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On the rigidity and analytical rigidity of two-connected regular surfaces of revolution for a given direction of displacement of edge points

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Abstract. In this paper, we investigate infinitesimal bends of two-connected surfaces of revolution on which are given conditions that allow the points of one of the boundary parallels to move only along a given constant direction. We formulated the obtained results in the form of a theorem.

1. Introduction

With the development of the theory of bending of surfaces, studies of infinitesimal bends of higher orders are very important since they are closely related to the problem of the continuation of infinitesimal bends into continuous (in particular, analytical) bending of surfaces. This task, set by S. Cohn-Vossen in 1929 [4, 5], has not yet received a final solution. This is explained by the fact that the study of infinitesimal bends of higher orders leads to the study of rather cumbersome systems of partial differential equations.

Among the results obtained in this area, the result of N. V. Efimov deserves particular attention [6, 7]. He proved that the rigidity of the first and second orders of surfaces implies its analytical rigidity. In this regard, the question naturally arises about the possibility of continuing infinitesimal bends into continuous bends, provided that certain connections are imposed on the surface.

The great interest shown in the theory of infinitesimal bends of surfaces is explained, on the one hand, by deep connections with such branches of mathematics as the theory of differential and integral equations, the theory of generalized analytic functions, etc. On the other hand, the important applications that this theory has received in mechanics, especially in the theory of thin shells, since it is known that every non-trivial infinitesimal bending of the middle surface of the shell corresponds to an infinite stress state of this shell, unloaded by surface load, and per revolution, every momentarily stressed state of an unloaded shell corresponds to a field of rotation of an infinitesimal bending.

Important results in the theory of infinitesimal bends were obtained in the works of W. Blaschke [2], E. Rembs [25, 26], T. Minagawa, T. Rado [19], K.P. Grotemeyer [8], S. Hellwing [9], S. Baudoin-Gohier [3], A.V.

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Pogorelov [22, 23], E.G. Poznyak [24], V.I. Mihailovskii and others [12–16]. Nowadays, e.g., Lj.S. Velimirović and others [1, 11, 20, 21, 28]. Recently, more general infinitesimal transformations have been studied, for example, in works by J. Mikeš and others [10, 17, 18, 27]. Many above results are devoted to the rigidity of surfaces under different conditions, which are of great practical importance.

The work of these scientists greatly stimulated the further development of the theory of bending surfaces. Thus, it follows from the above that this article is of interest, both from the point of view of its deep geometric content and from the point of view of applications. Therefore it is one of the very relevant topics of modern differential geometry. At the same time, it should be noted that, at present, there are few papers in this field in which boundary value problems for infinitesimal bends of higher orders have been studied.

In our work, we continue to study the bending of surfaces of revolution and their rigidity under interesting non-standard boundary conditions.

2. Main Theorem

In the proposed work, we investigate infinitesimal bends of the first and second orders of two-connected regular surfaces of revolution Φ, on which connections are imposed that allow moving points of one of the boundary parallels only along a given constant direction $\boldsymbol{v}$.

Theorem 2.1. *Let Φ be an arbitrary regular two-connected surface of revolution (of differentiability class C^2), whose meridian has no points, whose tangents are perpendicular to the axis of rotation, and γ_0 is one of the two parallels bounding it. If we impose connections on such a surface that allow the movement of parallel points γ_0 only in the same constant direction $\boldsymbol{v}$, then the surface Φ in such a class of deformations will have rigidity no higher than the second order, and therefore will be analytically rigid.*

As is known [14], if the vector $\boldsymbol{v}$ is not parallel to the axis of rotation, then the surface Φ in the class of deformations under consideration has a rigidity of the first order, and therefore [6, 7], is analytically rigid. If the vector $\boldsymbol{v}$ is parallel to the axis of rotation, then the surface remains non-rigid, i.e. admits non-trivial infinitesimal bends of the first order. We prove that these infinitesimal first-order bends of the surface Φ cannot be continued into non-trivial infinitesimal second-order bends. Hence, according to the theorem of N.V. Efimov [6, 7], it follows that the surface Φ in the deformation class under consideration is analytically rigid.

3. Infinitesimal bending of the second order

Let $\boldsymbol{x} = \boldsymbol{x}(u, v)$, $(u, v) \in D$, be a regular parametrization of the surface Φ.

Assume that the surface Φ in the process of *infinitesimal deformation of the second order* to go to the surface Φ^* [6]:

$$\boldsymbol{x}^*(u, v, t) = \boldsymbol{x}(u, v) + 2\varepsilon \overset{1}{\boldsymbol{z}}(u, v) + 2\varepsilon^2 \overset{2}{\boldsymbol{z}}(u, v), \tag{1}$$

where $\overset{1}{\boldsymbol{z}}(u, v)$ and $\overset{2}{\boldsymbol{z}}(u, v)$ are regular vector functions, ε is a deformation parameter of the surface Φ.

In order for $\overset{1}{\boldsymbol{z}}(u, v)$ and $\overset{2}{\boldsymbol{z}}(u, v)$ to be some fields that define an infinitesimal second-order bending of the surface Φ, it is necessary and sufficient that the vector functions $\overset{1}{\boldsymbol{z}}(u, v)$ and $\overset{2}{\boldsymbol{z}}(u, v)$ satisfy the system of equations [7]:

$$\text{(a)} \quad (d\boldsymbol{x}, d\overset{1}{\boldsymbol{z}}) = 0, \qquad \text{(b)} \quad (d\boldsymbol{x}, d\overset{2}{\boldsymbol{z}}) + d\overset{1}{\boldsymbol{z}}{}^2 = 0. \tag{2}$$

As you know [7] that the vector-functions $\overset{1}{\boldsymbol{z}}(u, v)$ and $\overset{2}{\boldsymbol{z}}(u, v)$ correspond to these vector functions $\overset{1}{\boldsymbol{y}}(u, v)$ and $\overset{2}{\boldsymbol{y}}(u, v)$, there are equalities:

$$d\overset{1}{\boldsymbol{z}} = [\overset{1}{\boldsymbol{y}}, d\boldsymbol{x}] \quad \text{and} \quad d\overset{2}{\boldsymbol{z}} = [\overset{2}{\boldsymbol{y}}, d\boldsymbol{x}] + [\overset{1}{\boldsymbol{y}}, d\overset{1}{\boldsymbol{z}}]. \tag{3}$$

The system of vector fields $\overset{1}{y}(u,v)$ and $\overset{2}{y}(u,v)$ is uniquely determined by the system of vector fields $\overset{1}{z}(u,v)$ and $\overset{2}{z}(u,v)$. Conversely, if there is a set of functions $\overset{1}{y}(u,v)$ and $\overset{2}{y}(u,v)$ satisfying equations (3), then equations (2) will be also fulfilled and, consequently, the deformation (1) will be infinitesimal the second order bends of the surface Φ. If it follows from system (3) for the surface Φ that $\overset{1}{y} = \mathbf{const}$, then the surface Φ has a *rigidity of the second order*. Then it follows [6, 7] that the surface Φ in the class of deformations under consideration will be *analytically rigid*.

It is not difficult to show that in order for $\overset{1}{z}(u,v)$ and $\overset{2}{z}(u,v)$ to be bending fields of the surface Φ on which connections are imposed that allow the points of the curve to move on the surface Φ, only in a given constant direction $\boldsymbol{v}$, it is necessary and sufficient that the systems of differential equations (2) and the following boundary conditions are satisfied

$$\overset{1}{z}(u,v)_{|g} = \lambda_1 \boldsymbol{v} \quad \text{and} \quad \overset{2}{z}(u,v)_{|g} = \lambda_2 \boldsymbol{v}, \tag{4}$$

where λ_1, λ_2 are arbitrary functions of the points of the line g.

So, let the vector $\boldsymbol{v}$ be parallel to the axis of rotation and the surface of revolution Φ is given by the equation [5]:

$$\boldsymbol{x}(u,v) = u\,\boldsymbol{k} + \varrho(u)\,\boldsymbol{a}(v), \quad 0 \le u \le b,\ 0 \le v \le 2\pi, \tag{5}$$

with respect to the moving frame $\boldsymbol{k}, \boldsymbol{a}(v), \boldsymbol{a}'(v)$, where $\boldsymbol{k}$ is unit vector field of the axis of rotation, $\boldsymbol{a}(v)$ is a unit vector field perpendicular to the axis of rotation, which with the change in v defines a circle of radius one. The vectors $\boldsymbol{k}$ and $\boldsymbol{a}(v)$ define the meridian plane, and $\varrho = \varrho(u)$ is the equation of the meridian of the surface of revolution Φ. The lines u = const are parallels, and the lines v = const are meridians.

4. Proof of Theorem 1

We consider surfaces of revolution Φ with respect to infinite small bending of the second order (1), which are defined by the vector fields $\overset{1}{z}(u,v)$ and $\overset{2}{z}(u,v)$ in the moving frame $\boldsymbol{k}, \boldsymbol{a}(v), \boldsymbol{a}'(v)$:

$$\text{(a)}\ \overset{1}{z} = \overset{1}{\varphi}(u,v)\,\boldsymbol{k} + \overset{1}{\psi}(u,v)\,\boldsymbol{a}(v) + \overset{1}{\chi}(u,v)\,\boldsymbol{a}'(v), \qquad \text{(b)}\ \overset{2}{z} = \overset{2}{\varphi}(u,v)\,\boldsymbol{k} + \overset{2}{\psi}(u,v)\,\boldsymbol{a}(v) + \overset{2}{\chi}(u,v)\,\boldsymbol{a}'(v), \tag{6}$$

where $\overset{1}{\varphi}(u,v), \overset{1}{\psi}(u,v), \overset{1}{\chi}(u,v)$ and $\overset{2}{\varphi}(u,v), \overset{2}{\psi}(u,v), \overset{2}{\chi}(u,v)$ are coordinate functions in this frame.

If the system of equations (2), which describes the infinitesimal bendings of the second order for surfaces of revolution (5), is equivalent to the system of differential equations of the first order of this type [5, 17, 24]:

$$\begin{aligned}
&\overset{1}{\varphi}_u(u,v) + \varrho'(u)\,\overset{1}{\psi}_u(u,v) = 0, \quad \overset{1}{\psi}(u,v) + \overset{1}{\chi}_v(u,v) = 0,\\
&\overset{1}{\varphi}_v(u,v) + \varrho'(u)\,[\overset{1}{\psi}_v(u,v) - \overset{1}{\chi}(u,v)] + \varrho(u)\,\overset{1}{\chi}_u(u,v) = 0,
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\overset{2}{\varphi}_u(u,v) + \varrho'(u)\,\overset{2}{\psi}_u(u,v) = -[\overset{1}{\varphi}{}^2_u(u,v) + \overset{1}{\psi}{}^2_u(u,v) + \overset{1}{\chi}{}^2_u(u,v)],\\
&\overset{2}{\psi}(u,v) + \overset{2}{\chi}_v(u,v) = -\frac{1}{\varrho(u)}\left[\overset{2}{\varphi}{}^2_v(u,v) + (\overset{1}{\psi}_v(u,v) - \overset{1}{\chi}(u,v))^2\right],\\
&\overset{2}{\varphi}_v(u,v) + \varrho'(u)\,[\overset{2}{\psi}_v(u,v) - \overset{2}{\chi}(u,v)] + \varrho(u)\,\overset{2}{\chi}_u(u,v) =\\
&\qquad -2\left[\overset{1}{\varphi}_u(u,v)\,\overset{1}{\varphi}_v(u,v) + \overset{1}{\psi}_u(u,v)\,(\overset{1}{\psi}_v(u,v) - \overset{1}{\chi}(u,v))\right].
\end{aligned} \tag{8}$$

As for surfaces of revolution Φ functions $\overset{i}{\varphi}(u,v), \overset{i}{\psi}(u,v), \overset{i}{\chi}(u,v)$ $(i = 1, 2)$ are periodic functions with respect to 2π, the solution of (7) can be found in the form of Fourier series:

$$\overset{1}{\varphi}(u,v) = \sum_{m=0}^{\infty}[\overset{1}{\varphi}_{m1}(u)\,\cos mv + \overset{1}{\varphi}_{m2}(u)\,\sin mv],$$
$$\overset{1}{\psi}(u,v) = \sum_{m=0}^{\infty}[\overset{1}{\psi}_{m1}(u)\,\cos mv + \overset{1}{\psi}_{m2}(u)\,\sin mv],$$
$$\overset{1}{\chi}(u,v) = \sum_{m=0}^{\infty}[\overset{1}{\chi}_{m1}(u)\,\cos mv + \overset{1}{\chi}_{m2}(u)\,\sin mv].$$

Then from (7) for the Fourier coefficients $\overset{1}{\varphi}_{mi}(u)$, $\overset{1}{\psi}_{mi}(u)$, $\overset{1}{\chi}_{mi}(u)$ $(i = 1,2; m = 0,1,2,\dots)$ of these functions we obtain the following system of ordinary differential equations (for $m = 0,1,2,\dots$):

$$\begin{aligned}
&\overset{1}{\varphi}{}'_{m1}(u) + \varrho'(u)\,\overset{1}{\psi}{}'_{m1}(u) = 0, \qquad \overset{1}{\psi}_{m2}(u) + m\,\overset{1}{\chi}_{m2}(u) = 0,\\
&m\,\overset{1}{\varphi}_{m1}(u) + \varrho'(u)\,[m\,\overset{1}{\psi}_{m1}(u) + \overset{1}{\chi}_{m2}(u)] - \varrho(u)\,\overset{1}{\chi}{}'_{m2}(u) = 0,\\
&\overset{1}{\varphi}{}'_{m2}(u) + \varrho'(u)\,\overset{1}{\psi}{}'_{m2}(u) = 0, \qquad \overset{1}{\psi}_{m2}(u) - m\,\overset{1}{\chi}_{m1}(u) = 0,\\
&m\,\overset{1}{\varphi}_{m2}(u) + \varrho'(u)\,[m\,\overset{1}{\psi}_{m2}(u) - \overset{1}{\chi}_{m1}(u)] - \varrho(u)\,\overset{1}{\chi}{}'_{m1}(u) = 0.
\end{aligned} \tag{9}$$

The solution to this system, corresponding to the number m, determines the infinitesimal bending $\overset{1}{\boldsymbol{z}}_m = \overset{1}{\varphi}_m(u,v)\,\boldsymbol{k} + \overset{1}{\psi}_m(u,v)\,\boldsymbol{a}(v) + \overset{1}{\chi}_m(u,v)\,\boldsymbol{a}'(v)$ of a surface of revolution Φ. Here, by $\overset{1}{\varphi}_m(u)$, $\overset{1}{\psi}_m(u)$, $\overset{1}{\chi}_m(u)$ are denoted

$$\begin{aligned}
\overset{1}{\varphi}_m(u,v) &= \overset{1}{\varphi}_{m1}(u)\cos mv + \overset{1}{\varphi}_{m2}(u)\sin mv,\\
\overset{1}{\psi}_m(u,v) &= \overset{1}{\psi}_{m1}(u)\cos mv + \overset{1}{\psi}_{m2}(u)\sin mv,\\
\overset{1}{\chi}_m(u,v) &= \overset{1}{\chi}_{m1}(u)\cos mv + \overset{1}{\chi}_{m2}(u)\sin mv.
\end{aligned} \tag{10}$$

Any infinitesimal bending of the first class of surface Φ is represented as a linear combination of fundamental infinitesimal bends. Excluding from the system (9) for each fixed m the functions $\overset{1}{\varphi}_{mi}(u)$, $\overset{1}{\psi}_{mi}(u)$, we obtain the differential equation

$$\varrho(u)\,\overset{1}{\chi}{}''_{mi}(u) + (m^2 - 1)\,\varrho''(u)\,\overset{1}{\chi}_{mi}(u) = 0. \tag{11}$$

It is known [5] that every non-identically zero solution of equation (11) for integer $m \geq 2$ corresponds to a non-trivial bending field of the surface under consideration.

In the case when equation (11) does not admit limited non-trivial solutions for integer values $m \geq 2$, the corresponding surface of revolution will be rigid with respect to infinitesimal bends of the first order.

If equation (11) admits a non-trivial regular solution on the interval $0 \leq u \leq b$, only for $m = n$ $(m \geq 2)$ and does not allow for such solutions when $m \neq n$, then the solution of the system (8) can be sought in the form of Fourier polynomials consisting of the free term and members containing trigonometric functions of order $2n$ [5, 24]:

$$\begin{aligned}
\overset{2}{\varphi}_n(u,v) &= \overset{2}{\varphi}_{2n,1}(u)\cos 2nv + \overset{2}{\varphi}_{2n,2}(u)\sin 2nv,\\
\overset{2}{\psi}_n(u,v) &= \overset{2}{\psi}_{2n,1}(u)\cos 2nv + \overset{2}{\psi}_{2n,2}(u)\sin 2nv,\\
\overset{2}{\chi}_n(u,v) &= \overset{2}{\chi}_{2n,1}(u)\cos 2nv + \overset{2}{\chi}_{2n,2}(u)\sin 2nv.
\end{aligned} \tag{12}$$

Then from the system (8), taking into account (10), (12) to determine the functions $\overset{2}{\varphi}_{2n,i}(u)$, $\overset{2}{\psi}_{2n,i}(u)$, $\overset{2}{\chi}_{2n,i}(u)$, $\overset{2}{\varphi}_0(u)$, $\overset{2}{\psi}_0(u)$, $\overset{2}{\chi}_0(u)$ $(i = 1, 2)$, we obtain the following system of equations:

$$\begin{aligned}
&\overset{2}{\varphi}'_{2n,1} + \varrho' \overset{2}{\psi}'_{2n,1} = -\frac{1}{2}\left[\overset{1}{\varphi}'^2_{n1} + \overset{1}{\psi}'^2_{n1} + \overset{1}{\chi}'^2_{n1} - \overset{1}{\varphi}'^2_{n2} - \overset{1}{\psi}'^2_{n2} - \overset{1}{\chi}'^2_{n2}\right],\\
&\overset{2}{\psi}_{2n,1} + 2n\overset{2}{\chi}_{2n,2} = -\frac{1}{2\varrho}\left[-n^2\overset{1}{\varphi}^2_{n1} - n^2\overset{1}{\psi}^2_{n1} - 2n\overset{1}{\psi}_{n1}\overset{1}{\chi}_{n2} - \overset{1}{\chi}^2_{n2} + n^2\overset{1}{\varphi}^2_{n2} + n^2\overset{1}{\psi}^2_{n2} - 2n\overset{1}{\psi}_{n2}\overset{1}{\chi}_{n1} + \overset{1}{\chi}^2_{n1}\right],\\
&-2n\overset{2}{\varphi}_{2n,1} - \varrho'(2n\overset{2}{\psi}'_{2n,1} + \overset{2}{\chi}_{2n,2}) + \varrho\overset{2}{\chi}'_{2n,2} = -\Big[-n\overset{1}{\varphi}_{n1}\overset{1}{\varphi}'_{n1} +\\
&\qquad n\overset{1}{\varphi}_{n2}\overset{1}{\varphi}'_{n2} - n\overset{1}{\psi}_{n1}\overset{1}{\psi}'_{n1} + n\overset{1}{\psi}_{n2}\overset{1}{\psi}'_{n2} - \overset{1}{\chi}_{n1}\overset{1}{\psi}'_{n2} - \overset{1}{\psi}'_{n1}\overset{1}{\chi}_{n2}\Big],\\
&\overset{2}{\varphi}'_{2n,2} + \varrho'\overset{2}{\psi}'_{2n,2} = -\left[\overset{1}{\varphi}'_{n2}\overset{1}{\varphi}'_{n1} + \overset{1}{\psi}'_{n2}\overset{1}{\psi}'_{n1} + \overset{1}{\chi}'_{n2}\overset{1}{\chi}'_{n1}\right],\\
&\overset{2}{\psi}_{2n,2} - 2n\overset{2}{\chi}_{2n,1} = -\frac{1}{\varrho}\left[n^2\overset{1}{\varphi}_{n1}\overset{1}{\varphi}_{n2} - n^2\overset{1}{\psi}_{n1}\overset{1}{\psi}_{n2} + n\overset{1}{\chi}_{n1}\overset{1}{\psi}_{n1} - n\overset{1}{\psi}_{n2}\overset{1}{\chi}_{n2} + \overset{1}{\chi}_{n1}\overset{1}{\chi}_{n2}\right],\\
&2n\overset{2}{\varphi}_{2n,2} - \varrho'(2n\overset{2}{\psi}'_{2n,2} - \overset{2}{\chi}_{2n,1}) + \varrho\overset{2}{\chi}'_{2n,1} = -\Big[n\overset{1}{\varphi}'_{n2}\overset{1}{\varphi}_{n1} +\\
&\qquad n\overset{1}{\psi}'_{n2}\overset{1}{\psi}_{n1} + \overset{1}{\psi}'_{n2}\overset{1}{\chi}_{n2} + n\overset{1}{\varphi}'_{n1}\overset{1}{\varphi}_{n2} + n\overset{1}{\psi}'_{n1}\overset{1}{\psi}_{n2} - \overset{1}{\psi}'_{n1}\overset{1}{\chi}'_{n1}\Big].
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\overset{2}{\varphi}'_0 + \varrho'\overset{2}{\psi}'_0 = -\frac{1}{2}\left(\overset{1}{\varphi}'^2_{n1} + \overset{1}{\psi}'^2_{n1} + \overset{1}{\chi}'^2_{n1} + \overset{1}{\varphi}'^2_{n2} + \overset{1}{\psi}'^2_{n2} + \overset{1}{\chi}'^2_{n2}\right),\\
&\overset{2}{\psi}_0 = -\frac{1}{2\varrho}\left(n^2\overset{1}{\varphi}^2_{n1} + n^2\overset{1}{\psi}^2_{n1} + 2n\overset{1}{\psi}_{n1}\overset{1}{\chi}_{n2} + \overset{1}{\chi}^2_{n2} + n^2\overset{1}{\varphi}^2_{n2} + n^2\overset{1}{\psi}^2_{n2} - 2n\overset{1}{\psi}_{n2}\overset{1}{\chi}_{n1} + \overset{1}{\chi}^2_{n1}\right),\\
&\varrho\overset{2}{\chi}'_0 + \varrho'\overset{2}{\chi}_0 = n\overset{1}{\varphi}_{n1}\overset{1}{\varphi}'_{n2} + n\overset{1}{\psi}_{n1}\overset{1}{\psi}'_{n2} + \overset{1}{\chi}_{n2}\overset{1}{\psi}'_{n2} - n\overset{1}{\varphi}_{n2}\overset{1}{\varphi}'_{n1} - n\overset{1}{\psi}_{n2}\overset{1}{\psi}'_{n1} + \overset{1}{\psi}_{n1}\overset{1}{\chi}'_{n1},
\end{aligned} \tag{14}$$

Excluding the functions $\overset{2}{\varphi}_{2n,i}(u)$, $\overset{2}{\psi}_{2n,i}(u)$ from the system (13), we obtain the differential equations

$$\varrho\overset{2}{\chi}''_{2n,i} + (4n^2 - 1)\,\varrho''\,\overset{2}{\chi}_{2n,i} = R_i \qquad (i = 1, 2), \tag{15}$$

where

$$R_1 = 2n\varrho''(\overset{1}{\varphi}_{n2}\overset{1}{\psi}'_{n1} + \overset{1}{\varphi}_{n1}\overset{1}{\psi}'_{n2})$$

and

$$R_2 = 2n\varrho''(\overset{1}{\varphi}_{n2}\overset{1}{\psi}'_{n2} - \overset{1}{\varphi}_{n1}\overset{1}{\psi}'_{n1}).$$

If equations (14) and (15) have a regular solution on the interval $0 \le u \le b$, then the corresponding surface of revolution admits a non-trivial infinitesimal bending of the second order, which is a continuation of the fundamental infinitesimal bending $\overset{1}{z}_n(u, v)$ of the first order [5, 24].

So, let the vector $\boldsymbol{v}$ be parallel to the axis of rotation and the surface Φ is given by equation (6). Without loss of generality, we assume that $\boldsymbol{v} = \boldsymbol{k}$ and the parallel γ_0 is described by equation $u = 0$, then the radius vector $\boldsymbol{x}(0, v)$ of an arbitrary parallel point γ_0 can be written as:

$$\boldsymbol{x}(0, v) = \varrho(0)\,\boldsymbol{a}(v), \ \ 0 \le v \le 2\pi. \tag{16}$$

The vector functions $\overset{1}{z}(u, v)$ and $\overset{2}{z}(u, v)$ will be bending fields of the surface Φ in the specified class of deformations if and only if they are solutions of the system of equations (2) and along the parallel γ_0 satisfy the conditions (4). These conditions for the case under consideration will be written as follows:

$$\overset{1}{z}(0, v) = \lambda_1(v)\,\boldsymbol{k}, \ \ 0 \le v \le 2\pi, \tag{17}$$

$$\overset{2}{z}(0,v) = \lambda_2(v)\,\boldsymbol{k},\ \ 0 \le v \le 2\pi, \tag{18}$$

where $\lambda_1(v)$ and $\lambda_2(v)$ are arbitrary differentiable functions.

Since equality (18) must be fulfilled for all $v \in [0, 2\pi]$, then for all v from the same interval, the following equality will be fulfilled:

$$d\overset{2}{z}(0,v) = d\lambda_2(v)\,\boldsymbol{k},\ \ 0 \le v \le 2\pi. \tag{19}$$

Then from equation (2b), given the equalities (16) and (19), we find $(d\overset{1}{z}(0,v))^2 = 0,\ \ 0 \le v \le 2\pi$, hence it follows that

$$\overset{1}{z}(0,v) = \boldsymbol{c},\ \ 0 \le v \le 2\pi, \tag{20}$$

where $\boldsymbol{c}$ is an arbitrary constant vector.

Comparing formulas (17) and (20), we get

$$\overset{1}{z}(0,v) = c\,\boldsymbol{k},\ \ 0 \le v \le 2\pi, \tag{21}$$

where c is the length of the vector $\boldsymbol{c}$ in (20).

Let the bending field $\overset{1}{z}(u,v)$ of the surface Φ in the basis $\boldsymbol{k}, \boldsymbol{a}(v), \boldsymbol{a}'(v)$ has expression (6a). Then along the parallel γ_0, if we take into account (21), we obtain the following restrictions for the functions $\overset{1}{\varphi}(u,v)$, $\overset{1}{\psi}(u,v)$, $\overset{1}{\chi}(u,v)$:

$$\overset{1}{\varphi}(0,v) = c,\ \ \overset{1}{\psi}(0,v) = 0,\ \ \overset{1}{\chi}(0,v) = 0.$$

Hence, taking into account the decomposition of the functions $\overset{1}{\varphi}(u,v)$, $\overset{1}{\psi}(u,v)$, $\overset{1}{\chi}(u,v)$ into Fourier series and the system of differential equations (9), which connects the Fourier coefficients of these functions, we obtain

$$\overset{1}{\chi}_{mi}(0) = 0,\ \ \overset{1}{\chi}{}'_{mi}(0) = 0,\ \ (m \ge 2;\ i = 1, 2). \tag{22}$$

Since the meridian of the surface of revolution Φ has no points whose tangents are perpendicular to the axis of rotation, then, as is known, the only regular solution of equation (11) that satisfies condition (22) is $\overset{1}{\chi}_{mi}(u) = 0$.

Then, as is known, [5], $\overset{1}{\boldsymbol{y}}(u,v) = \mathbf{const}$ on the entire surface Φ. Thus, for the considered class of deformations of the surface Φ, it follows from the system of equations (3) that $\overset{1}{\boldsymbol{y}}(u,v) = \mathbf{const}$, which means that the surface Φ in the specified class of deformations has second-order rigidity, and therefore is analytically rigid. Therefore, the theorem is fully proved.

5. Conclusion

Investigating the infinitesimal bendings of regular unfolding surfaces and doubly connected regular surfaces of revolution, which are fixed along a curve on a surface with respect to a point and a plane, we came to the conclusion that these surfaces under analytic boundary conditions are analytically rigid.

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