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Certain aspects of rough $\mathcal{I}$-statistical convergence in probabilistic n-normed space

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Abstract. The main aim of this investigation is to introduce rough $\mathcal{I}$-statistical convergence in probabilistic n-normed spaces (briefly Pr-n-spaces). We establish some results on rough $\mathcal{I}$-statistical convergence and also we introduce the notion of rough $\mathcal{I}$-statistical limit set in Pr-n-spaces and discuss some topological aspects on this set. Moreover, we define rough $\mathcal{I}$-lacunary statistical convergent, rough lacunary $\mathcal{I}$-convergent, rough lacunary $\mathcal{I}$-Cauchy and rough lacunary $\mathcal{I}^*$-convergent sequences in Pr-n-spaces. We obtain several significant results related to these notions.

1. Introduction

The concept of probabilistic metric spaces was put forward by Menger [15]. The idea of Menger was to utilize distribution function instead of non-negative real numbers as values of the metric. In the early 1960s, A. N. Šerstnev [27] generalized usual normed spaces and investigated probabilistic normed spaces (briefly RNS), and put forward to questions regarding the completeness and the completion of RNS, then examined the problem of best approximation in RNS. The theory of PNS had gone through significant advancements before Alsina et al. [1] investigated a new and wider recognized definition of PN spaces. The theory of PNS supplies an significant method of generalising the conclusions of normed linear spaces. It has beneficial implementation, in different fields such as continuity features [2], topological spaces [6], boundedness [9], convergence of random variables [10] etc. A comprehensive study in this direction can be examined from the book by Guillen and Harikrishnan [11]. The theory of 2-norm and n-norm on a linear space was presented by Gahler [7, 8] which was later studied by Tripathy and Borgohain [28], Tripathy and Dutta [29] and many others. The concepts of statistical convergence $\mathcal{I}$-convergence were defined at the initial stage by Fast [5] and Kostyrko et al. [12], respectively. Later on it was further studied by several authors (see [3, 4, 13, 14, 16–26, 30–34]).

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2. Preliminaries

Now, we recall some notations and definitions which is utilized in this study.

Definition 2.1. *A probabilistic n-normed linear space or in short Pr-n-space is a triplet* $(Y, \vartheta, *)$ *where* Y *is a real linear space of dimension greater than one,* ϑ *is a mapping from* Y^n *into* D *and* $*$*, a continuous t-norm supplying the following conditions for all* $s_1, s_2, ..., s_{n-1} \in Y$ *and* $k, l > 0$;
(i) $\vartheta((s_1, s_2, ..., s_n), l) = 1$ *iff* $s_1, s_2, ..., s_n$ *are linearly dependent,*
(ii) $\vartheta((s_1, s_2, ..., s_n), l)$ *is invariant under any permutation of* $s_1, s_2, ..., s_n$,
(iii) $\vartheta((s_1, s_2, ..., \alpha s_n), l) = \vartheta\left((s_1, s_2, ..., s_n), \frac{l}{|\alpha|}\right)$ *if* $\alpha \neq 0$, $\alpha \in \mathbb{R}$,
(iv) $\vartheta((s_1, s_2, ..., s_n + s_n'), k + l) \geq \vartheta((s_1, s_2, ..., s_n), k) * \vartheta((s_1, s_2, ..., s_n'), l)$.

Definition 2.2. *Assume* $(Y, \vartheta, *)$ *be a Pr-n-space. A sequence* $w = (w_p)$ *in* Y *is said to be convergent to* $\sigma \in Y$ *w.r.t. the probabilistic n-norm* ϑ^n *provided that for each* $\rho > 0$, $\eta \in (0, 1)$, $s_1, s_2, ..., s_{n-1} \in Y$*, there is* $p_0 \in \mathbb{N}$ *such that*

$$\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), \rho\right) > 1 - \eta$$

for all $p \geq p_0$*. Symbolically we write,* $\vartheta^n - \lim_{p\to\infty} w_p = \sigma$ *or* $w_p \stackrel{\vartheta^n}{\to} \sigma$.

Definition 2.3. *A sequence* $w = (w_p)$ *in* Y *is named to Cauchy sequence w.r.t. the probabilistic n-norm* ϑ^n *provided that given* $\rho > 0$, $\eta \in (0, 1)$, $s_1, s_2, ..., s_{n-1} \in Y$*, there is* $p_0 \in \mathbb{N}$ *such that*

$$\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - w_m\right), \rho\right) > 1 - \eta$$

for all $p, m \geq p_0$.

Definition 2.4. *A sequence* $w = (w_p)$ *in* Y *is named to be* $\mathcal{I}$*-convergent to* $\sigma \in Y$ *w.r.t. the probabilistic n-norm* ϑ^n *provided that for each* $\rho > 0$, $\eta \in (0, 1)$, $s_1, s_2, ..., s_{n-1} \in Y$*, the set*

$$\left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), \rho\right) \leq 1 - \eta\right\} \in \mathcal{I}.$$

Symbolically we write, $\mathcal{I}_{\vartheta^n} - \lim_{p\to\infty} w_p = \sigma$ *or* $w_p \stackrel{\mathcal{I}_{\vartheta^n}}{\to} \sigma$.

Definition 2.5. *By a lacunary sequence we mean an increasing integer sequence* $\theta = (k_u)$, $u = 1, 2, ...$ *such that* $k_0 = 0$ *and* $h_u = k_u - k_{u-1} \to \infty$ *as* $u \to \infty$*. The intervals determined by* θ *will be demonstrated by* $I_u = (k_{u-1}, k_u]$ *and the ratio* $\frac{k_u}{k_{u-1}}$ *will be abbreviated by* q_u.

3. Main Results

At the beginning, we examine the conceptions rough convergence and rough $\mathcal{I}$-statistical convergence in Pr-n-space as follows:

Definition 3.1. *Assume* $(Y, \vartheta, *)$ *be a Pr-n-space. A sequence* $w = (w_p)$ *in* Y *is said to be rough convergent to* $\sigma \in Y$ *w.r.t. the probabilistic n-norm* ϑ^n *provided that for each* $\rho > 0$, $\eta \in (0, 1)$, $s_1, s_2, ..., s_{n-1} \in Y$ *and some* $r > 0$ *there exists* $p_0 \in \mathbb{N}$ *such that*

$$\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) > 1 - \eta$$

for all $p \geq p_0$*. Symbolically we write* $r_{\vartheta^n} - \lim_{p\to\infty} w_p = \sigma$ *or* $w_p \stackrel{r_{\vartheta^n}}{\to} \sigma$.

Definition 3.2. *Assume $(Y, \vartheta, *)$ be a Pr-n-space. A sequence $w = \left(w_p\right)$ in Y is said to be rough $\mathcal{I}$-statistically convergent to $\sigma \in Y$ w.r.t. the probabilistic n-norm ϑ^n provided that for each ρ, $\delta > 0$, $\eta \in (0, 1)$, $s_1, s_2, ..., s_{n-1} \in Y$ and some $r > 0$*

$$\left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \leq t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) \leq 1 - \eta\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $r - S^{\vartheta^n}(\mathcal{I}) - \lim_{p\to\infty} w_p = \sigma$ or $w_p \overset{r-S^{\vartheta^n}(\mathcal{I})}{\to} \sigma$.

Remark 3.3. *For the case $r = 0$, the concept rough $\mathcal{I}$-statistical convergence w.r.t. the probabilistic n-norm ϑ^n agrees with the $\mathcal{I}$-statistical convergence w.r.t. the probabilistic n-norm ϑ^n.*

From the above definition it is clear that the r-$\mathcal{I}$-statistical limit of a sequence w.r.t. the probabilistic n-norm ϑ^n is not unique. So, we consider the set of r-$\mathcal{I}$-statistical limits of a sequence w and we use the notation $S^{\vartheta^n}(\mathcal{I}) - LIM^r_w$ to indicate the set of all r-$\mathcal{I}$-statistical limits of a sequence w. We say that a sequence w is r-$\mathcal{I}$-statistically convergent when $S^{\vartheta^n}(\mathcal{I}) - LIM^r_w \neq \emptyset$. So, we examine $r - S^{\vartheta^n}(\mathcal{I})$-limit set of a sequence $w = \left(w_p\right)$ as

$$S^{\vartheta^n}(\mathcal{I}) - LIM^r_w = \left\{\sigma : w_p \overset{r-S^{\vartheta^n}(\mathcal{I})}{\to} \sigma\right\}.$$

The sequence $w = \left(w_p\right)$ is r_{ϑ^n}-convergent when $LIM^{r_{\vartheta^n}}_w \neq \emptyset$ where $LIM^{r_{\vartheta^n}}_w = \left\{\sigma^* \in Y : w_p \overset{r_{\vartheta^n}}{\to} \sigma\right\}$. For unbounded sequence $LIM^{r_{\vartheta^n}}_w$ is always empty.

Definition 3.4. *Assume $(Y, \vartheta, *)$ be a Pr-n-space. A sequence $w = \left(w_p\right)$ in Y is said to be $\mathcal{I}$-statistically bounded w.r.t. the probabilistic n-norm ϑ^n provided that for all $\delta > 0$, $\eta \in (0, 1)$ there is a real number $T > 0$ such that*

$$A = \left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \leq t : \vartheta\left(s_1, s_2, ..., s_{n-1}, w_p, T\right) \leq 1 - \eta\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In the light of above definitions, we obtain the following significant results on rough $\mathcal{I}$-statistical convergence in Pr-n-space.

Theorem 3.5. *A sequence $w = \left(w_p\right)$ is $\mathcal{I}$-statistically bounded in a Pr-n-space $(Y, \vartheta, *)$ iff $S^{\vartheta^n}(\mathcal{I}) - LIM^r w \neq \emptyset$ for some $r > 0$.*

Proof. Necessity:
Assume that the sequence $w = \left(w_p\right)$ is $\mathcal{I}$-statistically bounded in a Pr-n-space $(Y, \vartheta, *)$. Then, for all $\delta > 0$, $\eta \in (0, 1)$ and $s_1, s_2, ..., s_{n-1} \in Y$, there is a real number $T > 0$ such that

$$A = \left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \leq t : \vartheta\left(s_1, s_2, ..., s_{n-1}, w_p, T\right) \leq 1 - \eta\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

For $t \in A^c$ we get $\vartheta\left(s_1, s_2, ..., s_{n-1}, w_p, M\right) > 1 - \eta$. Also

$$\begin{aligned} \vartheta\left(s_1, s_2, ..., s_{n-1}, w_p, r + T\right) &\geq \vartheta\left(s_1, s_2, ..., s_{n-1}, 0, r\right) * \vartheta\left(s_1, s_2, ..., s_{n-1}, w_p, T\right) \\ &> 1 * (1 - \eta) = 1 - \eta. \end{aligned}$$

So, $0 \in S^{\vartheta^n}(\mathcal{I}) - LIM^r_w$. As a result, $S^{\vartheta^n}(\mathcal{I}) - LIM^r_w \neq \emptyset$.

Sufficiency:
Assume that $S^{\vartheta^n}(\mathcal{I}) - LIM^r_w \neq \emptyset$ for some $r > 0$. Then, there is $\sigma \in Y$ such that $\sigma \in S^{\vartheta^n}(\mathcal{I}) - LIM^r_w$. For $\rho, \delta > 0$, $\eta \in (0, 1)$, $s_1, s_2, ..., s_{n-1} \in Y$ and some $r > 0$ we get,

$$\left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \leq t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) \leq 1 - \eta\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

As a result, almost all w_p's are included in some ball with center σ which means that sequence $w = \left(w_p\right)$ is $\mathcal{I}$-statistically bounded in a Pr-n-space $(Y, \vartheta, *)$. □

Next, we put forward the the algebraic characterization of rough $\mathcal{I}$-statistically convergent sequences in Pr-n-spaces $(Y, \vartheta, *)$.

Theorem 3.6. *Let $w = (w_p)$ and $q = (q_p)$ be two sequences in Pr-n-spaces $(Y, \vartheta, *)$. Then, for some $r > 0$, the followings hold:*
(i) If $w_p \overset{r-S^{\vartheta^n}(\mathcal{I})}{\rightarrow} \sigma$ and $\alpha \in \mathbb{N}$, then $\alpha w_p \overset{r-S^{\vartheta^n}(\mathcal{I})}{\rightarrow} \alpha\sigma$,
(ii) If $w_p \overset{r-S^{\vartheta^n}(\mathcal{I})}{\rightarrow} \sigma_1$ and $q_p \overset{r-S^{\vartheta^n}(\mathcal{I})}{\rightarrow} \sigma_2$ then $(w_p + q_p) \overset{r-S^{\vartheta^n}(\mathcal{I})}{\rightarrow} \sigma_1 + \sigma_2$.

Proof. Proof of above results are obvious so we are omitting them. □

Theorem 3.7. *Assume $(Y, \vartheta, *)$ be a Pr-n- space. If a sequence $w = (w_p)$ be $\mathcal{I}$-statistical convergent w.r.t. the probabilistic n-norm ϑ^n, then $r - S^{\vartheta^n}(\mathcal{I})$-limit is unique.*

Proof. Let us presume that $r - S^{\vartheta^n}(\mathcal{I}) - \lim_{p\to\infty} w_p = \sigma_1$ and $r - S^{\vartheta^n}(\mathcal{I}) - \lim_{p\to\infty} w_p = \sigma_2$. For given $\lambda > 0$, select $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > 1 - \lambda$. Then, for any $\rho, \delta > 0$, $s_1, s_2, ..., s_{n-1} \in Y$ and for some $r > 0$, we determine the following sets

$$K_{\vartheta^n,1}(\rho, \eta, \delta) = \left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \le t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma_1\right), r + \rho\right) > 1 - \eta\right\}\right| < \delta\right\}$$

and

$$K_{\vartheta^n,2}(\rho, \eta, \delta) = \left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \le t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma_2\right), r + \rho\right) > 1 - \eta\right\}\right| < \delta\right\}.$$

Since $r - S^{\vartheta^n}(\mathcal{I}) - \lim_{p\to\infty} w_p = \sigma_1$, so $K_{\vartheta^n,1}(\rho, \eta, \delta) \in \mathcal{F}(\mathcal{I})$, for all $\rho, \delta > 0$. Also, $K_{\vartheta^n,2}(\rho, \eta, \delta) \in \mathcal{F}(\mathcal{I})$, for all $\rho, \delta > 0$. Assume

$$K_{\vartheta^n}(\rho, \eta, \delta) = K_{\vartheta^n,1}(\rho, \eta, \delta) \cap K_{\vartheta^n,2}(\rho, \eta, \delta).$$

Then, $K_{\vartheta^n}(\rho, \eta, \delta) \in \mathcal{F}(\mathcal{I})$.
Now if $p \in K_{\vartheta^n}(\rho, \eta, \delta)$, then we get

$$\begin{aligned}\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, \sigma_1 - \sigma_2\right), r + \rho\right) &\ge \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma_1\right), r + \frac{\rho}{2}\right) * \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma_2\right), r + \frac{\rho}{2}\right)\\ &> (1-\eta) * (1-\eta) > 1 - \lambda.\end{aligned}$$

Since $\lambda > 0$ was arbitrary, we obtain $\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, \sigma_1 - \sigma_2\right), r + \rho\right) = 1$ for all $\rho > 0$ and for some $r > 0$, which yields $\sigma_1 = \sigma_2$. □

Theorem 3.8. *The set $S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$ of a sequence $w = (w_p)$ in a Pr-n-spaces $(Y, \vartheta, *)$ is a closed set.*

Proof. We have nonthing to demonstrate as $S^{\vartheta^n}(\mathcal{I}) - LIM_w^r = \emptyset$.

Assume $S^{\vartheta^n}(\mathcal{I}) - LIM_w^r \neq \emptyset$ for some $r > 0$ and consider $q = (q_p)$ be a convergent sequence in $S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$ w.r.t. the norm ϑ^n to $q_0 \in Y$. For $t \in (0,1)$ select $\lambda \in (0,1)$ such that $(1-\lambda) * (1-\lambda) > 1 - t$. Then, for all $\rho > 0$, $\lambda \in (0,1)$ there is a $p_1 \in \mathbb{N}$ such that

$$\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, q_p - q_0\right), \frac{\rho}{2}\right) > 1 - \lambda, \text{ for all } p \ge p_1.$$

Choose $q_m \in S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$ with $m \ge p_1$ such that

$$A = \left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \le t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_m\right), r + \frac{\rho}{2}\right) \le 1 - \eta\right\}\right| \ge \delta\right\} \in \mathcal{I}. \tag{1}$$

Assume

$$j \in \left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \leq t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_m\right), r + \frac{\rho}{2}\right) > 1 - \eta\right\}\right| < \delta\right\}$$

we get

$$\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_j - q_m\right), r + \frac{\rho}{2}\right) > 1 - \eta.$$

Then, we obtain

$$\begin{aligned}\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_j - q_0\right), r + \rho\right) &\geq \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_j - q_m\right), r + \frac{\rho}{2}\right) * \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, q_m - q_0\right), \frac{\rho}{2}\right)\\ &> (1 - \lambda) * (1 - \lambda) > 1 - t.\end{aligned}$$

So

$$j \in \left\{p \leq t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_0\right), r + \rho\right) \geq 1 - t\right\}.$$

So, we acquire the following inclusion

$$\begin{aligned}&\left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \leq t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_m\right), r + \frac{\rho}{2}\right) > 1 - \eta\right\}\right| < \delta\right\}\\ &\subseteq \left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \leq t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_0\right), r + \rho\right) > 1 - t\right\}\right| < \delta\right\}\end{aligned}$$

i.e.

$$\begin{aligned}&\left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \leq t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_0\right), r + \rho\right) \leq 1 - t\right\}\right| \geq \delta\right\}\\ &\subseteq \left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \leq t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_m\right), r + \frac{\rho}{2}\right) \leq 1 - \eta\right\}\right| \geq \delta\right\}.\end{aligned}$$

Utilizing equation (1) we obtain

$$\left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \leq t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_0\right), r + \rho\right) \leq 1 - t\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

As a result, we get $q_0 \in S^{\vartheta^n}(\mathcal{I}) - LIM^r_w$. □

In the next result, we examine the convexity of the set $S^{\vartheta^n}(\mathcal{I}) - LIM^r_w$.

Theorem 3.9. *Let $w = \left(w_p\right)$ be a sequence in a Pr-n- space $(Y, \vartheta, *)$. Then, rough $\mathcal{I}$-statistical limit set $S^{\vartheta^n}(\mathcal{I}) - LIM^r_w$ w.r.t. the norm ϑ^n is convex for some $r > 0$.*

Proof. Suppose $\sigma_1, \sigma_2 \in S^{\vartheta^n}(\mathcal{I}) - LIM^r_w$. For the convexity of the set $S^{\vartheta^n}(\mathcal{I}) - LIM^r_w$, we have to indicate that $[(1-\alpha)\sigma_1 + \alpha\sigma_2] \in S^{\vartheta^n}(\mathcal{I}) - LIM^r_w$ for some $\alpha \in (0, 1)$. For $t \in (0, 1)$ let $\lambda \in (0, 1)$ such that $(1-\lambda)*(1-\lambda) > 1-t$. Then, for all $\rho, \delta > 0$, $\lambda \in (0, 1)$, we determine

$$K_1 = \left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \leq t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma_1\right), \frac{r + \rho}{2(1-\alpha)}\right) \leq 1 - \lambda\right\}\right| \geq \delta\right\} \in \mathcal{I},$$

$$K_2 = \left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \leq t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma_2\right), \frac{r + \rho}{2\alpha}\right) \leq 1 - \lambda\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

$M=\mathbb{N}\setminus(K_1\cup K_2)\in\mathcal{F}(\mathcal{I})$ and so M have to be infinite set. Assume $u\in M$ then $\delta(T_1)=0$, where

$$T_1=\left\{p\leq u:\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma_1\right),r+\rho\right)\leq 1-\lambda\right\}$$

and $\delta(T_2)=0$, where

$$T_2=\left\{p\leq u:\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma_2\right),r+\rho\right)\leq 1-\lambda\right\}.$$

Now, for all $p\in T_1^c\cap T_2^c$ and each $\lambda\in(0,1)$,

$$\begin{aligned}&\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-[(1-\alpha)\sigma_1+\alpha\sigma_2]\right),r+\rho\right)\\&\geq\vartheta\left(\left(s_1,s_2,...,s_{n-1},(1-\alpha)\left(w_p-\sigma_1\right)+\alpha\left(w_p-\sigma_2\right)\right),r+\rho\right)\\&\geq\vartheta\left(\left(s_1,s_2,...,s_{n-1},(1-\alpha)\left(w_p-\sigma_1\right)\right),\frac{r+\rho}{2}\right)*\vartheta\left(\left(s_1,s_2,...,s_{n-1},\alpha\left(w_p-\sigma_2\right)\right),\frac{r+\rho}{2}\right)\\&\geq\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma_1\right),\frac{r+\rho}{2(1-\alpha)}\right)*\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma_2\right),\frac{r+\rho}{2\alpha}\right)\\&>(1-\lambda)*(1-\lambda)>1-t.\end{aligned}$$

Thus, we obtain

$$\left\{t\in\mathbb{N}:\frac{1}{t}\left|\left\{p\leq t:\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-[(1-\alpha)\sigma_1+\alpha\sigma_2]\right),r+\rho\right)\leq 1-t\right\}\right|\geq\delta\right\}\in\mathcal{I}.$$

This proves that, $[(1-\alpha)\sigma_1+\alpha\sigma_2]\in S^{\vartheta^n}(\mathcal{I})-LIM_w^r$ i.e. $S^{\vartheta^n}(\mathcal{I})-LIM_w^r$ is a convex set. □

Theorem 3.10. *A sequence $w=\left(w_p\right)$ in a Pr-n-space $(Y,\vartheta,*)$ is rough $\mathcal{I}$-statistically convergent to $\sigma\in Y$ w.r.t. the norm ϑ^n for some $r>0$ if there is a sequence $q=\left(q_p\right)$ in Y, which is $\mathcal{I}$-statistically convergent to $\sigma\in Y$ w.r.t. the norm ϑ^n and for all $\lambda\in(0,1)$ have $\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-q_p\right),r\right)>1-\lambda$, for all $p\in\mathbb{N}$.*

Proof. Take $\rho>0$ and $\lambda\in(0,1)$. Examine $q_p\overset{S^{\vartheta^n}(\mathcal{I})}{\rightarrow}\sigma$ and $\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-q_p\right),r\right)>1-\lambda$ for all $p\in\mathbb{N}$. For given $\lambda\in(0,1)$ select $t\in(0,1)$ such that $(1-t)*(1-t)>1-\lambda$. Identify

$$K_1=\left\{t\in\mathbb{N}:\frac{1}{t}\left|\left\{p\leq t:\vartheta\left(\left(s_1,s_2,...,s_{n-1},q_p-\sigma\right),\rho\right)\leq 1-t\right\}\right|\geq\delta\right\}\in\mathcal{I},$$

$$K_2=\left\{t\in\mathbb{N}:\frac{1}{t}\left|\left\{p\leq t:\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-q_p\right),r\right)\leq 1-t\right\}\right|\geq\delta\right\}\in\mathcal{I}.$$

$M=\mathbb{N}\setminus(K_1\cup K_2)\in\mathcal{F}(\mathcal{I})$ and so M have to be infinite set. For $p\in M$, we obtain

$$\begin{aligned}\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma\right),r+\rho\right)&\geq\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-q_p\right),r\right)*\vartheta\left(\left(s_1,s_2,...,s_{n-1},q_p-\sigma\right),\rho\right)\\&>(1-t)*(1-t)>1-\lambda.\end{aligned}$$

Then $\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma\right),r+\rho\right)>1-\lambda$ for all $p\in M=K_1^c\cap K_2^c$.
This gives that

$$\left\{t\in\mathbb{N}:\frac{1}{t}\left|\left\{p\leq t:\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma\right),r+\rho\right)\leq 1-t\right\}\right|\geq\delta\right\}\subseteq K_1\cup K_2.$$

As $K_1\cup K_2\in\mathcal{I}$, so

$$\left\{t\in\mathbb{N}:\frac{1}{t}\left|\left\{p\leq t:\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma\right),r+\rho\right)\leq 1-t\right\}\right|\geq\delta\right\}\in\mathcal{I}.$$

As a result, we obtain $w_p\overset{r-S^{\vartheta^n}(\mathcal{I})}{\rightarrow}\sigma$. □

Theorem 3.11. *Assume $w = \left(w_p\right)$ be a sequence in a Pr-n-space $(Y, \vartheta, *)$. Then, there does not exist elements $y, z \in S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$ for some $r > 0$ and each $\lambda \in (0,1)$ such that $\vartheta\left((s_1, s_2, ..., s_{n-1}, y - z), ur\right) \leq 1 - \lambda$ for $u \geq 2$.*

Proof. We establish the result by method of contradiction. Assume there are elements $y, z \in S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$ such that

$$\vartheta\left((s_1, s_2, ..., s_{n-1}, y - z), ur\right) \leq 1 - \lambda, \tag{2}$$

for $u \geq 2$. For given $\lambda \in (0,1)$ select $t \in (0,1)$ such that $(1-t) * (1-t) > 1 - \lambda$. Then, for all $\rho > 0$, $t \in (0,1)$, we get $K_1 \in \mathcal{I}$ and $K_2 \in \mathcal{I}$ where

$$K_1 = \left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \leq t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - y\right), r + \frac{\rho}{2}\right) > 1 - t\right\}\right| < \delta\right\},$$

$$K_2 = \left\{t \in \mathbb{N} : \frac{1}{t}\left|\left\{p \leq t : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - z\right), r + \frac{\rho}{2}\right) > 1 - t\right\}\right| < \delta\right\}.$$

For $p \in K_1^c \cap K_2^c$ we get

$$\begin{aligned}\vartheta\left((s_1, s_2, ..., s_{n-1}, y - z), 2r + \rho\right) &\geq \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - z\right), r + \frac{\rho}{2}\right) * \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - y\right), r + \frac{\rho}{2}\right)\\ &> (1-t) * (1-t) > 1 - \lambda.\end{aligned}$$

Hence

$$\vartheta\left((s_1, s_2, ..., s_{n-1}, y - z), 2r + \rho\right) > 1 - \lambda. \tag{3}$$

Then, according to the (3), we have $\vartheta\left((s_1, s_2, ..., s_{n-1}, y - z), ur\right) > 1 - \lambda$ for $u \geq 2$ which is a contradiction to (2). Hence, there does not exists elements $y, z \in S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$ such that $\vartheta\left((s_1, s_2, ..., s_{n-1}, y - z), ur\right) \leq 1 - \lambda$ for $u \geq 2$. □

Definition 3.12. *Assume $(Y, \vartheta, *)$ be a Pr-n- space. Then, $\gamma \in Y$ is named rough $\mathcal{I}$-statistical cluster point of the sequence $w = \left(w_p\right)$ in Y w.r.t. the norm ϑ^n provided that for each $\rho > 0$, $\lambda \in (0,1)$ and some $r > 0$,*

$$\delta_{\mathcal{I}}\left(\left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \gamma\right), r + \rho\right) > 1 - \lambda\right\}\right) \neq 0$$

where

$$\delta_{\mathcal{I}}(A) = \mathcal{I}\text{-}\lim_{t \to \infty} \frac{1}{t}\left|\{p \leq t, p \in A\}\right|$$

if exists. In this case, γ is known as $r - S^{\vartheta^n}$-cluster points of a sequence $w = \left(w_p\right)$. Let $\Gamma_{w,\vartheta^n}^r(\mathcal{I})$ indicates the set of all r-S^{ϑ^n}-cluster points of a sequence $w = \left(w_p\right)$.

Theorem 3.13. *Assume $(Y, \vartheta, *)$ be a Pr-n- space. Then, the set $\Gamma_{w,\vartheta^n}^r(\mathcal{I})$ of any sequence $w = \left(w_p\right)$ is closed for some $r > 0$.*

Proof. If $\Gamma_{w,\vartheta^n}^r(\mathcal{I}) = \emptyset$, then we have to demonstrate nothing. If $\Gamma_{w,\vartheta^n}^r(\mathcal{I}) \neq \emptyset$, then take a sequence $q = \left(q_p\right) \subseteq \Gamma_{w,\vartheta^n}^r(\mathcal{I})$ such that $q_p \xrightarrow{\vartheta^n} q_*$. It is sufficient to prove that $q_* \in \Gamma_{w,\vartheta^n}^r(\mathcal{I})$. For $t \in (0,1)$ select $\lambda \in (0,1)$ such that $(1-\lambda) * (1-\lambda) > 1 - t$. Since $q_p \xrightarrow{\vartheta^n} q_*$, then for each $\rho > 0$ and $\lambda \in (0,1)$ there exists $p_\rho \in \mathbb{N}$ such that $\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, q_p - q_*\right), \frac{\rho}{2}\right) > 1 - \lambda$, for $p \geq p_\rho$. Now, select $p_0 \in \mathbb{N}$ such that $p_0 \geq p_\rho$. Then, we obtain $\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, q_{p_0} - q_*\right), \frac{\rho}{2}\right) > 1 - \lambda$. Since, $q = \left(q_p\right) \subseteq \Gamma_{w,\vartheta^n}^r(\mathcal{I})$, we get $q_{p_0} \in \Gamma_{w,\vartheta^n}^r(\mathcal{I})$. Namely,

$$\delta_{\mathcal{I}}\left(\left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_{p_0}\right), r + \frac{\rho}{2}\right) > 1 - \lambda\right\}\right) \neq 0. \tag{4}$$

Select

$$j \in \left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_{p_0}\right), r + \frac{\rho}{2}\right) > 1 - \lambda\right\},$$

then we obtain $\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_j - q_{p_0}\right), r + \frac{\rho}{2}\right) > 1 - \lambda$.

$$\begin{aligned}\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_j - q_*\right), r + \rho\right) &\geq \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_j - q_{p_0}\right), r + \frac{\rho}{2}\right) * \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, q_{p_0} - q_*\right), \frac{\rho}{2}\right) \\ &> (1 - \lambda) * (1 - \lambda) > 1 - t.\end{aligned}$$

Thus,

$$j \in \left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_*\right), r + \rho\right) > 1 - t\right\}.$$

Hence

$$\left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_{p_0}\right), r + \frac{\rho}{2}\right) > 1 - \lambda\right\} \subseteq \left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_*\right), r + \rho\right) > 1 - t\right\}.$$

Now

$$\begin{aligned}&\delta_{\mathcal{I}}\left(\left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_{p_0}\right), r + \frac{\rho}{2}\right) > 1 - \lambda\right\}\right) \\ &\leq \delta_{\mathcal{I}}\left(\left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_*\right), r + \rho\right) > 1 - t\right\}\right).\end{aligned} \tag{5}$$

By equation (4), we obtain that the set on left side of (5) has natural density more than 0. As a result, we get

$$\delta_{\mathcal{I}}\left(\left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q_*\right), r + \rho\right) > 1 - t\right\}\right) \neq 0.$$

Hence, $q_* \in \Gamma^r_{w,\vartheta^n}(\mathcal{I})$. □

Theorem 3.14. *Assume $\Gamma_{w,\vartheta^n}(\mathcal{I})$ be the set of all $\mathcal{I}$-statistical cluster point of the sequence $w = \left(w_p\right)$ in a Pr-n- space $(Y, \vartheta, *)$ and $r > 0$. Then, for an arbitrary $\gamma \in \Gamma_{w,\vartheta^n}(\mathcal{I})$ and $\lambda \in (0, 1)$ we get $\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, \xi - \gamma\right), r\right) > 1 - \lambda$, for all $\xi \in \Gamma^r_{w,\vartheta^n}(\mathcal{I})$.*

Proof. For $\lambda \in (0, 1)$ select $t \in (0, 1)$ such that $(1 - t) * (1 - t) > 1 - \lambda$. Let $\gamma \in \Gamma_{w,\vartheta^n}(\mathcal{I})$. Then, for each $\rho > 0$ and $t \in (0, 1)$, we get

$$\delta_{\mathcal{I}}\left(\left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \gamma\right), \rho\right) > 1 - t\right\}\right) \neq 0.$$

Now, we will denote that if for $\xi \in Y$ we obtain

$$\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, \xi - \gamma\right), r\right) > 1 - t \tag{6}$$

then $\xi \in \Gamma^r_{w,\vartheta^n}(\mathcal{I})$.
Let

$$j \in \left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \gamma\right), \rho\right) > 1 - t\right\}$$

then $\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_j - \gamma\right), \rho\right) > 1 - t$. Now

$$\begin{aligned}\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_j - \xi\right), r + \rho\right) &\geq \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_j - \gamma\right), \rho\right) * \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, \xi - \gamma\right), r\right) \\ &> (1 - t) * (1 - t) > 1 - \lambda.\end{aligned}$$

So, we get $\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_j - \xi\right), r + \rho\right) > 1 - \lambda$. Thus

$$j \in \left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \xi\right), r + \rho\right) > 1 - \lambda\right\}.$$

Now, the next inclusion supplies.

$$\left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \gamma\right), \rho\right) > 1 - t\right\} \subseteq \left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \xi\right), r + \rho\right) > 1 - \lambda\right\}.$$

Then

$$\delta_{\mathcal{I}}\left(\left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \gamma\right), \rho\right) > 1 - t\right\}\right) \leq \delta_{\mathcal{I}}\left(\left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \xi\right), r + \rho\right) > 1 - \lambda\right\}\right).$$

By equation (6), we have

$$\delta_{\mathcal{I}}\left(\left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \xi\right), r + \rho\right) > 1 - \lambda\right\}\right) \neq 0.$$

Hence, we obtain $\xi \in \Gamma^r_{w,\vartheta^n}(\mathcal{I})$. □

Theorem 3.15. *If* $\overline{B(c, \lambda, r)} = \{w \in Y : \vartheta((s_1, s_2, ..., s_{n-1}, w - c), r) \geq 1 - \lambda\}$ *represent the closure of open ball* $B(c, \lambda, r) = \{w \in Y : \vartheta((s_1, s_2, ..., s_{n-1}, w - c), r) > 1 - \lambda\}$ *for some* $r > 0$, $\lambda \in (0, 1)$ *and fixed* $c \in Y$ *then* $\Gamma^r_{w,\vartheta^n}(\mathcal{I}) = \bigcup_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)}$.

Proof. For $\lambda \in (0, 1)$ select $t \in (0, 1)$ such that $(1 - t) * (1 - t) > 1 - \lambda$. Take $\gamma \in \bigcup_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)}$ then there is $c \in \Gamma_{w,\vartheta^n}(\mathcal{I})$ for some $r > 0$ and each $t \in (0, 1)$ such that $\vartheta((s_1, s_2, ..., s_{n-1}, c - \gamma), r) \geq 1 - t$.
Fixed $\rho > 0$. As $c \in \Gamma_{w,\vartheta^n}(\mathcal{I})$ then there is a set

$$K = \left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - c\right), \rho\right) \geq 1 - t\right\}$$

with $\delta_{\mathcal{I}}(K) \neq 0$. For $p \in K$

$$\begin{aligned}\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \gamma\right), r + \rho\right) &\geq \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - c\right), \rho\right) * \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, c - \gamma\right), r\right) \\ &> (1 - t) * (1 - t) > 1 - \lambda.\end{aligned}$$

This implies

$$\delta_{\mathcal{I}}\left(\left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \gamma\right), r + \rho\right) > 1 - \lambda\right\}\right) \neq 0.$$

Hence $\gamma \in \Gamma^r_{w,\vartheta^n}(\mathcal{I})$. As a result, we obtain $\bigcup_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)} \subseteq \Gamma^r_{w,\vartheta^n}(\mathcal{I})$.

Conversely, suppose that $\gamma \in \Gamma^r_{w,\vartheta^n}(\mathcal{I})$. Next, we establish that $\gamma \in \bigcup_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)}$.

Suppose, $\gamma \notin \bigcup_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)}$, i.e. $\gamma \notin \overline{B(c, \lambda, r)}$, for all $c \in \Gamma_{w,\vartheta^n}(\mathcal{I})$. Then,

$$\vartheta((s_1, s_2, ..., s_{n-1}, \gamma - c), r) \leq 1 - \lambda$$

for each $c \in \Gamma_{w,\vartheta^n}(\mathcal{I})$. According to the Theorem 3.14 for arbitrary $c \in \Gamma_{w,\vartheta^n}(\mathcal{I})$ we get

$$\vartheta((s_1, s_2, ..., s_{n-1}, \gamma - c), r) > 1 - \lambda$$

for all $c \in \Gamma^r_{w,\vartheta^n}(\mathcal{I})$ which is a contradiction to the supposition. So, $\gamma \in \bigcup_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)}$. Hence, $\Gamma^r_{w,\vartheta^n}(\mathcal{I}) \subseteq \bigcup_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)}$. □

Theorem 3.16. *Assume $w = (w_p)$ be a sequence in a Pr-n- space $(Y, \vartheta, *)$. Then, for any $\lambda \in (0, 1)$,*
(i) If $c \in \Gamma_{w,\vartheta^n}(\mathcal{I})$, then $S^{\vartheta^n}(\mathcal{I}) - LIM_w^r \subseteq \overline{B(c, \lambda, r)}$.
(ii) $S^{\vartheta^n}(\mathcal{I}) - LIM_w^r = \bigcap_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)} = \left\{\xi \in Y : \Gamma_{w,\vartheta^n}(\mathcal{I}) \subseteq \overline{B(\xi, \lambda, r)}\right\}$.

Proof. (*i*) Let $\rho > 0$. For a given $\lambda \in (0, 1)$ select $t \in (0, 1)$ such that $(1 - t) * (1 - t) > 1 - \lambda$. Consider $\xi \in S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$ and $c \in \Gamma_{w,\vartheta^n}(\mathcal{I})$. For all $\rho > 0$ and $t \in (0, 1)$ identify sets

$$K = \left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \xi\right), r + \rho\right) > 1 - t\right\},$$

with $\delta_{\mathcal{I}}(K^c) = 0$, and

$$L = \left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - c\right), \rho\right) > 1 - t\right\},$$

with $\delta_{\mathcal{I}}(L^c) \neq 0$. Now, for $p \in K \cap L$ we obtain

$$\begin{aligned}\vartheta((s_1, s_2, ..., s_{n-1}, \xi - c), r) &\geq \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - c\right), \rho\right) * \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \xi\right), r + \rho\right)\\ &> (1 - t) * (1 - t) > 1 - \lambda.\end{aligned}$$

So, $\xi \in \overline{B(c, \lambda, r)}$. As a result, $S^{\vartheta^n}(\mathcal{I}) - LIM_w^r \subseteq \overline{B(c, \lambda, r)}$.
(*ii*) According to previous part we get $S^{\vartheta^n}(\mathcal{I}) - LIM_w^r \subseteq \bigcap_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)}$.
Presume $q \in \bigcap_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)}$ then $\vartheta((s_1, s_2, ..., s_{n-1}, \xi - c), r) \geq 1 - \lambda$, for all $c \in \Gamma_{w,\vartheta^n}(\mathcal{I})$. This implies

$$\Gamma_{w,\vartheta^n}(\mathcal{I}) \subseteq \overline{B(c, \lambda, r)}, \text{ i.e., } \bigcap_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)} = \left\{\xi \in Y : \Gamma_{w,\vartheta^n}(\mathcal{I}) \subseteq \overline{B(\xi, \lambda, r)}\right\}.$$

In addition, assume $q \notin S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$, then for all $\rho > 0$, we get

$$\delta_{\mathcal{I}}\left(\left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q\right), r + \rho\right) \leq 1 - \lambda\right\}\right) \neq 0,$$

which yields that the existence of a $\mathcal{I}$-statistical cluster point c of the sequence $w = (w_p)$ with

$$\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - q\right), r + \rho\right) \leq 1 - \lambda.$$

So, $\Gamma_{w,\vartheta^n}(\mathcal{I}) \nsubseteq \overline{B(c, \lambda, r)}$ and $q \notin \left\{\xi \in Y : \Gamma_{w,\vartheta^n}(\mathcal{I}) \subseteq \overline{B(\xi, \lambda, r)}\right\}$. This gives that

$$\left\{\xi \in Y : \Gamma_{w,\vartheta^n}(\mathcal{I}) \subseteq \overline{B(\xi, \lambda, r)}\right\} \subseteq S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$$

and we have $\bigcap_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)} \subseteq S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$. Hence, we have

$$S^{\vartheta^n}(\mathcal{I}) - LIM_w^r = \bigcap_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)} = \left\{\xi \in Y : \Gamma_{w,\vartheta^n}(\mathcal{I}) \subseteq \overline{B(\xi, \lambda, r)}\right\}.$$

□

Theorem 3.17. *Let $w = (w_p)$ be a sequence in a Pr-n- space $(Y, \vartheta, *)$ which is $\mathcal{I}$-statistically convergent to $\xi \in Y$ w.r.t. the norm ϑ^n there exists $\lambda \in (0, 1)$ such that $S^{\vartheta^n}(\mathcal{I}) - LIM_w^r = \overline{B(\xi, \lambda, r)}$, for some $r > 0$.*

Proof. Let $\rho > 0$. For a given $\lambda \in (0,1)$ choose $t \in (0,1)$ such that $(1-t) * (1-t) > 1-\lambda$. As $w_p \overset{S^{\vartheta^n}(\mathcal{I})}{\to} \xi$ then there exists a set

$$K = \left\{p \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \xi\right), \rho\right) \leq 1-t\right\}$$

with $\delta_{\mathcal{I}}(K) = 0$. Establish

$$q \in \overline{B(\xi, t, r)} = \{q \in Y : \vartheta((s_1, s_2, ..., s_{n-1}, y - \xi), r) \geq 1-t\}.$$

For $p \in K^c$

$$\begin{aligned}\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - y\right), r+\rho\right) &\geq \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \xi\right), \rho\right) * \vartheta((s_1, s_2, ..., s_{n-1}, y-\xi), r) \\ &> (1-t) * (1-t) > 1-\lambda.\end{aligned}$$

This implies $q \in S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$, i.e. $\overline{B(\xi, \lambda, r)} \subseteq S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$. In addition, $S^{\vartheta^n}(\mathcal{I}) - LIM_w^r \subseteq \overline{B(\xi, \lambda, r)}$. This results that, $S^{\vartheta^n}(\mathcal{I}) - LIM_w^r = \overline{B(\xi, \lambda, r)}$ for some $r > 0$. □

Theorem 3.18. *Assume $w = \left(w_p\right)$ be a sequence in a Pr-n- space $(Y, \vartheta, *)$ which converges $\mathcal{I}$-statistically w.r.t. the norm ϑ^n then $\Gamma^r_{w,\vartheta^n}(\mathcal{I}) = S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$ for some $r > 0$.*

Proof. Let $w_p \overset{S^{\vartheta^n}(\mathcal{I})}{\to} \xi$, then we have $\Gamma_{w,\vartheta^n}(\mathcal{I}) = \{\xi\}$. By Theorem 3.15, for some $r > 0$ and $\lambda \in (0,1)$ we get $\Gamma_{w,\vartheta^n}(\mathcal{I}) = \overline{B(\xi, \lambda, r)}$. Also, according to Theorem 3.17 we get $\overline{B(\xi, \lambda, r)} = S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$. Hence, $\Gamma^r_{w,\vartheta^n}(\mathcal{I}) = S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$.

Suppose $\Gamma^r_{w,\vartheta^n}(\mathcal{I}) = S^{\vartheta^n}(\mathcal{I}) - LIM_w^r$. According to Theorem 3.15 and Theorem 3.16(ii) we get

$$\bigcup_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)} = \bigcap_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)}.$$

This implies either $\Gamma_{w,\vartheta^n}(\mathcal{I}) = \emptyset$ or $\Gamma_{w,\vartheta^n}(\mathcal{I})$ is a singleton set. Then

$$S^{\vartheta^n}(\mathcal{I}) - LIM_w^r = \bigcap_{c \in \Gamma_{w,\vartheta^n}(\mathcal{I})} \overline{B(c, \lambda, r)} = \overline{B(\xi, \lambda, r)},$$

for some $\xi \in \Gamma_{w,\vartheta^n}(\mathcal{I})$, also according to the Theorem 3.17 we obtain $S^{\vartheta^n}(\mathcal{I}) - LIM_w^r = \{\xi\}$. □

Definition 3.19. *Let $(Y, \vartheta, *)$ be a Pr-n-space. A sequence $w = \left(w_p\right)$ in Y is said to be rough $\mathcal{I}$-lacunary statistically convergent to $\sigma \in Y$ w.r.t. the probabilistic n-norm ϑ^n provided that for each $\rho, \delta > 0$, $\eta \in (0,1)$, $s_1, s_2, ..., s_{n-1} \in Y$ and $r > 0$*

$$\left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\left\{p \in I_u : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r+\rho\right) \leq 1-\eta\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Symbolically written as, $r - S_\theta^{\vartheta^n}(\mathcal{I}) - \lim_{p\to\infty} w_p = \sigma$ or $w_p \overset{r-S_\theta^{\vartheta^n}(\mathcal{I})}{\to} \sigma$.

Definition 3.20. *Let $(Y, \vartheta, *)$ be a Pr-n-space. A sequence $w = \left(w_p\right)$ in Y is said to be rough $N_\theta^{\vartheta^n}(\mathcal{I})$-convergent to $\sigma \in Y$ w.r.t. the probabilistic n-norm ϑ^n provided that for each $\rho > 0$, $\eta \in (0,1)$, $s_1, s_2, ..., s_{n-1} \in Y$ and $r > 0$*

$$\left\{u \in \mathbb{N} : \frac{1}{h_u}\sum_{p \in I_u} \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r+\rho\right) \leq 1-\eta\right\} \in \mathcal{I}.$$

Symbolically written as, $r - N_\theta^{\vartheta^n}(\mathcal{I}) - \lim_{p\to\infty} w_p = \sigma$ or $w_p \overset{r-N_\theta^{\vartheta^n}(\mathcal{I})}{\to} \sigma$.

Theorem 3.21. *Let $(Y, \vartheta, *)$ be a Pr-n-space and θ be a lacunary sequence. Then*
(i) (a) $w_p \overset{r-N_\theta^{\vartheta^n}(\mathcal{I})}{\rightarrow} \sigma \Rightarrow w_p \overset{r-S_\theta^{\vartheta^n}(\mathcal{I})}{\rightarrow} \sigma$;
(b) $r - N_\theta^{\vartheta^n}(\mathcal{I})$ *is a proper subset of* $r - S_\theta^{\vartheta^n}(\mathcal{I})$.
(ii) $w = (w_p) \in \ell_\infty$ *and* $w_p \overset{r-S_\theta^{\vartheta^n}(\mathcal{I})}{\rightarrow} \sigma \Rightarrow w_p \overset{r-N_\theta^{\vartheta^n}(\mathcal{I})}{\rightarrow} \sigma$,
(iii) $r - S_\theta^{\vartheta^n}(\mathcal{I}) \cap \ell_\infty = r - N_\theta^{\vartheta^n}(\mathcal{I}) \cap \ell_\infty$.

Proof. *(i) (a)* If $\rho > 0$, $\eta \in (0,1)$ and $w_p \overset{r-N_\theta^{\vartheta^n}(\mathcal{I})}{\rightarrow} \sigma$, we can write

$$\sum_{p\in I_u} \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) \geq \sum_{p\in I_u, \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r+\rho\right) \leq 1-\eta} \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right)$$
$$\geq \eta. \left|\left\{p \in I_u : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) \leq 1 - \eta\right\}\right|$$

and so

$$\frac{1}{\eta h_u} \sum_{p\in I_u} \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) \geq \frac{1}{h_u} \left|\left\{p \in I_u : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) \leq 1 - \eta\right\}\right|$$

Then, for any $\delta > 0$, $\eta \in (0,1)$,

$$\left\{u \in \mathbb{N} : \tfrac{1}{h_u} \left|\left\{p \in I_u : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) \leq 1 - \eta\right\}\right| \geq \delta\right\}$$
$$\subseteq \left\{u \in \mathbb{N} : \tfrac{1}{h_u} \sum_{p\in I_u} \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) \leq (1 - \eta)\,\delta\right\} \in \mathcal{I}.$$

This establishes the result.

(ii) In order to show that the inclusion $r - N_\theta^{\vartheta^n}(\mathcal{I}) \subseteq r - S_\theta^{\vartheta^n}(\mathcal{I})$ is proper, let θ be given, and consider w_p to be $1, 2, ..., \left[\sqrt{h_u}\right]$ for the first $\left[\sqrt{h_u}\right]$ integers in I_u and $w_p = 0$ otherwise, for all $p = 1, 2, ...$ So, for any $\eta \in (0,1)$ and $\rho > 0$,

$$\frac{1}{h_u} \left|\left\{p \in I_u : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - 0\right), r + \rho\right) \leq 1 - \eta\right\}\right| \leq \frac{\left[\sqrt{h_u}\right]}{h_u},$$

and for any $\delta > 0$

$$\left\{u \in \mathbb{N} : \frac{1}{h_u} \left|\left\{p \in I_u : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - 0\right), r + \rho\right) \leq 1 - \eta\right\}\right| \geq \delta\right\} \subseteq \left\{u \in \mathbb{N} : \frac{\left[\sqrt{h_u}\right]}{h_u} \geq \delta\right\}.$$

Since the set on the right-hand side is a finite set and so belongs to $\mathcal{I}$, it shows that $w_p \overset{r-S_\theta^{\vartheta^n}(\mathcal{I})}{\rightarrow} 0$. However,

$$\frac{1}{h_u} \sum_{p\in I_u} \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - 0\right), r + \rho\right) = \frac{1}{h_u} . \frac{\left[\sqrt{h_u}\right]\left(\left[\sqrt{h_u}\right] + 1\right)}{2}.$$

Then

$$\left\{u \in \mathbb{N} : \frac{1}{h_u} \sum_{p\in I_u} \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - 0\right), r + \rho\right) \leq 1 - \frac{1}{4}\right\} = \left\{u \in \mathbb{N} : \frac{\left[\sqrt{h_u}\right]\left(\left[\sqrt{h_u}\right] + 1\right)}{h_u} \geq \frac{1}{2}\right\}$$

which belongs to $\mathcal{F}(\mathcal{I})$, as $\mathcal{I}$ is admissible. Hence $w_p \overset{r-N_\theta^{\vartheta^n}(\mathcal{I})}{\nrightarrow} 0$.
(ii) Suppose that $w_p \overset{r-S_\theta^{\vartheta^n}(\mathcal{I})}{\rightarrow} \sigma$ and $w = (w_p) \in \ell_\infty$. Then, there exists an $T > 0$ such that

$$\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) \geq 1 - T, \forall p \in \mathbb{N}.$$

Given $\eta \in (0,1)$, we obtain

$$\begin{aligned}&\tfrac{1}{h_u}\sum_{p\in I_u}\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) = \tfrac{1}{h_u}\sum_{p\in I_u, \vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma\right),r+\rho\right)\leq 1-\frac{\eta}{2}}\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right)\\&+\tfrac{1}{h_u}\sum_{p\in I_u, \vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma\right),r+\rho\right)> 1-\frac{\eta}{2}}\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right)\\&\leq \tfrac{M}{h_u}\left\{u \in \mathbb{N} : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - 0\right), r + \rho\right)\right\} + \tfrac{\eta}{2}.\end{aligned}$$

As a result, we obtain

$$\begin{aligned}&\left\{u \in \mathbb{N} : \frac{1}{h_u}\sum_{p\in I_u}\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) \leq 1 - \eta\right\}\\&\subseteq \left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\left\{p \in I_u : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) \leq 1 - \frac{\eta}{2}\right\}\right| \geq \frac{\eta}{2M}\right\} \in \mathcal{I}.\end{aligned}$$

This proves the result.
(iii) It follows from (i) and (ii). □

Theorem 3.22. *For any lacunary sequence θ, rough $\mathcal{I}$-statistical convergence w.r.t. the probabilistic norm ϑ^n gives rough $\mathcal{I}$-lacunary statistical convergence w.r.t. the probabilistic norm ϑ^n iff $\liminf_u q_u > 1$. When $\liminf_u q_u = 1$, then there is a bounded sequence $w = (w_p)$ which is rough $\mathcal{I}$-statistically convergent but not rough $\mathcal{I}$-lacunary statistically convergent.*

Proof. Assume first $\liminf_u q_u > 1$. Then, there exists $\alpha > 0$ such that $q_u > 1 + \alpha$ for sufficiently large u, which means that $\frac{h_u}{k_u} \geq \frac{\alpha}{1+\alpha}$. Since $w_p \overset{r-S^{\vartheta^n}(\mathcal{I})}{\rightarrow} \sigma$, for each $\rho > 0$, $\eta \in (0,1)$ and for sufficiently large u, we acquire

$$\begin{aligned}&\frac{1}{k_u}\left|\left\{p \leq k_u : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) \leq 1 - \eta\right\}\right|\\&\geq \frac{\alpha}{1+\alpha}\frac{1}{h_u}\left|\left\{p \in I_u : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) \leq 1 - \eta\right\}\right|.\end{aligned}$$

Then, for any $\delta > 0$, we have

$$\begin{aligned}&\left\{u \in \mathbb{N} : \frac{1}{h_u}\left|\left\{p \in I_u : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) \leq 1 - \eta\right\}\right| \geq \delta\right\}\\&\subseteq \left\{u \in \mathbb{N} : \frac{1}{k_u}\left|\left\{p \in I_u : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r + \rho\right) \leq 1 - \eta\right\}\right| \geq \frac{\delta\alpha}{1+\alpha}\right\} \in \mathcal{I}.\end{aligned}$$

This denotes the sufficiency.
Conversely, assume that $\liminf_u q_u = 1$. we can select a subsequence $\{k_{u(m)}\}$ of the lacunary sequence θ such that

$$\frac{k_{u(m)}}{k_{u(m)-1}} < 1 + \frac{1}{m} \text{ and } \frac{k_{u(m)-1}}{k_{u(m-1)}} > m$$

where $u(m) \geq u(m-1)+2$.
Now identify a bounded sequence $w = (w_p)$ by $w_p = 1$ for $p \in I_{u(m)}$ for some $m = 1, 2, ...$ and $w_p = 0$ otherwise. Then, for any real σ,

$$\frac{1}{h_u}\sum_{p\in I_u} \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r+\rho\right) = \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, 1-\sigma\right), r+\rho\right), \text{ for } p = 1, 2, ...$$

and

$$\frac{1}{h_u}\sum_{p\in I_u} \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r+\rho\right) = \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, \sigma\right), r+\rho\right), \text{ for } u \neq u(p).$$

Then it is obvious that w does not belong to $r - N_\theta^{\vartheta^n}(\mathcal{I})$. As w is bounded, Theorem 3.21 (ii) gives that $w_p \overset{r-S_\theta^{\vartheta^n}(\mathcal{I})}{\nrightarrow} \sigma$. $\square$

It is known that rough lacunary statistical convergence w.r.t. the probabilistic norm ϑ^n implies rough statistical convergence w.r.t the probabilistic norm ϑ^n iff $\lim_r \sup q_r < \infty$ (i.e. when $\mathcal{I} = \mathcal{I}_{fin}$ is the ideal of finite subsets of $\mathbb{N}$) But, for arbitrary admissible ideal $\mathcal{I}$, this is not clear, and we leave it as an open problem.

Problem 3.23. *When does rough $\mathcal{I}$-lacunary statistical convergence w.r.t. the probabilistic norm ϑ^n imply rough $\mathcal{I}$-statistical convergence w.r.t. the probabilistic norm ϑ^n?*

Theorem 3.24. *Let $\mathcal{I}$ be an admissible ideal satisfying condition (AP) and suppose $\theta \in \mathcal{F}(\mathcal{I})$. If $w_p \in \left[r - S^{\vartheta^n}(\mathcal{I})\right] \cap \left[r - S_\theta^{\vartheta^n}(\mathcal{I})\right]$, then $r - S^{\vartheta^n}(\mathcal{I}) - \lim w = r - S_\theta^{\vartheta^n}(\mathcal{I}) - \lim w$.*

Proof. Suppose $r - S^{\vartheta^n}(\mathcal{I}) - \lim w = \sigma_1$ and $r - S_\theta^{\vartheta^n}(\mathcal{I}) - \lim w = \sigma_2$ and $\sigma_1 \neq \sigma_2$. Take $\rho \in \left(0, \frac{1}{2}|\sigma_1 - \sigma_2|\right)$. Since $\mathcal{I}$ satisfies the condition *(AP)*, there is $M \in \mathcal{F}(\mathcal{I})$ (i.e., $\mathbb{N} \setminus M \in \mathcal{I}$) such that

$$\lim_{q\to\infty} \frac{1}{u_q}\left|\left\{p \leq u_q : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r+\rho\right) \leq 1-\eta\right\}\right| = 0,$$

where $M = \left\{u_q : q = 1, 2, ...\right\}$. Let

$$K = \left\{p \leq u_q : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma_1\right), r+\rho\right) \leq 1-\eta\right\}$$

and

$$L = \left\{p \leq u_q : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma_2\right), r+\rho\right) \leq 1-\eta\right\}.$$

Then $u_q \leq |K \cup L| \leq |K| + |L|$. This implies

$$1 \leq \frac{|K|}{u_q} + \frac{|L|}{u_q}.$$

Since $\frac{|L|}{u_q} \leq 1$ and $\lim_{q\to\infty} \frac{|K|}{u_q} = 0$, so we get $\lim_{q\to\infty} \frac{|L|}{u_q} = 1$.
Suppose, $M^* = \{(k_{\alpha_t}) : t = 1, 2, ...\} \cap \theta \in \mathcal{F}(\mathcal{I})$. Then, the (k_{α_t})th term of the statistical limit expression

$$\frac{1}{u_q}\left|\left\{p \leq u_q : \vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r+\rho\right) \leq 1-\eta\right\}\right|$$

is

$$\frac{1}{k_{\alpha_t}}\left|\left\{p\in\bigcup_{u=1}^{\alpha_t}I_u:\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma\right),r+\rho\right)\le 1-\eta\right\}\right|=\frac{1}{\sum_{u=1}^{\alpha_t}h_u}\sum_{u=1}^{\alpha_t}t_uh_u, \tag{7}$$

where

$$t_u=\frac{1}{h_u}\left|\left\{p\in I_u:\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma\right),r+\rho\right)\le 1-\eta\right\}\right|\xrightarrow{\mathcal{I}}0,$$

since $w_p\overset{r-S_\theta^{\vartheta^n}(\mathcal{I})}{\longrightarrow}\sigma$. Since θ is a lacunary sequence, (7) is a regular weighted mean transform of t_u's, and therefore it is also $\mathcal{I}$-convergent to 0 as $t\to\infty$, and hence it has a subsequence which is convergent to 0, since $\mathcal{I}$ satisfies the condition (AP). However since this is a subsequence of

$$\left\{\frac{1}{u}\left|\left\{1\le p\le u:\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma\right),r+\rho\right)\le 1-\eta\right\}\right|\right\}_{u\in M},$$

we deduce that $\left\{\frac{1}{u}\left|\left\{1\le p\le u:\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma\right),r+\rho\right)\le 1-\eta\right\}\right|\right\}_{u\in M}$ is not convergent to 1, which is a contradiction. This establishes the proof of the theorem. □

Definition 3.25. *A sequence $w=\left(w_p\right)$ in Y is said to be rough lacunary $\mathcal{I}$-Cauchy w.r.t. the probabilistic n-norm ϑ^n or $r-\mathcal{I}_\theta^{\vartheta^n}$-Cauchy sequence provided, for each $\rho>0$, $\eta\in(0,1)$, $s_1,s_2,...,s_{n-1}\in Y$ and some non-negative number r, there is a $q>0$ such that the set*

$$\left\{u\in\mathbb{N}:\frac{1}{h_u}\sum_{p\in I_u}\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-w_q\right),r+\rho\right)>1-\eta\right\}\in\mathcal{F}\left(\mathcal{I}\right).$$

Theorem 3.26. *If $w=\left(w_p\right)$ be any rough $N_\theta^{\vartheta^n}\left(\mathcal{I}\right)$-convergent sequence in a Pr-n-space $(Y,\vartheta,*)$, then it is $r-\mathcal{I}_\theta^{\vartheta^n}$-Cauchy sequence.*

Proof. Suppose that $w_p\overset{r-N_\theta^{\vartheta^n}(\mathcal{I})}{\longrightarrow}\sigma$. For a given $t>0$, select $\eta\in(0,1)$ such that $(1-\eta)*(1-\eta)>1-t$. Then, for any $\rho>0$, $s_1,s_2,...,s_{n-1}\in Y$ and $r>0$, determine the set

$$K=\left\{u\in\mathbb{N}:\frac{1}{h_u}\sum_{p\in I_u}\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma\right),r+\frac{\rho}{2}\right)>1-\eta\right\}.$$

Then, $K\in\mathcal{F}\left(\mathcal{I}\right)$. Select a fixed $q\in\mathbb{N}$. When $\eta\in K$, then we obtain

$$\frac{1}{h_u}\sum_{p\in I_u}\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-w_q\right),r+\rho\right)\ge\frac{1}{h_u}\sum_{p\in I_u}\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-\sigma\right),r+\frac{\rho}{2}\right)$$
$$*\frac{1}{h_u}\sum_{p\in I_u}\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_q-\sigma\right),r+\frac{\rho}{2}\right)>(1-\eta)*(1-\eta)>1-t.$$

This implies,

$$\left\{u\in\mathbb{N}:\frac{1}{h_u}\sum_{p\in I_u}\vartheta\left(\left(s_1,s_2,...,s_{n-1},w_p-w_q\right),r+\rho\right)>1-t\right\}\in\mathcal{F}\left(\mathcal{I}\right)$$

and so $\left(w_p\right)$ is a $r-\mathcal{I}_\theta^{\vartheta^n}$-Cauchy sequence. □

Definition 3.27. *A sequence $w = (w_p)$ in Y is said to be rough lacunary convergent to $\sigma \in Y$ w.r.t. the probabilistic n-norm ϑ^n provided that, for all $\rho > 0$, $\eta \in (0,1)$, $s_1, s_2, ..., s_{n-1} \in Y$ and some non-negative number r there exists $u_0 \in \mathbb{N}$ such that*

$$\frac{1}{h_u}\sum_{p\in I_u}\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r+\rho\right) > 1-\eta,$$

for all $u \geq u_0$ and we write $r - (\vartheta^n)^\theta - \lim w_p = \sigma$.

Definition 3.28. *A sequence $w = (w_p)$ in Y is said to be rough lacunary $\mathcal{I}^*$-convergent or briefly $r - \mathcal{I}_{\theta^*}^{\vartheta^n}$-convergent to $\sigma \in Y$ w.r.t. the probabilistic n-norm ϑ^n provided that there is a set $K = \{p_1 < p_2 < ... < p_m < ...\} \subset \mathbb{N}$ such that the set $\{u \in \mathbb{N} : p_m \in I_u\} \in \mathcal{F}(\mathcal{I})$ and $(\vartheta^n)^\theta - \lim w_{p_m} = \sigma$. Symbolically written as, $r - \mathcal{I}_{\theta^*}^{\vartheta^n} - \lim w_p = \sigma$.*

Theorem 3.29. *Assume $(Y, \vartheta, *)$ be a Pr-n-space and $\mathcal{I}$ be an admissible ideal. If $r - \mathcal{I}_{\theta^*}^{\vartheta^n} - \lim w_p = \sigma$, then $r - \mathcal{I}_{\theta}^{\vartheta^n} - \lim w_p = \sigma$.*

Proof. Suppose that $r - \mathcal{I}_{\theta^*}^{\vartheta^n} - \lim w_p = \sigma$. Then, there exists a set

$$K = \{p_m \in \mathbb{N} : p_m < p_{m+1}, \text{ for all } m \in \mathbb{N}\},$$

such that the followings are satisfied

$$L = \{u \in \mathbb{N} : p_m \in I_u\} \in \mathcal{F}(\mathcal{I})$$

and $r - (\vartheta^n)^\theta - \lim w_{p_m} = \sigma$. Then, for all $\rho > 0$, $\eta \in (0,1)$, $s_1, s_2, ..., s_{n-1} \in Y$ and $r > 0$ there exists $u_0 \in \mathbb{N}$ such that

$$\frac{1}{h_u}\sum_{p_m\in I_u}\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_{p_m} - \sigma\right), r+\rho\right) > 1-\eta,$$

for all $u \geq u_0$. Since $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$,

$$\left\{u \in \mathbb{N} : \frac{1}{h_u}\sum_{p_m\in I_u}\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_{p_m} - \sigma\right), r+\rho\right) \leq 1-\eta\right\} \in \mathcal{I}.$$

So

$$\left\{u \in \mathbb{N} : \frac{1}{h_u}\sum_{p\in I_u}\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r+\rho\right) \leq 1-\eta\right\} \in \mathcal{I},$$

for all $\rho > 0$, $\eta \in (0,1)$. Hence we have, $r - \mathcal{I}_{\theta}^{\vartheta^n} - \lim w_p = \sigma$. □

Theorem 3.30. *Let $(Y, \vartheta, *)$ be a Pr-n-space and $\mathcal{I}$ satisfies the condition (AP). If $w = (w_p)$ is a sequence in Y such that $r - \mathcal{I}_{\theta}^{\vartheta^n} - \lim w_p = \sigma$, then $r - \mathcal{I}_{\theta^*}^{\vartheta^n} - \lim w_p = \sigma$.*

Proof. Let $r - \mathcal{I}_{\theta}^{\vartheta^n} - \lim w_p = \sigma$, for all $\rho > 0$, $\eta \in (0,1)$, $s_1, s_2, ..., s_{n-1} \in Y$ and $r > 0$, the set

$$\left\{u \in \mathbb{N} : \frac{1}{h_u}\sum_{p\in I_u}\vartheta\left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma\right), r+\rho\right) \leq 1-\eta\right\} \in \mathcal{I}.$$

For $l \in \mathbb{N}$ and $\rho > 0$, we construct,

$$A_l = \left\{ u \in \mathbb{N} : 1 - \frac{1}{l} \leq \frac{1}{h_u} \sum_{p \in I_u} \vartheta \left(\left(s_1, s_2, ..., s_{n-1}, w_p - \sigma \right), r + \rho \right) < 1 - \frac{1}{l+1} \right\}.$$

So, it is obvious that $\{A_1, A_2, ...\}$ is a countable family of mutually disjoint sets belonging to $\mathcal{I}$ and hence according to the (AP) condition there is a countable family of sets $\{B_1, B_2, ...\}$ in $\mathcal{I}$ such that $A_p \Delta B_p$ is a finite set for all $p \in \mathbb{N}$ and $B = \bigcup_{p=1}^{\infty} B_p \in \mathcal{I}$. As $B \in \mathcal{I}$, there is a set K in $\mathcal{F}(\mathcal{I})$ such that $K = \mathbb{N} \setminus B$. Now, we demonstrate that the subsequence $\left(w_{p_m}\right)$, $p_m \in I_u$, $u \in K$ is convergent to $\sigma \in Y$ w.r.t. the probabilistic n-norm ϑ^n. For this, take $\eta \in (0,1)$ and $\rho > 0$. Select a $v > 0$ such that $\frac{1}{v} < \eta$. Then, we get

$$\left\{ u \in \mathbb{N} : \frac{1}{h_u} \sum_{p_m \in I_u} \vartheta \left(\left(s_1, s_2, ..., s_{n-1}, w_{p_m} - \sigma \right), r + \rho \right) \leq 1 - \eta \right\}$$
$$\subset \left\{ u \in \mathbb{N} : \frac{1}{h_u} \sum_{p_m \in I_u} \vartheta \left(\left(s_1, s_2, ..., s_{n-1}, w_{p_m} - \sigma \right), r + \rho \right) \leq 1 - \frac{1}{v} \right\} \subset \bigcup_{p=1}^{\infty} A_p.$$

Since $A_p \Delta B_p$ is a finite set for all $p = 1, 2, ..., v-1$, there exists a $p_0 > 0$, such that

$$\left(\bigcup_{p=1}^{\infty} B_p \right) \cap \{ u \in \mathbb{N} : u \geq u_0 \} = \left(\bigcup_{p=1}^{\infty} A_p \right) \cap \{ u \in \mathbb{N} : u \geq u_0 \}.$$

If $p > p_0$ and $u \in K$, then $u \notin B$. This means that $u \notin \bigcup_{p=1}^{\infty} B_p$ and so $u \notin \bigcup_{p=1}^{\infty} A_p$. So, for all $p \geq p_0$ and $u \in K$, we obtain

$$\frac{1}{h_u} \sum_{p_m \in I_u} \vartheta \left(\left(s_1, s_2, ..., s_{n-1}, w_{p_m} - \sigma \right), r + \rho \right) > 1 - \eta.$$

As a result, we get $r - \mathcal{I}_{\theta^*}^{\vartheta^n} - \lim w_p = \sigma$. □

4. Conclusion

The main goal of this article is to present the notion of rough $\mathcal{I}$-statistical convergence in Pr-n-spaces. To accomplish this goal, we mainly investigate some fundamental properties of the newly introduced notion. Then, we define the concept of rough $\mathcal{I}$-statistical limit set in Pr-n-spaces and obtain some algebraic and topological properties of this set. In addition, we investigate rough $\mathcal{I}$-lacunary statistical convergent, rough $\mathcal{I}$-convergent, rough lacunary $\mathcal{I}$-Cauchy and rough lacunary $\mathcal{I}^*$-convergent sequences in Pr-n-spaces. These ideas and results are expected to be a source for researchers in the area of rough convergence of sequences. Also, these concepts can be generalized and applied for further studies.

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