Filomat 37:25 (2023), 8425–8434
https://doi.org/10.2298/FIL2325425K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Pell and Pell-Lucas hybrid quaternions

Ferhat Kürüz[a], Ali Dağdeviren[b]

[a]*Department of Computer Engineering, Faculty of Engineering and Architecture, Istanbul Gelişim University*
[b]*Department of Computer Engineering, Faculty of Engineering, Khoja Akhmet Yassawi International Kazak-Turkish University*

Abstract. In this paper, we investigate Pell and Pell-Lucas numbers what new properties we can obtain by working on a new number system-hybrid quaternions. Firstly, we present some new identities of Pell and Pell-Lucas numbers and quaternions. Then, with the help of these identities and previously known identities, we get new identities on Pell and Pell-Lucas hybrid quaternions, including Binet's and Cassini's identities.

1. Introduction

Number sequences are a great subject studied by mathematicians and physicists for many years. Especially, Fibonacci numbers are a significant class of number sequences. These numbers have been studied from algebraic, combinatorial, and geometric perspectives. There are other number sequences that are similar to Fibonacci numbers with the differences in initial values and recurrence relations. The Pell and Pell-Lucas numbers are some of them. These numbers can be generated by the second order linear recurrence relation $F_n = 2F_{n-1} + F_{n-2}$ for $n \geq 2$ with initial conditions $P_0 = 0$, $P_1 = 1$ and $PL_0 = PL_1 = 2$ respectively. Generator functions of number sequences, Binet formulas, are necessary in some cases such as obtaining high index numbers. The Binet's formulas of the n^{th} Pell and Pell-Lucas numbers are

$$P_n = \frac{\varphi^n - \omega^n}{\varphi - \omega}, \qquad PL_n = \varphi^n + \omega^n \tag{1}$$

respectively, where φ and ω are roots of characteristic equation $x^2 - 2x - 1 = 0$. Many identities are given regarding the Pell and Pell-Lucas numbers in [3, 7, 12, 13]. For example, Horadam gave some identities of Pell numbers [13];

$$\begin{aligned} P_{n+r} &= P_r P_{n+1} + P_{r-1} P_n, \\ P_{2n} &= P_n P_{n+1} + P_n P_{n-1}, \\ P_{n+1} P_{n-1} - P_n^2 &= (-1)^n, \qquad \text{(Simson identity)} \\ P_{2n+1} &= P_n^2 + P_{n+1}^2. \end{aligned}$$

2020 *Mathematics Subject Classification*. Primary 11B37; Secondary 11B39, 15A66.
Keywords. Pell numbers; Pell-Lucas numbers; Hybrid numbers; Quaternions; Hybrid quaternions.
Received: 14 Jul 2022; Accepted: 02 December 2022
Communicated by Mića Stanković and Zoran Rakić
Email addresses: fkuruz@gelisim.edu.tr (Ferhat Kürüz), ali.dagdeviren@ayu.edu.kz (Ali Dağdeviren)

Hybrid numbers were defined as a general form of complex, hyperbolic(perplex) and dual numbers [1, 8, 9, 15]. After the number system was defined, it was studied extensively from various perspectives. A hybrid number is,

$$k = k_0 + \mathbf{i}k_1 + \boldsymbol{\varepsilon}k_2 + \mathbf{h}k_3$$

such that k_0, k_1, k_2 and k_3 are real coefficients and $\mathbf{i}^2 = -1$, $\mathbf{h}^2 = 1$, $\boldsymbol{\varepsilon}^2 = 0$ and $\mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i}$. Note that $\mathbf{i}$ is the unit of complex numbers, $\boldsymbol{\varepsilon}$ is the unit of dual numbers and $\mathbf{h}$ is the unit of hyperbolic numbers. The units of hybrid numbers are 1, $\mathbf{i}$, $\boldsymbol{\varepsilon}$ and $\mathbf{h}$.

Let $a = a_0 + \mathbf{i}a_1 + \boldsymbol{\varepsilon}a_2 + \mathbf{h}a_3$ and $b = b_0 + \mathbf{i}b_1 + \boldsymbol{\varepsilon}b_2 + \mathbf{h}b_3$ be hybrid numbers, then sum and subtraction of hybrid numbers are defined as

$$a \mp b = (a_0 \mp b_0) + \mathbf{i}(a_1 \mp b_1) + \boldsymbol{\varepsilon}(a_2 \mp b_2) + \mathbf{h}(a_3 \mp b_3).$$

Moreover, multiplication of hybrid numbers a and b is defined as

$$\begin{aligned} a.b = {} & (a_0b_0 - a_1b_1 - a_3b_3 + a_1b_2 + a_2b_1) + \mathbf{i}(a_1b_0 + a_0b_1 + a_1b_3 - a_3b_1) \\ & + \boldsymbol{\varepsilon}(a_2b_0 + a_0b_2 + a_1b_3 - a_3b_3 - a_3b_1 + a_3b_2) + \mathbf{h}(a_3b_0 + a_0b_3 - a_1b_2 + a_2b_1). \end{aligned}$$

For multiplication table of hybrid numbers' units, see Table 1.

$\cdot$	1	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
1	1	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1 - \mathbf{h}$	$\boldsymbol{\varepsilon} + \mathbf{i}$
$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}$	$1 + \mathbf{h}$	0	$-\boldsymbol{\varepsilon}$
$\mathbf{h}$	$\mathbf{h}$	$-\boldsymbol{\varepsilon} - \mathbf{i}$	$\boldsymbol{\varepsilon}$	1

Table 1: Multiplication of hybrid numbers' units

Quaternions were defined by W. R. Hamilton [11] and were studied widely in mathematics and physics [4, 10, 19, 20]. A quaternion is

$$q = q_0 + iq_1 + jq_2 + kq_3$$

such that q_0, q_1, q_2 and q_3 are real coefficients and i, j and k are the standart orthonormal basis of $\mathbb{R}^3$. Units of quaternions are 1, i, j and k.

Let $p = p_0 + ip_1 + jp_2 + kp_3$ and $q = q_0 + iq_1 + jq_2 + kq_3$ be quaternions, then sum and subtraction of any two quaternions is defined as

$$p \mp q = (p_0 \mp q_0) + i(p_1 \mp q_1) + j(p_2 \mp q_2) + k(p_3 \mp q_3).$$

In addition to this, multiplication of any two quaternions is defined as

$$\begin{aligned} p.q = {} & (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3) + i(p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2) \\ & + j(p_0q_2 + p_2q_0 - p_1q_3 + p_3q_1) + k(p_0q_3 + p_3q_0 + p_1q_2 - p_2q_1) \end{aligned}$$

with $i^2 = j^2 = k^2 = ijk = -1$ rules.

Hybrid quaternions were introduced by Dağdeviren in [5] as a new number system. A hybrid quaternion is

$$HQ = Q_0 + \mathbf{i}Q_1 + \boldsymbol{\varepsilon}Q_2 + \mathbf{h}Q_3$$

such that Q_0, Q_1, Q_2 and Q_3 are quaternions. The other presentation of a hybrid quaternion is

$$HQ = H_0 + iH_1 + jH_2 + kH_3$$

such that H_0, H_1, H_2 and H_3 are hybrid numbers. Sum, subtraction and multiplication operations can be easily done with the information up to here.

Some identities, properties and Binet's formulas of Pell and Pell-Lucas quaternions and related works have been given by some researchers [2, 4, 6, 18, 19]. The n^{th} Pell and Pell-Lucas quaternions are

$$PQ_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3},$$

$$PQL_n = PL_n + iPL_{n+1} + jPL_{n+2} + kPL_{n+3}$$

respectively.

Pell and Pell-Lucas hybrid numbers were defined by Liana [16] and have been studied in [14, 17]. Some identities, such as Cassini's identity, Binet's formula and some generalizations of these numbers have been given. The n^{th} Pell and Pell-Lucas hybrid numbers are, as can be easily inferred,

$$HP_n = P_n + \mathbf{i}P_{n+1} + \boldsymbol{\varepsilon}P_{n+2} + \mathbf{h}P_{n+3},$$

$$HPL_n = PL_n + \mathbf{i}PL_{n+1} + \boldsymbol{\varepsilon}PL_{n+2} + \mathbf{h}PL_{n+3},$$

respectively.

In the rest of the article, we introduce Pell and Pell-Lucas hybrid quaternions. Then, we give identities concerning Pell and Pell-Lucas numbers quaternions. Finally, we present Binet's formulas for these numbers. For the rest of the article, we will denote the n^{th} Pell, Pell-Lucas, Pell hybrid, Pell-Lucas hybrid numbers, Pell quaternions, Pell-Lucas quaternions with P_n, PL_n, HP_n, HPL_n, QP_n and QPL_n respectively.

2. Pell and Pell-Lucas Hybrid Quaternions

Definition 2.1. *The n^{th} Pell and Pell-Lucas hybrid quaternions are*

$$HQP_n = HP_n + iHP_{n+1} + jHP_{n+2} + kHP_{n+3},$$

$$HQPL_n = HPL_n + iHPL_{n+1} + jHPL_{n+2} + kHPL_{n+3}$$

for $n \geq 2$. In other form,

$$HQP_n = QP_n + \mathbf{i}QP_{n+1} + \boldsymbol{\varepsilon}QP_{n+2} + \mathbf{h}QP_{n+3},$$

$$HQPL_n = QPL_n + \mathbf{i}QPL_{n+1} + \boldsymbol{\varepsilon}QPL_{n+2} + \mathbf{h}QPL_{n+3}.$$

We firstly give some new properties about Pell numbers and Pell quaternions.

Proposition 2.2. *For all $n \geq 1$ integers,*

$$P_nP_{n+1} = 2\sum_{i=1}^{n} P_i^2.$$

Proof.

$$P_nP_{n+1} = P_n(2P_n + P_{n-1}) = 2P_n^2 + P_nP_{n-1} = 2P_n^2 + P_{n-1}(2P_{n-1} + P_{n-2}) = 2P_n^2 + 2P_{n-1}^2 + P_{n-2}P_{n-1}$$
$$= \ldots = 2(P_n^2 + P_{n-1}^2 + \ldots + P_1^2 + P_1P_0) = 2\sum_{i=1}^{n} P_i^2.$$

□

Proposition 2.3. *For $n \in \mathbb{N}$, $PL_{n+1} = 2(P_n + P_{n+1})$.*

Proof. We continue by induction on n. For $n = 1$, $P_1 = 1$ and $P_2 = 2$, then $PL_1 = 2(1 + 2)$ as required. Assume that the claim holds for $n = k$, $PL_{k+1} = 2(P_k + P_{k+1})$. For $n = k + 1$,

$$PL_{k+2} = 2PL_{k+1} + PL_k = 2(2P_k + 2P_{k+1}) + 2P_{k-1} + 2P_k = 6P_k + 4P_{k+1} + 2P_{k-1}$$
$$= 4P_{k+1} + 2P_k + 4P_k + 2P_{k-1} = 2(P_n + P_{n+1}).$$

□

Proposition 2.4. *For $n \in \mathbb{N}$, $PL_n = 2(P_{n+1} - P_n)$.*

Proof. If we use the previous proposition, then

$$2(P_{n+1} - P_n) = 2(2P_n + P_{n-1} - P_n) = 2(P_n + P_{n-1}) = PL_n.$$

□

Proposition 2.5. *For $n \in \mathbb{N}$, $PL_{n+1} - PL_n = 4P_n$.*

Proof. If we use the Proposition 2.3, then

$$PL_{n+1} - PL_n = 2(P_{n+1} + P_n) - 2(P_n + P_{n-1}) = 2(P_{n+1} - P_{n-1}) = 2(2P_n + P_{n-1} - P_{n-1}) = 4P_n.$$

□

We will give some identities about Pell-Lucas quaternions. These have been given in [4] but there are some differences in this article due to initial values affecting some identities and Binet's formula of Pell-Lucas quaternions.

Remark 2.6. *We can obtain the following identities about Pell and Pell-Lucas numbers via Proposition 2.3, 2.4 and 2.5, for $n \in \mathbb{N}$*

$$\begin{aligned} QPL_{n+1} &= 2(QP_n + QP_{n+1}), \\ QPL_n &= 2(QP_{n+1} - QP_n), \\ QPL_{n+1} - QPL_n &= 4QP_n. \end{aligned}$$

Proposition 2.7 (Cassini's identity for Pell quaternions). *Let φ and ω defined as in (1), then following identities are hold.*

$$\begin{aligned} QP_{n+1}QP_{n-1} - QP_n^2 &= (-1)^n \frac{[(\varphi^2+1)AB + (\omega^2+1)BA]}{(\varphi-\omega)^2}, \\ QPL_{n+1}QPL_{n-1} - QPL_n^2 &= (-1)^{n-1}[(\varphi^2+1)AB + (\omega^2+1)BA]. \end{aligned}$$

where $A = 1 + \varphi i + \varphi^2 j + \varphi^3 k$, $B = 1 + \omega i + \omega^2 j + \omega^3 k$ [4].

Proof. Using the Binet's formula for Pell quaternions, then

$$\begin{aligned} QP_{n+1}QP_{n-1} - QP_n^2 &= \Big(\frac{\varphi^{n+1}A - \omega^{n+1}B}{\varphi-\omega}\Big)\Big(\frac{\varphi^{n-1}A - \omega^{n-1}B}{\varphi-\omega}\Big) - \Big(\frac{\varphi^n A - \omega^n B}{\varphi-\omega}\Big)^2 \\ &= \frac{\varphi^{2n}A^2 - \varphi^{n+1}\omega^{n-1}AB - \varphi^{n-1}\omega^{n+1}BA + \omega^{2n}B^2}{(\varphi-\omega)^2} - \frac{\varphi^{2n}A^2 - \varphi^n\omega^n AB - \varphi^n\omega^n BA + \omega^{2n}B^2}{(\varphi-\omega)^2} \\ &= \frac{\varphi^{n-2}\omega^{n-2}(AB+BA)}{(\varphi-\omega)^2} + \frac{-\varphi^{n-1}\omega^{n-1}(\varphi^2 AB + \omega^2 BA)}{(\varphi-\omega)^2} = (-1)^n \frac{[(\varphi^2+1)AB + (\omega^2+1)BA]}{(\varphi-\omega)^2}. \end{aligned}$$

If we use the Binet's formula for Pell-Lucas quaternions, then

$$\begin{aligned} QPL_{n+1}QPL_{n-1} - QPL_n^2 &= (\varphi^{n+1}A + \omega^{n+1}B)(\varphi^{n-1}A + \omega^{n-1}B) - (\varphi^n A + \omega^n B)^2 \\ &= (\varphi^{n+1}\omega^{n-1}AB + \varphi^{n-1}\omega^{n+1}BA) - (\varphi^n\omega^n AB + \varphi^n\omega^n BA) \\ &= [\varphi^{n-1}\omega^{n-1}(\varphi^2 AB + \omega^2 BA)] - [\varphi^n\omega^n(AB+BA)] \\ &= [(-1)^{n-1}(\varphi^2 AB + \omega^2 BA)] - [(-1)^n(AB+BA)] \\ &= (-1)^{n-1}[(\varphi^2+1)AB + (\omega^2+1)BA]. \end{aligned}$$

□

In the rest of the article, A and B will be used as defined in the proposition (2.7).

Proposition 2.8. *Let HQP_n, $\overline{HQP_n}$ and HP_n be the n^{th} Pell hybrid quaternion, the conjugate of the n^{th} Pell hybrid quaternion and the n^{th} Pell hybrid number, respectively. Then the following identity holds for all $n \geq 2$ integers.*

$$HQP_n + \overline{HQP_n} = 2HP_n$$

Proof. We know that $QP_n + \overline{QP_n} = 2P_n$ from [4], then

$$\begin{aligned} HQP_n + \overline{HQP_n} &= QP_n + QP_{n+1}\mathbf{i} + QP_{n+2}\boldsymbol{\varepsilon} + QP_{n+3}\mathbf{h} + \overline{QP_n} + \overline{QP_{n+1}}\mathbf{i} + \overline{QP_{n+2}}\boldsymbol{\varepsilon} + \overline{QP_{n+3}}\mathbf{h} \\ &= 2P_n + 2P_{n+1}\mathbf{i} + 2P_{n+2}\boldsymbol{\varepsilon} + 2P_{n+3}\mathbf{h} = 2HP_n. \end{aligned}$$

□

Proposition 2.9. *Let HP_n and P_n be the n^{th} Pell hybrid number and the n^{th} Pell number respectively, then the following identity satisfies:*

$$HP_n^2 = P_n^2 - P_{n+1}^2 + P_{n+3}^2 + 2P_{n+1}P_{n+2} + \mathbf{i}(2P_nP_{n+1}) + \boldsymbol{\varepsilon}(2P_nP_{n+2}) + \mathbf{h}(2P_nP_{n+3}).$$

Proof.

$$\begin{aligned} HP_n^2 &= (P_n + P_{n+1}\mathbf{i} + P_{n+2}\boldsymbol{\varepsilon} + P_{n+3}h)(P_n + P_{n+1}\mathbf{i} + P_{n+2}\boldsymbol{\varepsilon} + P_{n+3}\mathbf{h}) \\ &= P_n^2 + P_nP_{n+1}\mathbf{i} + P_nP_{n+2}\boldsymbol{\varepsilon} + P_nP_{n+3}\mathbf{h} + P_nP_{n+1}\mathbf{i} - P_{n+1}^2 + P_{n+1}P_{n+2}(1-\mathbf{h}) + P_{n+1}P_{n+3}(\boldsymbol{\varepsilon}+\mathbf{i}) \\ &\quad + P_{n+2}P_n\boldsymbol{\varepsilon} + P_{n+1}P_{n+2}(1+\mathbf{h}) + P_{n+2}P_{n+3}(-\boldsymbol{\varepsilon}) + P_nP_{n+3}\mathbf{h} + P_{n+3}P_{n+1}(-\boldsymbol{\varepsilon}-\mathbf{i}) + P_{n+2}P_{n+3}\boldsymbol{\varepsilon} + P_{n+3}^2 \\ &= P_n^2 - P_{n+1}^2 + P_{n+3}^2 + 2P_{n+1}P_{n+2} + \mathbf{i}(2P_nP_{n+1}) + \boldsymbol{\varepsilon}(2P_nP_{n+2}) + \mathbf{h}(2P_nP_{n+3}). \end{aligned}$$

□

Proposition 2.10. *Let HQP_n, QP_n and HP_n be the n^{th} Pell hybrid quaternion, the n^{th} Pell quaternion and the n^{th} Pell hybrid number, respectively. Then the following identities satisfy:*

i) $HQP_n + 2HQP_{n+1} = HQP_{n+2}$,

ii) $HQP_n - HQP_{n+1}\mathbf{i} - HQP_{n+2}\boldsymbol{\varepsilon} - HQP_{n+3}\mathbf{h} = QP_n + QP_{n+2} - 2QP_{n+3} - QP_{n+6} + 2QP_{n+5}\boldsymbol{\varepsilon}$,

iii) $HQP_n - HQP_{n+1}i - HQP_{n+2}j - HQP_{n+3}k = HP_n + HP_{n+2} + HP_{n+4} + HP_{n+6}$.

Proof.

i) We know that $QP_n + 2QP_{n+1} = QP_{n+2}$ from Proposition 2 in [4]. Then,

$$\begin{aligned} HQP_n + 2HQP_{n+1} &= QP_n + QP_{n+1}\mathbf{i} + QP_{n+2}\boldsymbol{\varepsilon} + QP_{n+3}\mathbf{h} + 2QP_{n+1} + 2QP_{n+2}\mathbf{i} + 2QP_{n+3}\boldsymbol{\varepsilon} + 2QP_{n+4}\mathbf{h} \\ &= QP_{n+2} + \mathbf{i}QP_{n+3} + \varepsilon QP_{n+4} + \mathbf{h}QP_{n+5} = HQP_{n+2}. \end{aligned}$$

ii) We represent $HQP_n - HQP_{n+1}\mathbf{i} - HQP_{n+2}\boldsymbol{\varepsilon} - HQP_{n+3}\mathbf{h}$ as K in proof;

$$\begin{aligned} K &= HQP_n - (QP_{n+1} + QP_{n+2}\mathbf{i} + QP_{n+3}\boldsymbol{\varepsilon} + QP_{n+4}\mathbf{h})\mathbf{i} - (QP_{n+2} + QP_{n+3}\mathbf{i} + QP_{n+4}\boldsymbol{\varepsilon} + QP_{n+5}\mathbf{h})\boldsymbol{\varepsilon} \\ &\quad -(QP_{n+3} + QP_{n+4}\mathbf{i} + QP_{n+5}\boldsymbol{\varepsilon} + QP_{n+6}\mathbf{h})\mathbf{h} \\ &= QP_n + QP_{n+2} - 2QP_{n+3} - QP_{n+6} + 2QP_{n+5}\boldsymbol{\varepsilon}. \end{aligned}$$

iii) We represent $HQP_n - HQP_{n+1}i - HQP_{n+2}j - HQP_{n+3}k$ as T in proof;

$$\begin{aligned} T &= HP_n + HP_{n+1}i + HP_{n+2}j + HP_{n+3}k - (HP_{n+1} + HP_{n+2}i + HP_{n+3}j + QP_{n+4}k)i \\ &\quad -(HP_{n+2} + HP_{n+3}i + HP_{n+4}j + QP_{n+5}k)j - (HP_{n+3} + HP_{n+4}i + HP_{n+5}j + QP_{n+6}k)k \\ &= HP_n + HP_{n+2} + HP_{n+4} + HP_{n+6}. \end{aligned}$$

□

There are some special relationships between the Pell and Pell-Lucas hybrid quaternions. Following theorem gives some of them.

Theorem 2.11. *The following equations are hold:*

i) $2(HQP_n + HQP_{n+1}) = HQPL_{n+1}$,

ii) $2(HQP_{n+1} - HQP_n) = HQPL_n$,

iii) $HQP_{n-1} + HQP_{n+1} = HQPL_n$,

iv) $HQPL_{n+1} - HQPL_n = 4HQP_n$.

Proof.

i) We know that $2(QP_n + QP_{n+1}) = QPL_{n+1}$ from Remark 2.6.

$$\begin{aligned} 2(HQP_n + HQP_{n+1}) &= 2(QP_n + QP_{n+1}\mathbf{i} + QP_{n+2}\boldsymbol{\varepsilon} + QP_{n+3}\mathbf{h}) + 2(QP_{n+1} + QP_{n+2}\mathbf{i} + QP_{n+3}\boldsymbol{\varepsilon} + QP_{n+4}\mathbf{h}) \\ &= QPL_{n+1} + QPL_{n+2}\mathbf{i} + QPL_{n+3}\boldsymbol{\varepsilon} + QPL_{n+4}\mathbf{h} \\ &= HQPL_{n+1}. \end{aligned}$$

ii) We know that $2(QP_{n+1} - QP_n) = QPL_n$ from Remark 2.6.

$$\begin{aligned} 2(HQP_{n+1} - HQP_n) &= 2(QP_{n+1} + QP_{n+2}\mathbf{i} + QP_{n+3}\boldsymbol{\varepsilon} + QP_{n+4}\mathbf{h}) - 2(QP_n - QP_{n+1}\mathbf{i} - QP_{n+2}\boldsymbol{\varepsilon} - QP_{n+3}\mathbf{h}) \\ &= QPL_n + QPL_{n+1}\mathbf{i} + QPL_{n+2}\boldsymbol{\varepsilon} + QPL_{n+3}\mathbf{h} \\ &= HQPL_n. \end{aligned}$$

iii) We know that $QP_{n-1} + QP_{n+1} = QPL_n$ from [12]. Then,

$$\begin{aligned} HQP_{n-1} + HQP_{n+1} &= QP_{n-1} + QP_n\mathbf{i} + QP_{n+1}\boldsymbol{\varepsilon} + QP_{n+2}\mathbf{h} + QP_{n+1} + QP_{n+2}\mathbf{i} + QP_{n+3}\boldsymbol{\varepsilon} + QP_{n+4}\mathbf{h} \\ &= QPL_n + QPL_{n+1}\mathbf{i} + QPL_{n+2}\boldsymbol{\varepsilon} + QPL_{n+3}\mathbf{h} \\ &= HQPL_n. \end{aligned}$$

iv) We know that $QPL_{n+1} - QPL_n = 4QP_n$ from Remark 2.6. Then,

$$\begin{aligned} HQPL_{n+1} - HQPL_n &= QPL_{n+1} + QP_{n+2}\mathbf{i} + QP_{n+3}\boldsymbol{\varepsilon} + QP_{n+4}\mathbf{h} - QPL_n - QPL_{n+1}\mathbf{i} - QPL_{n+2}\boldsymbol{\varepsilon} - QPL_{n+3}\mathbf{h} \\ &= 4QP_n + 4QP_{n+1}\mathbf{i} + 4QP_{n+2}\boldsymbol{\varepsilon} + 4QP_{n+3}\mathbf{h} \\ &= 4HQP_n. \end{aligned}$$

□

Lemma 2.12. *For $n \geq 1$, the following identities satisfy:*

i) $QP_{n+1}QP_n - QP_nQP_{n+1} = (-1)^n(2i + 4j - 2k)$,

ii) $QP_{n+2}QP_n - QP_nQP_{n+2} = (-1)^n(4i + 11j - 4k)$,

iii) $QP_{n+3}QP_n - QP_nQP_{n+3} = (-1)^n(10i + 20j - 10k)$.

Proof. We will prove first identity, the others can be proven via similar calculations.

We can obtain $QP_{n+1}QP_n - QP_nQP_{n+1} = 2(P_{n+3}^2 i - P_{n+2}P_{n+4}i + P_{n+1}P_{n+4}j - P_{n+2}P_{n+3}j + P_{n+2}^2 k - P_{n+1}P_{n+3}k)$ by way of some basic calculations. We can acquire $QP_{n+1}QP_n - QP_nQP_{n+1} = 2(-1)^n i + 4(-1)^n j + 2(-1)^{n+1}k$ by using $(-1)^n P_aP_b = P_{n+a}P_{n+b} - P_nP_{n+a+b}$ and $P_{n-1}P_{n+1} - P_n^2 = (-1)^n$ identities [13]. □

Theorem 2.13. *Let HQP_n^C be the Hybrid conjugate of the n^{th} Pell hybrid quaternion HQP_n. Then,*

$$\begin{aligned} HQP_n.HQP_n^C &= QP_n^2 + QP_{n+1}^2 - QP_{n+3}^2 - QP_{n+1}QP_{n+2} - QP_{n+2}QP_{n+1} \\ &+ (-1)^n[(-2i - 7j + 2k)\mathbf{i} + (-2i - 4j + 2k)\boldsymbol{\varepsilon} + (12i + 24j - 12k)\mathbf{h}]. \end{aligned}$$

Proof.

$$\begin{aligned} HQP_n.HQP_n^C &= (QP_n + QP_{n+1}\mathbf{i} + QP_{n+2}\boldsymbol{\varepsilon} + QP_{n+3}\mathbf{h}) \quad .(QP_n - QP_{n+1}\mathbf{i} - QP_{n+2}\boldsymbol{\varepsilon} - QP_{n+3}\mathbf{h}) \\ &= QP_n^2 + QP_{n+1}^2 - QP_{n+3}^2 - QP_{n+1}QP_{n+2} - QP_{n+2}QP_{n+1} \\ &+ \mathbf{i}(QP_{n+1}QP_n - QP_nQP_{n+1} + QP_{n+3}QP_{n+1} - QP_{n+1}QP_{n+3}) \\ &+ \boldsymbol{\varepsilon}\big(QP_{n+2}QP_n - QP_nQP_{n+2} + QP_{n+3}QP_{n+1} - QP_{n+1}QP_{n+3} + QP_{n+2}QP_{n+3} - QP_{n+3}QP_{n+2}\big) \\ &+ \mathbf{h}(QP_{n+3}QP_n - QP_nQP_{n+3} + QP_{n+1}QP_{n+2} - QP_{n+2}QP_{n+1}) \\ &= QP_n^2 + QP_{n+1}^2 - QP_{n+3}^2 - QP_{n+1}QP_{n+2} - QP_{n+2}QP_{n+1} \\ &+ \mathbf{i}[(-1)^n(2i + 4j - 2k) + (-1)^{n+1}(4i + 11j - 4k)] \\ &+ \boldsymbol{\varepsilon}[(-1)^n(4i + 11j - 4k) + (-1)^{n+1}(4i + 11j - 4k) + (-1)^{n+2}(-2i - 4j + 2k)] \\ &+ \mathbf{h}[(-1)^n(10i + 20j - 10k) + (-1)^n(2i + 4j - 2k)] \\ &= QP_n^2 + QP_{n+1}^2 - QP_{n+3}^2 - QP_{n+1}QP_{n+2} - QP_{n+2}QP_{n+1} \\ &+ (-1)^n[(-2i - 7j + 2k)\mathbf{i} + (-2i - 4j + 2k)\boldsymbol{\varepsilon} + (12i + 24j - 12k)h]. \end{aligned}$$

□

It is known that the Binet's formula of the n^{th} Pell quaternion is

$$QP_n = \frac{\varphi^n A - \omega^n B}{\varphi - \omega}$$

[4]. In the following proposition the Pell-Lucas quaternions' Binet formula is presented.

Proposition 2.14. *The Binet's formula for the n^{th} Pell-Lucas quaternion is*

$$QPL_n = \varphi^n A + \omega^n B.$$

Proof. Using Theorem 5 in [4], we have $\varphi^n A + \omega^n B = (\varphi + \omega)QP_n + 2QP_{n-1}$. The following equation can be obtained easily

$$QPL_n = \varphi^n A + \omega^n B$$

because $\varphi + \omega = 2$ and $QPL_{n+1} = 2(QP_n + QP_{n+1})$. □

The Binet's formulas for the Pell and Pell-Lucas hybrid quaternions are given with the following theorem.

Theorem 2.15. *The Binet's formulas of the Pell and Pell-Lucas hybrid quaternion are*

$$HQP_n = \frac{\varphi^n\varphi^* A - \omega^n\omega^* B}{\varphi - \omega}, \quad HQPL_n = \varphi^n\varphi^* A + \omega^n\omega^* B$$

respectively. Here $\varphi^* = 1 + \mathbf{i}\varphi + \boldsymbol{\varepsilon}\varphi^2 + \mathbf{h}\varphi^3$ and $\omega^* = 1 + \mathbf{i}\omega + \boldsymbol{\varepsilon}\omega^2 + \mathbf{h}\omega^3$.

Proof.

$$\begin{aligned} HQP_n &= QP_n + \mathbf{i}QP_{n+1} + \boldsymbol{\varepsilon}QP_{n+2} + \mathbf{h}QP_{n+3} \\ &= \frac{\varphi^n A - \omega^n B}{\varphi - \omega} + \mathbf{i}\frac{\varphi^{n+1} A - \omega^{n+1} B}{\varphi - \omega} + \boldsymbol{\varepsilon}\frac{\varphi^{n+2} A - \omega^{n+2} B}{\varphi - \omega} + \mathbf{h}\frac{\varphi^{n+3} A - \omega^{n+3} B}{\varphi - \omega} \\ &= \frac{\varphi^n A(1 + \mathbf{i}\varphi + \boldsymbol{\varepsilon}\varphi^2 + \mathbf{h}\varphi^3) - \omega^n B(1 + \mathbf{i}\omega + \boldsymbol{\varepsilon}\omega^2 + \mathbf{h}\omega^3)}{\varphi - \omega} \\ &= \frac{\varphi^n\varphi^* A - \omega^n\omega^* B}{\varphi - \omega} \end{aligned}$$

$$\begin{aligned} HQPL_n &= QPL_n + \mathbf{i}QPL_{n+1} + \boldsymbol{\varepsilon}QPL_{n+2} + \mathbf{h}QPL_{n+3} \\ &= (\varphi^n A + \omega^n B) + \mathbf{i}(\varphi^{n+1}A + \omega^{n+1}B) + \boldsymbol{\varepsilon}(\varphi^{n+2}A + \omega^{n+2}B) + \mathbf{h}(\varphi^{n+3}A + \omega^{n+3}B) \\ &= \varphi^n A(1 + \mathbf{i}\varphi + \boldsymbol{\varepsilon}\varphi^2 + \mathbf{h}\varphi^3) + \omega^n B(1 + \mathbf{i}\omega + \boldsymbol{\varepsilon}\omega^2 + \mathbf{h}\omega^3) \\ &= \varphi^n\varphi^* A + \omega^n\omega^* B. \end{aligned}$$

□

We can obtain new identities using the Binet's formula of the Pell and Pell-Lucas hybrid quaternions. The next theorem is about a relation between the Pell and Pell-Lucas hybrid quaternions.

Theorem 2.16. $HQPL_n^2 - 8HQP_n^2 = (-1)^n 2(\varphi^*\omega^* AB + \omega^*\varphi^* BA)$.

Proof. Let denote $HQPL_n^2 - 2HQP_n^2$ by S. If we use the Binet's formulas for Pell and Pell-Lucas hybrid quaternions, then

$$\begin{aligned} S &= (\varphi^n\varphi^* A + \omega^n\omega^* B)^2 - 2\Big(\frac{\varphi^n\varphi^* A - \omega^n\omega^* B}{\varphi - \omega}\Big)^2 \\ &= (\varphi^{2n}(\varphi^*)^2 A^2 + \varphi^n\omega^n\varphi^*\omega^* AB + \varphi^n\omega^n\omega^*\varphi^* BA + \omega^{2n}(\omega^*)^2 B^2) \\ &\quad - 8\Big(\frac{\varphi^{2n}(\varphi^*)^2 A^2 - \varphi^n\omega^n\varphi^*\omega^* AB - \varphi^n\omega^n\omega^*\varphi^* BA + \omega^{2n}(\omega^*)^2 B^2}{8}\Big) \\ &= 2(\varphi^n\omega^n\varphi^*\omega^* AB + \varphi^n\omega^n\omega^*\varphi^* BA) = (-1)^n 2(\varphi^*\omega^* AB + \omega^*\varphi^* BA). \end{aligned}$$

□

In the following theorem we give a generalization of the Theorem 6 in [4].

Theorem 2.17. *Let a and b are integers such that $1 \le a \le b$. Then,*

$$\sum_{k=a}^{b} QP_k = \frac{QPL_{b+1} - QPL_a}{2}, \quad \sum_{k=a}^{b} QP_{2k} = \frac{QPL_{2b+1} - QPL_{2a-1}}{2}, \quad \sum_{k=a+1}^{b} QP_{2k-1} = \frac{QPL_{2b} - QPL_{2a}}{2}.$$

Proof. From identity $\sum_{k=1}^{b} QP_k = \frac{QPL_{b+1} - QPL_1}{2}$ in [4],

$$\sum_{k=a}^{b} QP_k = \sum_{k=1}^{b} QP_k - \sum_{k=1}^{a} QP_k = \frac{QPL_{b+1} - QPL_1}{2} - \frac{QPL_a - QPL_1}{2} = \frac{QPL_{b+1} - QPL_a}{2}.$$

The other identities can be proven similarly. □

A generalization of the previous theorem to Pell and Pell-Lucas hybrid quaternions can be given as following theorem.

Theorem 2.18. $\sum_{k=1}^{n} HQP_k = \frac{HQPL_{n+1} - HQPL_1}{2}$.

Proof.

$$\begin{aligned} \sum_{k=1}^{n} HQP_k &= QP_1 + QP_2\mathbf{i} + QP_3\boldsymbol{\varepsilon} + QP_4\mathbf{h} + QP_2 + QP_3\mathbf{i} + QP_4\boldsymbol{\varepsilon} + QP_5\mathbf{h} \\ &\quad + \ldots + QP_n + QP_{n+1}\mathbf{i} + QP_{n+2}\boldsymbol{\varepsilon} + QP_{n+3}\mathbf{h} \\ &= QP_1 + QP_2 + \ldots + QP_n + (QP_2 + QP_3 + \ldots + QP_{n+1})\mathbf{i} \\ &\quad + (QP_3 + QP_4 + \ldots + QP_{n+2})\boldsymbol{\varepsilon} + (QP_4 + QP_5 + \ldots + QP_{n+3})\mathbf{h} \\ &= (\frac{QPL_{n+1} - QPL_1}{2}) + (\frac{QPL_{n+2} - QPL_2}{2})\mathbf{i} + (\frac{QPL_{n+3} - QPL_3}{2})\boldsymbol{\varepsilon} + (\frac{QPL_{n+4} - QPL_4}{2})\mathbf{h} \\ &= \frac{HQPL_{n+1} - HQPL_1}{2}. \end{aligned}$$

□

The result we get when we make the starting point a random point in this series is as follows.

Corollary 2.19. *For the non-negative integers a and b, such that* $1 \leq a \leq b$ *the following identity holds:*

$$\sum_{k=a}^{b} HQP_k = \frac{HQPL_{b+1} - HQPL_a}{2}.$$

Proof. It can be easily proven by the previous two theorems. □

If the indices in the previous theorems are even or odd, we can see what kind of changes will happen in the following theorems.

Theorem 2.20. $\sum_{k=1}^{n} HQP_{2k} = \frac{HQP_{2n+1} - HQP_1}{2}$.

Proof.

$$\begin{aligned}\sum_{k=1}^{n} HQP_{2k} &= QP_2 + QP_3\mathbf{i} + QP_4\boldsymbol{\varepsilon} + QP_5\mathbf{h} + QP_4 + QP_5\mathbf{i} + QP_6\boldsymbol{\varepsilon} + QP_7\mathbf{h} \\ &\quad + \ldots + QP_{2n} + QP_{2n+1}\mathbf{i} + QP_{2n+2}\boldsymbol{\varepsilon} + QP_{2n+3}\mathbf{h} \\ &= QP_2 + QP_4 + \ldots + QP_{2n} + (QP_3 + QP_5 + \ldots + QP_{2n+1})\mathbf{i} \\ &\quad + (QP_4 + QP_6 + \ldots + QP_{2n+2})\boldsymbol{\varepsilon} + (QP_5 + QP_7 + \ldots + QP_{2n+3})\mathbf{h} \\ &= \frac{1}{2}[(QP_{2n+1} - QP_1) + (QP_{2n+2} - QP_2)\mathbf{i} + (QP_{2n+3} - QP_3)\boldsymbol{\varepsilon} + (QP_{2n+4} - QP_4)\mathbf{h}] = \frac{1}{2}(HQP_{2n+1} - HQP_1).\end{aligned}$$

□

Corollary 2.21. $\sum_{k=s}^{n} HQP_{2k} = \frac{HQP_{2n+1} - HQP_{2s-1}}{2}$.

Proof. It can be easily proven by the Theorem 2.17 and Theorem 2.20. □

Theorem 2.22. $\sum_{k=1}^{n} HQP_{2k-1} = \frac{HQP_{2n} - HQP_0}{2}$.

Proof.

$$\begin{aligned}\sum_{k=1}^{n} HQP_{2k-1} &= QP_1 + QP_2\mathbf{i} + QP_3\boldsymbol{\varepsilon} + QP_4\mathbf{h} + QP_3 + QP_4\mathbf{i} + QP_5\boldsymbol{\varepsilon} + QP_6\mathbf{h} \\ &\quad + \ldots + QP_{2n-1} + QP_{2n}\mathbf{i} + QP_{2n+1}\boldsymbol{\varepsilon} + QP_{2n+2}\mathbf{h} \\ &= QP_1 + QP_3 + \ldots + QP_{2n-1} + (QP_2 + QP_4 + \ldots + QP_{2n})\mathbf{i} + (QP_3 + QP_5 + \ldots + QP_{2n+1})\boldsymbol{\varepsilon} \\ &\quad + (QP_4 + QP_6 + \ldots + QP_{2n+2})\mathbf{h} \\ &= \frac{QP_{2n} - QP_0}{2} + \frac{QP_{2n+1} - QP_1}{2}\mathbf{i} + \frac{QP_{2n+2} - QP_2}{2}\boldsymbol{\varepsilon} + \frac{QP_{2n+3} - QP_3}{2}\mathbf{h} = \frac{HQP_{2n} - HQP_0}{2}.\end{aligned}$$

□

Corollary 2.23. $\sum_{k=s}^{n} HQP_{2k-1} = \frac{HQP_{2n} - HQP_{2k-2}}{2}$.

Proof. It can be easily proven by the Theorem 2.17 and Theorem 2.22. □

Theorem 2.24 (Cassini identities). *The following equations are hold:*

$$HQP_{n+1}HQP_{n-1} - HQP_n^2 = \frac{\varphi^{n-1}\omega^n\omega^*\varphi^*BA - \varphi^n\omega^{n-1}\varphi^*\omega^*AB}{\varphi - \omega},$$

$$HQPL_{n+1}HQPL_{n-1} - HQPL_n^2 = \varphi^{n-1}\omega^{n-1}(\varphi - \omega)(\varphi\varphi^*\omega^*AB - \omega\omega^*\varphi^*BA).$$

Proof. Let denote $HQP_{n+1}HQP_{n-1} - HQP_n^2$ by C.

$$\begin{aligned}
C &= \left(\frac{\varphi^{n+1}\varphi^*A - \omega^{n+1}\omega^*B}{\varphi-\omega}\right)\left(\frac{\varphi^{n-1}\varphi^*A - \omega^{n-1}\omega^*B}{\varphi-\omega}\right) - \left(\frac{\varphi^{n}\varphi^*A - \omega^{n}\omega^*B}{\varphi-\omega}\right)^2 \\
&= \frac{\varphi^{2n}(\varphi^*)^2A^2 - \varphi^{n+1}\omega^{n-1}\varphi^*\omega^*AB - \omega^{n+1}\varphi^{n-1}\omega^*\varphi^*BA + \omega^{2n}(\omega^*)^2\omega^2}{(\varphi-\omega)^2} \\
&\quad - \frac{\varphi^{2n}(\varphi^*)^2A^2 - \varphi^{n}\omega^{n}\varphi^*\omega^*AB - \omega^{n}\varphi^{n}\omega^*\varphi^*BA + \omega^{2n}(\omega^*)^2\omega^2}{(\varphi-\omega)^2} \\
&= \frac{\varphi^{n-1}\omega^n\omega^*\varphi^*(\varphi-\omega)BA - \varphi^n\omega^{n-1}\varphi^*\omega^*(\varphi-\omega)AB}{(\varphi-\omega)^2} = \frac{\varphi^{n-1}\omega^{n-1}(\omega\omega^*\varphi^*BA - \varphi\varphi^*\omega^*AB)}{\varphi-\omega}.
\end{aligned}$$

Let denote $HQPL_{n+1}HQPL_{n-1} - HQPL_n^2$ by D.

$$\begin{aligned}
D &= (\varphi^{n+1}\varphi^*A + \omega^{n+1}\omega^*B)(\varphi^{n-1}\varphi^*A + \omega^{n-1}\omega^*B) - (\varphi^n\varphi^*A + \omega^n\omega^*B)^2 \\
&= \varphi^{2n}(\varphi^*)^2A^2 + \varphi^{n+1}\omega^{n-1}\varphi^*\omega^*AB + \omega^{n+1}\varphi^{n-1}\omega^*\varphi^*BA + \omega^{2n}(\omega^*)^2\omega^2 \\
&\quad - \varphi^{2n}(\varphi^*)^2A^2 + \varphi^n\omega^n\varphi^*\omega^*AB + \omega^n\varphi^n\omega^*\varphi^*BA + \omega^{2n}(\omega^*)^2\omega^2 \\
&= \varphi^{n-1}\omega^n\omega^*\varphi^*(\omega-\varphi)BA + \varphi^n\omega^{n-1}\varphi^*\omega^*(\varphi-\omega)AB = \varphi^{n-1}\omega^{n-1}(\varphi-\omega)(\varphi\varphi^*\omega^*AB - \omega\omega^*\varphi^*BA).
\end{aligned}$$

□

3. Conclusion

In this paper, we have given some identities about Pell and Pell-Lucas numbers, and Pell and Pell-Lucas quaternions as a contribution to number sequences and quaternions. We have also introduced the notion of the Pell and Pell-Lucas hybrid quaternions. Then some significant identities and properties of the Pell and Pell-Lucas hybrid quaternions, including the Cassini identity and Binet's formula, have been given and proven. Also, the other number sequences, such as Padovan numbers and Jacobsthal numbers on hybrid quaternions are a future topic of interest.

References

[1] M. Akar, S. Yüce, S. Sahin, On the Dual Hyperbolic Numbers and the Complex Hyperbolic Numbers, Journal of Computer Science and Computational Mathematics (2018), 8.1, 1-6.

[2] H. Arslan, Gaussian Pell and Gaussian Pell-Lucas Quaternions, Filomat (2021), 35.5, 1609-1617.

[3] Z. Cerin, G. M. Gianella, On sums of Pell numbers, Acc. Sc. Torino-Atti Sc. Fis. (2007) 141, 23–31.

[4] B. C. Çimen, A. İpek, On pell quaternions and Pell-Lucas quaternion,. Adv. Appl. Clifford Algebras (2016), **26**(1), 39–51.

[5] A. Dağdeviren, A generalisation Of complex, dual and hyperbolic quaternion: hybrid quaternions (2021) Submitted.

[6] A. Dağdeviren, F. Kürüz, On the Horadam hybrid quaternions, Advanced Studies: Euro-Tbilisi Mathematical Journal (2022), (10), 293-306.

[7] B. G. Djordjevic, Convolutions of the generalized Pell and Pell-Lucas numbers, Filomat (2016), 30.1: 105-112.

[8] E. Erkan, F. Kürüz, A. Dağdeviren, A study on matrices with hybrid number entries, Advanced Studies: Euro-Tbilisi Mathematical Journal (2022), (10), 227-235.

[9] E. Erkan, A. Dağdeviren, k-Fibonacci and k-Lucas Hybrid Numbers, Tamap Journal of Mathematics and Statistics (2021), Article ID 125.

[10] S. Halıcı, On fibonacci quaternions, Adv. Appl. Clifford Algebras (2012), 22(2), 321–327.

[11] W. R. Hamilton, Elements of Quaternions, Longmans, Green and Co., London (1866).

[12] A. F. Horadam, P. Filipponi, Real Pell and Pell-Lucas numbers with real subscripts, The Fib. Quart (1995), 33(5), 398-406.

[13] A. F. Horadam, Pell identities, Fibonacci Quart (1971), 9(3), 245-252.

[14] M. Liana, A. Szynal-Liana, I. Wloch, On Pell hybrinomials, Miskolc Mathematical Notes (2019), **20**(2), 1051–1062.

[15] M. Özdemir, Introduction to hybrid numbers, Adv. Appl. Clifford Algebras (2018) **28**(1), 11.

[16] A. Szynal-Liana, The Horadam hybrid numbers, Discussiones Mathematicae-General Algebra and Applications (2018), **38**(1), 91–98.

[17] A. Szynal-Liana, I. Włoch, On Pell and Pell-Lucas Hybrid Numbers, Commentationes Mathematicae (2018), 58(1-2).

[18] A. Szynal-Liana, I. Włoch, The Pell quaternions and the Pell octonions, Adv. Appl. Clifford Algebras (2016), 26(1), 435-440.

[19] F. Torunbalcı Aydın, Generalized dual Pell quaternions, Notes on Number Theory and Discrete Mathematics (2017), **23**(4), 66–84.

[20] F. Zhang, Quaternions and matrices of quaternions, Linear algebra and its applications (1997), 251, 21–57.