{2}} \langle t \rangle^{\frac{m_1+m_2}{2}} \langle t \rangle^{-\frac{j+i}{2}}}{(q_1+q_2-j-i)!^{\alpha/2} (m_1+m_2)!^{\alpha/2}} \\ &\frac{(q_1+q_2-j-i)! (i+j)!}{(i+j)!^{\alpha/2} (q_1+q_2-j-i)!} \leq \left(\frac{C_2(h)}{A}\right)^{q_1+q_2+m_1+m_2} \sup_{\substack{q_1, q_2, m_1, m_2 \in \mathbb{N}_0^d \\ t \in \mathbb{R}_+^d}} \sum_{j=0}^{\lfloor \frac{q_1}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{q_2}{2} \rfloor} \frac{C_2^{i+j}}{(i+j)^{(\alpha/2-1)}} \end{aligned}$$

This implies that for given $h > 0$ there exist $A, C > 0$ such that for every $j \in \mathbb{N}_0$,

$$\|\varphi\|_{S_{A, \alpha/2}}^{\alpha/2} \leq C \|\psi\|_{G_{0, h, j, \alpha}}^{\alpha} \quad (\text{see Section 2}).$$

This completes the proof of the proposition. \square

We note that this proposition also follows from our result related to (8). Here we obtain this result by quite different methods.

4. Weyl pseudo-differential operators with radial symbols and hypoellipticity

Let $\alpha \geq 1$ and $f, g \in S_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$. We refer to [6] for the Wigner transform of $f, g \in S_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$. It is well known that the bilinear mapping $(f, g) \mapsto W(f, g), S_{\alpha/2}^{\alpha/2}(\mathbb{R}^d) \times S_{\alpha/2}^{\alpha/2}(\mathbb{R}^d) \rightarrow S_{\alpha/2}^{\alpha/2}(\mathbb{R}^{2d})$ is continuous.

The Weyl pseudo-differential operator with a symbol $\tilde{a} \in S_{\alpha/2}^{\alpha/2}(\mathbb{R}^{2d})$ defined by (1) is a continuous and linear mapping from $S_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$ into $S_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$ (see [20, Theorem 2]). Using the Hermite expansions of elements from $S_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$, we obtain

$$W(f, g) = \sum_{(m, k) \in \mathbb{N}_0^{2d}} \langle f, h_m \rangle \langle \tilde{g}, h_k \rangle W(h_m, h_k),$$

where the sum converges absolutely in $S_{\alpha/2}^{\alpha/2}(\mathbb{R}^{2d})$.

We introduce $\mathcal{A}_{\alpha}^{\alpha}(\mathbb{R}_+^d)$ as the space of all locally integrable functions $a(r), r \in \mathbb{R}_+^d$, with the properties

$$a(r) \in G_{\alpha}^{\alpha}(\mathbb{R}_+^d) \text{ and } \tilde{a}(x, \xi) = a(2x_1^2 + 2\xi_1^2, \dots, 2x_d^2 + 2\xi_d^2) \in S_{\alpha/2}^{\alpha/2}(\mathbb{R}^{2d}).$$

(Here we use the fact that the composition of a locally integrable function and a smooth function is locally integrable.)

We considered in [16] the symbols $a_w(x, \xi) := a(2x_1^2 + 2\xi_1^2, \dots, 2x_d^2 + 2\xi_d^2), a \in \mathcal{R}^{(\alpha); p}(\mathbb{R}_+^d)$ (see introduction). We obtained the characterisation of a in terms of the growths of their Laguerre coefficients for which the Weyl quantisation of a_w extends to a well-defined and continuous operator $a_w^W : S_{\alpha/2}^{\alpha/2}(\mathbb{R}^d) \rightarrow S_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$.

Example 4.1. Let $\alpha \geq 2$ and $s = \alpha/2$. With the following example, in the case when $d = 1$, we show that $\mathcal{R}^{(s); p}(\mathbb{R}_+)$ is a proper subset of $\mathcal{A}_{\alpha}^{\alpha}(\mathbb{R}_+)$. Let $f(x) = e^x, x \in [n, n + e^{-n}), n \in \mathbb{N}, n > N_0$ and $f(x) = 0$ in the rest of \mathbb{R} . Let $\varphi \in G_{\alpha}^{\alpha}(\mathbb{R}_+^d)$. We use the fact that $\varphi(x) \leq Ce^{h|x|^{2/\alpha}}, x > N_0$, for suitable $C, h, N_0 > 0$. With this, we have

$$\left| \int_{\mathbb{R}_+} f(x) \varphi(x) dx \right| \leq C \sum_{n > N_0}^{\infty} \frac{1}{e^{hn^{2/\alpha}}} < \infty,$$

so that $f \notin \mathcal{R}^{(\alpha/2); p}(\mathbb{R}_+^d)$ since $\int_{\mathbb{R}_+^d} e^{px^{2/\alpha} - hp x^{2/\alpha}} dx = \infty$ for $h = 1$. Thus, $f \in \mathcal{A}_{\alpha}^{\alpha}(\mathbb{R}_+^d) \setminus \mathcal{R}^{(\alpha/2); p}(\mathbb{R}_+^d)$.

Because of that, the next proposition has a sense although its proof is the same as in [16]; so, the proof is skipped.

Proposition 4.2. Let $a(r) \in \mathcal{A}_\alpha^\alpha(\mathbb{R}_+^d)$.

(i) The Weyl pseudo-differential operator defined as

$$(\bar{a}^W(x, D)\varphi)(\psi) = \langle \bar{a}^W(x, D)\varphi, \psi \rangle = \langle a(2x^2 + 2\xi^2), W(\varphi, \psi)(x, \xi) \rangle, \quad \varphi, \psi \in \mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$$

is a continuous bilinear mapping $\mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d) \times \mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d) \rightarrow \mathbb{C}$. It is given by $(\varphi, \psi) \mapsto \sum_{k \in \mathbb{N}_0^d} \varphi_k \psi_k a_k$, where $\varphi(x) = \sum_{k \in \mathbb{N}_0^d} \langle \varphi, h_k \rangle h_k(x)$, $\psi(x) = \sum_{k \in \mathbb{N}_0^d} \langle \psi, h_k \rangle h_k(x)$ and $a(\rho) = \sum_{k \in \mathbb{N}_0^d} a_k \ell_k(\rho)$, $a_k = (2\pi)^{d/2} (-1)^{|k|} 2^{-d} \langle a, \ell_k \rangle$.

(ii) It can be continuously extended as a bilinear mapping over $\mathcal{S}'_{\alpha/2}(\mathbb{R}^d) \times \mathcal{S}'_{\alpha/2}(\mathbb{R}^d)$, with values in \mathbb{C} .

4.1. Hypocoellipticity

We continue to assume $\alpha > 2$. We employ the notation $\mathcal{H}^\alpha(\mathbb{R}_+^d)$ for a subspace of $G_{0,\alpha}^\alpha(\mathbb{R}_+^d)$ consisting of smooth functions $a(r)$, $r \in \mathbb{R}_+^d$, which satisfy the next conditions: There exists $R > 0, h > 0, C > 0, m > 0$ such that

$$|a(r)| \geq Ce^{-m|r|^{1/\alpha}}, \quad |r^{p/2} \partial_r^p a(r)| \leq h^{|p|} |a(r)| p!^{\alpha/2}, \quad p \in \mathbb{N}_0^d, \quad r \in Q_R^c. \tag{15}$$

Proposition 4.3. Let $\tilde{a}(x, \xi) = a(2x^2 + 2\xi^2)$, $x, \xi \in \mathbb{R}^d$, where a function a belongs to $\mathcal{H}^\alpha(\mathbb{R}_+^d)$. Then the Weyl pseudo-differential operator $\tilde{a}^W(x, D)$ is $\Gamma_{\alpha/2}^{\alpha/2, \infty}$ -hypoelliptic.

Proof. We skip the part in which we prove that $\tilde{a}(x, \xi) \in \Gamma_{\alpha/2}^{\alpha/2, \infty}(\mathbb{R}^{2d})$.

Condition (6) is clear. We will prove (7). Let $(x, \xi) \in B_R^c$. To prove (7), we use our calculation in Proposition 3.4 (without the multiplication with powers of x and ξ). Let $c_A^{q_1, q_2} = A^{q_1 + q_2} (q_1 + q_2)!^{\alpha/2}$.

$$\begin{aligned} & \sup_{\substack{q_1, q_2 \in \mathbb{N}_0^d \\ x, \xi \in \mathbb{R}^d}} \frac{|\partial_x^{q_1} \partial_\xi^{q_2} \tilde{a}(x, \xi)|}{c_A^{q_1, q_2}} \leq \\ & \sup_{\substack{q_1, q_2 \in \mathbb{N}_0^d \\ t \in \mathbb{R}_+^d}} \sum_{j=0}^{\lfloor \frac{q_1}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{q_2}{2} \rfloor} \frac{q_1! q_2! |a^{(q_1 + q_2 - i - j)}(t)|}{c_A^{q_1, q_2} j! i! (q_1 - 2j)! (q_2 - 2i)!} \langle t \rangle^{\frac{q_1 + q_2 - 2i - 2j}{2}} 2^{2(q_1 + q_2) - 3(i + j)} \leq \\ & \left(\frac{C_1}{A}\right)^{q_1 + q_2} \sup_{\substack{q_1, q_2 \in \mathbb{N}_0^d \\ t \in \mathbb{R}_+^d}} \sum_{j=0}^{\lfloor \frac{q_1}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{q_2}{2} \rfloor} \frac{(q_1 + q_2 - j - i)! (i + j)! |a^{(q_1 + q_2 - j - i)}(t)| \langle t \rangle^{\frac{q_1 + q_2 - j - i}{2}} \langle t \rangle^{-\frac{j+i}{2}}}{(q_1 + q_2 - j - i)!^{\alpha/2} (i + j)!^{\alpha/2} (q_1 + q_2 - j - i)!} \\ & \leq \left(\frac{C_2(h)}{A}\right)^{q_1 + q_2} a(r) \sup_{\substack{q_1, q_2, m_1, m_2 \in \mathbb{N}_0^d \\ t \in \mathbb{R}_+^d}} \sum_{j=0}^{\lfloor \frac{q_1}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{q_2}{2} \rfloor} \frac{c_2^{i+j}}{(i + j)^{(\alpha/2 - 1)}} \end{aligned}$$

Now assumption (15) and the choice of $A > 0$ imply the assertion. \square

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