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Helicoid and curvature based functional variations

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Abstract. In view of the meaning of ruled surfaces in aesthetics, statics, scale and manufacturing technologies, we point out the possibility of a mathematical analysis in the case of infinitesimal deformations by considering the rigidity of surfaces. In case of bendable surfaces it is useful to discuss the variation of magnitudes such as the shape operator. The shape operator is a good way to measure how a regular surface S bends in R^3 by valuation how the surface normal ν changes from point to point. In this work we consider variations of the shape operator, the normal curvature and the principal curvatures of helicoid under infinitesimal bending of the surface.

1. Introduction

Ruled surfaces have wide application in civil engineering and architecture, but in many other sciences, too. The simplicity of production and very rich spectrum of shapes are the main reason for application of this kind of surfaces. Since DNA molecules is often studied as a double helix model, in this work we will consider curvature based functional variations of helicoid under infinitesimal bending.

Ruled surfaces and conoids were a subject of investigation of numerous books and papers from different point of view: [1], [8], [9], [10], [11], [12], [13], [19], [24].

Since the shape is an important feature of objects and can be immensely useful in characterizing objects, we point out the shape analysis considering the variation of quantities that characterize the shape itself. Fundamental functionals that measure the bending of a surface, are the shape operator, as a vector function, the normal curvature, as a real-valued function, with the principal curvatures as its extreme values.

Variation of some geometric magnitudes under infinitesimal bending was considered in [4], [5], [6], [7], [18], [19] and [21]. Gaudi surfaces at small deformations were analized in [14] and [16]. Aplication and variation of shape operator under infinitesimal bending of surface were considered in [15] and [20]. The variation of the Willmore energy under infinitesimal bending of a surface was studied in [17]. Infinitesimal bending field of DNA helices was determined in [22]. Curves on ruled surface under infinitesimal bending were presented in [23].

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2. Preliminary of Curvature Based Functions

Two rudimentary ways to characterize the shape of a surface S are to consider how the unit normal v behaves as we move around and to compare S to a sphere. The former of these methods is accomplished using the shape operator. It is a linear operator that calculates the bending of surface S. The calculus of variations studies the extreme and critical points of functions. It has its roots in many areas, from geometry to optimization to mechanics.

2.1. Shape Operator and Curvatures

Shape operator is a linear operator that calculates the bending of surface S. Marc L. Irons said that everything you had could want to know about a surface's curvature was locked up in the shape operator. The linear operator, the shape operator, applied to a tangent vector v_p is the negative of the derivative of ν in the direction v_p.

Definitions and preliminary of shape operator and curvature based functions are given in [20].

Definition 2.1. *Let $S \subset \mathcal{R}^3$ be a regular surface, and let ν be a surface normal to S defined in a neighborhood of a point $p \in S$. For a tangent vector v_p to S at p we put*

$$\underline{S}(v_p) = -D_v \nu. \tag{1}$$

Then $\underline{S}$ is called the ***shape operator***.

The shape operator of a plane is identically zero at all points of the plane. For a nonplanar surface, the surface normal ν will twist and turn from point to point, and $\underline{S}$ will be nonzero.

Now, we will show how to express the shape operator in terms of the coefficients of the first E, F, G and the second L, M, N fundamental form, in next Theorem that is proved in [1].

Theorem 2.2. *(The Weingarten equations) Let $\mathbf{r} : D \to \mathcal{R}^3, D \subset \mathcal{R}^2$ be a regular surface. Then the shape operator $\underline{S}$ of $\mathbf{r}$ is given in terms of the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$ by*

$$\begin{cases} -\underline{S}(\mathbf{r}_u) = \nu_u = \frac{MF-LG}{EG-F^2}\mathbf{r}_u + \frac{LF-ME}{EG-F^2}\mathbf{r}_v, \\ -\underline{S}(\mathbf{r}_v) = \nu_v = \frac{NF-MG}{EG-F^2}\mathbf{r}_u + \frac{MF-NE}{EG-F^2}\mathbf{r}_v. \end{cases} \tag{2}$$

While the shape operator is vector function that measures the bending of a surface, normal curvature is a real-valued function that does the same thing.

Definition 2.3. *[1] Let u_p is the tangent vector of regular surface $S \subset \mathcal{R}^3$ that is $\|u_p\| = 1$. Then the* ***normal curvature*** *of S in direction u_p will be equal:*

$$k_n(u_p) = \underline{S}(u_p) \cdot u_p. \tag{3}$$

In general case, if v_p is an arbitrary non-zero tangent vector of S in $p \in S$, then

$$k_n(v_p) = \frac{\underline{S}(v_p) \cdot v_p}{\|v_p\|^2}. \tag{4}$$

Normal curvature can be expressed as in [18]:

Lemma 2.4. *The normal curvature of S at a given point $p \in S$, in direction of tangent vector $(u(t))_s$, can be expressed in form:*

$$k_n(t) = L\cos^2 t + 2M \sin t \cos t + N \sin^2 t, \tag{5}$$

where $(u(t))_s$ represents all directions: $(u(t))_s = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}_s$, $t \in [0, 2\pi)$ and index $\mathbf{s}$ presents a vector in standard base $\{x_u, x_v\}$: $(x_u)_s = (1, 0)$, $(x_v)_s = (0, 1)$.

The principal curvatures measure the maximum and minimum bending of a regular surface S at each point $p \in S$.

Definition 2.5. *Let $S \subset \mathcal{R}^3$ is regular surface and $p \in S$. The maximal and the minimal value of normal curvature k_n of S at point p we call **the principal curvatures** of S at point p and denote k_1 and k_2.*

2.2. Connections among the curvatures

Among the curvatures and shape operator there are some connections.

Definition 2.6. *Gaussian and mean curvatures are the functions $K, H : S \to R$ defined as:*

$$K(p) = det(\underline{S}(p)) , \qquad H(p) = \frac{1}{2}tr(\underline{S}(p)) . \tag{6}$$

Theorem 2.7. *Let k_1 and k_2 are the principal curvatures of regular surface $S \subset \mathcal{R}^3$. Gaussian curvature of surface S is in form:*

$$K = k_1 k_2 . \tag{7}$$

Mean curvature of surface S is in form:

$$H = \frac{1}{2}(k_1 + k_2) . \tag{8}$$

Depending on the sign of curvatures, there are four types of surface points.

Definition 2.8. *Let p is a point of regular surface $S \subset \mathcal{R}^3$. The point p is:*

- *p is* elliptic *if* $K(p) > 0 \qquad (k_1k_2 > 0)$;
- *p is* hyperbolic *if* $K(p) < 0 \qquad (k_1k_2 < 0)$;
- *p is* parabolic *if* $K(p) = 0$ *and* $\underline{S}(p) \neq 0 \qquad (k_1 = 0 \veebar k_2 = 0)$;
- *p is* planar *if* $K(p) = 0$ *and* $\underline{S}(p) = 0 \qquad (k_1 = k_2 = 0)$.

2.3. The Main Types of Conoid Surfaces

Definition 2.9. ***Ruled surface** $S \subset \mathcal{R}^3$ is surface which contains at least one 1-parameter family of straight lines. Thus, a ruled surface has a parameterization $\mathbf{r} : D \to S$ of the form:*

$$\mathbf{r}(u, v) = \alpha(u) + v\gamma(u), \tag{9}$$

*where α and γ are curves in $\mathcal{R}^3$ with $\alpha' \neq 0$. We call $\mathbf{r}$ a **ruled patch**, the curve α is called the **directrix** or base curve of the ruled surface, and curve γ is called the **director curve**. The rulings of the ruled surface are the straight lines*

$$v \to \alpha(u) + v\gamma(u), \tag{10}$$

A ruled surface is called a ***conoid*** if it can be generated by moving of straight line parallel to a plane, intersecting a fixed straight line-axis of conoid and a fixed basic curve $\alpha(u)$.

The main kinds of conoid surfaces are hyperbolic paraboloid, helicoid, Plucker's conoid, generalized Plucker's conoid, cononeus or conical edge of Wallis, and sinusoidal conoid - famous as Gaudi surface, considered in [12], [14], [16].

In this work, we considered the helicoid, by calculating the curvatures and their variations and pointed out vanished curvatures.

3. Curvatures of Helicoid

Using the next parametrization for helicoid surface,

$$\mathbf{r}(u,v) = (u, v, c\arctan\frac{v}{u}) \ , \tag{11}$$

we can expressed the curvatures and its variation under the infinitesimal bending of the surface. Using the parametric equation (11) and calculating the coefficients E, F, G, L, M, N, we can get:

$$\underline{S} = \frac{1}{g^{\frac{3}{2}}}\begin{pmatrix} uv(2u^2+2v^2+1) & v^4-u^4+v^2 \\ v^4-u^4-u^2 & -uv(2u^2+2v^2+1) \end{pmatrix}, \tag{12}$$

where $g = (u^2+v^2+1)(u^2+v^2)$.
Also, the normal curvature of helicoid is equal to expression:

$$k_n(t) = \frac{1}{(g_0)^{\frac{3}{2}}}\Big(u_0v_0(2u_0^2+2v_0^2+1)\cos^2 t + (v_0^2-u_0^2)(2u_0^2+2v_0^2+1)\sin t\cos t - \\ - u_0v_0(2u_0^2+2v_0^2+1)\sin^2 t\Big) \ , \tag{13}$$

where $g_0 = (u_0^2+v_0^2+1)(u_0^2+v_0^2)$.
At points with $v_0 = 0$ the normal curvature of helicoid is equal:

$$k_n(t) = -\frac{2u_0^2+1}{u_0(1+u_0^2)^{\frac{3}{2}}}\sin t\cos t \ . \tag{14}$$

4. Variation of Curvatures of Helicoid

Using the basics of infinitesimal banding of a surface according to [4] and [7], as well as the standard machinery of differential geometry [26], it can be get the field of infinitesimal bending of helicoid.

Since we consider a regular surface S , parameterized by:

$$\mathbf{r}(u,v) = (u, v, f(u,v)) \ , \tag{15}$$

and infinitesimal bending field by:

$$\mathbf{z}(u,v) = (\xi(u,v), \eta(u,v), \zeta(u,v)), \tag{16}$$

we used the parametrisation of helicoid given with:

$$\mathbf{r}(u,v) = (u, v, c\arctan\frac{v}{u}) \ . \tag{17}$$

The field of infinitesimal bending of helicoid was determined in [22] and given with expression:

$$\mathbf{z}(u,v) = \Big(u(1-2\ln|u|)\sin v + c_1,\ u(\ln|u|-1)\cos v + c_2,\ c_3\Big) \ , \tag{18}$$

for $c = 1$.

According to [22], we can also consider the variation of DNA helices or, more precisely the flexibility of DNA molecule, i.e. the flexibility of double helix in infinitesimal bending theory. In that sense, we can present the parametric lines of helicoid.

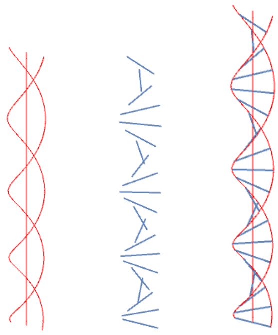

Figure 1: u and v parametric lines of DNA molecule

4.1. Shape operator under the infinitesimal bending of helicoid

Using the standard machinery of differential geometry [26] we can get that the coefficients of the first and the second fundamental form (see [21]) of surface S_ϵ. In [15] the variation of the shape operator of surface under infinitesimal bending was calculated. Since, from the bending field, $\zeta = 0$, we get:

Lemma 4.1. *Variation of shape operator under infinitesimal bending of helicoid:*

$$\begin{aligned} \delta\underline{S}(\mathbf{r}_u) &= \frac{1}{g^{\frac{3}{2}}}\Big(uv(2u^2+2v^2+1)\,\mathbf{z}_u + (v^4-u^4+v^2)\,\mathbf{z}_v\Big)\,, \\ \delta\underline{S}(\mathbf{r}_v) &= \frac{1}{g^{\frac{3}{2}}}\Big((v^4-u^4-u^2)\,\mathbf{z}_u - uv(2u^2+2v^2+1)\,\mathbf{z}_v\Big)\,, \end{aligned} \tag{19}$$

where $g = (u^2+v^2+1)(u^2+v^2)$.

4.2. Normal curvature under the infinitesimal bending of helicoid

Using the expression of variation of normal curvature and proved corollary in [18]:

Corollary 4.2. *Variation of the normal curvature of a surface under infinitesimal bending will be equal zero if the third coordinate of bending field* $\mathbf{z}$ *is linear function.*

We can conclude:

Lemma 4.3. *Variation of normal curvature under infinitesimal bending of helicoid is equal zero.*

4.3. Principal curvatures under the infinitesimal bending of helicoid

Also, from the same work [18], according to variation of principal curvatures and corollary:

Corollary 4.4. *Variation of the principal curvatures of a surface under infinitesimal bending of a surface, will be equal zero if the variation of the mean curvature vanishes.*

And it is expressed:

$$\delta H = \frac{(1+f_u^2)\zeta_{vv} - 2f_u f_v \zeta_{uv} + (1+f_v^2)\zeta_{uu}}{2\sqrt{1+f_u^2+f_v^2}}\ .$$

Since,

$$\zeta = 0 \quad \Rightarrow \quad \delta H = 0\ ,$$

there is conclusion:

Lemma 4.5. *Variation of principal curvatures under infinitesimal bending of helicoid is equal zero.*

4.4. Kinds of points under the infinitesimal bending of helicoid

Since the Gaussian curvature is stationary under infinitesimal bending of a surface, it is obviously that elliptic points stay elliptic and hyperbolic points stay hyperbolic. The thing is what happened with parabolic and planar points under the infinitesimal bending of a surface.

If we suppose that p is parabolic $K(p) = 0$ and $k_1 = 0 \underline{\vee} k_2 = 0$, for example, $k_1 = 0$ and $k_2 \neq 0$. Then, $\delta K = 0 \quad \Leftrightarrow \quad \delta(k_1 k_2) = k_1 \delta k_2 + k_2 \delta k_1 = 0$, and must be $\delta k_1 = 0$, which means that $\tilde{k}_1 = 0$.

The conditions when δk_2 will be vanished are given in next Lemma, presented in [20]:

Lemma 4.6. *If the variation of the mean curvature vanishes, parabolic point remains parabolic and planar point remains planar under the infinitesimal bending of the surface.*

As it is valid that the mean curvature of helicoid vanishes, parabolic point remains parabolic and planar point remains planar under the infinitesimal bending of the helicoid.

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