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SEP elements and solutions of related equations in a ring with involution

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Abstract. In this paper, we study many new characterizations of *SEP* elements in a rings with involution, mainly, we first discuss some properties of *SEP* elements by means of regular elements, {1,3}-inverses and the equality of certain set. Next, we give some necessity and sufficiency conditions of *SEP* elements by discussing the solutions of related equations. Finally, we characterize *SEP* elements by constructing the group inverses and *MP* inverses.

1. Introduction

Let R be a ring and $a \in R$. If there exists $b \in R$ such that $a = aba$, then a is called a regular element, and b is called the inner inverse element of a. Clearly, bab is also an inner inverse of a. We denote the set of all inner inverses of a by $a\{1\}$, and a^- is an inner inverse of a, and denote the set of all regular element of R by R^{reg}.

If there exists $a^\# \in R$ such that

$$aa^\#a = a,\ a^\#aa^\# = a^\#,\ aa^\# = a^\#a,$$

the element a is called group invertible element and $a^\#$ is called the group inverse of a [4, 8, 9], and it is uniquely determined by these equations. We write $R^\#$ to denote the set of all group invertible elements of R.

If a map $* : R \to R$ satisfies

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^*a^*,$$

then R is said to be an involution ring or a $*-$ring.

Let R be a $*$-ring and $a \in R$. If the following equations:

$$a = axa,\ ax = (ax)^*$$

have a common solution, then a is called $\{1,3\}$ invertible element of R, and such solution is called the $\{1,3\}$-inverse of a. If the following equations:

$$a = axa,\ xa = (xa)^*$$

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have a common solution, then a is called $\{1,4\}$ invertible element of R, and such solution is called the $\{1,4\}$-inverse of a.

Let R be a $*$-ring and $a \in R$. If there exists $a^+ \in R$ such that

$$a = aa^+a,\ a^+ = a^+aa^+,\ (aa^+)^* = aa^+,\ (a^+a)^* = a^+a,$$

then a is called Moore Penrose invertible element, and a^+ is called the Moore Penrose inverse of a [3, 6]. Let R^+ denote the set of all Moore Penrose invertible elements of R.

If $a \in R^\# \cap R^+$ and $a^\# = a^+$, then a is called *EP* element. On the studies of *EP*, the readers can refer to [2, 3, 5, 7, 10–14, 20, 21].

If $a = aa^*a$, then a is called partial isometry [15, 18]. It is known that $a \in R$ is a partial isometry if and only if $a \in R^+$ and $a^* = a^+$ [11].

$a \in R^\# \cap R^+$ is called *SEP* element if $a^\# = a^+ = a^*$. Clearly, $a \in R^\# \cap R^+$ is *SEP* if and only $a \in R^{EP}$ and $a \in R^{PI}$. Where R^{EP} , R^{PI} and R^{SEP} are denoted the set of all *EP* elements, all *PI* elements and all *SEP* elements of R respectively.

In [11], many characterizations of *SEP* elements are given. In [19], it is shown that $a \in R^\# \cap R^+$ is *SEP* if and only if the equation $xa^+a = xaa^*$ has at least one solution in $\chi_a =: \{a, a^\#, a^+, a^*, (a^+)^*, (a^\#)^*\}$. In [16], it is proved that $a \in R^\# \cap R^+$ is *SEP* if and only if the equation $yxa^* = yxa^\#$ has at least one solution in $\chi_a^2 =: \{(x, y)|x, y \in \chi_a\}$. In [18], it is shown that $a \in R^\# \cap R^+$ is *SEP* if and only if the equation $a^*xa = aa^+x$ has at least one solution in χ_a.

Motivated by these results, this paper mainly study the ways to characterize *SEP* elements.

2. Some characterizations of SEP elements

Let $a \in R^\# \cap R^+$. Then, clearly, $a \in R^{SEP}$ if and only if $a^2a^+ = a = (a^+)^*$, this induces us to give the following characterization of *SEP* elements.

Theorem 2.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $(a^2a^+ - (a^+)^*)^2 = a(a^2a^+ - (a^+)^*)$.*

Proof. ($\Rightarrow$) It is evident because $a^2a^+ - (a^+)^* = 0$.

($\Leftarrow$) Assume that $(a^2a^+ - (a^+)^*)^2 = a(a^2a^+ - (a^+)^*)$. Then

$$a^3a^+ - a(a^+)^* - (a^+)^*a^2a^+ + (a^+)^*(a^+)^* = a^3a^+ - a(a^+)^*,$$

this gives $(a^+)^*a^2a^+ = (a^+)^*(a^+)^*$. Multiplying the equality on the left by a^*, one has $a^+a^3a^+ = a^+a(a^+)^*$. Again multiply the last equality on the left by $aa^\#$, one gets

$$a^2a^+ = (a^+)^*,$$

it follows that $(a^+)^* = (a^+)^*aa^\# = (a^2a^+)aa^\# = a$. Hence $a \in R^{PI}$ and $a^2a^+ = (a^+)^* = a$, this infers $a \in R^{EP}$ by [11, Theorem 1.2.1]. Thus $a \in R^{SEP}$. □

The following corollary is an immediate result of Theorem 2.1.

Corollary 2.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^2a^+ = (a^+)^*$.*

Noting that $(a^2a^+)^+ = aa^\#a^+$. Hence Corollary 2.2 leads to the following theorem.

Theorem 2.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^\#a^+ = a^*$.*

Since $a \in R^{SEP}$ if and only if $a^* \in R^{SEP}$, replacing a in Corollary 2.2 by a^*, one obtains

Corollary 2.4. [11, Theorem 1.5.3] *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^*a^+a = a^+$.*

Noting that $a^* = a^+aa^*$. Then Corollary 2.4 induces the following result.

Corollary 2.5. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $aa^*a^+a = aa^+$.*

Theorem 2.6. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $(a^{\#})^*a^+ = aa^+$.*

Proof. Noting that $(aa^*a^+a)^+ = (a^{\#})^*a^+$ and $(aa^+)^+ = aa^+$. Hence we are done by Corollary 2.5. $\square$

Theorem 2.7. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $a^{\#}a(a^{\#})^*a^+aa^{\#} = aa^+$.*

Proof. Since $(aa^*a^+a)^{\#} = a^{\#}a(a^{\#})^*a^+aa^{\#}$ and $(aa^+)^{\#} = aa^+$, we are done by Corollary 2.5. $\square$

3. Construct regular elements to characterize SEP elements

From Theorem 2.3, we have the following lemma.

Lemma 3.1. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $a = aa^+(a^{\#})^*$.*

Theorem 3.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{EP}$ if and only if $\begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix}$ is a regular element and $\begin{pmatrix} aa^+a^* & 1-aa^+ \end{pmatrix} \in \begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix}\{1\}$.*

Proof. ($\Rightarrow$) If $a \in R^{EP}$, then $aaa^+ = aaa^{\#} = a$. It follows that

$$\begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix}\begin{pmatrix} aa^+a^* & 1-aa^+ \end{pmatrix}\begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix} = \begin{pmatrix} aa^+(a^{\#})^* \\ aaa^+ \end{pmatrix} = \begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix}.$$

Hence $\begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix}$ is a regular element and $\begin{pmatrix} aa^+a^* & 1-aa^+ \end{pmatrix} \in \begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix}\{1\}$.

($\Leftarrow$) If $\begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix}$ is a regular element and $\begin{pmatrix} aa^+a^* & 1-aa^+ \end{pmatrix} \in \begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix}\{1\}$, then

$$\begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix} = \begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix}\begin{pmatrix} aa^+a^* & 1-aa^+ \end{pmatrix}\begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix} = \begin{pmatrix} aa^+(a^{\#})^* \\ aaa^+ \end{pmatrix},$$

it follows that $a^2a^+ = a$. Hence $a \in R^{EP}$. $\square$

Theorem 3.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $\begin{pmatrix} a^+ & 1-aa^+ \end{pmatrix} \in \begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix}\{1\}$.*

Proof. ($\Rightarrow$) If $a \in R^{SEP}$, then $a \in R^{EP}$ and $a = aa^+(a^{\#})^*$ by Lemma 3.1.
This gives $\begin{pmatrix} aa^+a^* & 1-aa^+ \end{pmatrix} \in \begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix}\{1\}$ by Theorem 3.2.
Noting that $a^+ = (aa^+(a^{\#})^*)^+ = aa^+a^*$. Then $\begin{pmatrix} a^+ & 1-aa^+ \end{pmatrix} \in \begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix}\{1\}$.

($\Leftarrow$) From the assumption, one has

$$\begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix} = \begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix}\begin{pmatrix} a^+ & 1-aa^+ \end{pmatrix}\begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix} = \begin{pmatrix} aa^+(a^{\#})^*a^+(a^{\#})^* \\ aa^+(a^{\#})^* \end{pmatrix}.$$

This infers $a = aa^+(a^{\#})^*$. By Lemma 3.1, $a \in R^{SEP}$. $\square$

Theorem 3.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $\begin{pmatrix} aa^+a^* & 1-(a^+)^*a^{\#} \end{pmatrix} \in \begin{pmatrix} aa^+(a^{\#})^* \\ a \end{pmatrix}\{1\}$.*

Proof. (⇒) If $a \in R^{SEP}$, then $a = (a^+)^*$ and $a^\# = a^+$. It follows that $aa^+ = (a^+)^* a^\#$. Hence $\begin{pmatrix} aa^+a^* & 1-(a^+)^*a^\# \end{pmatrix} \in \begin{pmatrix} aa^+(a^\#)^* \\ a \end{pmatrix}\{1\}$ by Theorem 3.2.

(⇐) From the assumption, one has

$$\begin{pmatrix} aa^+(a^\#)^* \\ a \end{pmatrix} = \begin{pmatrix} aa^+(a^\#)^* \\ a \end{pmatrix}\begin{pmatrix} aa^+a^* & 1-(a^+)^*a^\# \end{pmatrix}\begin{pmatrix} aa^+(a^\#)^* \\ a \end{pmatrix}$$
$$= \begin{pmatrix} aa^+(a^\#)^* + aa^+(a^\#)^*(a-(a^+)^*) \\ a^2a^+ + a(a-(a^+)^*) \end{pmatrix}.$$

This gives

$$aa^+(a^\#)^* = aa^+(a^\#)^* + aa^+(a^\#)^*(a-(a^+)^*). \tag{3.1}$$

$$a = a^2a^+ + a(a-(a^+)^*). \tag{3.2}$$

Multiplying the Eq.(3.2) on the right by a^+a, one obtains $a^2a^+ = a^2a^+a^+a$, this gives

$$a^* = a^*aa^+ = a^*a^\#a^2a^+ = a^*a^\#a^2a^+a^+a = a^*a^+a,$$

and

$$(aa^\#)^* = (a^\#)^*a^* = (a^\#)^*(a^*a^+a) = a^+a.$$

Hence $a \in R^{EP}$ by [11, Theorem 1.1.3], this implies $aa^+(a^\#)^* = a^+a(a^\#)^* = (a^\#)^*$. From the Eq.(3.1), one gets $(a^\#)^*(a-(a^+)^*) = 0$, that is, $a^*a^\# = a^+a^\#$. Hence $a \in R^{PI}$ by [11, Theorem 1.5.2]. Thus $a \in R^{SEP}$. □

4. Using the equality of sets to characterize SEP elements

Let $a, b \in R$. We set $a \vee b = \{a^2, ab, ba, b^2\}$, and use this to characterize *SEP* elements as follows.

Theorem 4.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^+a^2 \vee a = a^+a^2 \vee aa^+(a^\#)^*$.*

Proof. (⇒) Assume that $a \in R^{SEP}$. Then $a = aa^+(a^\#)^*$ by Lemma 3.1. Hence $a^+a^2 \vee a = a^+a^2 \vee aa^+(a^\#)^*$.

(⇐) Noting that $a^+a^2 \vee a = \{a^+a^3, a^2\}$,

$$a^+a^2 \vee aa^+(a^\#)^* = \{a^+a^3,\ a^+a^3a^+(a^\#)^*,\ aa^+(a^\#)^*a^+a^2,\ aa^+(a^\#)^*(a^\#)^*\}.$$

By hypothesis, we have $\{a^+a^3, a^2\} = \{a^+a^3,\ a^+a^3a^+(a^\#)^*,\ aa^+(a^\#)^*a^+a^2,\ aa^+(a^\#)^*(a^\#)^*\}$.

(1) If $a^+a^3 = a^2$, then we have $a^+a^3 = a^+a^3a^+(a^\#)^* = aa^+(a^\#)^*a^+a^2 = aa^+(a^\#)^*(a^\#)^*$. Since $a^+a^3 = a^2$, $a \in R^{EP}$ by [11], this infers

$$a^2 = a^+a^3 = a^+a^3a^+(a^\#)^* = a^2a^+(a^\#)^* = a(a^\#)^*,$$

and

$$a = a^+a^2 = a^+a(a^\#)^* = (a^\#)^*.$$

Hence $a \in R^{SEP}$.

(2) If $a^+a^3 \neq a^2$, then we can discuss the following cases.

① $a^2 = aa^+(a^\#)^*a^+a^2$. In this case, we have

$$a^+a^3 = a^+a^3a^+(a^\#)^* \text{ and } aa^+(a^\#)^*a^+a^2 = aa^+(a^\#)^*(a^\#)^*,$$

or

$$a^+a^3 = a^+a^3a^+(a^\#)^* = aa^+(a^\#)^*(a^\#)^*,$$

or

$$a^+a^3 = aa^+(a^\#)^*(a^\#)^* \text{ and } a^+a^3a^+(a^\#)^* = aa^+(a^\#)^*a^+a^2,$$

or

$$a^+a^3a^+(a^\#)^* = aa^+(a^\#)^*(a^\#)^* = aa^+(a^\#)^*a^+a^2.$$

In the first case and the second case, we have

$$a^+a^3 = a^+a^3a^+(a^\#)^* = (a^+a^3a^+(a^\#)^*)aa^+ = a^+a^4a^+,$$

and

$$a = a^\#a^+a^3 = a^\#(a^+a^4a^+) = a^2a^+.$$

So $a \in R^{EP}$, one obtains $a^+a^3 = a^2$, which is a contradiction.

In the third case, we also have $a^+a^3 = aa^+(a^\#)^*(a^\#)^* = aa^+(a^\#)^*(a^\#)^*aa^+ = a^+a^4a^+$. So $a \in R^{EP}$, and $a^+a^3 = a^2$, which is also a contradiction.

In the fourth case, we have $a^2 = a^+a^3a^+(a^\#)^*$. Multiplying the equality on the left by $a^\#$, one has $a = aa^+(a^\#)^*$. Hence $a \in R^{SEP}$ by Lemma 3.1.

② $a^2 = aa^+(a^\#)^*(a^\#)^*$. Then $a^3a^+ = aa^+(a^\#)^*(a^\#)^*aa^+ = aa^+(a^\#)^*(a^\#)^* = a^2$. It follows that $a \in R^{EP}$, this gives $a^+a^3 = a^2$, which is a contradiction.

③ $a^2 = a^+a^3a^+(a^\#)^*$. From the proof of the fourth case in ① we get $a \in R^{SEP}$. Therefore, in any case, we have $a \in R^{SEP}$. □

Similarly, we have the following theorems.

Theorem 4.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^2a^+ \vee a = a^2a^+ \vee aa^+(a^\#)^*$.*

Theorem 4.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^*a^+a \vee a^* = a^*a^+a \vee a^\#aa^+$.*

5. Using the (1, 3)-inverse and (1, 4)−inverse to characterize SEP elements

Theorem 5.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $aa^+(a^\#)^* \in a^+\{1, 3\}$.*

Proof. (⇒) Assume that $a \in R^{SEP}$. Then $a = aa^+(a^\#)^*$ by Lemma 3.1. Noting that $a \in a^+\{1, 3\}$. Then $aa^+(a^\#)^* \in a^+\{1, 3\}$.

(⇐) If $aa^+(a^\#)^* \in a^+\{1, 3\}$, then

$$a^+(aa^+(a^\#)^*)a^+ = a^+. \tag{5.1}$$

$$a^+(aa^+(a^\#)^*) = (a^+(aa^+(a^\#)^*))^*. \tag{5.2}$$

By the Eq.(5.2), one has $a^+(a^\#)^* = a^\#(a^+)^*$. Multiplying the equality on the right by a^*, one obtains $a^+ = a^\#aa^+$. Hence $a \in R^{EP}$ by [11, Theorem 1.2.1]. By the Eq.(5.1), one gets $a^+ = a^+(a^\#)^*a^+$, this gives $a = aa^+a = aa^+(a^\#)^*a^+a = (a^+)^*$. Hence $a \in R^{PI}$ and so $a \in R^{SEP}$. □

Corollary 5.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^+ = a^\#(a^+)^*a^+$.*

Proof. (⇒) If $a \in R^{SEP}$, then $aa^+(a^\#)^* \in a^+\{1, 3\}$ by Theorem 5.1. It follows that $a^+ = (aa^+(a^\#)^*)^*(a^+)^*a^+ = a^\#aa^+(a^+)^*a^+ = a^\#(a^+)^*a^+$.

(⇐) Assume that $a^+ = a^\#(a^+)^*a^+$. Then

$$aa^+ = aa^\#(a^+)^*a^+ = (a^+)^*a^+ \textit{ and } a^* = a^*aa^+ = a^*(a^+)^*a^+ = a^+.$$

It follows that $a^+a = (a^\#(a^+)^*a^+)a = a^\#(a^+)^* = a^\#a$. Hence $a \in R^{EP}$ and so $a \in R^{SEP}$. □

Theorem 5.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $a^+ \in aa^+(a^\#)^*\{1, 4\}$.*

Proof. (⇒) If $a \in R^{SEP}$. Then $a = aa^+(a^\#)^*$ by Lemma 3.1. Noting that $a^+ \in a\{1,4\}$. Then $a^+ \in aa^+(a^\#)^*\{1,4\}$.

(⇐) If $a^+ \in aa^+(a^\#)^*\{1,4\}$, then

$$aa^+(a^\#)^* = aa^+(a^\#)^*a^+aa^+(a^\#)^*. \tag{5.3}$$

$$(a^+aa^+(a^\#)^*)^* = a^+aa^+(a^\#)^*. \tag{5.4}$$

By the Eq.(5.3), one yields $aa^+(a^\#)^* = (a^+)^*a^+(a^\#)^*$. Multiplying the equality on the right by a^*, one has $aa^+ = (a^+)^*a^+$. Hence $a \in R^{PI}$. By the Eq.(5.4), one obtains

$$a^\#(a^+)^* = a^+(a^\#)^*.$$

By Theorem 5.1, we have $a \in R^{EP}$. Hence $a \in R^{SEP}$. □

Theorem 5.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $(a^\#)^*a^+ - aa^+ \in l(a^+) = \{x|x \in R, xa^+ = 0\}$.*

Proof. (⇒) If $a \in R^{SEP}$, then we have $a = aa^+(a^\#)^*$ by Lemma 3.1. Noting that

$$aa^+(a^\#)^*a^+ = (a^+)^*a^+ \text{ and } a^+ = a^\#.$$

Then $(a^\#)^*a^+ - aa^+ = aa^+(a^\#)^*a^+ - aa^+ = aa^+ - aa^+ = 0 \in l(a^+)$.

(⇐) If $(a^\#)^*a^+ - aa^+ \in l(a^+)$, then $(a^\#)^*a^+a^+ = aa^+a^+$. By [2, Lemma 2.10], we have $(a^\#)^*a^+ = aa^+$. It follows that $a^+ = a^*(a^\#)^*a^+ = a^*aa^+ = a^*$, and so

$$(aa^\#)^* = (a^\#)^*a^* = (a^\#)^*a^+ = aa^+.$$

Hence $a \in R^{SEP}$ by [1, Theorem 1.3.1]. □

6. Characterizing SEP elements by the solution of univariate equations in a given set

When $a \in R^{SEP}$, we have $aa^+(a^\#)^* = a = aa^+a$ by Lemma 3.1. Hence we can construct the following equation:

$$xa^+(a^\#)^* = aa^+x. \tag{6.1}$$

Theorem 6.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(6.1) has at least one solution in $\rho_a = \{a, a^\#, a^+, a^*, (a^+)^*, (a^\#)^*, (a^+)^\#, (a^\#)^+\}$.*

Proof. (⇒) If $a \in R^{SEP}$, then $x = a$ is a solution by Lemma 3.1.

(⇐) (1) If $x = a$ is a solution, then $aa^+(a^\#)^* = aa^+a = a$. Hence $a \in R^{SEP}$ by Lemma 3.1;

(2) If $x = a^\#$, then $a^\#a^+(a^\#)^* = aa^+a^\# = a^\#$. Multiplying the equality by a^2 on the left, we have $a = aa^+(a^\#)^*$. Hence $a \in R^{SEP}$ by Lemma 3.1;

(3) If $x = a^+$, then $a^+a^+(a^\#)^* = aa^+a^+$. Multiplying the equality on the right by a^* , one has $a^+a^+ = aa^+a^+a^*$. Again multiply the last equality by a^+ on the left, one gets

$$a^+a^+a^+ = a^+a^+a^*.$$

Hence $a^+a^+ = a^+a^* = a^+a^+aa^*$ and $a^+ = a^+aa^* = a^*$ by [2, Lemma 2.10]. It follows that $a^+ = a^+a^*(a^\#)^* = a^+a^+(a^\#)^* = aa^+a^+$, and $a^* = a^+aa^* = (aa^+a^+)aa^* = aa^+a^*$. Taking involution of the above equality, we obtains that $a = a^2a^+$. Hence $a \in R^{EP}$ by [11, Theorem 1.2.1]. Thus $a \in R^{SEP}$;

(4) If $x = a^*$, then $a^*a^+(a^\#)^* = aa^+a^*$. Multiplying the equality on the left by $(aa^\#)^*$, one yields $a^*a^+(a^\#)^* = a^*$. It follows that $a^* = aa^+a^*$. Hence $a \in R^{EP}$. Now we have $aa^+ = (a^+)^*a^* = (a^+)^*a^*a^+(a^\#)^* = aa^+a^+(a^\#)^* = a^+(a^\#)^*$, and $a = a^2a^+ = aa^+(a^\#)^*$. Thus $a \in R^{SEP}$ by Lemma 3.1;

(5) If $x = (a^+)^*$, then $(a^+)^*a^+(a^\#)^* = aa^+(a^+)^* = (a^+)^*$. Multiplying the equality on the left by aa^*, one gets $aa^+(a^\#)^* = a$. Hence $a \in R^{SEP}$ by Lemma 3.1;

(6) If $x = (a^\#)^*$, then $(a^\#)^* a^+ (a^\#)^* = aa^+(a^\#)^*$. Multiplying the equality on the left by a^*a^*, one yields $a^*a^+(a^\#)^* = a^*$. Hence $a \in R^{SEP}$ by (4);

(7) If $x = (a^+)^\# = (aa^\#)^* a(aa^\#)^*$, then $(aa^\#)^* a(aa^\#)^* a^+ (a^\#)^* = aa^+(aa^\#)^* a(aa^\#)^*$, e.g., $(a^\#)^* = a(aa^\#)^*$. Multiplying the equality on the left by a^*a^+, one obtains $a^*a^+(a^\#)^* = a^*$. Hence $a \in R^{SEP}$ by (4);

(8) If $x = (a^\#)^+ = a^+a^3a^+$, then $a^+a^3a^+a^+(a^\#)^* = aa^+a^+a^3a^+$. Multiplying the equality on the left by a^+a, one has $aa^+a^+a^3a^+ = a^+a^2a^+a^+a^3a^+$. Again multiply the last equality on the right by $a^\#$, one gets $aa^+a^+a = a^+a^2a^+a^+a$. It follows that

$$aa^+ = aa^+(a^+aa^*(a^\#)^*) = a^+a^2a^+a^+a(a^*(a^\#)^*) = a^+a^2a^+.$$

Hence $a \in R^{EP}$, this infers $x = (a^\#)^+ = a^+a^3a^+ = a$. Thus $a \in R^{SEP}$ by (1). □

Noting that $a \in R^{SEP}$ if and only if $a^* \in R^{SEP}$. Then replace a by a^* in Eq.(6.1), one gets

$$x(a^+)^* a^\# = a^+ax. \tag{6.2}$$

Theorem 6.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(6.2) has at least one solution in ρ_a.*

Since $a \in R^{SEP}$ if and only if $a \in R^{EP}$ and $(a^\#)^* = (a^\#)^*(a^\#)^* a^* = a^2a^* = a^3a^+a^*$. Hence we can revise Eq.(6.1) as follows.

$$xa^+a^3a^+a^* = aa^+x. \tag{6.3}$$

Theorem 6.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(6.3) has at least one solution in ρ_a.*

Noting that when $a \in R^{EP}$, we have $a^+a^3a^+ = (a^+a)^* a(aa^+)^* = (aa^\#)^* a(aa^\#)^*$. Then Eq.(6.3) can be changed as follows.

$$x(aa^\#)^* aa^* = aa^+x. \tag{6.4}$$

Theorem 6.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(6.4) has at least one solution in ρ_a.*

7. The general solution of bivariate equations

Now we generalize Eq.(6.1) as follows

$$xa^+(a^\#)^* = aa^+y. \tag{7.1}$$

Theorem 7.1. *Let $a \in R^\# \cap R^+$. Then the general solution of Eq.(7.1) is given by*

$$\begin{cases} x = aa^+p + u - ua^+a \\ y = pa^+(a^\#)^* + v - aa^+v \end{cases}, \ p,\ u,\ v \in R. \tag{7.2}$$

Proof. First $(aa^+p + u - ua^+a)a^+(a^\#)^* = aa^+pa^+(a^\#)^* = aa^+(pa^+(a^\#)^* + v - aa^+v)$. It follows that the formula (7.2) is the solution of Eq.(7.1).

Next, let $\begin{cases} x = x_0 \\ y = y_0 \end{cases}$ be any solution of Eq.(7.1). Then $x_0a^+(a^\#)^* = aa^+y_0$.

Choose $p = x_0$, $v = y_0$ and $u = x_0 - aa^+p$. Then

$$\begin{aligned} ua^+a &= (x_0 - aa^+p)a^+a = x_0a^+a - aa^+x_0a^+a = x_0a^+a - aa^+x_0(a^+(a^\#)^*a^*)a \\ &= x_0a^+a - aa^+(x_0a^+(a^\#)^*)a^*a = x_0a^+a - aa^+(aa^+y_0)a^*a = x_0a^+a - aa^+y_0a^*a \\ &= x_0a^+a - (x_0a^+(a^\#)^*)a^*a = x_0a^+a - x_0a^+a = 0. \end{aligned}$$

It follows that $x_0 = aa^+p + (x_0 - aa^+p) = aa^+p + u = aa^+p + u - ua^+a$. Noting that

$$aa^+v = aa^+y_0 = x_0a^+(a^\#)^* = pa^+(a^\#)^*.$$

Then $y_0 = pa^+(a^\#)^* + y_0 - aa^+v = pa^+(a^\#)^* + v - aa^+v$. Hence the general solution of Eq.(7.1) is given by the formula (7.2). □

Theorem 7.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if the general solution of Eq.(7.1) is given by*

$$\begin{cases} x = aa^{+}p + u - ua^{+}a \\ y = pa^{+}a + v - aa^{+}v \end{cases}, p, u, v \in R. \tag{7.3}$$

Proof. (⇒) If $a \in R^{SEP}$, then $(a^{\#})^{*} = a$. By Theorem 7.1, one gets the general solution of Eq.(7.1) is given by the formula (7.3).

(⇐) From the assumption, we get $(aa^{+}p + u - ua^{+}a)a^{+}(a^{\#})^{*} = aa^{+}(pa^{+}a + v - aa^{+}v)$, e.g.,

$$aa^{+}pa^{+}(a^{\#})^{*} = aa^{+}pa^{+}a \text{ for all } p \in R.$$

Choose $p = (a^{+})^{*}$, then $(a^{+})^{*}a^{+}(a^{\#})^{*} = (a^{+})^{*}$. Taking involution of the above equality, one obtains $a^{\#}(a^{+})^{*}a^{+} = a^{+}$. Multiplying the equality on the right by aa^{*}, one has $aa^{\#}a^{+} = a^{*}$. Again multiply the last equality on the right by a, one gets $aa^{\#} = a^{*}a$. Hence $a \in R^{SEP}$ by [11, Theorem 1.5.3]. □

Theorem 7.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if the general solution of Eq.(7.1) is given by*

$$\begin{cases} x = a^{*}ap + u - ua^{+}a \\ y = pa^{+}(a^{\#})^{*} + v - aa^{+}v \end{cases}, \ p, \ u, \ v \in R. \tag{7.4}$$

Proof. (⇒) If $a \in R^{SEP}$, then $a^{*}a = a^{\#}a = aa^{\#} = aa^{+}$. It follows from Theorem 7.1 that the general solution of Eq.(7.1) is given by the formula (7.4).

(⇐) From the assumption, we get $(a^{*}ap + u - ua^{+}a)a^{+}(a^{\#})^{*} = aa^{+}(pa^{+}(a^{\#})^{*} + v - aa^{+}v)$, e.g.,

$$a^{*}apa^{+}(a^{\#})^{*} = aa^{+}pa^{+}(a^{\#})^{*} \text{ for all } p \in R.$$

Choose $p = a$, then $a^{*}a^{2}a^{+}(a^{\#})^{*} = aa^{+}(a^{\#})^{*}$. Multiplying the equality on the right by $a^{*}aa^{\#}a^{+}$, one has $a^{*} = aa^{\#}a^{+}$. Hence $a \in R^{SEP}$. □

We don't know the general solution of which equation is given by formula (7.4), for this we can construct the equation as follows.

$$xa^{+}(a^{\#})^{*} = a^{*}a(aa^{\#})^{*}y. \tag{7.5}$$

Theorem 7.4. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution of Eq.(7.5) is given by*

$$\begin{cases} x = a^{*}ap + u - ua^{+}a \\ y = pa^{+}(a^{\#})^{*} + v - aa^{+}v \end{cases}, \text{where } p, u, v \in R \text{ with } a^{+}p = a^{+}a^{+}ap. \tag{7.6}$$

Proof. When $a^{+}p = a^{+}a^{+}ap$, we have $a^{*}a(a^{\#})^{*}a^{*}p = a^{*}ap$. So

$$(a^{*}ap + u - ua^{+}a)a^{+}(a^{\#})^{*} = a^{*}a(a^{\#})^{*}a^{*}pa^{+}(a^{\#})^{*} = a^{*}a(aa^{\#})^{*}(pa^{+}(a^{\#})^{*} + v - aa^{+}v).$$

It follows that the formula (7.6) is the solution of Eq.(7.5).

Next, let $\begin{cases} x = x_0 \\ y = y_0 \end{cases}$ be any solution of Eq.(7.5). Then $x_0a^{+}(a^{\#})^{*} = a^{*}a(aa^{\#})^{*}y_0$.

Choose $p = (aa^{\#})^{*}y_0a^{*}a$, $v = y_0 - (aa^{\#})^{*}y_0$ and $u = x_0$. Then

$$a^{+}p = a^{+}(aa^{\#})^{*}y_0a^{*}a = a^{+}(a^{+}a(aa^{\#})^{*}y_0a^{*}a) = a^{+}a^{+}ap,$$

$$ua^{+}a = x_0a^{+}a = x_0a^{+}(a^{\#})^{*}a^{*}a = a^{*}a(aa^{\#})^{*}y_0a^{*}a = a^{*}ap.$$

It follows that $x_0 = u = a^{*}ap + u - ua^{+}a$. Noting that

$$aa^{+}v = aa^{+}(y_0 - (aa^{\#})^{*}y_0) = aa^{+}y_0 - aa^{+}(aa^{\#})^{*}y_0 = aa^{+}y_0 - aa^{+}y_0 = 0,$$

and

$$pa^+(a^\#)^* = (aa^\#)^* y_0 a^* aa^+(a^\#)^* = (aa^\#)^* y_0 (aa^\#)^* = a^+a(aa^\#)^* y_0 (aa^\#)^*$$
$$= a^+(a^\#)^* a^* a(aa^\#)^* y_0 (aa^\#)^* = a^+(a^\#)^* x_0 a^+(a^\#)^*(aa^\#)^* = a^+(a^\#)^* x_0 a^+(a^\#)^*$$
$$= a^+(a^\#)^* a^* a(aa^\#)^* y_0 = a^+a(aa^\#)^* y_0 = (aa^\#)^* y_0.$$

Then $y_0 = pa^+(a^\#)^* + y_0 - (aa^\#)^* y_0 = pa^+(a^\#)^* + v - aa^+v$. Hence the general solution of Eq.(7.5) is given by formula (7.6). □

Theorem 7.5. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(7.1) and Eq.(7.5) have the same solution.*

Proof. (⇒) If $a \in R^{SEP}$. Then by Theorem 7.3, one has the general solution of Eq.(7.1) is given by formula (7.4). Since $a \in R^{EP}$, then $a^+a^+a = a^+$. Hence formula (7.6) is consistent with formula (7.4). By Theorem 7.4, one obtains the general solution of Eq.(7.5) is given by formula (7.4). Hence Eq.(7.1) and Eq.(7.5) have the same solution.

(⇐) If Eq.(7.1) and Eq.(7.5) have the same solution. By Theorem 7.1, one gets the general solution of Eq.(7.5) is given by formula (7.2). Then

$$(aa^+p + u - ua^+a)a^+(a^\#)^* = a^*a(aa^\#)^*(pa^+(a^\#)^* + v - aa^+v),$$

e.g.,

$$aa^+pa^+(a^\#)^* = a^*a(aa^\#)^* pa^+(a^\#)^* \text{ for all } p \in R.$$

Choose $p = a$, then $aa^+(a^\#)^* = a^*a(a^\#)^*$. Multiplying the equality on the right by a^*a^+, one has $aa^+a^+ = a^*$. Again multiply the last equality on the left by a^+, one gets

$$a^+a^+ = a^+a^* = a^+a^+aa^*.$$

By [2, Lemma 2.10], we have $a^+ = a^+aa^* = a^*$. Hence $a \in R^{PI}$. By Theorem 7.4, one obtains the general solution of Eq.(7.1) is given by formula (7.6). Then

$$(a^*ap + u - ua^+a)a^+(a^\#)^* = aa^+(pa^+(a^\#)^* + v - aa^+v),$$

e.g.,

$$a^*apa^+(a^\#)^* = aa^+pa^+(a^\#)^* \text{ for } p \in R \text{ with } a^+a^+ap = a^+p.$$

Choose $p = a^+$, one yields $a^*a^+(a^\#)^* = aa^+a^+a^+(a^\#)^*$. Noting that $a^+ = a^*$, then we have $a^* = aa^*a^*$, so $a = a^2a^*$. Hence $a \in R^{SEP}$ by [11, Theorem 1.5.3].

□

8. The solution of bivariate equations in a fixed set

Lemma 8.1. *Let $a \in R^\# \cap R^+$, $y \in R$, $x \in \rho_a$. If $x(a^\#)^* y = 0$, then $(a^\#)^* y = 0$.*

Proof. Noting that $x^\# x = \begin{cases} aa^\# & ,x \in \tau_a =: \{a, a^\#, (a^+)^*\} \\ (aa^\#)^* & ,x \in \theta_a =: \{a^+, a^*, (a^\#)^*, (a^+)^\#, (a^\#)^+\} \end{cases}$. Then we have

(1) if $x \in \tau_a$, then $(a^\#)^* y = a^+a(a^\#)^* y = a^+aaa^\#(a^\#)^* y = a^+ax^\# x(a^\#)^* y = 0$.

(2) if $x \in \theta_a$, then $(a^\#)^* y = (aa^\#)^*(a^\#)^* y = x^\# x(a^\#)^* y = 0$.

Thus, in any case, we have $(a^\#)^* y = 0$. □

Since $a \in R^{SEP}$, $aa^+(a^\#)^* = a = a(aa^\#)^* a^+a$. Hence we can give the following equation.

$$xy(a^\#)^* = x(aa^\#)^* ya. \tag{8.1}$$

Theorem 8.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(8.1) has at least one solution in $\rho_a^2 \triangleq \{(x, y) | x, y \in \rho_a\}$.*

Proof. (⇒) Clearly $(x, y) = (a, a^+)$ is a solution.

(⇐) (1) If $y = a$, then we have $xa(a^\#)^* = x(aa^\#)^*a^2$.

① If $x = a$, then $a^2(a^\#)^* = a(aa^\#)^*a^2$. Multiplying the equality on the right by a^+a, one yields $a^2(a^\#)^* = a^2(a^\#)^*a^+a$. Again multiply the last equality on the left by $a^+a^*a^+a^\#$, one gets

$$a^+ = a^+a^+a.$$

Hence $a \in R^{EP}$, it follows that $a^2(a^\#)^* = a(aa^\#)^*a^2 = a(aa^+)^*a^2 = a^3$. Thus $a^2 = a^\#a^3 = a^\#a^2(a^\#)^* = a(a^\#)^*$. Hence $a \in R^{SEP}$;

② If $x = a^\#$, then $a^\#a(a^\#)^* = a^\#(aa^\#)^*a^2$. Multiplying the equality on the left by a^2, one has $a^2(a^\#)^* = a(aa^\#)^*a^2$. Hence $a \in R^{SEP}$ by ①;

③ If $x = a^+$, then $a^+a(a^\#)^* = a^+(aa^\#)^*a^2$, e.g., $(a^\#)^* = a^+a^2$. This gives $a(a^\#)^* = a^2$. Hence $a \in R^{SEP}$;

④ If $x = a^*$, then $a^*a(a^\#)^* = a^*(aa^\#)^*a^2 = a^*a^2$. Multiplying the equality on the left by $(a^+)^*$, one obtains $a(a^\#)^* = a^2$. Hence $a \in R^{SEP}$;

⑤ If $x = (a^+)^*$, then $(a^+)^*a(a^\#)^* = (a^+)^*(aa^\#)^*a^2$. Multiplying the equality on the left by aa^*, one has $a^2(a^\#)^* = a(aa^\#)^*a^2$. Hence $a \in R^{SEP}$ by ①;

⑥ If $x = (a^\#)^*$, then $(a^\#)^*a(a^\#)^* = (a^\#)^*(aa^\#)^*a^2$. Multiplying the equality on the left by a^*a^*, one gets $a^*a(a^\#)^* = a^*(aa^\#)^*a^2$. Hence $a \in R^{SEP}$ by ④;

⑦ If $x = (a^+)^\# = (aa^\#)^*a(aa^\#)^*$, then $(aa^\#)^*a(aa^\#)^*a(a^\#)^* = (aa^\#)^*a(aa^\#)^*(aa^\#)^*a^2$. Multiplying the equality on the left by a^*a^+, one obtains $a^*a(a^\#)^* = a^*(aa^\#)^*a^2$. Hence $a \in R^{SEP}$ by ④;

⑧ If $x = (a^\#)^+ = a^+a^3a^+$, then $a^+a^3a^+a(a^\#)^* = a^+a^3a^+(aa^\#)^*a^2$, e.g., $a^+a^3(a^\#)^* = a^+a^4$. This induces $a(a^\#)^* = a^\#a^+a^3(a^\#)^* = a^\#a^+a^4 = a^2$. Hence $a \in R^{SEP}$;

(2) If $y = a^\#$, then we have $xa^\#(a^\#)^* = x(aa^\#)^*a^\#a$.

⑨ If $x = a$, then $aa^\#(a^\#)^* = a(aa^\#)^*a^\#a$. Multiplying the equality on the right by aa^+, one obtains

$$a(aa^\#)^*a^\#a = a(aa^\#)^*aa^+ = a(aa^\#)^*.$$

Again multiply the last equality on the left by aa^+a^+, one has $aa^\# = aa^+$. Hence $a \in R^{EP}$, it follows that $(a^\#)^* = a^+a(a^\#)^* = aa^\#(a^\#)^* = a(aa^\#)^*a^\#a = aaa^+a^\#a = a$. Hence $a \in R^{SEP}$;

⑩ If $x = a^\#$, then $a^\#a^\#(a^\#)^* = a^\#(aa^\#)^*a^\#a$. Multiplying the equality on the left by a^2, one has $aa^\#(a^\#)^* = a(aa^\#)^*a^\#a$. Hence $a \in R^{SEP}$ by ⑨;

⑪ If $x = a^+$, then $a^+a^\#(a^\#)^* = a^+(aa^\#)^*a^\#a = a^+a^\#a$. Multiplying the equality on the left by a^3, one gets $a(a^\#)^* = a^2$. Hence $a \in R^{SEP}$;

⑫ If $x = a^*$, then $a^*a^\#(a^\#)^* = a^*(aa^\#)^*a^\#a = a^*a^\#a$. Multiplying the equality on the left by $a^2(a^+)^*$, one has $a(a^\#)^* = a^2$. Hence $a \in R^{SEP}$;

⑬ If $x = (a^+)^*$, then $(a^+)^*a^\#(a^\#)^* = (a^+)^*(aa^\#)^*a^\#a$. Multiplying the equality on the left by aa^*, one obtains $aa^\#(a^\#)^* = a(aa^\#)^*a^\#a$. Hence $a \in R^{SEP}$ by ⑨;

⑭ If $x = (a^\#)^*$, then $(a^\#)^*a^\#(a^\#)^* = (a^\#)^*(aa^\#)^*a^\#a$. Multiplying the equality on the left by a^*a^*, one gets $a^*a^\#(a^\#)^* = a^*(aa^\#)^*a^\#a = a^*a^\#a$. Hence $a \in R^{SEP}$ by ⑫;

⑮ If $x = (a^+)^\#$, then $(aa^\#)^*a(aa^\#)^*a^\#(a^\#)^* = (aa^\#)^*a(aa^\#)^*(aa^\#)^*a^\#a$. Multiplying the equality on the left by a^*a^+, one has $a^*a^\#(a^\#)^* = a^*a^\#a$. Hence $a \in R^{SEP}$ by ⑫;

⑯ If $x = (a^\#)^+$, then $a^+a^3a^+a^\#(a^\#)^* = a^+a^3a^+(aa^\#)^*a^\#a$, e.g., $(a^\#)^* = a^+a^2$. This gives $a(a^\#)^* = a^2$. Hence $a \in R^{SEP}$;

(3) If $y = a^+$, then we have $xa^+(a^\#)^* = x(aa^\#)^*a^+a$, that is, $xa^+(a^\#)^* = xa^+a$.

⑰ If $x = a$, then $aa^+(a^\#)^* = aa^+a = a$. Hence $a \in R^{SEP}$ by Lemma 3.1;

⑱ If $x = a^\#$, then $a^\#a^+(a^\#)^* = a^\#a^+a = a^\#$. Multiplying the equality on the left by a^2, one gets $aa^+(a^\#)^* = a$. Hence $a \in R^{SEP}$ by Lemma 3.1;

⑲ If $x = a^+$, then $a^+a^+(a^\#)^* = a^+a^+a$. By [17, Lemma 2.10], we get $a^+(a^\#)^* = a^+a$. Multiplying the equality on the left by a, one obtains $aa^+(a^\#)^* = a$. Hence $a \in R^{SEP}$ by Lemma 3.1;

⑳ If $x = a^*$, then $a^*a^+(a^\#)^* = a^*a^+a$. Multiplying the equality on the left by $(a^\#)^*$, one has $a^+(a^\#)^* = a^+a$. Hence $a \in R^{SEP}$ by ⑲;

㉑ If $x = (a^+)^*$, then $(a^+)^*a^+(a^\#)^* = (a^+)^*a^+a = (a^+)^*$. Multiplying the equality on the left by aa^*, one gets $aa^+(a^\#)^* = a$. Hence $a \in R^{SEP}$ by Lemma 3.1;

㉒ If $x = (a^\#)^*$, then $(a^\#)^*a^+(a^\#)^* = (a^\#)^*a^+a$. Multiplying the equality on the left by aa^*, one obtains $aa^+(a^\#)^* = a$. Hence $a \in R^{SEP}$ by Lemma 3.1;

㉓ If $x = (a^+)^\#$, then $(aa^\#)^*a(aa^\#)^*a^+(a^\#)^* = (aa^\#)^*a(aa^\#)^*a^+a$. Multiplying the equality on the left by a^*a^+, one gets $a^*a^+(a^\#)^* = a^*a^+a$. Hence $a \in R^{SEP}$ by ⑳;

㉔ If $x = (a^\#)^+$, then $a^+a^3a^+a^+(a^\#)^* = a^+a^3a^+a^+a$. Multiplying the equality on the left by $a^+a^\#a^\#a$, one has $a^+a^+(a^\#)^* = a^+a^+a$. Hence $a \in R^{SEP}$ by ⑲;

(4) If $y = a^*$, then we have $xa^*(a^\#)^* = x(aa^\#)^*a^*a$, e.g., $xa^*(a^\#)^* = xa^*a$. Multiplying the equation on the right by a^+, one obtains $xa^+ = xa^*$. Hence $a \in R^{PI}$ by [18, Lemma 2.2]. It follows that $y = a^* = a^+$. Hence $a \in R^{SEP}$ by (3).

(5) If $y = (a^+)^*$, then we have $x(a^+)^*(a^\#)^* = x(aa^\#)^*(a^+)^*a$.

㉕ If $x = a$, then $a(a^+)^*(a^\#)^* = a(aa^\#)^*(a^+)^*a$. Multiplying the equality on the right by a^+a, one yields

$$a(a^+)^*(a^\#)^* = a(a^+)^*(a^\#)^*a^+a.$$

Again multiply the last equality on the left by $a^*a^*a^\#$, one has

$$a^*(a^\#)^* = a^+a.$$

Applying the involution to the last equality, we get $a^\#a = a^+a$. Hence $a \in R^{EP}$, it follows that $a(a^+)^*(a^\#)^* = a(a^+)^*a$. Taking involution of the above equality, we obtain

$$a^\#a^+a^* = a^*a^+a^*.$$

Multiplying the equality on the right by $(a^\#)^*a$, one gets $a^*a^+a = a^\# = a^+$. Hence $a \in R^{SEP}$ by [11, Theorem 1.5.3];

㉖ If $x = a^\#$, then $a^\#(a^+)^*(a^\#)^* = a^\#(aa^\#)^*(a^+)^*a$. Multiplying the equality on the left by a^2, one has $a(a^+)^*(a^\#)^* = a(aa^\#)^*(a^+)^*a$. Hence $a \in R^{SEP}$ by ㉕;

㉗ If $x = a^+$, then $a^+(a^+)^*(a^\#)^* = a^+(aa^\#)^*(a^+)^*a$. Multiplying the equality on the right by a^+a, one yields

$$a^+(a^+)^*(a^\#)^* = a^+(a^+)^*(a^\#)^*a^+a.$$

Again multiply the last equality on the left by a, one has

$$(a^+)^*(a^\#)^* = (a^+)^*(a^\#)^*a^+a.$$

Applying the involution to the last equality, we get $a^\#a^+ = a^+aa^\#a^+$. Multiplying the equality on the right by a^2, one gets $a^\#a = a^+a$. Hence $a \in R^{EP}$. It follows that $x = a^+ = a^\#$. Hence $a \in R^{SEP}$ by ㉖;

㉘ If $x = a^*$, then $a^*(a^+)^*(a^\#)^* = a^*(aa^\#)^*(a^+)^*a$, e.g., $(a^\#)^* = a^+aa$. Multiplying the equality on the left by a, one yields $a(a^\#)^* = a^2$. Hence $a \in R^{SEP}$;

㉙ If $x = (a^+)^*$, then $(a^+)^*(a^+)^*(a^\#)^* = (a^+)^*(aa^\#)^*(a^+)^*a$. Multiplying the equality on the left by aa^*, one has $a(a^+)^*(a^\#)^* = a(aa^\#)^*(a^+)^*a$. Hence $a \in R^{SEP}$ by ㉕;

㉚ If $x = (a^\#)^*$, then $(a^\#)^*(a^+)^*(a^\#)^* = (a^\#)^*(aa^\#)^*(a^+)^*a$. Multiplying the equality on the left by a^+a^*, one obtains $a^+(a^+)^*(a^\#)^* = a^+(aa^\#)^*(a^+)^*a$. Hence $a \in R^{SEP}$ by ㉗;

㉛ If $x = (a^+)^\#$, then $(aa^\#)^*a(aa^\#)^*(a^+)^*(a^\#)^* = (aa^\#)^*a(aa^\#)^*(aa^\#)^*(a^+)^*a$. Multiplying the equality on the left by a^*a^+, one gets $(a^\#)^* = a^+aa$. Hence $a \in R^{SEP}$ by ㉘;

㉜ If $x = (a^\#)^+$, then $a^+a^3a^+(a^+)^*(a^\#)^* = a^+a^3a^+(aa^\#)^*(a^+)^*a$, e.g.,

$$a^+a^2(a^+)^*(a^\#)^* = a^+a^2(a^+)^*a.$$

Multiplying the equality on the left by $a^+a^\#a^\#a$, one has $a^+(a^+)^*(a^\#)^* = a^+(aa^\#)^*(a^+)^*a$. Hence $a \in R^{SEP}$ by ㉗;

(6) If $y = (a^\#)^*$, then we have $x(a^\#)^*(a^\#)^* = x(aa^\#)^*(a^\#)^*a$. Multiplying the equality on the right by $1 - aa^+$, one yields

$$x(a^\#)^*a(1 - aa^+) = 0.$$

By Lemma 8.1, we get

$$(a^\#)^*a(1 - aa^+) = 0.$$

Again multiply the last equality on the left by aa^+a^*, one has $a = a^2a^+$. Hence $a \in R^{EP}$. It follows that $y = (a^\#)^* = (a^+)^*$. Hence $a \in R^{SEP}$ by (5).

(7) If $y = (a^+)^\#$, then we have $x(a^+)^\#(a^\#)^* = x(aa^\#)^*(a^+)^\# a$, e.g.,

$$x(aa^\#)^*a(aa^\#)^*(a^\#)^* = x(aa^\#)^*(aa^\#)^*a(aa^\#)^*a.$$

Multiplying the equality on the right by $(1 - a^+a)$, one obtains

$$x(aa^\#)^*a(a^\#)^*(1 - a^+a) = 0.$$

By Lemma 8.1, one has $(aa^\#)^*a(a^\#)^*(1 - a^+a) = 0$. Again multiply the last equality on the left by $a^+a^*a^+$, one yields $a^+ = a^+a^+a$. Hence $a \in R^{EP}$. This infers $y = (a^+)^\# = (a^\#)^\# = a$. Hence $a \in R^{SEP}$ by (1).

(8) If $y = (a^\#)^+$, then we have $xa^+a^3a^+(a^\#)^* = x(aa^\#)^*a^+a^3a^+a$, that is

$$xa^+a^3a^+(a^\#)^* = xa^+a^3.$$

Multiplying the equality on the right by $1 - aa^+$, one obtains

$$xa^+a^3(1 - aa^+) = 0.$$

Noting that $a^+ = (a^\#)^*a^*a^+$. By Lemma 8.1, one gets

$$a^+a^3(1 - aa^+) = 0.$$

Again multiply the last equality on the left by $a^\#$, one has $a = a^2a^+$. Hence $a \in R^{EP}$. This infers $y = (a^\#)^+ = (a^+)^+ = a$. Hence $a \in R^{SEP}$ by (1). □

It is well known that $a \in R^{SEP}$ if and only if $a^* \in R^{SEP}$. Hence instead a in Eq.(8.1) by a^*, one obtains the following equation and theorem.

$$xya^\# = xaa^\#ya^*. \tag{8.2}$$

Theorem 8.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(8.2) has at least one solution in ρ_a^2.*

Now we establish the following equation.

$$xya^+ + a^\# = xaa^\#ya^* + a^+. \tag{8.3}$$

Theorem 8.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(8.3) has at least one solution in ρ_a^2.*

Proof. (⇒) Assume that $a \in R^{SEP}$. Then $a^\# = a^+$, this infers Eq.(8.1) has the same solution as Eq.(8.3). By Theorem 8.1, we are done.

(⇐) Assume that Eq.(8.3) has at least one solution in ρ_a^2. Let $(x, y) = (x_0, y_0)$ be any solution of Eq.(8.3). Then we have

$$x_0y_0a^+ + a^\# = x_0aa^\#y_0a^* + a^+.$$

Multiplying the equality on the right by $1 - aa^+$, one gets $a^\#(1 - aa^+) = 0$. This gives $a^\# = a^\#aa^+$. Hence $a \in R^{EP}$ by [11, Theorem 1.2.1]. Thus Eq.(8.1) has the same solution as Eq.(8.3). By Theorem 8.1, $a \in R^{SEP}$. □

9. The consistency of equations and SEP elements

Consider the following equation

$$ax(a^{\#})^{*} = a. \tag{9.1}$$

Theorem 9.1. *Let $a \in R^{\#} \cap R^{+}$. Then Eq.(9.1) is consistent if and only if $a \in R^{EP}$. In this case, the general solution of Eq.(9.1) is given by*

$$x = a^{*} + u - a^{+}aua^{+}a, \text{ where } u \in R. \tag{9.2}$$

Proof. (⇒) Assume that Eq.(9.1) is consistent. Then $a = a^2a^{+}$, it follows that $a \in R^{EP}$.

(⇐) If $a \in R^{EP}$, then $aa^{*}(a^{\#})^{*} = a(aa^{\#})^{*} = aaa^{\#} = a$ by [1, Theorem 1.3.1]. Hence $x = a^{*}$ is a solution, so Eq.(9.1) is consistent. Now if Eq.(9.1) is consistent, then $x = a^{*}$ is a solution and $a \in R^{EP}$. Hence the formula (9.2) is the solution of Eq.(9.1). Let $x = x_0$ be any solution of Eq.(9.1). Then $ax_0(a^{\#})^{*} = a$. Clearly

$$a^{+}ax_0aa^{+} = a^{+}(ax_0(a^{\#})^{*})a^{*}a^{+}a = a^{+}aa^{*}a^{+}a = a^{*}a^{+}a = a^{*}.$$

Hence $x_0 = a^{*} + x_0 - a^{+}ax_0a^{+}a$. Thus the general solution of Eq.(9.1) is given by formula (9.2). □

Theorem 9.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if Eq.(9.1) is consistent and the general solution is given by*

$$x = a^{+} + u - a^{+}aua^{+}a, \text{ where } u \in R. \tag{9.3}$$

Proof. (⇒) If $a \in R^{SEP}$, then $a \in R^{EP}$ and $a^{+} = a^{*}$. It follows that the formula (9.2) is equal to the formula (9.3). By Theorem 9.1, we get Eq.(9.1) is consistent and the general solution is given by the formula (9.3).

(⇐) From the assumption, we get $a(a^{+} + u - a^{+}aua^{+}a)(a^{\#})^{*} = a$, e.g.,

$$aa^{+}(a^{\#})^{*} + au(a^{\#})^{*} - aua^{+}a(a^{\#})^{*} = a \text{ for all } a \in R.$$

This gives $aa^{+}(a^{\#})^{*} = a$. Hence $a \in R^{SEP}$ by Lemma 3.1. □

It is easy to show that the general solution of the following equation is given by (9.3).

$$a(aa^{\#})^{*}axa^{+}a = a. \tag{9.4}$$

Hence we have the following corollary.

Corollary 9.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if Eq.(9.1) has the same solution as Eq.(9.4).*

Now we change Eq.(9.4) as follows.

$$a(aa^{\#})^{*}axa^{+}a = (a^{\#})^{*}. \tag{9.5}$$

Lemma 9.4. *Let $a \in R^{\#} \cap R^{+}$. Then Eq.(9.5) is consistent if and only if $a \in R^{EP}$.*

In this case, the general solution of Eq.(9.5) is given by

$$x = a^{+}a^{+}(a^{\#})^{*} + u - a^{+}aua^{+}a, \text{ where } u \in R. \tag{9.6}$$

Theorem 9.5. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if Eq.(9.5) is consistent and the general solution of Eq.(9.5) is given by $x = a^{+} + u - a^{+}aua^{+}a$, where $u \in R$.*

Proof. (⇒) If $a \in R^{SEP}$, then $a^{\#} = a^{+} = a^{*}$, this gives $a^{+}a^{+}(a^{\#})^{*} = a^{+}$. By Lemma 9.4, we get Eq.(9.5) is consistent and the general solution of Eq.(9.5) is given by $x = a^{+} + u - a^{+}aua^{+}a$, where $u \in R$.

(⇐) From the assumption, we get $a(aa^{\#})^{*}a(a^{+} + u - a^{+}aua^{+}a)a^{+}a = (a^{\#})^{*}$, this gives $a = (a^{\#})^{*}$. Hence $a \in R^{SEP}$ by [11, Theorem 1.5.3]. □

10. Constructions of group invertible elements and Moore Penrose invertible elements

We need the following lemma which proof is routine.

Lemma 10.1. *Let $a \in R^{\#} \cap R^{+}$. Then*
(1) $x(aa^{\#})^{*}x^{+} = aa^{+}$, where $x \in \tau_a$;
(2) $x(aa^{\#})^{*}x^{+} = a^{+}a$, where $x \in \theta_a$.

Noting that $a \in R^{EP}$ if and only if $a \in R^{+}$ and $aa^{+} = a^{+}a$ and $a \in R^{SEP}$ if and only if $a \in R^{\#} \cap R^{+}$ and $aa^{+} = a^{*}a$ or $a^{+}a = aa^{*}$. Hence the following theorem is an immediate corollary of Lemma 10.1.

Theorem 10.2. *Let $a \in R^{\#} \cap R^{+}$. Then the following conditions are equivalent:*
(1) $a \in R^{EP}$;
(2) $x(aa^{\#})^{*}x^{+} = a^{+}a$ for some $x \in \tau_a$;
(3) $x(aa^{\#})^{*}x^{+} = aa^{+}$ for some $x \in \theta_a$;
(4) $x(aa^{\#})^{*}x^{+} = a^{*}a$ for some $x \in \tau_a$;
(5) $x(aa^{\#})^{*}x^{+} = aa^{*}$ for some $x \in \theta_a$.

Theorem 10.3. *Let $a \in R^{\#} \cap R^{+}$. Then*
(1) $(xa^{+}(a^{\#})^{*})^{\#} = \begin{cases} aa^{+}a^{*}a(aa^{\#})^{*}x^{+}, x \in \tau_a \\ a^{*}ax^{\#} \quad , x \in \theta_a \end{cases}$;
(2) $((a^{\#})^{+})^{\#} = (aa^{\#})^{*}a^{\#}(aa^{\#})^{*}$;
(3) $(xa^{+}(a^{\#})^{*})(xa^{+}(a^{\#})^{*})^{\#} = \begin{cases} aa^{+} \quad , x \in \tau_a \\ (aa^{\#})^{*}, x \in \theta_a \end{cases}$;
(4) $a \in R^{SEP}$ if and only if $(xa^{+}(a^{\#})^{*})(xa^{+}(a^{\#})^{*})^{\#} = a^{*}a$ for some $x \in \tau_a$;
(5) $a \in R^{SEP}$ if and only if $(xa^{+}(a^{\#})^{*})(xa^{+}(a^{\#})^{*})^{\#} = aa^{*}$ for some $x \in \theta_a$.

Proof. (1) If $x \in \tau_a$, then we have $aa^{+}x = x$, this gives

$$(xa^{+}(a^{\#})^{*})(aa^{+}a^{*}a(aa^{\#})^{*}x^{+}) = x(aa^{\#})^{*}x^{+} = aa^{+};$$

$$(aa^{+}a^{*}a(aa^{\#})^{*}x^{+})(xa^{+}(a^{\#})^{*}) = aa^{+}a^{*}a(aa^{\#})^{*}a^{+}aa^{+}(a^{\#})^{*} = aa^{+};$$

$$(xa^{+}(a^{\#})^{*})(aa^{+}a^{*}a(aa^{\#})^{*}x^{+})(xa^{+}(a^{\#})^{*}) = aa^{+}xa^{+}(a^{\#})^{*} = xa^{+}(a^{\#})^{*};$$

and

$$(aa^{+}a^{*}a(aa^{\#})^{*}x^{+})(xa^{+}(a^{\#})^{*})(aa^{+}a^{*}a(aa^{\#})^{*}x^{+}) = aa^{+}(aa^{+}a^{*}a(aa^{\#})^{*}x^{+} = aa^{+}a^{*}a(aa^{\#})^{*}x^{+}.$$

Hence $(xa^{+}(a^{\#})^{*})^{\#} = aa^{+}a^{*}a(aa^{\#})^{*}x^{+}$;

If $x \in \theta_a$, then we have $(aa^{\#})^{*}x = x$, this gives

$$(xa^{+}(a^{\#})^{*})(a^{*}ax^{\#}) = xa^{+}ax^{\#} = (aa^{\#})^{*};$$

$$(a^{*}ax^{\#})(xa^{+}(a^{\#})^{*}) = a^{*}a(aa^{\#})^{*}a^{+}(a^{\#})^{*} = (aa^{\#})^{*};$$

$$(xa^{+}(a^{\#})^{*})(a^{*}ax^{\#})(xa^{+}(a^{\#})^{*}) = (aa^{\#})^{*}xa^{+}(a^{\#})^{*} = xa^{+}(a^{\#})^{*};$$

and

$$(a^{*}ax^{\#})(xa^{+}(a^{\#})^{*})(a^{*}ax^{\#}) = (aa^{\#})^{*}(a^{*}ax^{\#}) = a^{*}ax^{\#}.$$

Hence $(xa^{+}(a^{\#})^{*})^{\#} = a^{*}ax^{\#}$;

(2) Noting that $(a^{\#})^{+} = a^{+}a^{3}a^{+}$. Then

$$(a^{\#})^{+}((aa^{\#})^{*}a^{\#}(aa^{\#})^{*}) = (a^{+}a^{3}a^{+})((aa^{\#})^{*}a^{\#}(aa^{\#})^{*}) = (aa^{\#})^{*};$$

$$((aa^{\#})^{*}a^{\#}(aa^{\#})^{*})(a^{\#})^{+} = ((aa^{\#})^{*}a^{\#}(aa^{\#})^{*})(a^{+}a^{3}a^{+}) = (aa^{\#})^{*};$$

$$(a^{\#})^{+}((aa^{\#})^{*}a^{\#}(aa^{\#})^{*})(a^{\#})^{+} = (aa^{\#})^{*}(a^{+}a^{3}a^{+}) = a^{+}a^{3}a^{+};$$

and

$$((aa^{\#})^*a^{\#}(aa^{\#})^*)(a^{\#})^+((aa^{\#})^*a^{\#}(aa^{\#})^*) = (aa^{\#})^*((aa^{\#})^*a^{\#}(aa^{\#})^*) = (aa^{\#})^*a^{\#}(aa^{\#})^*.$$

Hence $((a^{\#})^+)^{\#} = (aa^{\#})^*a^{\#}(aa^{\#})^*$;

(3) It is an immediate result of (1).

Noting that $a \in R^{SEP}$ if and only if $aa^+ = a^*a$ if and only if $(aa^{\#})^* = aa^*$. Hence (4) and (5) follows from (3). □

Theorem 10.4. *Let $a \in R^{\#} \cap R^+$. Then the following conditions are equivalent:*
(1) $a \in R^{SEP}$;
(2) $(xa^+(a^{\#})^*)(xa^+(a^{\#})^*)^+ = a^*a$ for some $x \in \tau_a$;
(3) $(xa^+(a^{\#})^*)(xa^+(a^{\#})^*)^+ = aa^*$ for some $x \in \theta_a$.

Proof. First, we have $(xa^+(a^{\#})^*)^+ = aa^+a^*a(aa^{\#})^*x^+$. Hence

$$(xa^+(a^{\#})^*)(xa^+(a^{\#})^*)^+ = x(aa^{\#})^*x^+.$$

Next, using Theorem 10.2, we can complete the proof. □

Similar to Theorem 10.3, we can show the following theorem.

Theorem 10.5. *Let $a \in R^{\#} \cap R^+$. Then*
(1) $(aa^+x)^+ = x^+(aa^{\#})^*$, where $x \in \rho_a$;
(2) $(aa^+x)^{\#} = \begin{cases} x^{\#}aa^{\#} & ,x \in \tau_a \\ x^+(aa^{\#})^*, & x \in \theta_a \end{cases}$.

The following theorem follows from Theorem 6.1, Theorem 10.3, Theorem 10.4 and Theorem 10.5.

Theorem 10.6. *Let $a \in R^{\#} \cap R^+$. Then the following conditions are equivalent:*
(1) $a \in R^{SEP}$;
(2) $aa^+a^*a(aa^{\#})^*x^+ = x^+(aa^{\#})^*$ for some $x \in \rho_a$;
(3) $aa^+a^*a(aa^{\#})^*x^+ = x^{\#}aa^{\#}$ for some $x \in \tau_a$;
(4) $a^*ax^{\#} = x^+(aa^{\#})^*$ for some $x \in \theta_a$.

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