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An extension of the Euclidean Berezin number

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Abstract. The Berezin transform $\widetilde{A}$ of an operator A, acting on the reproducing kernel Hilbert space $\mathbb{H} = \mathbb{H}(\Theta)$ over some (non-empty) set Θ, is defined by $\widetilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle$ $(\lambda \in \Theta)$, where $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ is the normalized reproducing kernel of $\mathbb{H}$. The Berezin number of an operator A is defined by $\mathbf{ber}(A) = \sup\limits_{\lambda\in\Theta}\left|\widetilde{A}(\lambda)\right| = \sup\limits_{\lambda\in\Theta}\left|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|$. In this paper, by using the definition of g-generalized Euclidean Berezin number, we obtain some possible relations and inequalities. It is shown, among other inequalities, that if $A_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(i = 1, \ldots, n)$, then

$$\mathbf{ber}_g(A_1, ..., A_n) \le g^{-1}\left(\sum_{i=1}^n g\left(\mathbf{ber}(A_i)\right)\right) \le \sum_{i=1}^n \mathbf{ber}(A_i),$$

in which $g : [0, \infty) \to [0, \infty)$ is a continuous increasing convex function such that $g(0) = 0$.

1. Introduction

Let $\mathbb{L}(\mathbb{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathbb{H}$ with an inner product $\langle\,.\,,\,.\,\rangle$ and the corresponding norm $\|\,.\,\|$. An operator $A \in \mathbb{L}(\mathbb{H})$ is called positive if $\langle Ax, x\rangle \ge 0$ for all $x \in \mathbb{H}$, and then we write $A \ge 0$. For $A \in \mathbb{L}(\mathbb{H})$, let $A = \Re(A) + i\Im(A)$ be the Cartesian decomposition of A, where the Hermitian matrices $\Re(A) = \frac{A+A^*}{2}$ and $\Im(A) = \frac{A-A^*}{2i}$ are called the real and the imaginary parts of A, respectively.
A functional Hilbert space $\mathbb{H} = \mathbb{H}(\Theta)$ is a Hilbert space of complex valued functions on a(nonempty) set Θ, which has the property that point evaluations are continuous i.e. for each $\lambda \in \Theta$ the map $f \mapsto f(\lambda)$ is a continuous linear functional on $\mathbb{H}$. The Riesz representation theorem ensure that for each $\lambda \in \Theta$ there is a unique element $k_\lambda \in \mathbb{H}$ such that $f(\lambda) = \langle f, k_\lambda\rangle$ for all $f \in \mathbb{H}$. The collection $\{k_\lambda : \lambda \in \Theta\}$ is called the reproducing kernel of $\mathbb{H}$. If $\{e_n\}$ is an orthonormal basis for a functional Hilbert space $\mathbb{H}$, then the reproducing kernel of $\mathbb{H}$ is given by $k_\lambda(z) = \sum_n \overline{e_n(\lambda)}e_n(z)$; (see [12, Problem 37]). For $\lambda \in \Theta$, let $\hat{k_\lambda} = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathbb{H}$. For a bounded linear operator A on $\mathbb{H}$, the function $\widetilde{A}$ defined on Θ by $\widetilde{A}(\lambda) = \langle A\hat{k_\lambda}, \hat{k_\lambda}\rangle$ is the Berezin symbol of A, which firstly have been introduced by Berezin [4, 5]. The Berezin set and the Berezin number of the operator A are defined by

$$\mathbf{Ber}(A) := \{\widetilde{A}(\lambda) : \lambda \in \Theta\} \qquad \text{and} \qquad \mathbf{ber}(A) := \sup\{|\widetilde{A}(\lambda)| : \lambda \in \Theta\},$$

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respectively, (see [14]). In [3], the authors show that

$$\mathbf{ber}(A) = \sup_{\theta\in\mathbb{R}} \mathbf{ber}\left(\Re(e^{i\theta}A)\right) = \sup_{\alpha^2+\beta^2=1} \mathbf{ber}\left(\alpha\Re A + \beta\Im A\right).$$

The Berezin symbol and the Berezin number has large application in the study of various questions of operator theory in the functional Hilbert space, quantum physics and non-commutative geometry. These are the important tools to study operators on Hardy and Bergman spaces, especially for Toeplitz and Hankel operators. Recall that the Hardy space $\mathbb{H}_2(\mathbb{D})$ of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is a RKHS of analytic functions on $\mathbb{D}$ with reproducing kernel $k_\tau(z) = \frac{1}{1-\bar{\tau}z}$ (see, Paulsen and Raghupati [19]). Since, the collection of normalized reproducing kernel of $\mathbb{H}$ is a subset of the unit sphere of $\mathbb{H}$, so the numerical radius and the Berezin number of an operator on $\mathbb{H}$ may not be equal. The Berezin number inequalities have been studied by many mathematicians over the years, interested readers can see [11, 15, 18, 20]. Namely, the Berezin symbol have been investigated in detail for the Toeplitz and Hankel operators on the Hardy and Bergman spaces; it is widely applied in the various questions of analysis and uniquely determines the operator(i.e., for all $\lambda \in \Theta, \widetilde{A}(\lambda) = \widetilde{B}(\lambda)$ implies $A = B$). For further information about Berezin symbol we refer the reader to [2, 9–11, 15, 18, 20–22] and references therein.

Moreover, The Berezin number of operators A, B satisfy the following properties:
(i) $\mathbf{ber}(\alpha \mathrm{A}) = |\alpha|\mathbf{ber}(\mathrm{A})$ for all $\alpha \in \mathbb{C}$;
(ii) $\mathbf{ber}(A + B) \leq \mathbf{ber}(A) + \mathbf{ber}(B)$.

The numerical radius of $A \in \mathbb{L}(\mathbb{H}(\Theta))$ is defined by

$$w(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathbb{H}, \|x\| = 1\}.$$

It is clear that

$$\mathbf{ber}(A) \leq w(A) \leq \|A\| \qquad \text{for all } A \in \mathbb{L}(\mathbb{H}(\Theta)).$$

Let $A_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(1 \leq i \leq n)$. The generalized Euclidean Berezin number of $A_1, \ldots, A_n$ is defined in [2] as follows:

$$\mathbf{ber_p}(A_1, \cdots, A_n) := \sup_{\lambda\in\Theta} \left(\sum_{i=1}^{n} |\widetilde{A}_i(\lambda)|^p\right)^{\frac{1}{p}} \quad \text{for all} \quad p \geq 1.$$

In the case $p = 2$, we have the Euclidean Berezin number and denote by

$$\mathbf{ber_e}(A_1, \ldots, A_n) := \sup_{\lambda\in\Theta} \left(\sum_{i=1}^{n} |\widetilde{A}_i(\lambda)|^2\right)^{\frac{1}{2}}.$$

For $p = 1$ if $A_1 = \cdots = A_n = A$, then $\mathbf{ber_1}(A, \cdots, A) = n\mathbf{ber}(A)$.
The generalized Euclidean Berezin number $\mathbf{ber_p}(\cdot)$ $(p \geq 1)$ has the following properties:
(i) $\mathbf{ber_p}(A_1, \cdots, A_n) = 0$ if and if $A_i = 0$ $(i = 1, \ldots, n)$;
(ii) $\mathbf{ber_p}(\alpha A_1, \cdots, \alpha A_n) = |\alpha|\mathbf{ber_p}(A_1, \cdots, A_n)$ for all $\alpha \in \mathbb{C}$;
(iii) $\mathbf{ber_p}(A_1 + B_1, \cdots, A_n + B_n) \leq \mathbf{ber_p}(A_1, \cdots, A_n) + \mathbf{ber_p}(B_1, \cdots, B_n)$;
(iv) $\mathbf{ber_p}(A_1, A_2, \cdots, A_n) = \mathbf{ber_p}(A_1^*, A_2^*, \cdots, A_n^*)$,
where $A_i, B_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(i = 1, \ldots, n)$.
The proof of the properties (i) – (iv) immediately comes from definition of generalized Berezin number.
In [7], the author obtained the following inequality

$$\mathbf{ber_p^p}(A_1|A_1|^{s+t-1}, \ldots, A_n|A_n|^{s+t-1}) \leq \mathbf{ber}\left(\sum_{i=1}^{n}\left(\frac{|A_i|^{2s} + |A_i^*|^{2t}}{2}\right)^p\right), \tag{1}$$

in which $A_1, \ldots, A_n \in \mathbb{L}(\mathbb{H}(\Theta))$, $p > 1$ and $s, t \in [0, 1]$ such that $s + t \geq 1$.
The following we define an extension of the generalized Euclidean Berezin number as follows:

Definition 1.1. *Assume that $A_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(1 \le i \le n)$ and $g : [0,\infty) \to [0,\infty)$ be a continuous increasing convex function such that $g(0) = 0$. We define the g-generalized Euclidean Berezin number of $A_1, \cdots, A_n$ by*

$$\boldsymbol{ber}_g(A_1, ..., A_n) := \sup_{\lambda \in \Theta} g^{-1}\left(\sum_{i=1}^{n} g\left(|\widetilde{A_i}(\lambda)|\right)\right).$$

For $g(t) = t^p$ $(p \ge 1)$ we have $\mathbf{ber}_g(\cdot) = \mathbf{ber}_\mathbf{p}(\cdot)$ and for $g(t) = t^2$ we have $\mathbf{ber}_g(\cdot) = \mathbf{ber}_\mathbf{e}(\cdot)$.

A function $g : [0,\infty) \to [0,\infty)$ is convex if $g((1-\lambda)a + \lambda b) \le (1-\lambda)g(a) + \lambda g(b)$ for all $\lambda \in [0,1]$ and $a, b \in [0,\infty)$. If $g : [0,\infty) \to [0,\infty)$ is convex such that $g(0) = 0$, then

$$g(x) + g(y) \le g(x+y) \quad \text{(superadditive)}$$

for all $x, y \in [0,\infty)$. The recent inequality is reversed if g is concave.

Dragomir [8] provided a generalization of Furuta's inequality

$$\left|(\widetilde{DCBA})(\lambda)\right|^2 \le (\widetilde{A^*|B|^2A})(\lambda)(\widetilde{D|C^*|^2D^*})(\lambda), \tag{2}$$

where $A, B, C, D \in \mathbb{L}(\mathbb{H}(\Theta))$ and $\lambda \in \Theta$.

In this paper, by using the definition of g-generalized Euclidean Berezin number, we show some possible relations and inequalities. For these goals, we will apply some methods from [1].

2. Main results

In this section, we would like to check some properties about the g-generalized Euclidean Berezin number and then we state some inequalities related to this concept.

First we need the following lemmas:

Lemma 2.1. *[6] Let g be a convex function on a real interval J and let $A \in \mathbb{L}(\mathbb{H}(\Theta))$ be a self-adjoint operator with spectrum in J. Then*

$$g\left(\widetilde{A}(\lambda)\right) \le \widetilde{g(A)}(\lambda) \quad \text{for all} \quad \lambda \in \Theta.$$

The inequality is reversed if g is concave.

The following lemma is a simple consequence of the classical Jensen and Young inequalities(see [13]).

Lemma 2.2. *Let $a, b \ge 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \le \frac{a}{p} + \frac{b}{q} \le \left(\frac{a^r}{p} + \frac{b^r}{q}\right)^{\frac{1}{r}} \qquad \text{for all} \quad r \ge 1. \tag{3}$$

Lemma 2.3. *[16] Let $A \in \mathbb{L}(\mathbb{H}(\Theta))$ and $\lambda \in \Theta$. If $0 \le s \le 1$, then*

$$|\widetilde{A}(\lambda)|^2 \le \widetilde{|A|^{2s}}(\lambda)\, \widetilde{|A^*|^{2(1-s)}}(\lambda),$$

where $|A| = (A^*A)^{\frac{1}{2}}$ is the absolute value of A.

Proposition 2.4. *Assume that $A_i, B_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(i = 1, \ldots, n)$ and $g : [0,\infty) \to [0,\infty)$ be a continuous increasing function such that $g(0) = 0$. Then*

(i) $\boldsymbol{ber}_g(A_1, ..., A_n) = 0$ *if and only if* $A_i = 0$ $(i = 1, \ldots, n)$;
(ii) $\boldsymbol{ber}_g(\alpha A_1, ..., \alpha A_n) = |\alpha| \boldsymbol{ber}_g(A_1, ..., A_n)$ *for all* $\alpha \in \mathbb{C}$ *if* g *is multiplicative;*
(iii) $\boldsymbol{ber}_g(A_1, A_2, ..., A_n) = \boldsymbol{ber}_g(A_1^*, A_2^*, ..., A_n^*)$;

(iv) $\boldsymbol{ber}_g(A_1+B_1,...,A_n+B_n) \le \boldsymbol{ber}_g(A_1,...,A_n)+\boldsymbol{ber}_g(B_1,...,B_n)$, *if* g *is geometrically convex i.e.* $g(\sqrt{xy}) \le \sqrt{g(x)g(y)}$.

Proof. The parts (i), (ii) and (iii) immediately come from the definition of the g-generalized Euclidean Berezin number. For the part (iv) if g is increasing, then we have

$$\sum_{i=1}^{n} g\left(|(\widetilde{A}_i+\widetilde{B}_i)(\lambda)|\right) = \sum_{i=1}^{n} g\left(|\widetilde{A}_i(\lambda)+\widetilde{B}_i(\lambda)|\right) \le \sum_{i=1}^{n} g\left(|\widetilde{A}_i(\lambda)|+|\widetilde{B}_i(\lambda)|\right),$$

whence by the monotonicity of g^{-1} and the geometrically convexity condition of g we get

$$\begin{aligned} g^{-1}\left(\sum_{i=1}^{n} g\left(|\widetilde{A}_i(\lambda)+\widetilde{B}_i(\lambda)|\right)\right) &\le g^{-1}\left(\sum_{i=1}^{n} g\left(|\widetilde{A}_i(\lambda)|+|\widetilde{B}_i(\lambda)|\right)\right) \\ &\le g^{-1}\left(\sum_{i=1}^{n} g\left(|\widetilde{A}_i(\lambda)|\right)\right) + g^{-1}\left(\sum_{i=1}^{n} g\left(|\widetilde{B}_i(\lambda)|\right)\right), \end{aligned}$$

where the last inequality follows from [17, Corollary 1.1]. Hence by taking the supremum on $\lambda \in \Theta$ we get

$$\mathbf{ber}_g(A_1+B_1,\cdots,A_n+B_n) \le \mathbf{ber}_g(A_1,\cdots,A_n) + \mathbf{ber}_g(B_1,...,B_n).$$

□

Now, we obtain a result for the g-generalized Euclidean Berezin number.

Theorem 2.5. *Assume that* $A_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(i=1,\dots,n)$ *and* $g:[0,\infty)\to[0,\infty)$ *be a continuous increasing convex function such that* $g(0)=0$. *Then*

$$\boldsymbol{ber}_g(A_1,...,A_n) \le g^{-1}\left(\sum_{i=1}^{n} g\left(\boldsymbol{ber}(A_i)\right)\right) \le \sum_{i=1}^{n} \boldsymbol{ber}(A_i). \tag{4}$$

Proof. It follows from g is increasing convex that g^{-1} is increasing concave, and so g is superadditive and g^{-1} is subadditive. By the definition of $\mathbf{ber}(\cdot)$ we have

$$|\widetilde{A}_i(\lambda)| \le \mathbf{ber}(A_i) \quad \text{for all } i=1,\dots,n.$$

Hence by the monotonicity of g and g^{-1} we have

$$\sum_{i=1}^{n} g\left(|\widetilde{A}_i(\lambda)|\right) \le \sum_{i=1}^{n} g\left(\mathbf{ber}(A_i)\right) \;\Rightarrow\; g^{-1}\left(\sum_{i=1}^{n} g\left(|\widetilde{A}_i(\lambda)|\right)\right) \le g^{-1}\left(\sum_{i=1}^{n} g\left(\mathbf{ber}(A_i)\right)\right).$$

Taking the supremum on $\lambda \in \Theta$ we get

$$\begin{aligned} \mathbf{ber}_g(A_1,...,A_n) &= \sup_{\lambda\in\Theta} g^{-1}\left(\sum_{i=1}^{n} g\left(|\widetilde{A}_i(\lambda)|\right)\right) \\ &\le g^{-1}\left(\sum_{i=1}^{n} g\left(\mathbf{ber}(A_i)\right)\right) \\ &\le \sum_{i=1}^{n} g^{-1}\left(g\left(\mathbf{ber}(A_i)\right)\right) \qquad \text{(by the subadditivity of } g^{-1}\text{)} \\ &= \sum_{i=1}^{n} \mathbf{ber}(A_i). \end{aligned}$$

□

In the next result, we show an inequality for concave functions.

Theorem 2.6. *Assume that $A_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(i = 1,\dots,n)$ and $g : [0,\infty) \to [0,\infty)$ be a continuous increasing concave function such that $g(0) = 0$. Then*

$$\boldsymbol{ber}_g(A_1,...,A_n) \geq \boldsymbol{ber}\left(\sum_{i=1}^n A_i\right).$$

Proof. It follows from g is increasing concave that g^{-1} is increasing convex, and so g^{-1} is superadditive. Hence

$$\begin{aligned}\left|\left(\widetilde{\sum_{i=1}^n A_i}\right)(\lambda)\right| &= \left|\sum_{i=1}^n \widetilde{A}_i(\lambda)\right| \\ &\leq \sum_{i=1}^n \left|\widetilde{A}_i(\lambda)\right| \\ &= \sum_{i=1}^n g^{-1}\left(g\left(\left|\widetilde{A}_i(\lambda)\right|\right)\right) \\ &\leq g^{-1}\left(\sum_{i=1}^n g\left(\left|\widetilde{A}_i(\lambda)\right|\right)\right) \quad \text{(by the superadditivity of } g^{-1}\text{)} \\ &\leq \mathbf{ber}_g(A_1,\dots,A_n).\end{aligned}$$

Take the supremum on $\lambda \in \Theta$ we get the desired result. □

In the next theorem, we present another lower bound for $\mathbf{ber}_g(\cdot)$

Theorem 2.7. *Assume that $A_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(i = 1,\dots,n)$ and $g : [0,\infty) \to [0,\infty)$ be a continuous increasing convex function. Then*

$$\boldsymbol{ber}_g(A_1,...,A_n) \geq \sup_{|\mu_i|\leq 1} \boldsymbol{ber}\left(\sum_{i=1}^n \frac{\mu_i}{n} A_i\right).$$

In particular,

$$\boldsymbol{ber}_g(A_1,...,A_n) \geq \frac{1}{n}\max\left\{\boldsymbol{ber}\left(\sum_{i=1}^n \pm A_i\right)\right\}.$$

Proof. The convexity of g implies that

$$\begin{aligned}\left|\left(\widetilde{\sum_{i=1}^n \frac{\mu_i}{n} A_i}\right)(\lambda)\right| &= \left|\sum_{i=1}^n \left(\widetilde{\frac{\mu_i}{n} A_i}\right)(\lambda)\right| \\ &\leq \sum_{i=1}^n \frac{1}{n}\left|\widetilde{A}_i(\lambda)\right| \\ &= g^{-1}\left(g\left(\sum_{i=1}^n \frac{1}{n}\left|\widetilde{A}_i(\lambda)\right|\right)\right) \\ &\leq g^{-1}\left(\sum_{i=1}^n \frac{1}{n} g\left(\left|\widetilde{A}_i(\lambda)\right|\right)\right) \\ &\leq g^{-1}\left(\sum_{i=1}^n g\left(\left|\widetilde{A}_i(\lambda)\right|\right)\right) \quad \text{(by the monotonicity of } g^{-1}\text{)},\end{aligned}$$

in which $\lambda \in \Theta$ and $\mu_i \in \mathbb{C}$ such that $|\mu_i| \leq 1$. Taking the supremum on $\lambda \in \Theta$ yields

$$\mathbf{ber}\left(\sum_{i=1}^{n} \frac{\mu_i}{n} A_i\right) \leq \mathbf{ber}_g(A_1, ..., A_n).$$

Therefore,

$$\sup_{|\mu_i| \leq 1} \mathbf{ber}\left(\sum_{i=1}^{n} \frac{\mu_i}{n} A_i\right) \leq \mathbf{ber}_g(A_1, ..., A_n),$$

which we reach the first inequality. If we put $\mu_i = \pm 1$, then we get the second inequality. □

As a consequence, we have the next result.

Corollary 2.8. *Assume that $A_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(i = 1, \ldots, n)$ and $g : [0, \infty) \to [0, \infty)$ be continuous increasing convex. Then*

$$\boldsymbol{ber}_g(A_1, ..., A_n) \geq \frac{1}{n} \max\{\boldsymbol{ber}(A_1), \ldots, \boldsymbol{ber}(A_n)\}.$$

Proof. If for any j $(j = 1, \ldots, n)$ we assume that $\mu_j = 1$ and $\mu_i = 0$ when $i \neq j$ in Theorem 2.7, then

$$\mathbf{ber}_g(A_1, ..., A_n) \geq \frac{1}{n} \mathbf{ber}(A_j) \qquad \text{for all} \quad j = 1, \ldots, n.,$$

whence

$$\mathbf{ber}_g(A_1, ..., A_n) \geq \frac{1}{n} \max\{\mathbf{ber}(A_1), \ldots, \mathbf{ber}(A_n)\}.$$

□

Remark 2.9. *Assume that $A_1 = A_2 = \cdots = A_n = A$. Using Theorem 2.5 we have*

$$\boldsymbol{ber}_g(A, \ldots, A) \leq n\boldsymbol{ber}(A).$$

Moreover, applying Corollary 2.8 we get

$$\boldsymbol{ber}_g(A, \ldots, A) \geq \boldsymbol{ber}(A).$$

Therefore,

$$\boldsymbol{ber}(A) \leq \boldsymbol{ber}_g(A, \ldots, A) \leq n\boldsymbol{ber}(A).$$

Theorem 2.10. *Assume that $A_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(i = 1, \ldots, n)$, $g : [0, \infty) \to [0, \infty)$ be a continuous increasing convex function. Then*

$$\boldsymbol{ber}_g(A_1, \ldots, A_n) \leq g^{-1}\left(\boldsymbol{ber}\left(\sum_{i=1}^{n}\left(\frac{g(|A_i|^{2s}) + g(|A_i^*|^{2(1-s)})}{2}\right)\right)\right),$$

where $s \in [0, 1]$.

Proof. It follows from Lemma 2.3 and the arithmetic geometric mean inequality that

$$|\widetilde{A}_i(\lambda)|^2 \leq \widetilde{|A_i|^{2s}}(\lambda)\widetilde{|A_i^*|^{2(1-s)}}(\lambda) \leq \left(\frac{\widetilde{|A_i|^{2s}}(\lambda) + \widetilde{|A_i^*|^{2(1-s)}}(\lambda)}{2}\right)^2. \tag{5}$$

Hence for the increasing function g we have

$$\sum_{i=1}^{n} g\left(|\widetilde{A}_i(\lambda)|\right) \le \sum_{i=1}^{n} g\left(\left(\frac{\widetilde{|A_i|^{2s}(\lambda)} + \widetilde{|A_i^*|^{2(1-s)}(\lambda)}}{2}\right)\right)$$

(by inequality (5))

$$\le \sum_{i=1}^{n} \frac{g\left(\widetilde{|A_i|^{2s}}(\lambda)\right) + g\left(\widetilde{|A_i^*|^{2(1-s)}}(\lambda)\right)}{2}$$

(by the convexity of g)

$$\le \sum_{i=1}^{n} \frac{g\widetilde{(|A_i|^{2s})}(\lambda) + g\widetilde{\left(|A_i^*|^{2(1-s)}\right)}(\lambda)}{2}$$

(by Lemma 2.1)

$$= \sum_{i=1}^{n} \left(\frac{\widetilde{g\left(|A_i|^{2s}\right) + g\left(|A_i^*|^{2(1-s)}\right)}}{2}\right)(\lambda),$$

whence

$$g^{-1}\left(\sum_{i=1}^{n} g\left(|\widetilde{A}_i(\lambda)|\right)\right) \le g^{-1}\left(\sum_{i=1}^{n} \frac{\widetilde{g\left(|A_i|^{2s}\right) + g\left(|A_i^*|^{2(1-s)}\right)}}{2}(\lambda)\right).$$

If we take the supremum on $\lambda \in \Theta$, then we get the desired result. □

Proposition 2.11. *Assume that $A_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(i = 1, \ldots, n)$, $g : [0,\infty) \to [0,\infty)$ be a continuous increasing geometrically convex function. Then*

$$\boldsymbol{ber}_g(A_1, \ldots, A_n) \le g^{-1}\left(\left[\sum_{i=1}^{n} \boldsymbol{ber}^p g\left((|A_i|^{2s})\right)\right]^{\frac{1}{2p}} \left[\sum_{i=1}^{n} \boldsymbol{ber}^q \left(g(|A_i^*|^{2(1-s)})\right)\right]^{\frac{1}{2q}}\right),$$

in which $s \in [0,1]$ and $p, q > 1$ such that $p^{-1} + q^{-1} = 1$.

Proof. It follows from Lemma 2.3 and the monotonicity and the geometrically convexity of g, respectively, that

$$g\left(|\widetilde{A}_i(\lambda)|\right) \le g\left(\widetilde{|A_i|^{2s}(\lambda)}^{\frac{1}{2}} \widetilde{|A_i^*|^{2(1-s)}}(\lambda)^{\frac{1}{2}}\right) \le \sqrt{g\left(\widetilde{|A_i|^{2s}(\lambda)}\right) g\left(\widetilde{|A_i^*|^{2(1-s)}}(\lambda)\right)}.$$

The last inequality follows from the geometrically convexity of g. Hence

$$\sum_{i=1}^{n} g\left(|\widetilde{A}_i(\lambda)|\right) \le \sum_{i=1}^{n} \sqrt{g\left(\widetilde{|A_i|^{2s}(\lambda)}\right) g\left(\widetilde{|A_i^*|^{2(1-s)}}(\lambda)\right)}$$

$$\le \left[\sum_{i=1}^{n} g\left(\widetilde{|A_i|^{2s}(\lambda)}\right)^p\right]^{\frac{1}{2p}} \left[\sum_{i=1}^{n} g\left(\widetilde{|A_i^*|^{2(1-s)}}(\lambda)\right)^q\right]^{\frac{1}{2q}}$$

(by the Cauchy Schwarz inequality)

$$\le \left[\sum_{i=1}^{n} \left(g\widetilde{(|A_i|^{2s})}(\lambda)\right)^p\right]^{\frac{1}{2p}} \left[\sum_{i=1}^{n} \left(g\widetilde{(|A_i^*|^{2(1-s)})}(\lambda)\right)^q\right]^{\frac{1}{2q}}$$

(by Lemma 2.1).

Hence

$$g^{-1}\left(\sum_{i=1}^{n} g\left(|\widetilde{A}_i(\lambda)|\right)\right) \leq g^{-1}\left(\left[\sum_{i=1}^{n}\left(g(\widetilde{|A_i|^{2s})}(\lambda)\right)^p\right]^{\frac{1}{2p}}\left[\sum_{i=1}^{n}\left(g(\widetilde{|A_i^*|^{2(1-s)}})(\lambda)\right)^q\right]^{\frac{1}{2q}}\right)$$
$$\leq g^{-1}\left(\left[\sum_{i=1}^{n}\mathbf{ber}^p\left(g(|A_i|^{2s})\right)\right]^{\frac{1}{2p}}\left[\sum_{i=1}^{n}\mathbf{ber}^q\left(g(|A_i^*|^{2(1-s)})\right)\right]^{\frac{1}{2q}}\right).$$

The last inequality follows from the monotonicity of g^{-1} and the definition of $\mathbf{ber}(\cdot)$. By taking the supremum on $\lambda \in \Theta$ we get the desired result. □

The following, we present some results for the g-generalized Euclidean Berezin number.

Theorem 2.12. *Assume that $A_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(i = 1, \ldots, n)$, $g : [0, \infty) \to [0, \infty)$ be a continuous increasing convex and super-multiplicative function. Then*

$$\boldsymbol{ber}_g(A_1, \ldots, A_n) \leq g^{-1}\left(\boldsymbol{ber}\left(\sqrt{n\sum_{i=1}^{n} g\left(\frac{A_i^*A_i + A_iA_i^*}{2}\right)}\right)\right).$$

Proof. By the convexity of $h(t) = t^2$ we have

$$|\widetilde{A}_i(\lambda)|^2 = \left(\widetilde{\Re(A_i)}(\lambda)\right)^2 + \left(\widetilde{\Im(A_i)}(\lambda)\right)^2$$
$$\leq \widetilde{(\Re(A_i))^2}(\lambda) + \widetilde{(\Im(A_i))^2}(\lambda)$$
$$= \left(\widetilde{(\Re(A_i))^2 + (\Im(A_i))^2}\right)(\lambda),$$

which implies that

$$g^2\left(|\widetilde{A}_i(\lambda)|\right) \leq g\left(|\widetilde{A}_i(\lambda)|^2\right)$$
$$\leq g\left(\left(\widetilde{(\Re(A_i))^2 + (\Im(A_i))^2}\right)(\lambda)\right)$$
$$\leq \left(g\left(\widetilde{(\Re(A_i))^2 + (\Im(A_i))^2}\right)\right)(\lambda),$$

whence

$$\sum_{i=1}^{n} g^2\left(|\widetilde{A}_i(\lambda)|\right) \leq \sum_{i=1}^{n}\left(g\left(\widetilde{(\Re(A_i))^2 + (\Im(A_i))^2}\right)(\lambda)\right).$$

Moreover, the Jensen inequality for the function $h(t) = t^2$ implies that

$$\left(\frac{1}{n}\sum_{i=1}^{n} g\left(|\widetilde{A}_i(\lambda)|\right)\right)^2 \leq \frac{1}{n}\sum_{i=1}^{n} g^2\left(|\widetilde{A}_i(\lambda)|\right)$$
$$\leq \frac{1}{n}\sum_{i=1}^{n}\left(g\left(\widetilde{(\Re(A_i))^2 + (\Im(A_i))^2}\right)(\lambda)\right),$$

which equivalent to

$$\sum_{i=1}^{n} g\left(|\widetilde{A}_i(\lambda)|\right) \leq \left(n\sum_{i=1}^{n}\left(g\left(\widetilde{(\Re(A_i))^2 + (\Im(A_i))^2}\right)(\lambda)\right)\right)^{\frac{1}{2}}.$$

It follows from g^{-1} is increasing that

$$g^{-1}\left(\sum_{i=1}^{n} g\left(|\widetilde{A_i}(\lambda)|\right)\right) \leq g^{-1}\left(\left(n\sum_{i=1}^{n}\left(g\left(\widetilde{(\Re(A_i))^2+(\Im(A_i))^2}\right)(\lambda)\right)\right)^{\frac{1}{2}}\right)$$
$$= g^{-1}\left(\sqrt{n}\left(\sum_{i=1}^{n}\left(g\left(\frac{\widetilde{A_i^*A_i+A_iA_i^*}}{2}\right)(\lambda)\right)\right)^{\frac{1}{2}}\right).$$

If we take the supremum on $\lambda \in \Theta$, then we get the desired result. □

Corollary 2.13. *Assume that $A_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(i = 1, \ldots, n)$ and $p \geq 1$. Then*

$$\mathbf{ber}_p^p(A_1, \ldots, A_n) \leq \frac{\sqrt{n}}{2^{\frac{p}{2}}}\mathbf{ber}\left(\sqrt{\sum_{i=1}^{n}\left(A_i^*A_i + A_iA_i^*\right)^p}\right).$$

In particular,

$$\mathbf{ber}_e^2(A_1, \ldots, A_n) \leq \frac{\sqrt{n}}{2}\mathbf{ber}\left(\sqrt{\sum_{i=1}^{n}\left(A_i^*A_i + A_iA_i^*\right)^2}\right).$$

Proof. Employing Theorem 2.12 for the convex function $g(t) = t^p$ $(p \geq 1)$ we have the first inequality. For the second inequality put $p = 2$ in the first inequality. □

Theorem 2.14. *Assume that $A_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(i = 1, \ldots, n)$ and $g : [0, \infty) \to [0, \infty)$ be a continuous increasing convex function such that $g(0) = 0$. Then*

$$\mathbf{ber}_g(A_1, \ldots, A_n) \leq g^{-1}\left(\sum_{i=1}^{n}\mathbf{ber}\big(g\left(|\Re(A_i)| + |\Im(A_i)|\right)\big)\right).$$

Proof. Let $\lambda \in \Theta$. We have

$$\begin{aligned}
\sum_{i=1}^{n} g\left(|\widetilde{A_i}(\lambda)|\right) &= \sum_{i=1}^{n} g\left(\sqrt{\left|\widetilde{\Re(A_i)}(\lambda)\right|^2 + \left|\widetilde{\Im(A_i)}(\lambda)\right|^2}\right)\\
&\leq \sum_{i=1}^{n} g\left(\left|\widetilde{\Re(A_i)}(\lambda)\right| + \left|\widetilde{\Im(A_i)}(\lambda)\right|\right)\\
&\leq \sum_{i=1}^{n} g\left(\widetilde{|\Re(A_i)|}(\lambda) + \widetilde{|\Im(A_i)|}(\lambda)\right) \quad \text{(by Lemma 2.1)}\\
&= \sum_{i=1}^{n} g\left(\left(\widetilde{|\Re(A_i)| + |\Im(A_i)|}\right)(\lambda)\right)\\
&\leq \sum_{i=1}^{n} g\left(\widetilde{|\Re(A_i)| + |\Im(A_i)|}\right)(\lambda) \quad \text{(by Lemma 2.1)},
\end{aligned}$$

whence it follows from g^{-1} is increasing that

$$g^{-1}\left(\sum_{i=1}^{n} g\left(|\widetilde{A_i}(\lambda)|\right)\right) \leq g^{-1}\left(\sum_{i=1}^{n} g\left(\widetilde{|\Re(A_i)| + |\Im(A_i)|}\right)(\lambda)\right)$$
$$\leq g^{-1}\left(\sum_{i=1}^{n}\mathbf{ber}\big(g\left(|\Re(A_i)| + |\Im(A_i)|\right)\big)\right).$$

Taking the supremum on $\lambda \in \Theta$ on the last term we get the desired result. □

Remark 2.15. *If $A_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(i = 1, \ldots, n)$ and $p \geq 1$, then Theorem 2.14 concludes that*

$$\boldsymbol{ber}_p^p(A_1, \ldots, A_n) \leq \sum_{i=1}^{n} \boldsymbol{ber}\Big(\big(|\Re(A_i)| + |\Im(A_i)|\big)^p\Big).$$

In the next theorem, we present the g-generalized Euclidean Berezin number for product of operators.

Theorem 2.16. *Assume that $A_i, B_i, C_i, D_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(i = 1, \ldots, n)$ and $g : [0, \infty) \to [0, \infty)$ be a continuous increasing geometrically convex function such that $g(0) = 0$. Then*

$$\boldsymbol{ber}_g(D_1C_1B_1A_1, \ldots, D_nC_nB_nA_n) \leq g^{-1}\left(\boldsymbol{ber}\left(\sum_{i=1}^{n}\left(\frac{1}{p}g^{\frac{p}{2}}(A_i^*|B_i|^2A_i) + \frac{1}{q}g^{\frac{q}{2}}(D_i|C_i^*|^2D_i^*)\right)\right)\right),$$

in which $p, q > 1$ such that $p^{-1} + q^{-1} = 1$.

Proof. If $\lambda \in \Theta$, then by applying (2) we have

$$\begin{aligned}
&\sum_{i=1}^{n} g\left(\left|(\widetilde{D_iC_iB_iA_i})(\lambda)\right|\right) \\
&\leq \sum_{i=1}^{n} g\left(\sqrt{(\widetilde{A_i^*|B_i|^2A_i})(\lambda)(\widetilde{D_i|C_i^*|^2D_i^*})(\lambda)}\right) \\
&\qquad \text{(by inequality (2))} \\
&\leq \sum_{i=1}^{n} g^{\frac{1}{2}}\left((\widetilde{A_i^*|B_i|^2A_i})(\lambda)\right) g^{\frac{1}{2}}\left((\widetilde{D_i|C_i^*|^2D_i^*})(\lambda)\right) \\
&\qquad \text{(by the geometrically convexity)} \\
&\leq \left(\sum_{i=1}^{n} g^{\frac{p}{2}}\left((\widetilde{A_i^*|B_i|^2A_i})(\lambda)\right)\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} g^{\frac{q}{2}}\left((\widetilde{D_i|C_i^*|^2D_i^*})(\lambda)\right)\right)^{\frac{1}{q}} \\
&\qquad \text{(by the Cauchy Schwarz inequality)} \\
&\leq \frac{1}{p}\left(\sum_{i=1}^{n} g^{\frac{p}{2}}\left((\widetilde{A_i^*|B_i|^2A_i})(\lambda)\right)\right) + \frac{1}{q}\left(\sum_{i=1}^{n} g^{\frac{q}{2}}\left((\widetilde{D_i|C_i^*|^2D_i^*})(\lambda)\right)\right) \\
&\qquad \text{(by the Young inequality (3))} \\
&\leq \frac{1}{p}\left(\sum_{i=1}^{n}\left(g^{\frac{p}{2}}(\widetilde{A_i^*|B_i|^2A_i})(\lambda)\right)\right) + \frac{1}{q}\left(\sum_{i=1}^{n}\left(g^{\frac{q}{2}}(\widetilde{D_i|C_i^*|^2D_i^*})(\lambda)\right)\right) \\
&\qquad \text{(by Lemma 2.1)} \\
&= \sum_{i=1}^{n}\left(\frac{1}{p}g^{\frac{p}{2}}(\widetilde{A_i^*|B_i|^2A_i}) + \frac{1}{q}g^{\frac{q}{2}}(\widetilde{D_i|C_i^*|^2D_i^*})\right)(\lambda),
\end{aligned}$$

whence it follows from g^{-1} is increasing that

$$g^{-1}\left(\sum_{i=1}^{n} g\left(\left|(\widetilde{D_iC_iB_iA_i})(\lambda)\right|\right)\right) \leq g^{-1}\left(\sum_{i=1}^{n}\left(\frac{1}{p}g^{\frac{p}{2}}(\widetilde{A_i^*|B_i|^2A_i}) + \frac{1}{q}g^{\frac{q}{2}}(\widetilde{D_i|C_i^*|^2D_i^*})\right)(\lambda)\right).$$

By taking the supremum on $\lambda \in \Theta$ we get

$$\mathbf{ber}_g(D_1C_1B_1A_1, \ldots, D_nC_nB_nA_n) \leq g^{-1}\left(\mathbf{ber}\left(\sum_{i=1}^{n}\left(\frac{1}{p}g^{\frac{p}{2}}(A_i^*|B_i|^2A_i) + \frac{1}{q}g^{\frac{q}{2}}(D_i|C_i^*|^2D_i^*)\right)\right)\right).$$

□

Corollary 2.17. *Assume that* $T_i \in \mathbb{L}(\mathbb{H}(\Theta))$ $(i = 1, \ldots, n)$, $g : [0, \infty) \to [0, \infty)$ *be a continuous increasing convex function such that* $g(0) = 0$ *and* $s, t \in [0, 1]$, *where* $s + t \geq 1$. *Then*

$$\boldsymbol{ber}_g(T_1|T_1|^{s+t-1}, \ldots, T_n|T_n|^{s+t-1}) \leq g^{-1}\left(\boldsymbol{ber}\left(\sum_{i=1}^{n}\left(\frac{1}{p}g^{\frac{p}{2}}(|T_i|^{2s}) + \frac{1}{q}g^{\frac{q}{2}}(|T_i^*|^{2t})\right)\right)\right),$$

in which $p, q > 1$ *such that* $p^{-1} + q^{-1} = 1$. *In particular,*

$$\boldsymbol{ber}_r^r(T_1|T_1|^{s+t-1}, \ldots, T_n|T_n|^{s+t-1}) \leq \boldsymbol{ber}\left(\sum_{i=1}^{n}\left(\frac{1}{p}(|T_i|^{rsp} + \frac{1}{q}|T_i^*|^{rtp}\right)\right),$$

where $r \geq 1$.

Proof. Let $D_i = U_i$, $B_i = 1_{\mathbb{H}}$, $C_i = |T_i|^t$ and $A_i = |T_i|^s$ such that $s + t \geq 1$ in Theorem 4, where T_i and U_i are in the polar decomposition of $T_i = U_i |T_i|$ $(i = 1, \ldots, n)$. Then we have

$$D_iC_iB_iA_i = U_i |T_i|^t |T_i|^s = U_i |T_i| |T_i|^{s+t-1} = T_i |T_i|^{s+t-1},$$

also, we have $A_i^* |B_i|^2 A_i = |T_i|^{2s}$ and $D_i |C_i^*|^2 D_i^* = U_i |T_i|^{2t} U_i^* = |T_i^*|^{2t}$. If we take $g(t) = t^r$ $(r \geq 1)$ in the first inequality, then we have

$$\mathbf{ber}_r(T_1|T_1|^{s+t-1}, \ldots, T_n|T_n|^{s+t-1}) \leq \mathbf{ber}^{\frac{1}{r}}\left(\sum_{i=1}^{n}\left(\frac{1}{p}|T_i|^{rsp} + \frac{1}{q}|T_i^*|^{rtp}\right)\right).$$

□

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