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On the isometries of homogeneous nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$

Miloš B. Djorić[a], Mirjana Djorić[a]

[a]*University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Belgrade, Serbia*

Abstract. In this article we prove some properties of the isometry groups of manifold $\mathbb{S}^3 \times \mathbb{S}^3$, both with respect to the standard Euclidean product metric $\langle \cdot, \cdot \rangle$ and nearly Kähler metric g. We also investigate the action of these isometries on certain classes of hypersufaces of $\mathbb{S}^3 \times \mathbb{S}^3$.

1. Introduction

A nearly Kähler manifold is one of the 16 different almost Hermitian manifolds whose covariant derivative of its almost complex structure J is skew symmetric, namely $G = \nabla J$ satisfies $G(X, Y) = -G(Y, X)$, or equivalently, $G(X, X) = (\nabla_X J)X = 0$. Since the Kähler manifolds are defined by the condition $G \equiv 0$, a non-Kähler nearly Kähler manifold (i.e. G does not vanish identically) is called strict. The interest in nearly Kähler manifolds has increased since these manifolds are examples of geometries with torsion, and therefore they have applications in mathematical physics. The lowest dimension in which a strict nearly Kähler manifold can exist is six. Moreover, the case of 6−dimensional strict nearly Kähler manifolds is of particular importance since they are building blocks of arbitrary nearly Kähler manifolds and their research leads to better understanding of the whole class of manifolds. It is known that the only homogeneous, complete, strict nearly Kähler manifolds in dimension 6 are compact spaces: unit sphere $\mathbb{S}^6$, product of unit spheres $\mathbb{S}^3 \times \mathbb{S}^3$, complex projective space $\mathbb{C}P^3$ and the flag manifold $F_{1,2}(\mathbb{C}^3)$, where the last three are not endowed with the standard metric.

In several classification theorems for submanifolds of $\mathbb{S}^3 \times \mathbb{S}^3$, see for example [7], the authors have obtained three isometric examples. Therefore, it is of interest to investigate isometries of the homogeneous nearly Kähler manifold $\mathbb{S}^3 \times \mathbb{S}^3$, initiated in [8], [9]. Moreover, the classical symmetry approach in differential geometry has been based on the isometry group of a manifold. In fact, beginning from 1870, it became clear that the principle organizing geometry ought to be the study of its symmetry groups. In his inaugural lecture at the University of Erlangen in 1872, which later became known as the “Erlanger Programm”, Felix Klein said: *Let a manifold and on it a transformation group be given; the objects belonging to the manifold ought to be studied with respect to those properties which are not changed by the transformations of the group.*

In this article we investigate the action of the isometry groups of manifold $\mathbb{S}^3 \times \mathbb{S}^3$, both with respect to the standard Euclidean product metric $\langle \cdot, \cdot \rangle$ and nearly Kähler metric g and we prove some properties of these isometries on certain classes of hypersufaces of $\mathbb{S}^3 \times \mathbb{S}^3$.

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Email addresses: milosdj@matf.bg.ac.rs (Miloš B. Djorić), mdjoric@matf.bg.ac.rs (Mirjana Djorić)

2. Preliminaries

Let us shortly present the nearly Kähler structure on $\mathbb{S}^3 \times \mathbb{S}^3$ (see [1] for more details). Considering $\mathbb{S}^3$ in $\mathbb{R}^4$ as the set of all unit quaternions in $\mathbb{H}$, we conclude that the sphere $\mathbb{S}^3$ has a Lie group structure with respect to the standard quaternionic multiplication and it is isomorphic to the group $SU(2)$ or $Sp(1)$, with left and right translations being ordinary quaternionic multiplication from left and right. Its Lie algebra $T_1(\mathbb{S}^3)$ is isomorphic to $\mathfrak{su}(2) \cong \mathrm{Im}(\mathbb{H})$, so the tangent space $T_p(\mathbb{S}^3)$ is identified with $p\,\mathrm{Im}(\mathbb{H})$, meaning that a tangent vector in $T_p\mathbb{S}^3$ can be expressed as $p\alpha$, where α is an arbitrary imaginary quaternion. The fixed basis $[\mathbf{i}, \mathbf{j}, \mathbf{k}]$ of $T_1(\mathbb{S}^3)$ ($\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the imaginary units of $\mathbb{H}$) is sent via left translation to the basis $[p\mathbf{i}, p\mathbf{j}, p\mathbf{k}]$ of $T_p(\mathbb{S}^3)$, defining one global moving orthonormal frame on $\mathbb{S}^3$.

For an arbitrary Lie group G and its closed subgroup H, which is of course a Lie subgroup, quotient space G/H has the Lie group structure because there is a naturally defined right action of a group H on G. Furthermore, quotient mapping $\pi : G \to G/H$ is a smooth submersion, and ordered quadruple $(G, G/H, H, \pi)$ is a smooth bundle, with a total space G, base space G/H and fibers H. This action also preserves the fibers, acting on it freely and transitively, so $(G, G/H, H, \pi, H)$ has a principal bundle structure.

In this way, by taking $G = \mathbb{S}^3\times\mathbb{S}^3\times\mathbb{S}^3$, $H = \mathbb{S}^3$, we obtain that $\mathbb{S}^3\times\mathbb{S}^3$ is both a Lie group and a homogeneous manifold. Namely, if we consider the product of three unit spheres $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$, with the usual structure induced from $\mathbb{H}^3$, then for tangent vector fields $V = (g_1V_1, g_2V_2, g_3V_3)$, $W = (g_1W_1, g_2W_2, g_3W_3)$ at the point (g_1, g_2, g_3), where $V_1, V_2, V_3, W_1, W_2, W_3$ are imaginary quaternions, we have the following induced metric (the set of imaginary quaternions is identified with $\mathbb{R}^3$)

$$\langle (g_1V_1, g_2V_2, g_3V_3), (g_1W_1, g_2W_2, g_3W_3)\rangle = \sum_{i=1}^{3} \mathrm{Re}(g_iV_i\overline{W_i}\bar{g}_i) = \sum_{i=1}^{3}\langle V_i, W_i\rangle. \tag{1}$$

Furthermore, an ordered triple of unit quaternions (g_1, g_2, g_3) acts on $\mathbb{S}^3 \times \mathbb{S}^3$ in the following way:

$$((g_1, g_2, g_3), (p, q)) \mapsto (g_1pg_3^{-1}, g_2qg_3^{-1}) = (g_1p\bar{g}_3, g_2q\bar{g}_3).$$

This action of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ on $\mathbb{S}^3 \times \mathbb{S}^3$ is transitive and the stabilizer of a unit element $(1, 1)$ is a group of all ordered triples (h, h, h), where h is a unit quaternion. If we denote this isotropy subgroup with $SU(2)_\triangle$, using the orbit-stabilizer theorem we obtain that $\mathbb{S}^3 \times \mathbb{S}^3$ is a smooth homogeneous manifold and Lie group, diffeomorphic to $(SU(2) \times SU(2) \times SU(2))/SU(2)_\triangle$. The quotient mapping $\pi : \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$ is then given with

$$\pi : (g_1, g_2, g_3) \mapsto (p, q) = (g_1\bar{g}_3, g_2\bar{g}_3).$$

Hence, it is clear that $\pi(g_1, g_2, g_3) = \pi(g_1', g_2', g_3')$ if and only if $(g_1', g_2', g_3') = (g_1a, g_2a, g_3a)$, for some unit quaternion $a \in \mathbb{S}^3$, meaning that the fibres are precisely the spheres $\mathbb{S}^3$. The mapping π is a smooth submersion from $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 = SU(2) \times SU(2) \times SU(2)$ onto

$$\mathbb{S}^3 \times \mathbb{S}^3 = (SU(2) \times SU(2) \times SU(2))/SU(2)_\triangle,$$

since it is easily shown that it holds

$$d\pi_{(g_1,g_2,g_3)}(g_1\alpha, g_2\beta, g_3\gamma) = (g_1\bar{g}_3(g_3(\alpha-\gamma)\bar{g}_3), g_2\bar{g}_3(g_3(\beta-\gamma)\bar{g}_3)) = (g_1(\alpha-\gamma)\bar{g}_3, g_2(\beta-\gamma)\bar{g}_3).$$

This also implies the definition of vertical and horizontal distribution given in [8], as a kernel of $d\pi$ and its orthogonal complement, respectively. Therefore, π is also a Riemannian submersion, since π preserves the fibres and respects the horizontal and vertical distributions. The metric $\langle\cdot,\cdot\rangle$ on horizontal distribution defines via π a metric g_s on $\mathbb{S}^3 \times \mathbb{S}^3$, which is shown to be a nearly Kähler metric.

It is enough to define a nearly Kähler structure on $\mathbb{S}^3 \times \mathbb{S}^3$ first at $(1, 1)$, and then to transfer it via left translations over the whole $\mathbb{S}^3 \times \mathbb{S}^3$. By the natural identification $T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3) \cong T_p\mathbb{S}^3 \oplus T_q\mathbb{S}^3$, we will write a tangent vector at (p, q) as $X_{(p,q)} = (pU_{(p,q)}, qV_{(p,q)})$ for imaginary quaternions U, V, or simply $X = (pU, qV)$, when there is no risk of confusion. The almost complex structure J on $\mathbb{S}^3 \times \mathbb{S}^3$ is defined by

$$J(pU, qV) = \frac{1}{\sqrt{3}}(p(2V - U), q(-2U + V)), \quad (pU, qV) \in T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3). \tag{2}$$

The standard product metric $\langle\cdot,\cdot\rangle$ on $\mathbb{S}^3 \times \mathbb{S}^3$ is not compatible with the almost complex structure J. Corresponding nearly Kähler metric g is defined, up to a scalar multiplication, as the mean of $\langle\cdot,\cdot\rangle$ and $\langle J\cdot, J\cdot\rangle$

$$g(Z, Z') = \frac{1}{2}(\langle Z, Z'\rangle + \langle JZ, JZ'\rangle) = \frac{4}{3}(\langle U, U'\rangle + \langle V, V'\rangle) - \frac{2}{3}(\langle U, V'\rangle + \langle U', V\rangle), \tag{3}$$

where $Z = (pU, qV)$ and $Z' = (pU', qV')$, and $\langle\cdot,\cdot\rangle$ stands for both the usual Euclidean product metric on $\mathbb{S}^3 \times \mathbb{S}^3$ and the usual Euclidean metric on $\mathbb{S}^3$. It is a direct check that this metric is the same as the previously defined metric g_s, up to a scalar multiplication $g = 2g_s$.

Using definitions (2) and (3), the almost complex structure J is compatible with the metric g. It is now straightforward to check that the $(1, 2)$-tensor $G = \nabla J$, where ∇ is the Levi-Civita connection of g on $\mathbb{S}^3 \times \mathbb{S}^3$, is skew-symmetric. Consequently, $\mathbb{S}^3 \times \mathbb{S}^3$ equipped with g and J, is a nearly Kähler manifold.

Every linear mapping of the tangent bundle of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ which preserves the metric, horizontal and vertical distributions and fibers of submersion π, induces linear mapping of the tangent bundle of $\mathbb{S}^3 \times \mathbb{S}^3$. In this way we can define not only the almost complex structure, but also the existence of an almost product structure P, which is a property that is characteristic specifically for $\mathbb{S}^3 \times \mathbb{S}^3$ and not for the whole class of nearly Kähler manifolds, with

$$PZ = (pV, qU), \quad Z = (pU, qV) \in T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3). \tag{4}$$

This definition is, similar to the definition of the almost complex structure, compatible with the Lie group structure of $\mathbb{S}^3 \times \mathbb{S}^3$. It is straightforward to show directly that the symmetric endomorphism P is compatible with the metric g and that it anticommutes with J (see [1] for more details).

This almost product structure P is not integrable, so it is not a product structure. Transformations $\tilde{P}_i$, $i = 1, 2, 3$, of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$, given with

$$\begin{aligned}
&\tilde{P}_1(g_1V_1, g_2V_2, g_3V_3) = (g_1V_2, g_2V_1, g_3V_3),\\
&\tilde{P}_2(g_1V_1, g_2V_2, g_3V_3) = (g_1V_3, g_2V_2, g_3V_1),\\
&\tilde{P}_3(g_1V_1, g_2V_2, g_3V_3) = (g_1V_1, g_2V_3, g_3V_2),
\end{aligned}$$

actually define three different almost product structures on $\mathbb{S}^3 \times \mathbb{S}^3$, we will denote this set by $\mathcal{P} = \{P_1, P_2, P_3\}$. The almost product structures $P_1 = P$, $P_2 = -\frac{1}{2}P - \frac{\sqrt{3}}{2}JP$, $P_3 = -\frac{1}{2}P + \frac{\sqrt{3}}{2}JP$ are also not integrable. Almost product structures on $\mathbb{S}^3 \times \mathbb{S}^3$ have been extensively studied in [8], where the authors classified the called nearly productlike structures on $\mathbb{S}^3 \times \mathbb{S}^3$. They showed that the only possible nearly productlike structures on $\mathbb{S}^3 \times \mathbb{S}^3$ are precisely the following ones

$$P_{l+1} = \cos\left(\frac{2\pi l}{3}\right)P - \sin\left(\frac{2\pi l}{3}\right)JP, \qquad l = 0, 1, 2. \tag{5}$$

3. Isometry group of $\mathbb{S}^3 \times \mathbb{S}^3$

It is clear that each isometry of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ with respect to $\langle\cdot,\cdot\rangle$, defined by (1), which preserves the horizontal and vertical distribution and the fibres of submersion π, induces the isometry of $\mathbb{S}^3 \times \mathbb{S}^3$. The isometry group of $\mathbb{S}^3$ can be identified with the group $\mathbb{O}(4)$ of all orthogonal 4×4 matrices. Nevertheless, the isometry group of the product manifold $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ is larger than the product of isometry groups $\mathbb{O}(4) \times \mathbb{O}(4) \times \mathbb{O}(4)$ since there is also an action of symmetry group $\mathbb{S}_3$ on $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ obtained by permuting the 3 components, so the isometry group of $\mathbb{S}^3\times\mathbb{S}^3\times\mathbb{S}^3$ has $(\mathbb{O}(4)\times\mathbb{O}(4)\times\mathbb{O}(4))\rtimes\mathbb{S}_3$ as a subgroup. The authors are not familiar with the proof whether this isometry group is actually equal to $(\mathbb{O}(4)\times\mathbb{O}(4)\times\mathbb{O}(4))\rtimes\mathbb{S}_3$.

Recall the structure of Lie group $\mathbb{SO}(4)$, the special orthogonal group of order 4 of all the rotations in $\mathbb{E}^4$ which fix the origin. For each such rotation $R\in\mathbb{SO}(4)$, there is at least one pair of orthogonal 2-planes each of which is invariant under R and whose direct sum is the entire space (they have only one point in common, the origin). Hence, R operating on either of these planes produces an ordinary rotation of that plane, with the angles of these rotations determined up to a sign since there are two possible choices of orientations in these planes that are (jointly) consistent with the orientation of the whole space. If exactly one of the rotations is an identity transformation, then such rotation R is called a simple rotation, else it is called a double rotation. In the case of a double rotation, the angles of two rotations in the orthogonal planes can be either equal, up to a sign, or distinct. If the rotation angles of a double rotation are equal, then there are infinitely many invariant planes instead of just two, while in the case od distinct angles, these planes are unique. Such rotations with equal angles (up to a sign) are called isoclinic or equiangular rotations. Furthermore, isoclinic rotations with the same angles are denoted as left-isoclinic; those with opposite angles are right-isoclinic. The left and right isoclinic rotations are represented respectively by left and right multiplication by unit quaternions. Therefore, all left-isoclinic rotations form a noncommutative subgroup isomorphic to the multiplicative group of unit quaternions $\mathbb{S}^3_L$, which is a subgroup of $\mathbb{SO}(4)$. The same holds for the group $\mathbb{S}^3_R$ of right isoclinic rotations and both these subgroups are maximal subgroups of $\mathbb{SO}(4)$. Their intersection consists of the identity transformation I and the so-called central inversion $-I$, with $\{I,-I\}$ being the center of $\mathbb{SO}(4)$ and both $\mathbb{S}^3_L$, $\mathbb{S}^3_R$. Each left-isoclinic rotation commutes with each right-isoclinic rotation and the conjugation with a reflection transforms a left-isoclinic rotation into a right-isoclinic rotation and vice versa. Hence, distinct subgroups $\mathbb{S}^3_L$, $\mathbb{S}^3_R$ are conjugate to each other in $\mathbb{O}(4)$, but not in $\mathbb{SO}(4)$. This is a special property of $\mathbb{SO}(4)$ among rotation groups in general: all even-dimensional rotation groups $\mathbb{SO}(2n)$, $n\geqslant 2$, contain isoclinic rotations, but unlike $\mathbb{SO}(4)$, in all higher even-dimensional rotation groups any two isoclinic rotations through the same angle are conjugate. The set of all isoclinic rotations is even not a subgroup of $\mathbb{SO}(2n)$, $n\geqslant 3$, let alone a normal subgroup. Each rotation $R\in\mathbb{SO}(4)$ can be transformed in two ways into the product of left and right isoclinic rotations, which are together determined up to the central inversion, i.e. when both are multiplied by the central inversion, their product is R again. This implies that $\mathbb{S}^3_L\times\mathbb{S}^3_R$ is the universal covering group of $\mathbb{SO}(4)$ and it is its unique double cover. Also, $\mathbb{S}^3_L$, $\mathbb{S}^3_R$ are the normal subgroups of $\mathbb{SO}(4)$ and it holds

$$\mathbb{S}^3_L/\{I,-I\}\cong\mathbb{SO}(3),\qquad \mathbb{S}^3_R/\{I,-I\}\cong\mathbb{SO}(3),\qquad (\mathbb{S}^3_L\times\mathbb{S}^3_R)/\{I,-I\}\cong\mathbb{SO}(4).$$

With respect to any orthonormal basis, every element R of $\mathbb{SO}(4)$ is represented by a 4×4 orthogonal matrix A with a determinant equal to $+1$, so that $X'=AX$, where X,X' represent the columns of coordinates of the point $P(x_1,x_2,x_3,x_4)$ and its image $P'(x_1',x_2',x_3',x_4')$ under R. The isoclinic decomposition of the rotation matrix A is given with:

$$A=A_LA_R=\begin{pmatrix} a & -b & -c & -d\\ b & a & -d & c\\ c & d & a & -b\\ d & -c & b & a\end{pmatrix}\begin{pmatrix} p & -q & -r & -s\\ q & p & s & -r\\ r & -s & p & q\\ s & r & -q & p\end{pmatrix}.$$

The first factor in this decomposition is a left-isoclinic rotation $X\mapsto A_LX$, while the second factor is a right-isoclinic rotation $X\mapsto A_RX$ and the factors are determined up to the negative 4×4 identity matrix, i.e. the central inversion. Namely, there are exactly two sets of a,b,c,d and p,q,r,s, opposite to each other, such that decomposition holds and $a^2+b^2+c^2+d^2=1$, $p^2+q^2+r^2+s^2=1$. In the quaternionic language, this means that

$$x_1'+x_2'\mathbf{i}+x_3'\mathbf{j}+x_4'\mathbf{k}=(a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k})(x_1+x_2\mathbf{i}+x_3\mathbf{j}+x_4\mathbf{k})(p+q\mathbf{i}+r\mathbf{j}+s\mathbf{k}).$$

From the formula (1), it is clear that every isometry of $\mathbb{S}^3\times\mathbb{S}^3\times\mathbb{S}^3$ that belongs to the subgroup $\mathbb{O}(4)\times\mathbb{O}(4)\times\mathbb{O}(4)$ consists of three isometries of $\mathbb{S}^3$ on each of the components. Consequently, among all the isometries of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$, the most natural ones to investigate are the direct isometries that consist of the direct isometries of $\mathbb{S}^3$ on each component.

According to the isoclinic decomposition, we can express such isometry as $\tilde{\mathcal{F}}_{a'a''b'b''c'c''} : \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$, coming from the left and right quaternionic multiplication on each component

$$\tilde{\mathcal{F}}_{a'a''b'b''c'c''}(g_1, g_2, g_3) = (a'g_1a'', b'g_2b'', c'g_3c''), \quad a', a'', b', b'', c', c'' \in \mathbb{S}^3.$$

Since for arbitrary unit quaternion d we have

$$\pi(\tilde{\mathcal{F}}_{a'a''b'b''c'c''}(g_1d, g_2d, g_3d)) = \pi(a'g_1da'', b'g_2db'', c'g_3dc'') = (a'g_1da''\bar{c}''\bar{d}\bar{g}_3\bar{c}', b'g_2db''\bar{c}''\bar{d}\bar{g}_3\bar{c}'),$$

the fibers are preserved iff $\pi(\tilde{\mathcal{F}}_{a'a''b'b''c'c''}(g_1d, g_2d, g_3d)) = \pi(\tilde{\mathcal{F}}_{a'a''b'b''c'c''}(g_1, g_2, g_3))$, namely $a'' = b'' = c'' = 1$. In that way we obtain the well known isometries (see [9]) $\mathcal{F}_{abc} : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$ of $\mathbb{S}^3 \times \mathbb{S}^3$, given by

$$\mathcal{F}_{abc}(p, q) = (ap\bar{c}, bq\bar{c}), \quad a, b, c \in \mathbb{S}^3,$$

coming from the isometries $\tilde{\mathcal{F}}_{abc} : \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$, given by

$$\tilde{\mathcal{F}}_{abc}(g_1, g_2, g_3) = (ag_1, bg_2, cg_3),$$

where $\pi((g_1, g_2, g_3)) = (p, q)$. Let us denote by $\mathbb{F}$ the subgroup of all isometries $\mathcal{F}_{abc}$ of $\mathbb{S}^3 \times \mathbb{S}^3$:

$$\mathbb{F} = \{\mathcal{F}_{abc} \mid a, b, c \in \mathbb{S}^3\}.$$

Notice that the triplets of unit quaternions (a, b, c) and $(-a, -b, -c)$ induce the same isometry $\mathcal{F}_{abc}$. Hence, using the above study, we have proved the following proposition.

Proposition 3.1. *The only isometries of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ that belong to $\mathbb{SO}(4) \times \mathbb{SO}(4) \times \mathbb{SO}(4)$ and define an isometry of $\mathbb{S}^3 \times \mathbb{S}^3$ via submersion π are precisely the isometries $\tilde{\mathcal{F}}_{abc}$. The corresponding isometries $\mathcal{F}_{abc}$ of $\mathbb{S}^3 \times \mathbb{S}^3$ form the subgroup $\mathbb{F}$ of the group of all isometries of nearly Kähler $\mathbb{S}^3 \times \mathbb{S}^3$, isomorphic to $(\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3)/\{I, -I\}$.*

Remark 3.2. *The group $\mathbb{F}$ is in the literature frequently taken to be isomorphic to $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$, which is not correct, having in mind Proposition 3.1.*

Moreover, if arbitrary isometry $\tilde{\mathcal{F}}$ of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ has indirect part of $\mathbb{O}(4)$ on some component, it will not preserve the fibres. It is enough to prove this for the conjugation, since every indirect isometry of $\mathbb{O}(4)$ is a composition of the conjugation and the element of $\mathbb{SO}(4)$. If we take conjugation $\tilde{\mathcal{F}}(g_1, g_2, g_3) = (\bar{g}_1, \bar{g}_2, \bar{g}_3)$ on each component, we derive

$$\pi(\tilde{\mathcal{F}}(g_1d, g_2d, g_3d)) = \pi(\bar{d}\bar{g}_1, \bar{d}\bar{g}_2, \bar{d}\bar{g}_3) = (\bar{d}\bar{g}_1g_3d, \bar{d}\bar{g}_2g_3d).$$

By permuting the components of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$, we define the isometries $\tilde{\mathcal{F}}_i : \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$, $i = 1 \dots, 5$

$$\begin{aligned} &\tilde{\mathcal{F}}_1(g_1, g_2, g_3) = (g_2, g_1, g_3), && \tilde{\mathcal{F}}_2(g_1, g_2, g_3) = (g_3, g_2, g_1), && \tilde{\mathcal{F}}_3(g_1, g_2, g_3) = (g_3, g_1, g_2), \\ &\tilde{\mathcal{F}}_4(g_1, g_2, g_3) = (g_2, g_3, g_1), && \tilde{\mathcal{F}}_5(g_1, g_2, g_3) = (g_1, g_3, g_2), \end{aligned}$$

which induce isometries $\mathcal{F}_i : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$, $i = 1 \dots, 5$, given by

$$\begin{aligned} &\mathcal{F}_1(p, q) = (q, p), && \mathcal{F}_2(p, q) = (\bar{p}, q\bar{p}), && \mathcal{F}_3(p, q) = (\bar{q}, p\bar{q}), \\ &\mathcal{F}_4(p, q) = (q\bar{p}, \bar{p}), && \mathcal{F}_5(p, q) = (p\bar{q}, \bar{q}). \end{aligned}$$

Lemma 3.3. *The set*

$$\mathbb{G} = \{\mathcal{E}, \mathcal{F}_1, \dots, \mathcal{F}_5\},$$

where $\mathcal{E}$ denotes the identity transformation on $\mathbb{S}^3 \times \mathbb{S}^3$, is a subgroup of the isometry group of $\mathbb{S}^3 \times \mathbb{S}^3$, isomorphic to the group $\mathbb{S}_3$. Moreover, the groups $\mathbb{G}$ and $\mathbb{F}$ commute.

Proof. From the defining relations, we obtain all the compositions of $\mathcal{F}_i$, presented in the following table:

$$\begin{array}{|c||c|c|c|c|c|c|}
\hline
\circ & \mathcal{E} & \mathcal{F}_1 & \mathcal{F}_2 & \mathcal{F}_3 & \mathcal{F}_4 & \mathcal{F}_5 \\
\hline\hline
\mathcal{E} & \mathcal{E} & \mathcal{F}_1 & \mathcal{F}_2 & \mathcal{F}_3 & \mathcal{F}_4 & \mathcal{F}_5 \\
\hline
\mathcal{F}_1 & \mathcal{F}_1 & \mathcal{E} & \mathcal{F}_4 & \mathcal{F}_5 & \mathcal{F}_2 & \mathcal{F}_3 \\
\hline
\mathcal{F}_2 & \mathcal{F}_2 & \mathcal{F}_3 & \mathcal{E} & \mathcal{F}_1 & \mathcal{F}_5 & \mathcal{F}_4 \\
\hline
\mathcal{F}_3 & \mathcal{F}_3 & \mathcal{F}_2 & \mathcal{F}_5 & \mathcal{F}_4 & \mathcal{E} & \mathcal{F}_1 \\
\hline
\mathcal{F}_4 & \mathcal{F}_4 & \mathcal{F}_5 & \mathcal{F}_1 & \mathcal{E} & \mathcal{F}_3 & \mathcal{F}_2 \\
\hline
\mathcal{F}_5 & \mathcal{F}_5 & \mathcal{F}_4 & \mathcal{F}_3 & \mathcal{F}_2 & \mathcal{F}_1 & \mathcal{E} \\
\hline
\end{array}. \tag{6}$$

Now it is clear that the set $\mathbb{G}$ is a subgroup of order 6 of the isometry group of $\mathbb{S}^3 \times \mathbb{S}^3$. Moreover, using relations $\mathcal{F}_1^2 = \mathcal{F}_2^2 = (\mathcal{F}_1 \circ \mathcal{F}_2)^3 = \mathcal{F}_4^3 = \mathcal{E}$, we conclude that this group is isomorphic to the group $\mathbb{S}_3$, which is expected, since their definition comes from the $\mathbb{S}_3$ action on $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$.

Also, since it holds

$$\begin{aligned}
&\mathcal{F}_{abc} \circ \mathcal{F}_1 = \mathcal{F}_1 \circ \mathcal{F}_{bac}, \quad \mathcal{F}_{abc} \circ \mathcal{F}_2 = \mathcal{F}_2 \circ \mathcal{F}_{cba}, \quad \mathcal{F}_{abc} \circ \mathcal{F}_3 = \mathcal{F}_3 \circ \mathcal{F}_{cab}, \\
&\mathcal{F}_{abc} \circ \mathcal{F}_4 = \mathcal{F}_4 \circ \mathcal{F}_{bca}, \quad \mathcal{F}_{abc} \circ \mathcal{F}_5 = \mathcal{F}_5 \circ \mathcal{F}_{acb},
\end{aligned} \tag{7}$$

the groups $\mathbb{F}$ and $\mathbb{G}$ commute. □

Using (6) it is straightforward to prove the following lemma, which is needed for the results that follow.

Lemma 3.4. *Differentials of the elements of the groups* $\mathbb{F}$ *and* $\mathbb{G}$ *are given with*

$$\begin{aligned}
d\mathcal{F}_{abc}(p\alpha, q\beta) &= (ap\bar{c}(c\alpha\bar{c}), bq\bar{c}(c\beta\bar{c})), & d\mathcal{F}_1(p\alpha, q\beta) &= (q\beta, p\alpha), \\
d\mathcal{F}_2(p\alpha, q\beta) &= (\bar{p}(p(-\alpha)\bar{p}), q\bar{p}(p(\beta-\alpha)\bar{p})), & d\mathcal{F}_3(p\alpha, q\beta) &= (\bar{q}(q(-\beta)\bar{q}), p\bar{q}(q(\alpha-\beta)\bar{q})), \\
d\mathcal{F}_4(p\alpha, q\beta) &= (q\bar{p}(p(\beta-\alpha)\bar{p}), \bar{p}(p(-\alpha)\bar{p})), & d\mathcal{F}_5(p\alpha, q\beta) &= (p\bar{q}(q(\alpha-\beta)\bar{q}), \bar{q}(q(-\beta)\bar{q})).
\end{aligned} \tag{8}$$

Similar to the situation with the isometries of $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$ with respect to the standard Euclidean product metric, the group of isometries of $\mathbb{S}^3 \times \mathbb{S}^3$ with respect to the standard Euclidean product metric $\langle \cdot, \cdot \rangle$ is at least $(\mathbb{O}(3) \times \mathbb{O}(3)) \rtimes \mathbb{S}_2$. Now a natural question arises: which of these are the isometries of both $(\mathbb{S}^3 \times \mathbb{S}^3, \langle \cdot, \cdot \rangle)$ and $(\mathbb{S}^3 \times \mathbb{S}^3, g)$?

Those isometries from the subgroup $\mathbb{O}(3) \times \mathbb{O}(3)$, which are indirect on one of the components, are not isometries of $\mathbb{S}^3 \times \mathbb{S}^3$ with respect to nearly Kähler metric g. This can be illustrated for the transformation $\bar{\mathcal{F}} : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$, $\bar{\mathcal{F}}(p, q) = (\bar{p}, \bar{q})$, by using (3) and the following formula for the differential

$$d\bar{\mathcal{F}}(p\alpha, q\beta) = (\bar{p}(p\bar{\alpha}\bar{p}), \bar{q}(q\bar{\beta}\bar{q})).$$

Also, all the isometries of $(\mathbb{S}^3 \times \mathbb{S}^3, \langle \cdot, \cdot \rangle)$ that belong to $\mathbb{SO}(3) \times \mathbb{SO}(3)$, are given with $\mathcal{F}_{abcd} : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$, $\mathcal{F}_{abcd}(p, q) = (ap\bar{c}, bq\bar{d})$, due to isoclinic decomposition. These mappings are also isometries of $(\mathbb{S}^3 \times \mathbb{S}^3, g)$ if and only if $c = d$, which directly follows from (3) and the following relation

$$d\mathcal{F}_{abcd}(p\alpha, q\beta) = (ap\bar{c}(c\alpha\bar{c}), bq\bar{d}(d\beta\bar{d})).$$

In this way we obtain the already mentioned family of isometries $\mathcal{F}_{abc}$, so they are compatible both with Euclidean product metric $\langle \cdot, \cdot \rangle$ and nearly Kähler metric g. For example, mapping $((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) \mapsto ((x_2, x_3, x_1, x_4), (y_2, y_1, y_3, y_4))$, is the isometry of $(\mathbb{S}^3 \times \mathbb{S}^3, \langle \cdot, \cdot \rangle)$, but not of $(\mathbb{S}^3 \times \mathbb{S}^3, g)$, while $((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) \mapsto ((x_2, x_3, x_1, x_4), (y_2, y_3, y_1, y_4))$ is the isometry of both $(\mathbb{S}^3 \times \mathbb{S}^3, \langle \cdot, \cdot \rangle)$ and $(\mathbb{S}^3 \times \mathbb{S}^3, g)$.

Also, from the $\mathbb{S}_2$ action on $\mathbb{S}^3 \times \mathbb{S}^3$ by permuting the components, we obtain one more isometry of both $(\mathbb{S}^3 \times \mathbb{S}^3, \langle \cdot, \cdot \rangle)$ and $(\mathbb{S}^3 \times \mathbb{S}^3, g)$: it is the (previously defined) isometry $\mathcal{F}_1$. Using Lemma 3.4, we can check that other isometries $\mathcal{F}_i$, $i = 2, 3, 4, 5$, of $(\mathbb{S}^3 \times \mathbb{S}^3, g)$ from $\mathbb{G}$ are not isometries of $(\mathbb{S}^3 \times \mathbb{S}^3, \langle \cdot, \cdot \rangle)$. Hence, we have proved the following theorem.

Theorem 3.5. *The only isometries of* $(\mathbb{S}^3 \times \mathbb{S}^3, g)$ *that belong to the group* $(\mathbb{O}(3) \times \mathbb{O}(3)) \rtimes \mathbb{S}_2$ *of isometries of* $(\mathbb{S}^3 \times \mathbb{S}^3, \langle\cdot,\cdot\rangle)$ *are precisely the isometries from the group* $\mathbb{F}$. *The only isometries of* $(\mathbb{S}^3 \times \mathbb{S}^3, \langle\cdot,\cdot\rangle)$ *that belong to either of isometry subgroups* $\mathbb{F}$ *and* $\mathbb{G}$ *of* $(\mathbb{S}^3 \times \mathbb{S}^3, g)$ *are the isometry* $\mathcal{F}_1$ *and the isometries from the group* $\mathbb{F}$.

Having in mind that $\mathcal{F}_1$ is the isometry of both $(\mathbb{S}^3 \times \mathbb{S}^3, \langle\cdot,\cdot\rangle)$ and $(\mathbb{S}^3 \times \mathbb{S}^3, g)$, we consider the geodesic lines of both $(\mathbb{S}^3 \times \mathbb{S}^3, \langle\cdot,\cdot\rangle)$ and $(\mathbb{S}^3 \times \mathbb{S}^3, g)$.

Proposition 3.6. *The only geodesic lines of both* $(\mathbb{S}^3 \times \mathbb{S}^3, \langle\cdot,\cdot\rangle)$ *and* $(\mathbb{S}^3 \times \mathbb{S}^3, g)$ *are given by*

$$\gamma(t) = (\cos(\|a\|t) + \sin(\|a\|t)a, \cos(\|a\|t) \pm \sin(\|a\|t)a), \quad t \in \mathbb{R}, \tag{9}$$

for arbitrary unit imaginary quaternion a.

Proof. Using the definition of geodesic lines $\nabla^E_{\gamma'}\gamma' = \nabla_{\gamma'}\gamma' = 0$ and formula

$$\nabla^E_X Y = \nabla_X Y + \frac{1}{2}(JG(X, PY) + JG(Y, PX))$$

relating the Levi-Civita connections ∇^E and ∇ of metrics $\langle\cdot,\cdot\rangle$ and g, respectively, we can obtain relation (9). For more details see [3].

However, on this occasion we point out that there is another proof. Using the general form of the geodesic lines of $(\mathbb{S}^3 \times \mathbb{S}^3, \langle\cdot,\cdot\rangle)$

$$\gamma(t) = (\cos(\|a\|t) + \sin(\|a\|t)a, \cos(\|\tilde{a}\|t) + \sin(\|\tilde{a}\|t)\tilde{a}), \quad t \in \mathbb{R},$$

where $a, \tilde{a}$ are arbitrary unit imaginary quaternions, it follows that the only geodesic lines γ that are invariant under $\mathcal{F}_1$ are the ones where $a, \tilde{a}$ are collinear, i.e. $\tilde{a} = \pm a$. Now it is easy to check that these lines are really geodesic for both metrics. □

Furthermore, it is interesting to see whether almost complex and almost product structures are compatible with these isometries.

Proposition 3.7. *For the differentials of the isometries* $\mathcal{F}_{abc}, \mathcal{F}_1, \ldots, \mathcal{F}_5$, *the almost complex structure* J *and almost product structures* P_1, P_2, P_3, *the following relations hold*

$$\begin{array}{ccc}
d\mathcal{F}_{abc} \circ J = J \circ d\mathcal{F}_{abc}, & d\mathcal{F}_1 \circ J = -J \circ d\mathcal{F}_1, & d\mathcal{F}_2 \circ J = -J \circ d\mathcal{F}_2, \\
d\mathcal{F}_3 \circ J = J \circ d\mathcal{F}_3, & d\mathcal{F}_4 \circ J = J \circ d\mathcal{F}_4, & d\mathcal{F}_5 \circ J = -J \circ d\mathcal{F}_5, \\
d\mathcal{F}_{abc} \circ P_1 = P_1 \circ d\mathcal{F}_{abc}, & d\mathcal{F}_1 \circ P_1 = P_1 \circ d\mathcal{F}_1, & d\mathcal{F}_2 \circ P_1 = P_3 \circ d\mathcal{F}_2, \\
d\mathcal{F}_{abc} \circ P_2 = P_2 \circ d\mathcal{F}_{abc}, & d\mathcal{F}_1 \circ P_2 = P_3 \circ d\mathcal{F}_1, & d\mathcal{F}_2 \circ P_2 = P_2 \circ d\mathcal{F}_2, \\
d\mathcal{F}_{abc} \circ P_3 = P_3 \circ d\mathcal{F}_{abc}, & d\mathcal{F}_1 \circ P_3 = P_2 \circ d\mathcal{F}_1, & d\mathcal{F}_2 \circ P_3 = P_1 \circ d\mathcal{F}_2, \\
d\mathcal{F}_3 \circ P_1 = P_3 \circ d\mathcal{F}_3, & d\mathcal{F}_4 \circ P_1 = P_2 \circ d\mathcal{F}_4, & d\mathcal{F}_5 \circ P_1 = P_2 \circ d\mathcal{F}_5, \\
d\mathcal{F}_3 \circ P_2 = P_1 \circ d\mathcal{F}_3, & d\mathcal{F}_4 \circ P_2 = P_3 \circ d\mathcal{F}_4, & d\mathcal{F}_5 \circ P_2 = P_1 \circ d\mathcal{F}_5, \\
d\mathcal{F}_3 \circ P_3 = P_2 \circ d\mathcal{F}_3, & d\mathcal{F}_4 \circ P_3 = P_1 \circ d\mathcal{F}_4, & d\mathcal{F}_5 \circ P_3 = P_3 \circ d\mathcal{F}_5.
\end{array} \tag{10}$$

Proof. The proof follows directly from formulae (8) and relations (2), (5). □

Corollary 3.8. *The set* $\mathcal{P}$ *is obviously invariant under each of the isometries* $\mathcal{F}_i$, $i = 1, \ldots, 5$, *since they only permute three almost product structures. Consequently, in many known classification theorems regarding submanifolds of* $\mathbb{S}^3 \times \mathbb{S}^3$, *there are* 3 *different isometric examples, with similar almost product structures. From the compatibility with almost complex structure (up to a sign), it also follows that both groups of isometries preserve holomorphic planes.*

4. Hypersurfaces of $\mathbb{S}^3 \times \mathbb{S}^3$ and isometries

In the paper [4], the authors introduced the notions of $\mathcal{P}-$principal and $\mathcal{P}-$isotropic vector fields on $\mathbb{S}^3 \times \mathbb{S}^3$, motivated by the similar notions of $\mathcal{A}-$principal and $\mathcal{A}-$isotropic vector fields on the complex quadrics. Namely, since the distribution $\mathcal{D}_Z = \text{span}\{Z, JZ, PZ, JPZ\}$ of $T(\mathbb{S}^3 \times \mathbb{S}^3)$ is $J-$invariant and $P-$invariant, it has even dimension 2 or 4, for an arbitrary tangent vector field Z of $\mathbb{S}^3 \times \mathbb{S}^3$. The corresponding tangent vector field Z in the former case is $\mathcal{P}-$**principal**, while in the latter case it is $\mathcal{P}-$**isotropic**, with an additional condition in the latter case that all the vectors in $\mathcal{D}_Z$ are mutually orthogonal. The authors have given several useful characterisations of these vector fields, for example: Z is $\mathcal{P}-$principal if and only if $PZ = \cos\theta Z + \sin\theta JZ$, for smooth angle function θ; Z is $\mathcal{P}-$isotropic if and only if there exist orthonormal vector fields X, Y such that $PX = X$, $PY = Y$, $Z = \frac{X+JY}{\sqrt{2}}$. Moreover, the holomorphic sectional curvature of the holomorphic plane $\text{span}\{Z, JZ\}$ is minimal possible and equal to 0 if Z is $\mathcal{P}-$principal, while it is maximal possible and equal to $\frac{2}{3}$ if Z is $\mathcal{P}-$isotropic.

Let Z be an arbitrary unit tangent vector field on $\mathbb{S}^3 \times \mathbb{S}^3$. We will decompose PZ along the holomorphic plane $\Pi_Z = \text{span}\{Z, JZ\}$ and its orthogonal complement $\Pi_Z^{\perp}$. If we denote by $Z^{\perp}$ the unit vector field collinear with the orthogonal projection of the vector field PZ onto the subspace $\Pi_Z^{\perp}$, then both $Z^{\perp}$ and $J(Z^{\perp})$ are orthogonal to Π_Z and we can write

$$PZ = aZ + bJZ + cZ^{\perp}, \quad a^2 + b^2 + c^2 = 1,$$

where a, b, c are smooth functions on $\mathbb{S}^3 \times \mathbb{S}^3$. There exist smooth angle functions φ, θ, defined along the trajectories of Z, such that $a = \sin\varphi\cos\theta$, $b = \sin\varphi\sin\theta$, $c = \cos\varphi$, so it holds

$$PZ = \sin\varphi\cos\theta\, Z + \sin\varphi\sin\theta\, JZ + \cos\varphi\, Z^{\perp}. \tag{11}$$

Let M be a hypersurface of $\mathbb{S}^3 \times \mathbb{S}^3$ with unit normal vector field ξ. If ν is one unit vector field collinear with the projection of $P\xi$ onto the orthogonal complement in $T(\mathbb{S}^3 \times \mathbb{S}^3)$ of the holomorphic plane spanned with ξ and $J\xi$, then it holds

$$P\xi = a\xi + bJ\xi + c\nu = \sin\varphi\cos\theta\,\xi + \sin\varphi\sin\theta\, J\xi + \cos\varphi\,\nu. \tag{12}$$

Besides the general characterisations of $\mathcal{P}-$principal and $\mathcal{P}-$isotropic vector fields, it is also true that the vector field ξ is:

- $\mathcal{P}-$principal iff $c = 0$, i.e. $\varphi = \frac{\pi}{2}$;
- $\mathcal{P}-$isotropic iff $a = b = 0$, i.e. $\varphi = 0$.

In [4] the authors classified all the hypersurfaces with $\mathcal{P}-$isotropic normal vector field. Also, in [2] the author investigated the hypersurfaces of $\mathbb{S}^3 \times \mathbb{S}^3$ such that $P\xi$ is collinear with ξ or $J\xi$, i.e. ξ is a special $\mathcal{P}-$principal vector field. We call a hypersurface M of $\mathbb{S}^3 \times \mathbb{S}^3$ a $\mathcal{P}-$**slant** hypersurface if the angle φ between $P\xi$ and holomorphic plane spanned by ξ and $J\xi$ is constant. Even though the angle functions θ and φ are not always constant, most of the known examples of hypersurfaces of $\mathbb{S}^3 \times \mathbb{S}^3$ actually have constant angle functions.

The first explicit examples of hypersurfaces of $\mathbb{S}^3 \times \mathbb{S}^3$ appeared in [6] and are defined by the immersions $f_{i,r} : \mathbb{S}^3 \times \mathbb{S}^2 \to \mathbb{S}^3 \times \mathbb{S}^3$, $i = 1, 2, 3$, $r \in (0, 1]$. The parametrisations of $M_{i,r} = f_{i,r}(\mathbb{S}^3 \times \mathbb{S}^2)$ are:

- $f_{1,r}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, (ry_1, ry_2, ry_3, \sqrt{1-r^2}))$, $\mathbf{x} \in \mathbb{S}^3$, $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{S}^2$;
- $f_{2,r}(\mathbf{x}, \mathbf{y}) = ((ry_1, ry_2, ry_3, \sqrt{1-r^2}), \mathbf{x})$, $\mathbf{x} \in \mathbb{S}^3$, $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{S}^2$;
- $f_{3,r}(\mathbf{x}, \mathbf{y}) = (\bar{\mathbf{x}}, (ry_1, ry_2, ry_3, \sqrt{1-r^2})\bar{\mathbf{x}})$, $\mathbf{x} \in \mathbb{S}^3$, $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{S}^2$.

These hypersurfaces of $\mathbb{S}^3 \times \mathbb{S}^3$ are immersions of the product of $\mathbb{S}^3$ and the sphere $\mathbb{S}^2$, as the simplest hypersurface of $\mathbb{S}^3$. If we replace $\mathbb{S}^2$ with Clifford tori $\mathbb{S}^1 \times \mathbb{S}^1$ embedded in $\mathbb{S}^3$ (see [7]), we obtain three more families $M_{i,k,l} = f_{i,k,l}(\mathbb{S}^3 \times \mathbb{S}^1 \times \mathbb{S}^1)$, $i = 1,2,3$, of hypersurfaces of $\mathbb{S}^3 \times \mathbb{S}^3$, defined by the immersions $f_{i,k,l}$, $f_{i,k,l} : \mathbb{S}^3 \times \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^3 \times \mathbb{S}^3$, $i = 1,2,3$, $k^2 + l^2 = 1$, $0 < k, l < 1$:

- $f_{1,k,l}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{y})$, $\mathbf{x} \in \mathbb{S}^3$, $\mathbf{y} = (y_1, y_2, y_3, y_4) \in \mathbb{S}^1(k) \times \mathbb{S}^1(l) \subset \mathbb{S}^3$;
- $f_{2,k,l}(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$, $\mathbf{x} \in \mathbb{S}^3$, $\mathbf{y} = (y_1, y_2, y_3, y_4) \in \mathbb{S}^1(k) \times \mathbb{S}^1(l) \subset \mathbb{S}^3$;
- $f_{3,k,l}(\mathbf{x}, \mathbf{y}) = (\bar{\mathbf{x}}, \mathbf{y}\bar{\mathbf{x}})$, $\mathbf{x} \in \mathbb{S}^3$, $\mathbf{y} = (y_1, y_2, y_3, y_4) \in \mathbb{S}^1(k) \times \mathbb{S}^1(l) \subset \mathbb{S}^3$.

The families $M_{i,r}$ and $M_{i,k,l}$, $i = 1,2,3$, have many similar properties - they are all Hopf, they have constant principal curvatures (3 for $M_{i,r}$ and 5 for $M_{i,k,l}$) and the almost product structure P satisfies one of the following relations: $P\xi = \frac{1}{2}\xi + \frac{\sqrt{3}}{2}J\xi$, $P\xi = \frac{1}{2}\xi - \frac{\sqrt{3}}{2}J\xi$ or $P\xi = -\xi$. Therefore, all these hypersurfaces have $\mathcal{P}-$principal normal vector field, with constant angle functions $\theta = \frac{\pi}{3}$, $\theta = -\frac{\pi}{3}$ or $\theta = \pi$. Actually, it holds that arbitrary hypersurface M with $\mathcal{P}-$principal normal belongs to one of the following three sets, whose normal ξ behaves in the same manner under the action of the almost product structure P as it was in the examples above (see the preprint [5]). Let $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ denote the sets of hypersurfaces with $\mathcal{P}-$principal normal, satisfying $P\xi = \frac{1}{2}\xi + \frac{\sqrt{3}}{2}J\xi$, $P\xi = \frac{1}{2}\xi - \frac{\sqrt{3}}{2}J\xi$, $P\xi = -\xi$, respectively. Observe that all these three families are fixed under the action of the group $\mathbb{F}$, since $\mathbb{F}$ is compatible with the structures J and P (see Proposition 3.7). Also, these three families are either fixed or mutually isometric under the action of the group $\mathbb{G}$. Namely, the following lemma holds:

Lemma 4.1.

$$\begin{array}{lllll} \mathcal{F}_1(\mathcal{M}_1) = \mathcal{M}_2, & \mathcal{F}_2(\mathcal{M}_1) = \mathcal{M}_3, & \mathcal{F}_3(\mathcal{M}_1) = \mathcal{M}_2, & \mathcal{F}_4(\mathcal{M}_1) = \mathcal{M}_3, & \mathcal{F}_5(\mathcal{M}_1) = \mathcal{M}_1, \\ \mathcal{F}_1(\mathcal{M}_2) = \mathcal{M}_1, & \mathcal{F}_2(\mathcal{M}_2) = \mathcal{M}_2, & \mathcal{F}_3(\mathcal{M}_2) = \mathcal{M}_3, & \mathcal{F}_4(\mathcal{M}_2) = \mathcal{M}_1, & \mathcal{F}_5(\mathcal{M}_2) = \mathcal{M}_3, \\ \mathcal{F}_1(\mathcal{M}_3) = \mathcal{M}_3, & \mathcal{F}_2(\mathcal{M}_3) = \mathcal{M}_1, & \mathcal{F}_3(\mathcal{M}_3) = \mathcal{M}_1, & \mathcal{F}_4(\mathcal{M}_3) = \mathcal{M}_2, & \mathcal{F}_5(\mathcal{M}_3) = \mathcal{M}_2. \end{array} \tag{13}$$

Proof. We will illustrate the proof of $\mathcal{F}_2(\mathcal{M}_1) = \mathcal{M}_3$ and $\mathcal{F}_3(\mathcal{M}_3) = \mathcal{M}_1$.

Let $M \in \mathcal{M}_1$ be an arbitrary hypersurface with the normal ξ such that $P\xi = \frac{1}{2}\xi + \frac{\sqrt{3}}{2}J\xi$ and let $M' = \mathcal{F}_2(M)$ be its image under the isometry $\mathcal{F}_2$. Then $\xi' = d\mathcal{F}_2(\xi)$ is a unit normal vector field of M'. Using formulae (10), we obtain $P\xi' = d\mathcal{F}_2\left(-\frac{1}{2}P\xi + \frac{\sqrt{3}}{2}JP\xi\right) = -\xi'$, so a hypersurface M' satisfies $M' \in \mathcal{M}_3$. Similarly, if $\tilde{M} \in \mathcal{M}_3$ is an arbitrary hypersurface with the normal $\tilde{\xi}$ which satisfies $P\tilde{\xi} = -\tilde{\xi}$ and $M'' = \mathcal{F}_2(\tilde{M})$ its image under isometry $\mathcal{F}_2^{-1} = \mathcal{F}_2$, then $\xi'' = d\mathcal{F}_2(\tilde{\xi})$ is its unit normal. Analogously as for M', we derive $P\xi'' = d\mathcal{F}_2\left(-\frac{1}{2}P\tilde{\xi} + \frac{\sqrt{3}}{2}JP\tilde{\xi}\right) = \frac{1}{2}\xi'' + \frac{\sqrt{3}}{2}J\xi''$, so a hypersurface M'' satisfies $M'' \in \mathcal{M}_1$.

For arbitrary hypersurface $M \in \mathcal{M}_3$ with the normal ξ such that $P\xi = -\xi$, let $M' = \mathcal{F}_3(M)$. Using formulae (10), for $\xi' = d\mathcal{F}_3(\xi)$, we obtain $P\xi' = d\mathcal{F}_3\left(-\frac{1}{2}P\xi - \frac{\sqrt{3}}{2}JP\xi\right) = \frac{1}{2}\xi' + \frac{\sqrt{3}}{2}J\xi'$, namely $M' \in \mathcal{M}_1$. Also, for a hypersurface $\tilde{M} \in \mathcal{M}_1$ and its normal $\tilde{\xi}$ which satisfies $P\tilde{\xi} = \frac{1}{2}\tilde{\xi} + \frac{\sqrt{3}}{2}J\tilde{\xi}$, we conclude that for its image $M'' = \mathcal{F}_4(\tilde{M})$ under the isometry $\mathcal{F}_4 = \mathcal{F}_3^{-1}$, its unit normal is $\xi'' = d\mathcal{F}_4(\tilde{\xi})$. Analogously as for M', we derive $P\xi'' = d\mathcal{F}_4\left(-\frac{1}{2}P\tilde{\xi} + \frac{\sqrt{3}}{2}JP\tilde{\xi}\right) = -\xi''$, so a hypersurface M'' satisfies $M'' \in \mathcal{M}_3$. $\square$

Consequently, in order to classify certain hypersurfaces of $\mathbb{S}^3 \times \mathbb{S}^3$ with $\mathcal{P}-$principal normal, it is enough to do that with one of the sets $\mathcal{M}_i$, $i = 1,2,3$, since using Lemma 4.1, the following proposition holds.

Proposition 4.2. *The set of all hypersurfaces of $\mathbb{S}^3 \times \mathbb{S}^3$ with $\mathcal{P}-$principal normal is preserved under the action of the groups $\mathbb{F}$ and $\mathbb{G}$.*

Now we will study the behaviour of the angle functions on arbitrary hypersurface M of $\mathbb{S}^3 \times \mathbb{S}^3$, under the action of the groups $\mathbb{F}$ and $\mathbb{G}$.

Theorem 4.3. *Let M be a hypersurface of $\mathbb{S}^3 \times \mathbb{S}^3$ with unit normal vector field ξ. The angle functions defined in* (12) *are preserved under the action of the group $\mathbb{F}$ on M, while under the action of the group $\mathbb{G}$ the angle function φ is preserved (up to a sign) and the angle function θ changes in the following way:*

$$\mathcal{F}_1 : \theta \mapsto -\theta, \quad \mathcal{F}_2 : \theta \mapsto \frac{4\pi}{3} - \theta, \quad \mathcal{F}_3 : \theta \mapsto \frac{4\pi}{3} + \theta, \quad \mathcal{F}_4 : \theta \mapsto \frac{2\pi}{3} + \theta, \quad \mathcal{F}_5 : \theta \mapsto \frac{2\pi}{3} - \theta. \tag{14}$$

Proof. We will illustrate the proof for $M' = \mathcal{F}_5(M)$, where M is an arbitrary hypersurface with a unit normal ξ and $\xi' = d\mathcal{F}_5(\xi)$ is a unit normal vector field of M'. From (10), we have $d\mathcal{F}_5 \circ J = -J \circ d\mathcal{F}_5$ and $d\mathcal{F}_5 \circ P_1 = P_2 \circ d\mathcal{F}_5$, so we can compute $d\mathcal{F}_5(P_1\xi)$, using (12) and $P_1 = P$, in two ways:

$$d\mathcal{F}_5(P\xi) = -\frac{1}{2}P\xi' - \frac{\sqrt{3}}{2}JP\xi' = a\xi' - bJ\xi' + c\tilde{\nu}, \tag{15}$$

where $\tilde{\nu} = d\mathcal{F}_5(\nu)$. Applying J on (15), we obtain

$$P\xi' = \left(-\frac{1}{2}a + \frac{\sqrt{3}}{2}b\right)\xi' + \left(\frac{\sqrt{3}}{2}a + \frac{1}{2}b\right)J\xi' + c\left(-\frac{1}{2}\tilde{\nu} + \frac{\sqrt{3}}{2}J\tilde{\nu}\right).$$

It is obvious that $\nu' = -\frac{1}{2}\tilde{\nu} + \frac{\sqrt{3}}{2}J\tilde{\nu}$ is a unit vector field orthogonal to the holomorphic plane $\mathrm{span}\{\xi', J\xi'\}$ and the following relation holds

$$P\xi' = \sin\varphi\cos\left(\frac{2\pi}{3} - \theta\right)\xi' + \sin\varphi\sin\left(\frac{2\pi}{3} - \theta\right)J\xi' + \cos\varphi\,\nu'.$$

□

The proof of Theorem 4.3 can be adapted to an arbitrary holomorphic plane, not only to the $\mathrm{span}\{\xi, J\xi\}$. Therefore, computing how the angle functions θ, φ defined in (11) change, we conclude that the angle function φ is preserved and obtain the following corollary.

Corollary 4.4. *The set of all $\mathcal{P}-$slant hypersurfaces of $\mathbb{S}^3 \times \mathbb{S}^3$ is preserved under the action of the groups $\mathbb{F}$ and $\mathbb{G}$. Specially, the set of hypersurfaces of $\mathbb{S}^3 \times \mathbb{S}^3$ with $\mathcal{P}-$isotropic normal is also preserved under these isometries.*

Since in the Corollary 3.8 we have already mentioned that the holomorphic planes are preserved under the isometries, using the formula

$$H(Z) = \frac{2}{3} - \frac{2}{3}(g^2(PZ, Z) + g^2(JPZ, Z)) = \frac{2}{3}\cos^2\varphi$$

derived in [2], we obtain the next corollary.

Corollary 4.5. *Holomorphic sectional curvature of arbitrary holomorphic plane of $\mathbb{S}^3 \times \mathbb{S}^3$ is preserved under the action of the groups $\mathbb{F}$ and $\mathbb{G}$.*

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