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Characterizations of weakly star-type Rothberger and Menger properties in hyperspaces

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Abstract. In this paper, we introduce the selection principles $\mathbf{wSS}^*_1(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$, $\mathbf{wSS}^*_{\mathsf{fin}}(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$, $\mathbf{wS}^*_1(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$ and $\mathbf{wS}^*_{\mathsf{fin}}(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$ to characterize the properties of weakly strong-star Rothberger (Menger) and weakly star-Rothberger (Menger) in the hyperspace (Λ, τ^+_Δ), respectively. Furthermore, we introduce the notions $H(\mathbb{C}_\Delta(\Lambda))$ and $\mathbf{I}_{\mathsf{fin}}(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$ to characterize, respectively, the H-separability and the principle $\mathbf{U}_{\mathsf{fin}}(\mathscr{D}, \mathscr{D})$, in the same hyperspace.

1. Introduction and preliminaries

The study of hyperspace theory started in the first half of the 20th century (see [13, 21, 23, 28]). We denote by $\mathrm{CL}(X)$ the family of all nonempty closed subsets of a topological space X. The family $\mathrm{CL}(X)$, endowed with some topology, is known as hyperspace of X. Numerous relations between properties of the space X and their hyperspaces have been widely studied. Otherwise, the research on selection principles started in [4, 14, 20, 24, 25]. Some researchers have studied selection principles concerning weaker versions of Rothberger and Menger properties and star type selection principles [1, 15, 16, 18, 22, 27].

The relationships between selection principles and hyperspaces have been intensely studied. In [10] the authors used π-networks to characterize topological spaces whose hyperspaces, endowed with the upper Fell topology, satisfy the Rothberger property. Then, in [19] are defined the concepts of π_F-network, π_V-network, k_F-cover and c_V-cover and they are used to study the $\mathbf{S}_1(\mathscr{A}, \mathscr{B})$ and $\mathbf{S}_{\mathsf{fin}}(\mathscr{A}, \mathscr{B})$ principles in $\mathrm{CL}(X)$ endowed with the Fell and Vietoris topologies, for different families $\mathscr{A}$ and $\mathscr{B}$. Later, in [5] the authors introduce the generic notions of $\pi_\Delta(\Lambda)$-networks (and $c_\Delta(\Lambda)$-covers), which are a generalization

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of π_F-networks and π_V-networks (and of k_F-cover and c_V-cover, respectively). These concepts are used to characterize Menger-type star selection principles [5], star and strong star-type versions of Rothberger and Menger principles [6], Hurewicz like properties [7] and weaker forms of Rothberger and Menger properties and groupability [8] in hyperspaces endowed with the hit-and-miss topology.

Next, we recall three known selection principles defined in 1996 by M. Scheepers [25]. Furthermore, we introduce the principle $\mathbf{I}_{\mathsf{fin}}(\mathscr{A},\mathscr{B})$. Given an infinite set X, let $\mathscr{A}$ and $\mathscr{B}$ be collections of families of subsets of X.

- $\mathbf{S}_1(\mathscr{A},\mathscr{B})$ denotes the principle: For any sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$, there is a sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $B_n \in \mathcal{A}_n$ and $\{B_n : n \in \mathbb{N}\}$ is an element of $\mathscr{B}$.

- $\mathbf{S}_{\mathsf{fin}}(\mathscr{A},\mathscr{B})$ denotes the principle: for each sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$ there is a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ such that $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n \in \mathscr{B}$.

- $\mathbf{U}_{\mathsf{fin}}(\mathscr{A},\mathscr{B})$ denotes the principle: for each sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$ there is a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and $\{\bigcup \mathcal{B}_n : n \in \mathbb{N}\} \in \mathscr{B}$.

- $\mathbf{I}_{\mathsf{fin}}(\mathscr{A},\mathscr{B})$ denotes the principle: for each sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$ there is a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and $\{\bigcap \mathcal{B}_n : n \in \mathbb{N}\} \in \mathscr{B}$.

We denote by $\mathscr{O} = \{\mathcal{A} \subseteq \tau : \bigcup \mathcal{A} = X\}$, $\mathcal{D} = \{D \subseteq X : cl_X(D) = X\}$ and $\mathscr{D}' = \{\mathcal{A} \subseteq \tau : \bigcup \mathcal{A} \in \mathcal{D}\}$. When we take $\mathbf{S}_1(\mathscr{O},\mathscr{O})$ and $\mathbf{S}_{\mathsf{fin}}(\mathscr{O},\mathscr{O})$, we get the well known *Rothberger property* [24] and the *Menger property* [14, 20], respectively. Moreover, $\mathbf{S}_1(\mathscr{O},\mathscr{D}')$ and $\mathbf{S}_{\mathsf{fin}}(\mathscr{O},\mathscr{D}')$ are known as *weakly Rothberger property* [9] and *weakly Menger property* [9], respectively.

On the other hand, in [17] Kočinac introduced the star version of some selection principles. Recall that given $A \subseteq X$ and any collection $\mathcal{U}$ of subsets of X, the star of A with respect to $\mathcal{U}$ is denoted by $\mathrm{St}(A,\mathcal{U})$ and defined as $\bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. As usual, we write $\mathrm{St}(x,\mathcal{U})$ instead of $\mathrm{St}(\{x\},\mathcal{U})$, for every $x \in X$.

Consider an infinite set X and let $\mathscr{A}$ and $\mathscr{B}$ be collections of families of subsets of X. We say that X satisfies the principle:

- $\mathbf{S}^*_1(\mathscr{A},\mathscr{B})$, if for any sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$, there is a sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $B_n \in \mathcal{A}_n$ and $\{\mathrm{St}(B_n,\mathcal{A}_n) : n \in \mathbb{N}\}$ is an element of $\mathscr{B}$.

- $\mathbf{S}^*_{\mathsf{fin}}(\mathscr{A},\mathscr{B})$, if for each sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$, there is a sequence $(\mathcal{B}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and $\bigcup_{n\in\mathbb{N}} \{\mathrm{St}(B,\mathcal{A}_n) : B \in \mathcal{B}_n\}$ is an element of $\mathscr{B}$.

- $\mathbf{SS}^*_1(\mathscr{A},\mathscr{B})$, if for every sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$, there is a sequence $(x_n : n \in \mathbb{N})$ of elements of X such that $\{\mathrm{St}(x_n,\mathcal{A}_n) : n \in \mathbb{N}\}$ is an element of $\mathscr{B}$.

- $\mathbf{SS}^*_{\mathsf{fin}}(\mathscr{A},\mathscr{B})$, if for any sequence $(\mathcal{A}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$, there is a sequence $(K_n : n \in \mathbb{N})$ of finite subsets of X such that $\{\mathrm{St}(K_n,\mathcal{A}_n) : n \in \mathbb{N}\}$ is an element of $\mathscr{B}$.

The particular cases $\mathbf{SS}^*_1(\mathscr{O},\mathscr{D}')$ and $\mathbf{SS}^*_{\mathsf{fin}}(\mathscr{O},\mathscr{D}')$ are known as *weakly strong star-Rothberger property* (WSSR) and *weakly strong star-Menger property* (WSSM), respectively (see [18]). Moreover, $\mathbf{S}^*_1(\mathscr{O},\mathscr{D}')$ and $\mathbf{S}^*_{\mathsf{fin}}(\mathscr{O},\mathscr{D}')$ are known as *weakly star-Rothberger property* (WSR) and *weakly star-Menger property* (WSM), respectively (see [22]).

Diagram 1 provides relationships between the properties defined previously. These follow immediately from the definitions.

$$\begin{array}{ccccccc}
\text{Menger} & \longrightarrow & \text{weakly Menger} & \longrightarrow & \text{WSSM} & \longrightarrow & \text{WSM} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\text{Rothberger} & \longrightarrow & \text{weakly Rothberger} & \longrightarrow & \text{WSSR} & \longrightarrow & \text{WSR}
\end{array}$$

Diagram 1: Relationships between selection principles

Now, we present some basic concepts about the theory of hyperspaces. All spaces are assumed to be Hausdorff noncompact and, even, nonparacompact. For a space (X, τ), we denote by $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ and $\mathbb{CS}(X)$ the family of all nonempty closed subsets, the family of all nonempty compact subsets, the family of all nonempty finite subsets of X and the family of all convergent sequences of X, respectively.

For any subset $U \subseteq X$ and every family $\mathcal{U}$ of subsets of X, we denote:

$$\begin{array}{rcl} U^- & = & \{A \in \mathrm{CL}(X) : A \cap U \neq \emptyset\}; \\ U^+ & = & \{A \in \mathrm{CL}(X) : A \subseteq U\}; \\ U^c & = & X \setminus U; \\ \mathcal{U}^c & = & \{U^c : U \in \mathcal{U}\}. \end{array}$$

Let Δ be a subfamily of $\mathrm{CL}(X)$ closed under finite unions and containing all singletons. Then, the *hit-and-miss topology on* $\mathrm{CL}(X)$ *respect to* Δ, denoted by τ_Δ^+, has as a base, the family

$$\left\{\left(\bigcap_{i=1}^{m} V_i^-\right) \cap (B^c)^+ : B \in \Delta \text{ and } V_i \in \tau \text{ for } i \in \{1, \ldots, m\}\right\}.$$

We use $(V_1, \ldots, V_m)_B^+$ to denote the basic element $(\bigcap_{i=1}^m V_i^-) \cap (B^c)^+$ (see [29]).

Two important known particular cases of the hit-and-miss topology are the *Vietoris topology*, τ_V, when $\Delta = \mathrm{CL}(X)$ (see [21, 28]), and the *Fell topology*, τ_F, when $\Delta = \mathbb{K}(X)$ (see [12]). Along this paper, unless we say the opposite, we will consider a subspace Λ of $(\mathrm{CL}(X), \tau_\Delta^+)$, which is closed under finite unions.

In another context, inspired by Li [19], we introduced in [5] the notions of $\pi_\Delta(\Lambda)$-network and $c_\Delta(\Lambda)$-cover of a space X. We remember them and a couple of lemmas which will be used along this work (see Lemmas 2.4 and 2.22 of [5]). As is common, $[A]^{<\omega}$ denotes the collection of all finite subsets of any set A.

Given a family $\Delta \subseteq \mathrm{CL}(X)$, we denote

$$\zeta_\Delta = \{(B; V_1, \ldots, V_n) : B \in \Delta \text{ and } V_1, \ldots, V_n \text{ are open subsets of } X \text{ with } V_i \cap B^c \neq \emptyset \ (1 \leq i \leq n),\ \ n \in \mathbb{N}\}.$$

Definition 1.1. A family $\mathcal{J} \subseteq \zeta_\Delta$ is called a $\pi_\Delta(\Lambda)$-*network of* X, if for each $U \in \Lambda^c$, there exist $(B; V_1, \ldots, V_n) \in \mathcal{J}$ with $B \subseteq U$ and $F \in [X]^{<\omega}$ such that $F \cap U = \emptyset$ and for each $i \in \{1, \ldots, n\}$, $F \cap V_i \neq \emptyset$. The family of all $\pi_\Delta(\Lambda)$-networks is denoted by $\Pi_\Delta(\Lambda)$.

Lemma 1.2. *Let* (X, τ) *be a topological space. Suppose that* $\mathcal{J} = \{(B_s; V_{1,s}, \ldots, V_{m_s,s}) : s \in S\}$ *and* $\mathscr{U} = \left\{(V_{1,s}, \ldots, V_{m_s,s})_{B_s}^+ : (B_s; V_{1,s}, \ldots, V_{m_s,s}) \in \mathcal{J}\right\}$. *Then,* $\mathcal{J}$ *is a* $\pi_\Delta(\Lambda)$*-network of* X *if and only if* $\mathscr{U}$ *is an open cover of* (Λ, τ_Δ^+).

Definition 1.3. Let (X, τ) be a topological space. A family $\mathcal{U} \subseteq \Lambda^c$ is called a $c_\Delta(\Lambda)$-*cover of* X, if for any $B \in \Delta$ and open subsets $V_1, \ldots, V_m$ of X, with $B^c \cap V_i \neq \emptyset$ for any $i \in \{1, \ldots, m\}$, there exist $U \in \mathcal{U}$ and $F \in [X]^{<\omega}$ such that $B \subseteq U$, $F \cap U = \emptyset$ and for each $i \in \{1, \ldots, m\}$, $F \cap V_i \neq \emptyset$. We denote by $\mathbb{C}_\Delta(\Lambda)$ the family of all $c_\Delta(\Lambda)$-covers of X.

Lemma 1.4. *Let* (X, τ) *be a topological space. A family* $\mathcal{U} \subseteq \Lambda^c$ *is a* $c_\Delta(\Lambda)$*-cover of* X *if and only if the family* $\mathcal{U}^c$ *is a dense subset of* (Λ, τ_Δ^+).

Continuing the research done in [5–8, 11], where the concepts of $\pi_\Delta(\Lambda)$-network and $c_\Delta(\Lambda)$-cover were used to characterize selection principles in hyperspaces, in this paper we introduce the selection principles $\mathbf{wSS}_1^*(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$, $\mathbf{wSS}_{\mathsf{fin}}^*(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$, $\mathbf{wS}_1^*(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$ and $\mathbf{wS}_{\mathsf{fin}}^*(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$ to characterize the properties weakly strong star-Rothberger (Theorem 2.2), weakly strong star-Menger (Theorem 2.7), weakly star-Rothberger (Theorem 3.2) and weakly star-Menger (Theorem 3.7) in the hyperspace (Λ, τ_Δ^+), respectively. Furthermore, in Section 4, we introduce the notions $H(\mathbb{C}_\Delta(\Lambda))$ and $\mathbf{I}_{\mathsf{fin}}(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$ to characterize, respectively, the H-separability (Theorem 4.2) and the principle $\mathbf{U}_{\mathsf{fin}}(\mathscr{D}, \mathscr{D})$ (Theorem 4.6), in the same hyperspace.

2. Weakly strong star-Rothberger and weakly strong star-Menger properties

In this section we introduce a couple of selection principles, applied to $\pi_\Delta(\Lambda)$-networks and $c_\Delta(\Lambda)$-covers, in order to characterize the weakly strong star-Rothberger and weakly strong star-Menger properties in the hyperspace (Λ, τ^+_Δ).

Definition 2.1. We say that the topological space (X, τ) satisfies the selection principle $\mathbf{wSS}^*_1(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$ if for every sequence $(\mathcal{J}_n : n \in \mathbb{N})$ in $\Pi_\Delta(\Lambda)$, there is a sequence $(A_n : n \in \mathbb{N})$ in Λ such that the family $\boldsymbol{\mathcal{U}}$ turns out to be a $c_\Delta(\Lambda)$-cover of X, where $\boldsymbol{\mathcal{U}} = \bigcup\{\boldsymbol{\mathcal{U}}_n : n \in \mathbb{N}\}$ and $\boldsymbol{\mathcal{U}}_n$ is the collection of every $U \in \Lambda^c$ for which there exist $(B; V_1, \ldots, V_m) \in \mathcal{J}_n$ and $F \in [X]^{<\omega}$ such that $A_n \cap V_i \neq \emptyset$, $A_n \cap B = \emptyset$, $B \subseteq U$, $F \cap V_i \neq \emptyset$ and $F \cap U = \emptyset$.

Theorem 2.2. *Let (X, τ) be a topological space. The following conditions are equivalent:*

(1) (Λ, τ^+_Δ) *is weakly strong star-Rothberger;*
(2) (X, τ) *satisfies the principle* $\mathbf{wSS}^*_1(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$.

Proof. (1) ⇒ (2) Let $(\mathcal{J}_n : n \in \mathbb{N})$ be a sequence of $\pi_\Delta(\Lambda)$-networks of X. Denote, for any $n \in \mathbb{N}$, $\mathscr{U}_n = \left\{(V_1, \ldots, V_m)^+_B : (B; V_1, \ldots, V_m) \in \mathcal{J}_n\right\}$. By Lemma 1.2, we have that for each $n \in \mathbb{N}$, $\mathscr{U}_n$ is an open cover of (Λ, τ^+_Δ). Applying (1) to the sequence $(\mathscr{U}_n : n \in \mathbb{N})$, there exists $A_n \in \Lambda$, for any $n \in \mathbb{N}$, such that $cl_\Lambda\left(\bigcup\{\mathrm{St}(A_n, \mathscr{U}_n) : n \in \mathbb{N}\}\right) = \Lambda$. As $\mathcal{F} = \bigcup\{\mathrm{St}(A_n, \mathscr{U}_n) : n \in \mathbb{N}\}$ is dense in Λ, then by Lemma 1.4, $\mathcal{F}^c$ is a $c_\Delta(\Lambda)$-cover of X. We claim that $\mathcal{F}^c$ is the same $\boldsymbol{\mathcal{U}}$ as in Definition 2.1. Indeed, $\mathcal{F}^c = \bigcup\{\mathrm{St}(A_n, \mathscr{U}_n)^c : n \in \mathbb{N}\}$. Furthermore, $U \in \mathrm{St}(A_n, \mathscr{U}_n)^c$ if and only if there exists $(B; V_1, \ldots, V_m) \in \mathcal{J}_n$ such that $A_n, U^c \in (V_1, \ldots, V_m)^+_B$. The last assertion means that there is $F \in [X]^{<\omega}$ such that $A_n \cap V_i \neq \emptyset$, $A_n \cap B = \emptyset$, $B \subseteq U$, $F \cap V_i \neq \emptyset$ and $F \cap U = \emptyset$. Hence, the claim follows.

(2) ⇒ (1): Consider $(\mathscr{U}_n : n \in \mathbb{N})$ a sequence of open covers of (Λ, τ^+_Δ), consisting in basic open sets. For each $n \in \mathbb{N}$, let $\mathcal{J}_n = \{(B; V_1, \ldots, V_m) : (V_1, \ldots, V_m)^+_B \in \mathscr{U}_n\}$. Then, by Lemma 1.2, $\mathcal{J}_n$ is a $\pi_\Delta(\Lambda)$-network of X. Applying (2) to the sequence $(\mathcal{J}_n : n \in \mathbb{N})$, there is a sequence $(A_n : n \in \mathbb{N})$ in Λ such that $\boldsymbol{\mathcal{U}} \in \mathbb{C}_\Delta(\Lambda)$, where $\boldsymbol{\mathcal{U}}$ is given as in Definition 2.1. Note that, as above, $\mathrm{St}(A_n, \mathscr{U}_n) = \boldsymbol{\mathcal{U}}^c_n$, which implies that $\boldsymbol{\mathcal{U}}^c = \bigcup\{\mathrm{St}(A_n, \mathscr{U}_n) : n \in \mathbb{N}\}$. So, by Lemma 1.4, $\boldsymbol{\mathcal{U}}^c$ is dense in Λ, and the proof follows. □

As an immediate consequence of Theorem 2.2, we obtain the following corollaries.

Corollary 2.3. *Let (X, τ) be a topological space. If Λ is any of the hyperspaces* $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$, *then* (Λ, τ^+_Δ) *is weakly strong star-Rothberger if and only if X satisfies the principle* $\mathbf{wSS}^*_1(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$.

Corollary 2.4. *Let (X, τ) be a topological space. Suppose that Λ is some of the spaces* $\mathbb{K}(X)$, $\mathbb{F}(X)$, $\mathbb{CS}(X)$, *we have:*

(a) If $\Delta = \mathbb{K}(X)$, then (Λ, τ_F) is weakly strong star-Rothberger if and only if X satisfies the selection principle $\mathbf{wSS}^*_1\left(\Pi_{\mathbb{K}(X)}(\Lambda), \mathbb{C}_{\mathbb{K}(X)}(\Lambda)\right)$.
(b) If $\Delta = \mathrm{CL}(X)$, then (Λ, τ_V) is weakly strong star-Rothberger if and only if X satisfies the selection principle $\mathbf{wSS}^*_1\left(\Pi_{\mathrm{CL}(X)}(\Lambda), \mathbb{C}_{\mathrm{CL}(X)}(\Lambda)\right)$.

Remember that $\Pi_{\mathbb{K}(X)}(\mathrm{CL}(X)) = \Pi_F$ and $\Pi_{\mathrm{CL}(X)}(\mathrm{CL}(X)) = \Pi_V$, (see [5, Remark 2.2] and [19]). Furthermore, $\mathbb{C}_{\mathbb{K}(X)}(\mathrm{CL}(X)) = \mathbb{K}_F$ and $\mathbb{C}_{\mathrm{CL}(X)}(\mathrm{CL}(X)) = \mathbb{C}_V$ (see [5, Remark 2.21] and [19]).

Corollary 2.5. *Let (X, τ) be a topological space, we have:*

(a) $(\mathrm{CL}(X), \tau_F)$ *is weakly strong star-Rothberger if and only if X satisfies the principle* $\mathbf{wSS}^*_1(\Pi_F, \mathbb{K}_F)$.
(b) $(\mathrm{CL}(X), \tau_V)$ *is weakly strong star-Rothberger if and only if X satisfies the principle* $\mathbf{wSS}^*_1(\Pi_V, \mathbb{C}_V)$.

Similarly, we can define the principle $\mathbf{wSS}^*_{\mathrm{fin}}(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$ to obtain a characterization of the weakly strong star-Menger property in (Λ, τ^+_Δ).

Definition 2.6. We say that the topological space (X, τ) satisfies the selection principle $\mathbf{wSS}^*_{\mathrm{fin}}(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$ if for every sequence $(\mathcal{J}_n : n \in \mathbb{N})$ in $\Pi_\Delta(\Lambda)$, there is a sequence $\mathcal{V}_n \in [\Lambda]^{<\omega}$ such that $\mathcal{U} \in \mathbb{C}_\Delta(\Lambda)$, where $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ and $\mathcal{U}_n$ is the collection of every $U \in \Lambda^c$ for which there exist $A \in \mathcal{V}_n$, $(B; V_1, \dots, V_m) \in \mathcal{J}_n$ and $F \in [X]^{<\omega}$ such that $A \cap V_i \neq \emptyset$, $A \cap B = \emptyset$, $B \subseteq U$, $F \cap V_i \neq \emptyset$ and $F \cap U = \emptyset$.

The proof of the next theorem follows the same structure as Theorem 2.2.

Theorem 2.7. *Let (X, τ) be a topological space. The following conditions are equivalent:*

(1) (Λ, τ^+_Δ) is weakly strong star-Menger;
(2) (X, τ) satisfies the principle $\mathbf{wSS}^*_{fin}(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$.

From Theorem 2.7, we obtain the following.

Corollary 2.8. *Let (X, τ) be a topological space. If Λ is any of the hyperspaces $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$, then (Λ, τ^+_Δ) is weakly strong star-Menger if and only if X satisfies the principle $\mathbf{wSS}^*_{fin}(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$.*

Corollary 2.9. *Let (X, τ) be a topological space. Suppose that Λ is some of the spaces $\mathbb{K}(X)$, $\mathbb{F}(X)$, $\mathbb{CS}(X)$, we have:*

(a) If $\Delta = \mathbb{K}(X)$, then (Λ, τ_F) is weakly strong star-Menger if and only if X satisfies the selection principle $\mathbf{wSS}^*_{fin}\Big(\Pi_{\mathbb{K}(X)}(\Lambda), \mathbb{C}_{\mathbb{K}(X)}(\Lambda)\Big)$.
(b) If $\Delta = \mathrm{CL}(X)$, then (Λ, τ_V) is weakly strong star-Menger if and only if X satisfies the selection principle $\mathbf{wSS}^*_{fin}\Big(\Pi_{\mathrm{CL}(X)}(\Lambda), \mathbb{C}_{\mathrm{CL}(X)}(\Lambda)\Big)$.

Corollary 2.10. *Let (X, τ) be a topological space, we have:*

(a) $(\mathrm{CL}(X), \tau_F)$ is weakly strong star-Menger if and only if X satisfies the principle $\mathbf{wSS}^*_{fin}(\Pi_F, \mathbb{K}_F)$.
(b) $(\mathrm{CL}(X), \tau_V)$ is weakly strong star-Menger if and only if X satisfies the principle $\mathbf{wSS}^*_{fin}(\Pi_V, \mathbb{C}_V)$.

3. Weakly star-Rothberger and weakly star-Menger properties

In this section we define two selection principles for $\pi_\Delta(\Lambda)$-networks and $c_\Delta(\Lambda)$-covers, in order to characterize the weakly star-Rothberger and weakly star-Menger properties in the hyperspace (Λ, τ^+_Δ).

Definition 3.1. We say that the topological space (X, τ) satisfies the selection principle $\mathbf{wS}^*_1(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$ if for every sequence $(\mathcal{J}_n : n \in \mathbb{N})$ in $\Pi_\Delta(\Lambda)$, we can choose $(B^n; V^n_1, \dots, V^n_{m_n}) \in \mathcal{J}_n$, for each $n \in \mathbb{N}$, such that the family $\mathcal{W}$ turns out to be a $c_\Delta(\Lambda)$-cover of X, where $\mathcal{W} = \bigcup\{\mathcal{W}_n : n \in \mathbb{N}\}$ and $\mathcal{W}_n$ is the collection of all $W \in \Lambda^c$ for which there exist $(B; V_1, \dots, V_l) \in \mathcal{J}_n$ and $H \in \Lambda$ such that $W^c \cap V_i \neq \emptyset$, $B \subseteq W$, $H \cap V_i \neq \emptyset$, $B \cap H = \emptyset$, $H \cap V^n_j \neq \emptyset$ and $B^n \cap H = \emptyset$, for $1 \leq i \leq l$ and $1 \leq j \leq m_n$.

Theorem 3.2. *Let (X, τ) be a topological space. The following conditions are equivalent:*

(1) (Λ, τ^+_Δ) is weakly star-Rothberger;
(2) (X, τ) satisfies the principle $\mathbf{wS}^*_1(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$.

Proof. (1) $\Rightarrow$ (2) Let $(\mathcal{J}_n : n \in \mathbb{N})$ be a sequence of $\pi_\Delta(\Lambda)$-networks of X. Denote, for any $n \in \mathbb{N}$, $\mathscr{U}_n = \Big\{(V_1, \dots, V_m)^+_B : (B; V_1, \dots, V_m) \in \mathcal{J}_n\Big\}$. By Lemma 1.2, we have that for each $n \in \mathbb{N}$, $\mathscr{U}_n$ is an open cover of (Λ, τ^+_Δ). Applying (1) to the sequence $(\mathscr{U}_n : n \in \mathbb{N})$, there exists $(V^n_1, \dots, V^n_{m_n})^+_{B^n} \in \mathscr{U}_n$, for any $n \in \mathbb{N}$, such that $cl_\Lambda\left(\bigcup\left\{\mathrm{St}\Big(\Big(V^n_1, \dots, V^n_{m_n}\Big)^+_{B^n}, \mathscr{U}_n\Big) : n \in \mathbb{N}\right\}\right) = \Lambda$.

As $\Big(B^n; V^n_1, \dots, V^n_{m_n}\Big) \in \mathcal{J}_n$ for any $n \in \mathbb{N}$, we define $\mathcal{W}$, as in Definition 3.1. To prove that $\mathcal{W}$ is a $c_\Delta(\Lambda)$-cover, in view of Lemma 1.4, we will show that $\mathcal{W}^c$ is dense in Λ. Indeed, it follows from the fact that for each $n \in \mathbb{N}$, $\mathcal{W}_n = \Big(\mathrm{St}\Big(\Big(V^n_1, \dots, V^n_{m_n}\Big)^+_{B^n}, \mathscr{U}_n\Big)\Big)^c$.

(2) ⇒ (1) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of (Λ, τ_Δ^+), consisting in basic open sets. For each $n \in \mathbb{N}$, let $\mathcal{J}_n = \{(B; V_1, \ldots, V_m) : (V_1, \ldots, V_m)_B^+ \in \mathcal{U}_n\}$. Then, by Lemma 1.2, $\mathcal{J}_n$ is a $\pi_\Delta(\Lambda)$-network of X. Applying (2) to the sequence $(\mathcal{J}_n : n \in \mathbb{N})$, there is $(B^n; V_1^n, \ldots, V_{m_n}^n) \in \mathcal{J}_n$, for every $n \in \mathbb{N}$ and $\mathcal{W}$, given as in Definition 3.1, such that $\mathcal{W} \in \mathbb{C}_\Delta(\Lambda)$. Hence, $\mathcal{W}^c$ is a dense subset of Λ.

As $\mathcal{W}^c = \bigcup\left\{\left(\mathrm{St}\left(V_1^n, \ldots, V_{m_n}^n\right)_{B^n}^+, \mathcal{U}_n\right) : n \in \mathbb{N}\right\}$, the result follows. □

As a consequence of Theorem 3.2, we obtain the next results.

Corollary 3.3. *Let (X, τ) be a topological space. If Λ is any of the hyperspaces* $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$, *then (Λ, τ_Δ^+) is weakly star-Rothberger if and only if X satisfies the principle* $\mathbf{wS}_1^*(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$.

Corollary 3.4. *Let (X, τ) be a topological space. Suppose that Λ is some of the spaces* $\mathbb{K}(X)$, $\mathbb{F}(X)$, $\mathbb{CS}(X)$, *we have:*

(a) Suppose that $\Delta = \mathbb{K}(X)$. Then (Λ, τ_F) is weakly star-Rothberger if and only if X satisfies the selection principle $\mathbf{wS}_1^*\left(\Pi_{\mathbb{K}(X)}(\Lambda), \mathbb{C}_{\mathbb{K}(X)}(\Lambda)\right)$.

(b) If $\Delta = \mathrm{CL}(X)$, then (Λ, τ_V) is weakly star-Rothberger if and only if X satisfies the selection principle $\mathbf{wS}_1^*\left(\Pi_{\mathrm{CL}(X)}(\Lambda), \mathbb{C}_{\mathrm{CL}(X)}(\Lambda)\right)$.

We have said that $\Pi_{\mathbb{K}(X)}(\mathrm{CL}(X)) = \Pi_F$, $\Pi_{\mathrm{CL}(X)}(\mathrm{CL}(X)) = \Pi_V$, $\mathbb{C}_{\mathbb{K}(X)}(\mathrm{CL}(X)) = \mathbb{K}_F$ and $\mathbb{C}_{\mathrm{CL}(X)}(\mathrm{CL}(X)) = \mathbb{C}_V$, so we have the following.

Corollary 3.5. *Let (X, τ) be a topological space, we have:*

(a) $(\mathrm{CL}(X), \tau_F)$ *is weakly star-Rothberger if and only if X satisfies the principle* $\mathbf{wS}_1^*(\Pi_F, \mathbb{K}_F)$.

(b) $(\mathrm{CL}(X), \tau_V)$ *is weakly star-Rothberger if and only if X satisfies the principle* $\mathbf{wS}_1^*(\Pi_V, \mathbb{C}_V)$.

Now, we define the principle $\mathbf{wS}_{\mathsf{fin}}^*(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$ in order to obtain a characterization of weakly star-Menger property in (Λ, τ_Δ^+).

Definition 3.6. We say that the topological space (X, τ) satisfies the selection principle $\mathbf{wS}_{\mathsf{fin}}^*(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$ if for every sequence $(\mathcal{J}_n : n \in \mathbb{N})$ in $\Pi_\Delta(\Lambda)$, we can choose $\mathcal{I}_n \in [\mathcal{J}_n]^{<\omega}$ for each $n \in \mathbb{N}$ such that $\mathcal{W} \in \mathbb{C}_\Delta(\Lambda)$, where $\mathcal{W} = \bigcup\{\mathcal{W}_n : n \in \mathbb{N}\}$, and $\mathcal{W}_n$ is the collection of all $W \in \Lambda^c$ for which there exist $(K; U_1, \ldots, U_m) \in \mathcal{I}_n$, $(B; V_1, \ldots, V_l) \in \mathcal{J}_n$ and $H \in \Lambda$ such that $W^c \cap V_i \neq \emptyset$, $B \subseteq W$, $H \cap V_i \neq \emptyset$, $B \cap H = \emptyset$, $H \cap U_j \neq \emptyset$ and $K \cap H = \emptyset$, for $1 \leq i \leq l$ and $1 \leq j \leq m$.

The proof of the next theorem is similar to the proof of Theorem 3.2.

Theorem 3.7. *Let (X, τ) be a topological space. The following conditions are equivalent:*

(1) (Λ, τ_Δ^+) *is weakly star-Menger;*

(2) (X, τ) *satisfies the principle* $\mathbf{wS}_{fin}^*(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$.

We obtain the next corollaries from Theorem 3.7.

Corollary 3.8. *Let (X, τ) be a topological space. If Λ is any of the hyperspaces* $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$, *then (Λ, τ_Δ^+) is weakly star-Menger if and only if X satisfies the selection principle* $\mathbf{wS}_{fin}^*(\Pi_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$.

Corollary 3.9. *Let (X, τ) be a topological space. Suppose that Λ is some of the spaces* $\mathbb{K}(X)$, $\mathbb{F}(X)$, $\mathbb{CS}(X)$, *we have:*

(a) Suppose that $\Delta = \mathbb{K}(X)$. Then (Λ, τ_F) is weakly star-Menger if and only if X satisfies the selection principle $\mathbf{wS}_{fin}^*\left(\Pi_{\mathbb{K}(X)}(\Lambda), \mathbb{C}_{\mathbb{K}(X)}(\Lambda)\right)$.

(b) Suppose that $\Delta = \mathrm{CL}(X)$. Then (Λ, τ_V) is weakly star-Menger if and only if X satisfies the selection principle $\mathbf{wS}_{fin}^*\left(\Pi_{\mathrm{CL}(X)}(\Lambda), \mathbb{C}_{\mathrm{CL}(X)}(\Lambda)\right)$.

Corollary 3.10. *Let (X, τ) be a topological space, we have:*

(a) $(\mathrm{CL}(X), \tau_F)$ *is weakly star-Menger if and only if X satisfies the selection principle* $\mathbf{wS}_{fin}^*(\Pi_F, \mathbb{K}_F)$.

(b) $(\mathrm{CL}(X), \tau_V)$ *is weakly star-Menger if and only if X satisfies the selection principle* $\mathbf{wS}_{fin}^*(\Pi_V, \mathbb{C}_V)$.

4. H-separable like properties

The definitions of R-separable, M-separable and H-separable spaces were considered in [2, 3, 26]. Characterizations of R-separability and M-separability for the hyperspace $(\mathrm{CL}(X), \tau^+_\Delta)$ were given respectively in [5, 6]. Now, following similar ideas, we provide a characterization for H-separable hyperspaces endowed with the hit-and-miss topology by means of $c_\Delta(\Lambda)$-covers.

Remember that a topological space (X, τ) is *H-separable* if for every sequence $(D_n : n \in \mathbb{N})$ of dense subsets of X, one can pick finite $F_n \subseteq D_n$ such that for every nonempty open set $U \subseteq X$, the intersection $U \cap F_n$ is nonempty for all but finitely many n.

Definition 4.1. A topological space (X, τ) satisfies the property $H(\mathbb{C}_\Delta(\Lambda))$ if for any sequence $(\mathcal{U}_n : n \in \mathbb{N})$ in $\mathbb{C}_\Delta(\Lambda)$, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and for any $B \in \Delta$ and open subsets $V_1, \dots, V_m$ of X, with $B^c \cap V_i \neq \emptyset$ for any $i \in \{1, \dots, m\}$, there exist $U_n \in \mathcal{V}_n$ and $F_n \in [X]^{<\omega}$ such that $B \subseteq U_n$, $F_n \cap U_n = \emptyset$ and for each $i \in \{1, \dots, m\}$, $F_n \cap V_i \neq \emptyset$ for every but finitely many n.

Theorem 4.2. *Let (X, τ) be a topological space. The following conditions are equivalent:*

(1) *(Λ, τ^+_Δ) is H-separable;*
(2) *(X, τ) satisfies the property $H(\mathbb{C}_\Delta(\Lambda))$.*

Proof. (1) ⇒ (2) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $c_\Delta(\Lambda)$-covers of X. For any $n \in \mathbb{N}$, we put $\mathcal{D}_n = \mathcal{U}_n^c$. By Lemma 1.4, we obtain that, for any $n \in \mathbb{N}$, $\mathcal{D}_n$ is a dense subset of (Λ, τ^+_Δ). Hence, applying (1) to the sequence $(\mathcal{D}_n : n \in \mathbb{N})$, we obtain, for each $n \in \mathbb{N}$, $\mathcal{F}_n \in [\mathcal{D}_n]^{<\omega}$ which witnesses the H-separability of Λ. Put $\mathcal{V}_n = \{U : U^c \in \mathcal{F}_n\}$ and let $B \in \Delta$ and $V_1, \dots, V_m$ open subsets of X, with $B^c \cap V_i \neq \emptyset$, for any $i \in \{1, \dots, m\}$. Consider the basic set $(V_1, \dots, V_m)^+_B$, then, there is $D_n \in \mathcal{F}_n \cap (V_1, \dots, V_m)^+_B$ for all but finitely many n. For those n choose an element $x^n_i \in V_i \cap D_n$ (for $i \in \{1, \dots, m\}$) and let $F_n = \{x^n_i, \dots, x^n_m\}$. As $U_n = D^c_n \in \mathcal{V}_n$, the result follows.

(2) ⇒ (1) Let $(\mathcal{D}_n : n \in \mathbb{N})$ be a sequence of dense subsets of (Λ, τ^+_Δ). For each $n \in \mathbb{N}$, we put $\mathcal{U}_n = \mathcal{D}^c_n$. By Lemma 1.4, we have that, for each $n \in \mathbb{N}$, $\mathcal{U}_n$ is a $c_\Delta(\Lambda)$-cover of X. Hence, applying (2) to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$, there exists $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$, for each $n \in \mathbb{N}$, which witnesses the property in (2). Let $\mathcal{F}_n = \{D : D^c \in \mathcal{V}_n\}$ and take a nonempty basic set $(V_1, \dots, V_m)^+_B$. So, since $B \in \Delta$ and the open sets $V_1, \dots, V_m$ satisfy $B^c \cap V_i \neq \emptyset$ (for $i \in \{1, \dots, m\}$), there exist $U_n \in \mathcal{V}_n$ and $F_n \in [X]^{<\omega}$ such that $B \subseteq U_n$, $F_n \cap U_n = \emptyset$ and for each $i \in \{1, \dots, m\}$, $F_n \cap V_i \neq \emptyset$, for every but finitely many n. It can be shown that for those n, $U^c_n \in (V_1, \dots, V_m)^+_B \cap \mathcal{F}_n$. We conclude that (Λ, τ^+_Δ) is H-separable. □

From Theorem 4.2, we obtain the following particular cases.

Corollary 4.3. *Let (X, τ) be a topological space. If Λ is any of the hyperspaces $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ or $\mathbb{CS}(X)$, then (Λ, τ^+_Δ) is H-separable if and only if X satisfies the property $H(\mathbb{C}_\Delta(\Lambda))$.*

Corollary 4.4. *Let (X, τ) be a topological space. Suppose that Λ is some of the spaces $\mathbb{K}(X)$, $\mathbb{F}(X)$, $\mathbb{CS}(X)$, we have:*

(a) If $\Delta = \mathbb{K}(X)$, then (Λ, τ_F) is H-separable if and only if X satisfies the property $H(\mathbb{C}_{\mathbb{K}(X)}(\Lambda))$.
(b) If $\Delta = \mathrm{CL}(X)$, then (Λ, τ_V) is H-separable if and only if X satisfies the property $H(\mathbb{C}_{\mathrm{CL}(X)}(\Lambda))$.

We will denote the property $H(\mathbb{C}_\Delta(\Lambda))$ by $H(\mathbb{K}_F)$, when $\Delta = \mathbb{K}(X)$ and $\Lambda = \mathrm{CL}(X)$. Also, we will write $H(\mathbb{C}_V)$, if $\Delta = \Lambda = \mathrm{CL}(X)$ (see [5, Remark 2.21]).

Corollary 4.5. *Let (X, τ) be a topological space, we have:*

(a) $(\mathrm{CL}(X), \tau_F)$ is H-separable if and only if X satisfies the property $H(\mathbb{K}_F)$.
(b) $(\mathrm{CL}(X), \tau_V)$ is H-separable if and only if X satisfies the property $H(\mathbb{C}_V)$.

Note that, in general, the selection principle $\mathbf{U}_{\mathrm{fin}}(\mathcal{D}, \mathcal{D})$ does not makes sense, where $\mathcal{D}$ is the family of dense subsets of a space X. However, it makes sense to consider this principle in hyperspaces Λ which are closed under finite unions. In this case we obtain the following characterization.

Theorem 4.6. *Let* (X, τ) *be a topological space and* $\mathscr{D} = \{\mathcal{A} \subseteq \Lambda : cl_\Lambda(\mathcal{A}) = \Lambda\}$*. The following conditions are equivalent:*

(1) (Λ, τ_Δ^+) *satisfies* $\mathbf{U}_{fin}(\mathscr{D}, \mathscr{D})$*;*
(2) (X, τ) *satisfies* $\mathbf{I}_{fin}(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$.

Proof. (1) ⇒ (2) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $c_\Delta(\Lambda)$-covers of X. We put, for each $n \in \mathbb{N}$, $\mathcal{A}_n = \mathcal{U}_n^c$. From Lemma 1.4, we obtain that, for each $n \in \mathbb{N}$, $\mathcal{A}_n$ is a dense subset of (Λ, τ_Δ^+). Thus, applying (1) to the sequence $(\mathcal{A}_n : n \in \mathbb{N})$, we obtain, for any $n \in \mathbb{N}$, $\mathcal{B}_n \in [\mathcal{A}_n]^{<\omega}$ such that the family $\{\bigcup \mathcal{B}_n : n \in \mathbb{N}\}$ is a dense subset of (Λ, τ_Δ^+). Hence, from Lemma 1.4, we have that $\{(\bigcup \mathcal{B}_n)^c : n \in \mathbb{N}\} \in \mathbb{C}_\Delta(\Lambda)$, that is, $\{(\bigcap \mathcal{B}_n^c) : n \in \mathbb{N}\} \in \mathbb{C}_\Delta(\Lambda)$. Since $\mathcal{B}_n^c$ is a finite subset of $\mathcal{U}_n$, $\mathbf{I}_{\mathsf{fin}}(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$ holds.

(2) ⇒ (1) Let $(\mathcal{D}_n : n \in \mathbb{N})$ be a sequence of dense subsets of (Λ, τ_Δ^+). For any $n \in \mathbb{N}$, let $\mathcal{U}_n = \mathcal{D}_n^c$. It follows from Lemma 1.4 that, for each $n \in \mathbb{N}$, $\mathcal{U}_n$ is a $c_\Delta(\Lambda)$-cover of X. Hence, applying (2) to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$, for each $n \in \mathbb{N}$, there exists $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ such that $\{\bigcap \mathcal{V}_n : n \in \mathbb{N}\}$ is a $c_\Delta(\Lambda)$-cover of X. Then, from Lemma 1.4, we have that $\{\bigcup \mathcal{V}_n^c : n \in \mathbb{N}\}$ is a dense subset of (Λ, τ_Δ^+). Therefore, $\mathbf{U}_{\mathsf{fin}}(\mathscr{D}, \mathscr{D})$ holds. □

As a consequence of Theorem 4.6, we have the next particular cases.

Corollary 4.7. *Let* (X, τ) *be a topological space. If* Λ *is any of the hyperspaces* $\mathrm{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ *or* $\mathbb{CS}(X)$*, then* (Λ, τ_Δ^+) *satisfies* $\mathbf{U}_{fin}(\mathscr{D}, \mathscr{D})$ *if and only if* X *satisfies the principle* $\mathbf{I}_{fin}(\mathbb{C}_\Delta(\Lambda), \mathbb{C}_\Delta(\Lambda))$.

Corollary 4.8. *Let* (X, τ) *be a topological space. Suppose that* Λ *is some of the spaces* $\mathbb{K}(X)$, $\mathbb{F}(X)$, $\mathbb{CS}(X)$*, we have:*

(a) If $\Delta = \mathbb{K}(X)$*, then* (Λ, τ_F) *satisfies* $\mathbf{U}_{fin}(\mathscr{D}, \mathscr{D})$ *if and only if* X *satisfies the principle* $\mathbf{I}_{fin}\Big(\mathbb{C}_{\mathbb{K}(X)}(\Lambda), \mathbb{C}_{\mathbb{K}(X)}(\Lambda)\Big)$.
(b) If $\Delta = \mathrm{CL}(X)$*, then* (Λ, τ_V) *satisfies* $\mathbf{U}_{fin}(\mathscr{D}, \mathscr{D})$ *if and only if* X *satisfies the principle* $\mathbf{I}_{fin}\Big(\mathbb{C}_{\mathbb{K}(X)}(\Lambda), \mathbb{C}_{\mathbb{K}(X)}(\Lambda)\Big)$.

From [5, Remark 2.21], we obtain the characterizations.

Corollary 4.9. *Let* (X, τ) *be a topological space, we have:*

(a) $(\mathrm{CL}(X), \tau_F)$ *satisfies* $\mathbf{U}_{fin}(\mathscr{D}, \mathscr{D})$ *if and only if* X *satisfies the property* $\mathbf{I}_{fin}(\mathbb{K}_F, \mathbb{K}_F)$.
(b) $(\mathrm{CL}(X), \tau_V)$ *satisfies* $\mathbf{U}_{fin}(\mathscr{D}, \mathscr{D})$ *if and only if* X *satisfies the property* $\mathbf{I}_{fin}(\mathbb{C}_V, \mathbb{C}_V)$.

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