



Modulus of continuity of normal derivative of a harmonic functions at a boundary point

Miloš Arsenović^a, Miodrag Mateljević^a

^aFaculty of mathematics, University of Belgrade, Studentski Trg 16, 11000 Belgrade, Serbia

Abstract. We give sufficient conditions which ensure that harmonic extension $u = P[f]$ to the upper half space $\{(x, y) \mid x \in \mathbb{R}^n, y > 0\}$ of a function $f \in L^p(\mathbb{R}^n)$ satisfies estimate $\frac{\partial u}{\partial y}(x, y) \leq C\omega(y)/y$ for every x in $E \subset \mathbb{R}^n$, where ω is a majorant. The conditions are expressed in terms of behaviour of the Riesz transforms $R_j f$ of f near points in E . We briefly investigate related questions for the cases of harmonic and hyperbolic harmonic functions in the unit ball.

1. Introduction and preliminaries

It is easily seen that if the majorant of $\varphi(x) \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is ω , and if $u(x, y)$ is the harmonic extension of φ to the upper half space, then for each $y > 0$ the function $\varphi_y(x) = u(x, y)$ has the same majorant as φ . This need not be true for functions $\varphi_x(y) = u(x, y)$ defined on $[0, +\infty)$. For example, if φ is Lipschitz continuous it is not necessarily true that u is also Lipschitz continuous. Therefore information on the vertical derivative of u is of interest in obtaining results on modulus of continuity of $u(x, y)$. We point out that in that respect hyperbolic Laplacian has better properties regarding preservation of Lipschitz continuity, see [5]. In addition, information on behaviour of normal derivatives is relevant when studying mappings which are at the same time harmonic and quasiconformal, see [4] for the case when the boundary is not flat.

In the case $n = 1$ important role is played by the harmonic conjugate of u and by the Hilbert transform of φ . In our general case the corresponding role is played by a conjugate system of harmonic functions, see (4) and by the Riesz transforms R_j , which are multi dimensional analogues of Hilbert transform.

We denote the upper half space by $\mathbb{H}^{n+1} = \{(x, y) \mid x \in \mathbb{R}^n, y > 0\}$, the boundary of \mathbb{H}^{n+1} is identified with \mathbb{R}^n . The surface measure of the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is $n\omega_n$, where ω_n is the volume of the unit ball \mathbb{B}^n in \mathbb{R}^n . The Poisson kernel for the upper half space is

$$P(x, y) = P_y(x) = c_n \frac{y}{(y^2 + |x|^2)^{(n+1)/2}}, \quad x \in \mathbb{R}^n, \quad y > 0$$

where

$$c_n = \Gamma\left(\frac{n+1}{2}\right) \pi^{-\frac{n+1}{2}}.$$

2020 *Mathematics Subject Classification.* Primary 42B15; Secondary 42B30.

Keywords. Poisson kernel; Harmonic functions; Modulus of continuity.

Received: 25 January 2023; Accepted: 03 April 2023

Communicated by Zoran Rakić and Mića Stanković

Email addresses: arsenovic@matf.bg.ac.rs (Miloš Arsenović), miodrag@matf.bg.ac.rs (Miodrag Mateljević)

The harmonic extension of a function φ on \mathbb{R}^n to \mathbb{H}^{n+1} is

$$P[\varphi](x, y) = c_n \int_{\mathbb{R}^n} \varphi(t) \frac{y}{(y^2 + |x - t|^2)^{(n+1)/2}} dt$$

Let $R_j, 1 \leq j \leq n$, be the Riesz operators. They are defined by the following formula

$$R_j f(x) = \lim_{\varepsilon \rightarrow 0} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}} \int_{|y| \geq \varepsilon} \frac{y_j}{|y|^{n+1}} f(x - y) dy, \quad f \in L^p(\mathbb{R}^n), \quad 1 \leq p < \infty.$$

These operators are bounded linear operators on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

We say that a function $\omega : [0, +\infty) \rightarrow \mathbb{R}$ is a majorant if it is continuous, concave and increasing on $[0, +\infty)$, strictly positive on $(0, +\infty)$ and $\omega(0) = 0$.

Lemma 1.1. *Let ω be a majorant. Set $M_\beta = (\beta - 1)^{-1} + (\beta - 2)^{-1}$, where $\beta > 2$. Then*

$$\int_1^\infty \frac{\omega(y s)}{s^\beta} ds \leq M_\beta \omega(y), \quad y > 0. \tag{1}$$

Proof. Set, for $C > 0$ and $t > 0$, $\omega_{C,t}(x) = Cx$ if $0 \leq x \leq t$ and $\omega_{C,t}(x) = Ct$ if $x > t$. Then $\omega_{C,t}$ is a majorant and we have

$$\int_1^\infty \frac{\omega_{C,t}(y s)}{s^\beta} ds = Ct \int_1^\infty \frac{ds}{s^\beta} = \frac{Ct}{\beta - 1} = \frac{1}{\beta - 1} \omega(y), \quad y \geq t.$$

If $0 < y < t$, then we have

$$\begin{aligned} \int_1^\infty \frac{\omega_{C,t}(y s)}{s^\beta} ds &= \int_1^{t/y} \frac{\omega_{C,t}(y s)}{s^\beta} ds + \int_{t/y}^\infty \frac{\omega_{C,t}(y s)}{s^\beta} ds \\ &= Cy \int_1^{t/y} \frac{ds}{s^{\beta-1}} + Ct \int_{t/y}^\infty \frac{ds}{s^\beta} \\ &= Cy \frac{1}{\beta - 2} \left(1 - \frac{y^{\beta-2}}{t^{\beta-2}} \right) + Ct \frac{1}{\beta - 1} \frac{y^{\beta-1}}{t^{\beta-1}} \\ &\leq Cy \frac{1}{\beta - 2} + Cy \frac{1}{\beta - 1} \frac{y^{\beta-2}}{t^{\beta-2}} \\ &< \left(\frac{1}{\beta - 2} + \frac{1}{\beta - 1} \right) Cy = M_\beta \omega(y). \end{aligned}$$

We proved (1) for $\omega = \omega_{C,t}$, clearly (1) also holds for functions of the form $\omega_{C_1,t_1} + \dots + \omega_{C_n,t_n}$, let us call them polygonal majorants. For arbitrary majorant ω there is an increasing sequence ω_n of polygonal majorants ω_n which converges pointwise to ω . Then, by the Monotone Convergence Theorem and already proved estimate (1) for polygonal majorants, we have

$$\int_1^\infty \frac{\omega(y s)}{s^\beta} ds = \lim_{n \rightarrow \infty} \int_1^\infty \frac{\omega_n(y s)}{s^\beta} ds \leq \lim_{n \rightarrow \infty} M_\beta \omega_n(y) = M_\beta \omega(y).$$

□

The case $\beta = 3$ is the only one that we need below.

2. An auxiliary result

Lemma 2.1. Let $\varphi \in L^p(\mathbb{R}^n)$ for some $1 \leq p \leq \infty$. Assume

$$|\varphi(t) - \varphi(x^0)| \leq \omega(|t - x^0|), \quad t \in \mathbb{R}^n \tag{2}$$

for some $x^0 \in \mathbb{R}^n$ and some majorant ω . Then the harmonic extension $g = P[\varphi]$ of φ satisfies the following estimate:

$$\left| \frac{\partial g}{\partial x_j}(x^0, y) \right| \leq C(n) \frac{\omega(y)}{y}, \quad 0 < y < +\infty, \quad 1 \leq j \leq n. \tag{3}$$

Proof. For all $(x, y) \in \mathbb{H}^{n+1}$ and all $j = 1, \dots, n$, we have

$$\begin{aligned} \frac{\partial g}{\partial x_j}(x, y) &= c_n \int_{\mathbb{R}^n} \varphi(t) \frac{\partial}{\partial x_j} \frac{y}{(y^2 + |x - t|^2)^{(n+1)/2}} dt \\ &= -(n + 1)c_n \int_{\mathbb{R}^n} \varphi(t) y \frac{x_j - t_j}{(y^2 + |x - t|^2)^{\frac{n+3}{2}}} dt \\ &= -(n + 1)c_n \int_{\mathbb{R}^n} [\varphi(t) - \varphi(x)] y \frac{x_j - t_j}{(y^2 + |x - t|^2)^{\frac{n+3}{2}}} dt. \end{aligned}$$

The last equality follows from the observation that $x_j - t_j$ is an odd function of the variable $x - t$. Therefore, using spherical coordinates centered at x^0 , we obtain

$$\begin{aligned} \left| \frac{\partial g}{\partial x_j}(x^0, y) \right| &\leq (n + 1)c_n y \int_{\mathbb{R}^n} \omega(|x^0 - t|) \frac{|x^0 - t|}{(y^2 + |x^0 - t|^2)^{\frac{n+3}{2}}} dt \\ &= n(n + 1)c_n \omega_n y \int_0^\infty \omega(r) \frac{r^n dr}{(y^2 + r^2)^{\frac{n+3}{2}}} \\ &= n(n + 1)c_n \omega_n \frac{1}{y} \int_0^\infty \frac{\omega(ys)s^n}{(1 + s^2)^{\frac{n+3}{2}}} ds \end{aligned}$$

Let us denote the above integral by $I(y)$. Then we have

$$\begin{aligned} I(y) &= \int_0^1 \frac{\omega(ys)s^n}{(1 + s^2)^{\frac{n+3}{2}}} ds + \int_1^\infty \frac{\omega(ys)s^n}{(1 + s^2)^{\frac{n+3}{2}}} ds \\ &\leq \int_0^1 \frac{\omega(y)s^n}{(1 + s^2)^{\frac{n+3}{2}}} ds + \int_1^\infty \frac{\omega(ys)}{s^3} ds \\ &\leq (1 + M_3)\omega(y) \end{aligned}$$

and this proves desired estimate (3). \square

The proof shows that one can take $C(n) = 5n(n + 1)c_n \omega_n / 2$.

3. Main result

We recall that a system of harmonic functions $u_j, 0 \leq j \leq n$, on \mathbb{H}^{n+1} is called a conjugate system if it satisfies the following system of equations

$$\sum_{j=0}^n \frac{\partial u_j}{\partial x_j} = 0, \quad \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \tag{4}$$

where $x_0 = y$, see [9]. When $n = 2$ this reduces to the classical Cauchy - Riemann equations. Given a function $\varphi = \varphi(x_0, x_1, \dots, x_n)$ harmonic in \mathbb{H}^{n+1} one gets a conjugate system by setting $u_j = \partial\varphi/\partial x_j$, $0 \leq j \leq n$. Conversely, given a conjugate system u_j , $0 \leq j \leq n$ of harmonic functions in \mathbb{H}^{n+1} , there is a (unique up to an additive constant) function φ on \mathbb{H}^{n+1} such that $u_j = \partial\varphi/\partial x_j$ for $j = 0, 1, \dots, n$.

The above system allows one to infer estimates of $\partial u_0/\partial x_0 = \partial u_0/\partial y$ from the estimates of $\partial u_j/\partial x_j$ for $1 \leq j \leq n$, this is how one proves the following proposition.

Proposition 3.1. *Let ω be a majorant, $E \subset \mathbb{R}^n \cong \partial\mathbb{H}^{n+1}$ and let f_j , $0 \leq j \leq n$, be a system of conjugate functions on \mathbb{H}^{n+1} . Assume*

$$\left| \frac{\partial f_j}{\partial x_j}(x, y) \right| \leq \frac{\omega(y)}{y} \quad x \in E, \quad y > 0, \quad 1 \leq j \leq n.$$

Then we have

$$\left| \frac{\partial f_0}{\partial y}(x, y) \right| \leq n \frac{\omega(y)}{y}, \quad x \in E, \quad y > 0.$$

Theorem 3.2. *Let $f \in L^p(\mathbb{R}^n)$ for some $1 < p < \infty$ and set $f_j = R_j f$ for $1 \leq j \leq n$. Let $u = P[f]$ be the harmonic extension of f to the upper half space \mathbb{H}^{n+1} . Let ω be a majorant and let E be a subset of $\mathbb{R}^n \cong \partial\mathbb{H}^{n+1}$. Assume*

$$|f_j(t) - f_j(x)| \leq \omega(|t - x|), \quad x \in E, \quad t \in \mathbb{R}^n, \quad 1 \leq j \leq n. \tag{5}$$

Then there is a constant $C = C(n)$ such that

$$\left| \frac{\partial u}{\partial y}(x, y) \right| \leq C(n) \frac{\omega(y)}{y}, \quad y > 0, \quad x \in E. \tag{6}$$

Proof. Let $u_0 = u$ and $u_j = P[f_j]$ for $1 \leq j \leq n$. The system $(u_j)_{j=0}^n$ is a conjugate system of harmonic functions. By Lemma 2.1 and our assumptions we have

$$\left| \frac{\partial u_j}{\partial x_j}(x, y) \right| \leq C(n) \frac{\omega(y)}{y}, \quad y > 0, \quad x \in E, \quad 1 \leq j \leq n.$$

Now the above proposition gives the desired estimate. \square

The proof shows that one can take $C = 5n^2(n + 1)c_n\omega_n/2$.

4. The unit ball setting: further results and remarks

Let $P_{\mathbb{B}}$ denote harmonic Poisson kernel for the unit ball \mathbb{B}^n and also the corresponding extension operator with the same kernel. Let Λ_α denote the class of Hölder continuous function with exponent $0 < \alpha \leq 1$ and set $\text{Lip} = \Lambda_1$. It is known that $P_{\mathbb{B}}$ maps $\Lambda_\alpha(\mathbb{S}^{n-1})$ into $\Lambda_\alpha(\mathbb{B}^n)$ whenever $0 < \alpha < 1$. However if $f \in \text{Lip}(\mathbb{S}^{n-1})$, then in general $P[f]$ is not in $\text{Lip}(\mathbb{B}^n)$.

It is natural to consider the corresponding question for the hyperbolic Poisson kernel P_h for the unit ball and the corresponding extension operator.

Problem 4.1. *Are the partial derivatives of $P_h[f]$ bounded for every f in $\text{Lip}(\mathbb{S}^{n-1})$?*

The answer is positive; see [1, 5]. More precisely, if $f \in \text{Lip}(\mathbb{S}^{n-1})$, then $P_h[f]$ is in $\text{Lip}(\mathbb{B}^n)$. This is not true in the standard (euclidean) harmonic case.

Let us introduce needed terminology and notation. If $x_0 \in G \subset \mathbb{R}^n$ is not an isolated point of G and $0 < \alpha \leq 1$ we introduce, for $f : G \rightarrow \mathbb{R}^m$,

$$H_\alpha f(x_0) = \limsup_{G \ni x \rightarrow x_0} |f(x) - f(x_0)|/|x - x_0|^\alpha$$

and write $Lf(x_0)$ instead of $H_1 f(x_0)$. We say that $f : G \rightarrow \mathbb{R}^m$ is locally Hölder (α -Hölder) continuous at $x_0 \in G$ if $H_\alpha f(x_0) < \infty$, for $\alpha = 1$ we use term locally Lipschitz continuous at x_0 .

The following results appears in [7]:

Theorem 4.2 ([7]). Assume $f : \mathbb{S}^{n-1} \rightarrow \mathbb{R}^n$ is locally Lipschitz continuous at $x_0 \in \mathbb{S}$, $f \in L^\infty(\mathbb{S}^{n-1})$ and set $h = P_{\mathbb{B}}[f]$. Then

S1)

$$|h'(rx_0)T| \leq M \tag{2'}$$

for every $0 \leq r < 1$ and every unit vector T tangent to $r\mathbb{S}^{n-1}$ at rx_0 , where M depends only on n , $\|f\|_\infty$ and $Lf(x_0)$.

If we suppose in addition that h is K -quasiregular (shortly K -qr) mapping along $[o, x_0)$, where o denotes the origin, then

S2)

$$|h'(rx_0)| \leq KM \tag{3'}$$

for every $0 \leq r < 1$.

The above result extends to the case of more general moduli functions which include $\omega(\delta) = \delta^\alpha$ ($0 < \alpha \leq 1$), and therefore includes earlier results on Hölder continuity (see [8]).

Let us review some results from [7]. In the proof of the next theorem we use Poisson integral representation of harmonic functions (see formula (12) below) by the Poisson kernel on the unit ball \mathbb{B}^n which is given by

$$P_{\mathbb{B}}(x, \eta) = \frac{1 - |x|^2}{n\omega_n|x - \eta|^n}, \quad x \in \mathbb{B}^n, \quad \eta \in \mathbb{S}^{n-1}.$$

Let $d\sigma$ denote positive Borel measure on \mathbb{S}^{n-1} invariant with respect to orthogonal group $O(n)$ normalized such that $\sigma(\mathbb{S}^{n-1}) = 1$.

For convenience of the reader we prove the following proposition which appeared in [7].

Proposition 4.3 ([7]). Suppose that $0 < \alpha < 1$ and $x = re_n$, $0 < r < 1$. Then

$$I_\alpha(re_n) =: \int_{\mathbb{S}^{n-1}} \frac{|e_n - t|^\alpha}{|x - t|^n} d\sigma(t) \leq \frac{c_{\alpha,n}}{(1-r)^{1-\alpha}}.$$

Proof. Since the integral is a continuous function of $0 \leq r < 1$, it suffices to prove the estimate under additional assumption $1/2 \leq r < 1$. The integrand depends only on the angle $\theta = \angle(t, e_n)$ so we can use integration in polar coordinates on the sphere \mathbb{S}^{n-1} . This gives

$$I_\alpha(re_n) \leq c_n \int_0^\pi \frac{|\theta|^{n-2}|\theta|^\alpha}{((1-r)^2 + \frac{4r}{\pi^2}\theta^2)^{n/2}} d\theta < \tag{7}$$

$$c_n \int_0^\infty \frac{\theta^{\alpha+n-2}}{((1-r)^2 + \frac{4r}{\pi^2}\theta^2)^{n/2}} d\theta. \tag{8}$$

Next using $(1 + \frac{4r}{\pi^2}u^2)^{-1} \leq C(1 + u^2)^{-1}$ for $\frac{1}{2} \leq r < 1$ and a change of variable $\theta = (1-r)u$, we find

$$I_\alpha(re_n) \leq C(1-r)^{\alpha-1} \int_0^\infty \frac{u^{\alpha+n-2}}{(1+u^2)^{n/2}} du. \tag{9}$$

Since the above improper integral is convergent the proof is completed. \square

If ω is a majorant satisfying the following condition

$$\int_0^\infty \frac{u^{n-2}}{(1+u^2)^{n/2}} \omega(\delta u) du \leq C\omega(\delta), \quad 0 < \delta \leq 1, \tag{10}$$

then one proves, by a similar argument, the following proposition.

Proposition 4.4.

$$I_\omega(re_n) =: \int_{\mathbb{S}^{n-1}} \frac{\omega(|e_n - t|)}{|x - t|^n} d\sigma(t) \leq c \cdot \frac{\omega(1 - r)}{1 - r}, \quad 0 \leq r < 1.$$

Theorem 4.5. Let ω be a majorant satisfying condition (10). Assume h is continuous on $\overline{\mathbb{B}^n}$, harmonic on \mathbb{B}^n and let x_0 be a point in \mathbb{S}^{n-1} . In addition suppose the following estimate is valid:

$$|h(x) - h(x_0)| \leq C\omega(|x - x_0|), \quad x \in \mathbb{S}^{n-1}. \tag{11}$$

Then there is a constant $M = M_{n,\omega}$ such that

$$|h'(rx_0)| \leq MC \frac{\omega(1 - r)}{1 - r}, \quad 0 \leq r < 1.$$

Proof. Since h is harmonic on \mathbb{B}^n and continuous on $\overline{\mathbb{B}^n}$ we have

$$h(x) = \int_{\mathbb{S}^{n-1}} P_{\mathbb{B}}(x, \eta)h(\eta)d\sigma(\eta), \quad x \in \mathbb{B}^n. \tag{12}$$

Set $d := 1 - |x|^2$. By computation $\partial_{x_k} P_{\mathbb{B}}(x, t) = -(\frac{2x_k}{|x-t|^n} + dn \frac{x_k - t_k}{|x-t|^{n+2}})$. Since $d \leq 2(1 - |x|) \leq 2|t - x|$ for all $t \in \mathbb{S}^{n-1}$ we obtain

$$|\partial_{x_k} P_{\mathbb{B}}(x, t)| \leq \frac{c_n}{|x - t|^n} \quad x \in \mathbb{B}^n, \quad t \in \mathbb{S}^{n-1}. \tag{13}$$

We can assume $x_0 = e_n$. Let $x = re_n$ and let θ be the angle between t and e_n . Note that $s := |x - t|^2 = 1 - 2r \cos \theta + r^2$ depends only on θ for fixed x . Next, since $\int_{\mathbb{S}^{n-1}} \partial_k P_{\mathbb{B}}(x, t)h(e_n)d\sigma(t) = 0$, we find

$$\partial_{x_k} h(x) = \int_{\mathbb{S}^{n-1}} \partial_k P_{\mathbb{B}}(x, t)(h(t) - h(e_n))d\sigma(t). \tag{14}$$

Hence by (13) and the hypothesis (11) we get

$$|\partial_{x_k} h(x)| \leq c_n C \int_{\mathbb{S}^{n-1}} \frac{\omega(|e_n - t|)}{|x - t|^n} d\sigma(t) \tag{15}$$

and the result follows from Proposition 4.4. \square

Remark 4.6. It is convenient to denote expressions that appear in formulae (7) and (8) without constants by $A(r, \alpha)$ and $B(r, \alpha)$ respectively. Note that $A(r, \alpha)$ is finite for $0 \leq r < 1$ and $0 < \alpha \leq 1$ and that $B(r, 1) = +\infty$. In order to estimate $A(r, \alpha)$ we used a change of variable $\theta = (1 - r)u$ and transformed the integral over $[0, \pi]$ into integral over $[0, a(r)]$ with respect to u , where $a(r) = \pi(1 - r)^{-1}$. Since $a(r) \rightarrow \infty$ as $r \rightarrow 1$, it is convenient to estimate integral $A(r, \alpha)$ by integral $B(r, \alpha)$ over interval $[0, \infty)$.

Remark 4.7. The above proof breaks down for $\alpha = 1$ because $B(r, 1) = \infty$. Moreover, for each $n = 2$, there is a Lipschitz continuous map $f : \mathbb{S}^{n-1} \rightarrow \mathbb{R}^n$ such that $u = P_{\mathbb{B}}[f]$ is not Lipschitz continuous. In the planar case, consider $f = u + iv$ such that $zf' = -\log(1 - z)$. u'_θ is bounded while its harmonic conjugate ru'_θ is not bounded. In the spatial case, consider $U(x_1, x_2, \dots, x_n) = u(x_1 + ix_2, x_3, \dots, x_n)$.

References

- [1] Chen J., Huang M., Rasila A., Wang X., *On Lipschitz continuity of solutions of hyperbolic Poisson's equation*, *Calc. Var. Partial Differ. Equ.*, **57**, no. 1, 2018, p. 1–32.
- [2] Ma L., *Hölder continuity of hyperbolic Poisson integral and hyperbolic Green integral*, *Monatshefte für Mathematik*, **199**, 2022,
- [3] M. MATELJEVIĆ, *The Lower Bound for the Modulus of the Derivatives and Jacobian of Harmonic Injective Mappings*. - *Filomat* 29:2, 2015, 221-244.
- [4] M. Mateljević, V. Božin, M. Knežević: *Quasiconformality of harmonic mappings between Jordan domains*, *Filomat*, Vol 24, No 3, 2010, 111-124.
- [5] M. Mateljević, N. Mutavdžić: *On Lipschitz continuity and smoothness up to the boundary of solutions of hyperbolic Poisson's equation*, arXiv:2208.06197v1 [math.CV] 12 Aug 2022
- [6] M. Mateljević, N. Mutavdžić, *The Boundary Schwarz lemma for harmonic and pluriharmonic mappings and some generalizations*, submitted for publication, accepted in *Bulletin of the Malaysian Mathematical Sciences Society*, June 2022
- [7] Mateljević M., Salimov R. and Sevost'yanov E., *Hölder and Lipschitz continuity in Orlicz-Sobolev classes, distortion and harmonic mappings*, *Filomat*, **36** no. 16 p. 5361–5392, 2022 .
- [8] NODLER, C.A. AND D.M. OBERLIN: *Moduli of continuity and a Hardy-Littlewood theorem*. - *Lecture Notes in Math.* 1351, p. 265-272, Springer-Verlag, Berlin etc., 1988.
- [9] E. M. Stein: *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
- [10] E. M. Stein, G. Weiss: *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, 1971.