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Quivers associated with finite rings - a cohomological approach

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Abstract. In this paper we present the ongoing research on connection between digraphs associated to finite (commutative) rings and quiver representations. Digraph associated to a finite ring A has the set of vertices $V = A^2$ and arrows (or edges) $E = \{(x, y) \to (x + y, xy), x, y \in A\}$. In another terminology, it is a finite quiver with loops. In addition to previous work to understand these graphs, the main goal of the present work is to introduce some new cohomological and quiver methods. These methods should provide us with better understanding of properties and classification of finite rings.

Consider the (small) category of directed graphs DG. We shall use the term "graph" interchangeably with "digraph", since we are all the time working in DG.

Definition 1. *The graph G is a functional graph if for every vertex $a \in V(G)$ there exist only one arrow or edge $v \in E(G)$, $v : a \to b$ from a.*

This means that there is a bijection $V(G) \to E(G)$, $a \mapsto v$. Let $FG \subset DG$ be the full subcategory of functional graphs.

In yet another terminology, digraph $G = Q$ is a quiver. A quiver is exactly a directed graph, possibly with multiple arrows, loops and cycles, $V(G) = Q_0$, $E(G) = Q_1$ and for an arrow $v : a \to b$, $a = s(v)$ and $b = h(v)$ (the *source* or tail and the *head* of arrow v). If G is functional, $Q_1 = Q_0$.

The problem in which we are interested is description of the structure of a special functional graph, introduced in [1]. Let A be a finite commutative ring with unity, with n elements. Then $A^2 = A \times A$ is also a commutative ring with unity in a standard way, with n^2 elements. Define a mapping

$$\varphi : A^2 \to A^2$$

by

$$\varphi(\alpha) = \varphi((x, y)) = (x + y, xy)$$

for $\alpha \in A^2$. There is a general feeling that this mapping should somehow reflect the ring structure of A. Functional graph $G(A)$ is the graph defined by vertices $V = A^2$ and arrows $E = \{(x, y) \to (x + y, xy), x, y \in A\} \cong V$. In the sequel it will be called a *ring graph*. Some preliminary results are published in [2,3]. For $A = \mathbb{Z}_n$ there is an interesting occurence of longer cycles in these graphs [3].

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1. Standard digraph homology

Let $Q = (V, E)$ be a directed graph (or a quiver) with set of vertices $V = Q_0$ and set of edges (or arrows) $E = Q_1$, with two standard mappings, source $s : E \to V$ and head $h : E \to V$. Let $F_0 = \mathbb{Z}Q_0$ be the free $\mathbb{Z}$-module (i.e. Abelian group) generated by $V = Q_0$ and $F_1 = \mathbb{Z}Q_1$ the free $\mathbb{Z}$-module generated by $E = Q_1$. If we consider G as a topological space in a standard way, there is a standard chain complex

$$... \longleftarrow 0 \xleftarrow{d_0} F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} 0 \longleftarrow ...$$

where the homomorphism d_1 is defined on generators by $d_1(v) = h(v) - s(v)$. Obviously, $d_0 = 0$ and $Kerd_0 = F_0$, and $a \in Kerd_1$ if and only if a is the sum of arrows in a cycle (i.e. a closed directed path) in G. The usual definition of homology gives us $H_0(G) = Kerd_0/Imd_1 = Coimd_1$, $H_1(G) = Kerd_1/Imd_2 = \mathbb{Z}[a, a \text{ is a cycle}]$ (and all other homology groups being 0). The group $H_1(G)$ is the cycle group. If the graph G is finite, with n vertices, m arrows, and c connected components, then $H_0(G) \cong \mathbb{Z}^c$ and $H_1(G) \cong \mathbb{Z}^{m-n+c}$. For a finite functional graph, $m = n$ and the rank of $H_1(G)$ is c. The number c of connected components is also the number of all cycles (including loops, i.e. cycles of length 1), since every component must end in a unique cycle. Actually, this is the only topological property of such graph that can be traced by its standard homology. Note that connected components of ring graph $G(A)$ are also functional graphs. For details one can see for example [4].

2. Digraph "line cohomology"

Homological approach is fruitful when there is a natural increase of topological dimension. However, in our case one deals only with line structures - paths. Therefore, for these very specific graphs we try to introduce concept similar to "cohomology", with purpose to achieve control over various path length functions.

Definition 2. *A path λ of length k is a sequence*

$$\alpha \to \varphi(\alpha) \to \varphi^2(\alpha) \to \cdots \to \varphi^k(\alpha).$$

where $\alpha \in A^2$. It is a path in the digraph G of length k. The set of all paths of a given length k is $L_k(A) = L_k$. Path of length k has $k+1$ vertices and k arrows. One has $L_0 = A^2 = V(G)$, $L_1 = E(G)$ ($\cong V(G)$ for a functional graph G). In the case of ring graph, all L_k are bijective with A^2, all have n^2 elements and each path λ is uniquely determined by its starting vertex α.

Standard concatenation introduces a partial binary operation $L \times L \to L$, $L_k \times L_m \to L_{k+m}$ (two paths λ and μ can be concatenated $\lambda\mu$ iff $h(\lambda) = s(\mu)$) on the set of all paths $L = \bigcup_{k \geq 0} L_k$. This operation can be extended to full binary operation on L by zero (if two paths λ and μ are such that $h(\lambda) \neq s(\mu)$, set $\lambda\mu = 0$). One can consider other operations on paths in a digraph: right cancellation $r = r_k : L_k \to L_{k-1}$, left cancellation $l = l_k : L_k \to L_{k-1}$ and, in the case of functional graphs, extension $d = d_k : L_k \to L_{k+1}$ (concatenation $L_k \times L_1 \to L_{k+1}$). Extension d and right cancellation r are mutually inverse: $d_k \circ r_{k+1} = id$, $r_{k+1} \circ d_k = id$, $L_k \cong Imd_k = L_{k+1}$, however composition of extension and left cancellation produces right shift of paths.

Definition 3. *A path $\lambda \in L_k$ is singular if it has two equal vertices, i.e. if for some $0 \leq i < j \leq k$, $\varphi^i(\alpha) = \varphi^j(\alpha)$. In the opposite case, i.e. when all vertices are distinct, a path is regular. Let i be the smallest such index in a singular path λ, and j the next one i.e. such that the difference $m = j - i$ is minimal. The path $\lambda' : \varphi^i(\alpha) \to \cdots \to \varphi^{m-1}(\alpha)$ is regular and its extension $\varphi^i(\alpha) \to \cdots \to \varphi^m(\alpha)$ is a subcycle of λ. A singular path $\lambda \in L_k$ is called cycle if the first and the last vertex are equal and all other vertices are distinct.*

Singular paths in L_k form a subset S_k, cycles - a subset $C_k \subset S_k$, regular paths - a subset $R_k \subset L_k$ and $L_k = S_k \cup R_k$, $S_k \cap R_k = \emptyset$, $C_k \subset S_k \subset L_k$. Define $S_0 = \emptyset$ and therefore $R_0 = V(G)$. It is easy to see that in a ring graph, $C_1 = S_1 = \{(x, 0) \to (x, 0) \mid x \in A\}$ has n elements and R_1 has $n^2 - n = n(n-1)$ elements. If we define $\sigma_k = \#S_k$, $\rho_k = \#R_k$, then for all k, $\rho_k + \sigma_k = n^2$ and, in particular, $\rho_k = \rho_{k+1} + \Delta\sigma_k$ where $\Delta\sigma_k = \sigma_{k+1} - \sigma_k$.

Lemma 1. *Cancellation maps send regular paths to regular paths $r_k, l_k : R_k \to R_{k-1}$. Extension map sends singular paths to singular paths $d_k : S_k \to S_{k+1}$. Moreover, if $\gamma_k = \#C_k$, then $C_{k+1} \subset d_k(R_k) \cap S_{k+1}$ and $\gamma_{k+1} \le \Delta\sigma_k$. It is clear that $\gamma_1 = n$, these are the n loops of length 1.* □

So, d_k may map regular paths to singular paths i.e. $d_k(R_k) \cap S_{k+1} \neq \emptyset$ and their number is $\Delta\sigma_k$. In the set-theoretic setting, $R_k = R_{k+1} \cup \Delta S_k$, $R_{k+1} \cap \Delta S_k = \emptyset$, where ΔS_k is the set of all new singular paths in L_{k+1} which have originated by d_k from regular paths in L_k. In other words, $\Delta S_k = d_k(R_k) \setminus R_{k+1}$.

In $\mathbb{Z}L_k$, the two subgroups generated by singular and by regular paths of length k are $\mathbb{Z}S_k$, $\mathbb{Z}R_k$, they are also free, and $\mathbb{Z}L_k = \mathbb{Z}S_k \oplus \mathbb{Z}R_k$ splits. The ranks of the two subgroups are σ_k and ρ_k, and $\rho_k + \sigma_k = n^2$. These groups form the *singular path sequence*

$$\mathbb{Z}S_0 \cong 0 \to \mathbb{Z}S_1 \cong \mathbb{Z}^n \to \mathbb{Z}S_2 \to \dots$$

and the *regular path sequence*

$$F_0 = \mathbb{Z}R_0 \cong \mathbb{Z}L_0/\mathbb{Z}S_0 \cong \mathbb{Z}^{n^2} \to F_1 = \mathbb{Z}R_1 \cong \mathbb{Z}L_1/\mathbb{Z}S_1 \cong \mathbb{Z}^{n(n-1)} \to F_2 = \mathbb{Z}R_2 \cong \mathbb{Z}L_2/\mathbb{Z}S_2 \to \dots$$

where homomorphisms (which are in the sequel also denoted by d_i) are compositions $\mathbb{Z}R_k \hookrightarrow \mathbb{Z}L_k \xrightarrow{d_i} \mathbb{Z}L_{k+1} \twoheadrightarrow \mathbb{Z}L_{k+1}/\mathbb{Z}S_{k+1} \cong \mathbb{Z}R_{k+1}$. After a finite number of steps, the first sequence stabilises and the second sequence becomes zero, because after a finite number of extensions every path becomes singular and the generators gradually dissapear. The biggest $k = m$ after which there are only zeros is the length of the longest regular path. In the next step the path would become singular, i.e. would end with a cycle. Note that these are not real cochain sequences, since in general $d^2 \neq 0$.

Proposition 1. *For rings $A = \mathbb{Z}_n$, $n = 2, ..., 9$, the regular path sequences $F_0 \cong \mathbb{Z}^{n^2} \to F_1 \cong \mathbb{Z}^{n(n-1)} \to F_2 \to \dots$ are:*
$\mathbb{Z}^4 \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$ *for $n = 2$ with the length $m = 2$.*
$\mathbb{Z}^9 \to \mathbb{Z}^6 \to \mathbb{Z}^4 \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$ *for $n = 3$ with the length $m = 4$.*
$\mathbb{Z}^{16} \to \mathbb{Z}^{12} \to \mathbb{Z}^6 \to \mathbb{Z}^2 \to 0$ *for $n = 4$. The length is $m = 3$.*
$\mathbb{Z}^{25} \to \mathbb{Z}^{20} \to \mathbb{Z}^{16} \to \mathbb{Z}^{12} \to \mathbb{Z}^5 \to \mathbb{Z}^2 \to 0$ *for $n = 5$. The length is $m = 5$.*
$\mathbb{Z}^{36} \to \mathbb{Z}^{30} \to \mathbb{Z}^{21} \to \mathbb{Z}^8 \to \mathbb{Z}^4 \to 0$ *for $n = 6$. The length is $m = 4$.*
$\mathbb{Z}^{49} \longrightarrow \mathbb{Z}^{42} \longrightarrow \mathbb{Z}^{36} \longrightarrow \mathbb{Z}^{30} \longrightarrow \mathbb{Z}^{25} \longrightarrow \mathbb{Z}^{21} \longrightarrow \mathbb{Z}^{17} \longrightarrow \mathbb{Z}^{12} \longrightarrow \mathbb{Z}^7 \longrightarrow 0$ *for $n = 7$. The length is $m = 8$.*
$\mathbb{Z}^{64} \longrightarrow \mathbb{Z}^{56} \longrightarrow \mathbb{Z}^{40} \longrightarrow \mathbb{Z}^{28} \longrightarrow \mathbb{Z}^{16} \longrightarrow \mathbb{Z}^8 \longrightarrow 0$ *for $n = 8$. The length is $m = 5$.*
$\mathbb{Z}^{81} \longrightarrow \mathbb{Z}^{72} \longrightarrow \mathbb{Z}^{54} \longrightarrow \mathbb{Z}^{36} \longrightarrow \mathbb{Z}^{15} \longrightarrow \mathbb{Z}^9 \longrightarrow 0$ *for $n = 9$. The length is $m = 5$.*

Proof. Proof is by direct calculation. □

3. Cycles

Let's look at the properties of left cancellation l.

Proposition 2. *Let the path $\lambda \in S_k$ be singular. Then λ is a cycle if and only if left cancellation sends $\lambda \in S_k$ to a regular path $l(\lambda) \in R_{k-1}$.*

Proof. Namely, if the path obtained by left cancellation is regular, then the source vertex of the original path has to coincide with the tail vertex $t(\lambda) = s(\lambda)$ and all other vertices are distinct (if not, then the next pair of vertices obtained by extension d would also coincide and the path $l(\lambda)$ is not regular) .

Therefore, for $\lambda \in S_k$, $l(\lambda) \in R_{k-1} \iff \lambda \in C_k$, or $C_k = l^{-1}(R_{k-1}) \cap S_k$. In a descriptive way, cycles of length k are exactly those singular paths of length k obtained from regular paths of length $k-1$ by their extension to the left of the source $s(\lambda)$. □

For $k \geq 1$ consider the sequences

$$... \to \mathbb{Z}L_{k-1} \overset{d_{k-1}}{\to} \mathbb{Z}L_k \overset{d_k}{\to} \mathbb{Z}L_{k+1} \to ...,$$
$$... \leftarrow \mathbb{Z}L_{k-1} \overset{r_k}{\leftarrow} \mathbb{Z}L_k \overset{r_{k+1}}{\leftarrow} \mathbb{Z}L_{k+1} \leftarrow ... \text{ and}$$
$$... \leftarrow \mathbb{Z}L_{k-1} \overset{l_k}{\leftarrow} \mathbb{Z}L_k \overset{l_{k+1}}{\leftarrow} \mathbb{Z}L_{k+1} \leftarrow ....$$

and the subgroup structure $\mathbb{Z}L_k = \mathbb{Z}S_k \oplus \mathbb{Z}R_k$. It is clear that $F_k = \mathbb{Z}R|_k \overset{d_k}{\to} \mathbb{Z}R_{k+1} = F_{k+1}$ is epi, and that the exact sequence $0 \to d_k^{-1}(\mathbb{Z}S_{k+1}) \cap \mathbb{Z}R_k \to \mathbb{Z}R_k \overset{d_k}{\to} \mathbb{Z}R_{k+1} \to 0$ splits by $\mathbb{Z}R_k \overset{r_{k+1}}{\leftarrow} \mathbb{Z}R_{k+1}$. Then $F_k = \mathbb{Z}R_k = d_k^{-1}(\mathbb{Z}S_{k+1}) \cap F_k \oplus F'_k$, where $F'_k \cong F_{k+1} = \mathbb{Z}R_{k+1}$ by r_{k+1}. Factoring out the "kernels" of d_k we obtain the sequence

$$... \to F_{k-1}/d_{k-1}^{-1}(\mathbb{Z}S_k) \cap F_{k-1} \overset{d'_{k-1}}{\to} F_k/d_k^{-1}(\mathbb{Z}S_{k+1}) \cap F_k \overset{d'_k}{\to} F_{k+1}/d_{k+1}^{-1}(\mathbb{Z}S_{k+2}) \cap F_{k+1} \to ... \text{ or}$$
$$... \to F'_{k-1} \overset{d'_{k-1}}{\to} F'_k \overset{d'_k}{\to} F'_{k+1} \to ... \text{ or}$$
$$... \to F_k \overset{d'_{k-1}}{\to} F_{k+1} \overset{d'_k}{\to} F_{k+2} \to ....$$

If $\beta \in L_{k+1}$ is a cycle, then it is singular and $r_{k+1}(\beta) = \alpha \in d_k^{-1}(\mathbb{Z}S_{k+1}) \cap F_k \subset \mathbb{Z}L_k$. One can see that α comes from a cycle $\Longleftrightarrow d_{k-1}(l_k(\alpha)) \notin S_k$ or $l_k(\alpha) \notin d_{k-1}^{-1}(\mathbb{Z}S_k) \cap F_{k-1}$, or $l_k(\alpha) \notin d_{k-1}^{-1}(\mathbb{Z}S_k) \cap F_{k-1}$ in $F_{k-1}/d_{k-1}^{-1}(\mathbb{Z}S_k) \cap F_{k-1} \cong F_k$. Exactly k elements in $d_k^{-1}(\mathbb{Z}S_{k+1}) \cap F_k$ come from the same cycle, since the composition $d_{k-1}l_k$, as already noted, produces a shift along cycle which has exactly k different elements. The k-cocycle group is $Z_k = Iml_{k+1}/d_k^{-1}(\mathbb{Z}S_{k+1}) \cap F_k < F_k/d_k^{-1}(\mathbb{Z}S_{k+1}) \cap F_k \cong F_{k+1}$ $(k = 0, 1, \dots)$. It is a subgroup of a free Abelian group, therefore free Abelian itself, of the rank less than or equal to $rankF_{k+1} = \rho_{k+1}$. Obviously, $Z_0 = Iml_1/Kerd_0 \cong F_1 \cong \mathbb{Z}^n$ since $l_1 : L_1 \to L_0$, $d_0 : L_0 \to L_1$ and Iml_1 is the set of all vertices which are in Imd_0.

Definition 4. *Z_k is the k-th line cohomology group of the ring graph $G(A)$.*

The generators of Z_k are exactly the paths which in the next step by extension d produce k-cycles. Its rank is divisible by k. Consider the graded group $Z = \bigoplus_{k\geq 0} Z_k$. It is clear that $Z_0 = \mathbb{Z}^n$ and $Z_k = 0$ for sufficiently big k.

Definition 5. *The graded group $Z(A) = Z = \bigoplus_{k\geq 0} Z_k$ is the (total) line cohomology group of the ring graph $G(A)$.*

We calculate this group for particular rings A.

Proposition 3. *For rings $A = \mathbb{Z}_n$, $n = 2, ..., 6$ one has the following:*
$A = \mathbb{Z}_2$. $Z_0 = \mathbb{Z}^2$, $Z_1 = 0$, two generators of length 1 (there are $2 = 2/1$ 1-cycles);
$A = \mathbb{Z}_3$. $Z_0 = \mathbb{Z}^3$,$Z_1 = 0$, three generators of length 1 (there are $3 = 3/1$ 1-cycles);
$A = \mathbb{Z}_4$. $Z_0 = \mathbb{Z}^4$, $Z_1 = \mathbb{Z}^2$, $Z_2 = 0$, four generators of length 1 (there are $4 = 4/1$ 1-cycles) and two generators of length 2 (there is $1 = 2/2$ 2-cycle);
$A = \mathbb{Z}_5$. $Z_0 = \mathbb{Z}^5$, $Z_1 = Z_2 = Z_3 = 0$, $Z_4 = \mathbb{Z}^4$, $Z_5 = 0$, 5 generators of length 1 (there are $5 = 5/1$ 1-cycles, there is $1 = 4/4$ 4-cycle);
$A = \mathbb{Z}_6$. $Z_0 = \mathbb{Z}^6$,$Z_1 = 0$, six generators of length 1 (there are $6 = 6/1$ 1-cycles).
The corresponding line cohomology graded groups are $Z(\mathbb{Z}_2) = \mathbb{Z}^2$, $Z(\mathbb{Z}_3) = \mathbb{Z}^3$, $Z(\mathbb{Z}_4) = \mathbb{Z}^4 \oplus \mathbb{Z}^2$, $Z(\mathbb{Z}_5) = \mathbb{Z}^5 \oplus 0 \oplus 0 \oplus 0 \oplus \mathbb{Z}^4$, $Z(\mathbb{Z}_6) = \mathbb{Z}^6$.
The Hilbert-Poincare polynomials corresponding to graded groups $Z(\mathbb{Z}_n)$ for $n - 2, 3, 4, 5, 6$ are $P_{Z(\mathbb{Z}_2)}(t) = 2$, $P_{Z(\mathbb{Z}_3)}(t) = 3$, $P_{Z(\mathbb{Z}_4)}(t) = 4 + 2t$, $P_{Z(\mathbb{Z}_5)}(t) = 5 + 4t^4$, $P_{Z(\mathbb{Z}_6)}(t) = 6$.

Proof. Proof is by direct calculation. □

4. Additivity and functoriality

Every ring graph is a disjoint union of its connected components, which itself are functional graphs. It is easy to see that there is certain additivity with respect to previous constructions.

Proposition 4. *The regular and the singular path sequences of a functional graph G are direct sums of the corresponding complexes of its components.*

Proof. Proof follows directly from two easy observations. First, if a generating set $L = L' \cup L''$ is a disjoint union, then $\mathbb{Z}L = \mathbb{Z}L' \oplus \mathbb{Z}L''$, and second, paths containing vertices in different components of a functional graph form disjoint path subsets. □

Example 1. *The graph $G(\mathbb{Z}_2)$ has two connected components: the A_1-type loop $\cdot \circlearrowleft$ and the A_3-type graph $\cdot \to \cdot \to \cdot \circlearrowleft$. Corresponding graded regular path sequences are $\mathbb{Z} \to 0$ and $\mathbb{Z}^3 \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$, and their sum is $\mathbb{Z}^4 \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$.*

Now, let's consider functoriality. Let $h : A \to B$ be a ring homomorphism and let the corresponding graphs be $G(A)$ and $G(B)$. Consider the functoriality of our constructions. Obviously, h induces homomorphism of graphs $G(h) : G(A) \to G(B)$.

Proposition 5. *The construction of regular and singular path sequences and the construction of line cohomology group are functorial with respect to $h : A \to B$.*

Proof. Proof is straightforward. □

Further investigation of functoriality gives us also the result concerning the products of rings.

Proposition 6. *The graph ring of products of rings is isomorphic to the product of corresponding graphs.*

Proof. Proof is straightforward, based on the previous considerations. □

In connection with classical Wedderburn-Artin theorem, we obtain further insight in ring graphs: every ring graph is a product of ring graphs of matrix rings over finite fields. If we apply commutativity, we obtain a product of ring graphs of finite fields (all matrix dimensions have to be = 1), which shows the importance of the above results for $\mathbb{Z}_n$. This also suggests the importance of graphs of matrix rings over finite fields, which should be more thoroughly investigated. It would be interesting to extend the notion of ring graph to noncommutative rings. The first attempt, based on the map $(x, y) \to (x + y, xy - yx)$, has already been considered in [5], but actually showed no great promises.

5. The path algebra

Definition 6. *The path algebra of the digraph G is the free Abelian group $\mathbb{Z}L$ generated by all paths, graded by subgroups of paths of fixed length $\mathbb{Z}L_k$, $\mathbb{Z}L = \bigoplus_{k\geq 0} \mathbb{Z}L_k$.*

For a ring digraph, all subgroups are isomorhic to $\mathbb{Z}^{n^2}$. It is a graded algebra over $\mathbb{Z}$ with concatenation $d : \mathbb{Z}L_k \times \mathbb{Z}L_m \to \mathbb{Z}L_{k+m}$ (if two paths λ and μ are such that $s(\lambda) \neq t(\mu)$, set $\mu\lambda = 0$). The extension homomorphism $d : \mathbb{Z}L \to \mathbb{Z}L$ will also be denoted by d. It is graded of degree 1.

If one interprets digraph as quiver $G = Q$, this is the quiver integer path algebra $\mathbb{Z}Q$. This quiver has directed cycles and is wild, and the rank of the corresponding algebra is infinite. However, we could also consider the *regular path algebra* $F = \mathbb{Z}R = \bigoplus_{k\geq 0} \mathbb{Z}R_k = \bigoplus_{k\geq 0} F_k$ where $rank \mathbb{Z}R_k = \rho_k$.

Proposition 7. *Hilbert-Poincare polynomials $P_A(t) = \sum_{k\in\mathbb{N}_0} \rho_k \cdot t^k$ of corresponding graded algebras F for $n = 2, ..., 9$ are as follows:*

$P_{\mathbb{Z}_2}(t) = 4 + 2t + t^2;$
$P_{\mathbb{Z}_3}(t) = 9 + 6t + 4t^2 + 2t^3 + t^4;$
$P_{\mathbb{Z}_4}(t) = 16 + 12t + 6t^2 + 2t^3;$
$P_{\mathbb{Z}_5}(t) = 25 + 20t + 16t^2 + 12t^3 + 5t^4 + 2t^5;$
$P_{\mathbb{Z}_6}(t) = 36 + 30t + 21t^2 + 8t^3 + 4t^4;$
$P_{\mathbb{Z}_7}(t) = 49 + 42t + 36t^2 + 30t^3 + 25t^4 + 21t^5 + 17t^6 + 12t^7 + 7t^8;$
$P_{\mathbb{Z}_8}(t) = 64 + 56t + 40t^2 + 28t^3 + 16t^4 + 8t^5;$
$P_{\mathbb{Z}_9}(t) = 81 + 72t + 54t^2 + 36t^3 + 15t^4 + 9t^5.$

All polynomials $P_A(t)$ have critical point in the interval $[-1, -1/2)$ and the first three of them exactly at $t = -1$.

Proof. Proof is straightforward calculation. □

Proposition 8. *The Hilbert-Poincare polynomial is additive with respect to connected components of the graph.*

Proof. Proof follows from the additivity of corresponding graded path sequences. □

Example 2. *The graph $G(\mathbb{Z}_2)$ has two connected components: the A_1-type loop $\cdot \circlearrowleft$ and the A_3-type graph $\cdot \to \cdot \to \cdot \circlearrowleft$. Corresponding Hilbert-Poincare polynomials are $P_{A_1}(t) = 1, P_{A_3}(t) = 3 + 2t + t^2$ and the sum $P_{\mathbb{Z}_2}(t) = P_{A_1}(t) + P_{A_3}(t) = 4 + 2t + t^2$. Note that polynomials which correspond to components do not have $t = -1$ as critical point.*

6. Quiver representations

Let K be an algebraically closed field. All previous constructions remain valid if one replaces integers $\mathbb{Z}$ with field K. Instead of free Abelian groups, one obtains K-vector spaces, instead of integer quiver algebra $\mathbb{Z}Q$, the quiver algebra KQ over the field K, the rank of free group is replaced by dimension of K-vector space. A *representation* of a quiver Q is a functor from Q (as small category) to K-vector spaces, or a Q-diagram in the category $Vect_K$. For more details see [6]. A few examples follow.

Example 3. *1) For the one-arrow A_2-quiver $\cdot \to \cdot$, a representation is a pair of vector spaces together with a linear mapping $L : V_1 \to V_2$. Two representations $L : V_1 \to V_2$ and $M : W_1 \to W_2$ are isomorphic if and only if* $\dim V_1 = \dim W_1$, $\dim V_2 = \dim W_2$ *and $rankL = rankM$.*

2) For the A_1-type loop quiver $\cdot \circlearrowleft$ (one vertex and one arrow), a representation is an endomorphism $L : V \to V$ and two such representations $L : V \to V$ and $M : V \to V$ are isomorphic if and only if L and M have the same Jordan normal form.

A representation is *trivial*, if all corresponding vector spaces are O. It is *reducible*, if it is a sirect sum of two nontrivial representations (as diagrams in $Vect_K$), and *irreducible* otherwise.

It would be interesting to find irreducible representations of ring quivers (i.e. ring graphs).

There are some other algebraic tools for the investigation of these quivers. Consider the asymmetric Ringel (or Euler) form of the quiver Q given by $\langle x, y\rangle := \sum_{v\in Q_0} x_v y_v - \sum_{e\in Q_1} x_{h(e)} y_{s(e)}$, $(x, y \in \mathbb{Z}^{Q_0})$, the symmetric Cartan form $(\alpha, \beta) := \langle\alpha, \beta\rangle + \langle\beta, \alpha\rangle$, and the corresponding Tits quadratic form $q_Q : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ $q_Q(x) := \sum_{v\in Q_0} x_v^2 - \sum_{e\in Q_1} x_{h(e)} x_{s(e)} = \sum_{v\in Q_0} x_\alpha^2 - \sum_{v\in Q_0} x_{\varphi(v)} x_v = \sum_{v\in Q_0} \left(x_v^2 - x_{\varphi(v)} x_v\right) = \sum_{v\in Q_0} x_v \left(x_{\varphi(v)} - x_v\right)$ where φ is the base mapping $v = (\alpha, \beta) \mapsto \varphi(v) = (\alpha + \beta, \alpha\beta)$. For small components of $G(A)$ (see [1]) this is

$q_Q(x) = 0$ for the loop A_1-quiver $\cdot \circlearrowleft$,
$q_Q(x) = x_1^2 - x_1 x_2$ for the A_2-quiver $\cdot \to \cdot \circlearrowleft$.

The Tits quadratic form is additive with respect to connected components of the graphs involved. Since the quivers are wild, the corresponding Tits quadratic forms are not definite (by Gabriel's theorem). Componentwise additivity holds also for corresponding quadratic forms.

7. Conclusions

The present paper is an expanded version of the talk [7]. Some new methods are introduced and developed. The authors do hope that the road marked by this paper would lead to some definite and interesting results on ring graphs.

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