



On holomorphic mappings with relatively p -compact range

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Abstract. Related to the concept of p -compact operators with $p \in [1, \infty]$ introduced by Sinha and Karn [20], this paper deals with the space $\mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$ of all Banach-valued holomorphic mappings on an open subset U of a complex Banach space E whose ranges are relatively p -compact subsets of F . We characterize such holomorphic mappings as those whose Mujica's linearisations on the canonical predual of $\mathcal{H}^\infty(U)$ are p -compact operators. This fact allows us to make a complete study of them. We show that $\mathcal{H}_{\mathcal{K}_p}^\infty$ is a surjective Banach ideal of bounded holomorphic mappings which is generated by composition with the ideal of p -compact operators and contains the Banach ideal of all right p -nuclear holomorphic mappings. We also characterize holomorphic mappings with relatively p -compact ranges as those bounded holomorphic mappings which factorize through a quotient space of ℓ_p^* or as those whose transposes are quasi p -nuclear operators (respectively, factor through a closed subspace of ℓ_p).

1. Introduction

Inspired by classical Grothendieck's characterization of a relatively compact subset of a Banach space as a subset of the convex hull of a norm null sequence of vectors [12], Sinha and Karn [20] introduced and studied p -compact sets and p -compact operators with $p \in [1, \infty]$.

Let E be a complex Banach space with closed unit ball B_E . Let $p \in (1, \infty)$ and let p^* denote the *conjugate index of p* given by $p^* = p/(p - 1)$. Given $p \in [1, \infty)$, $\ell_p(E)$ denotes the Banach space of all absolutely p -summable sequences (x_n) in E , equipped with the norm

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p},$$

and $c_0(E)$ stands for the Banach space of all norm null sequences (x_n) in E , endowed with the norm

$$\|(x_n)\|_\infty = \sup \{ \|x_n\| : n \in \mathbb{N} \}.$$

In the case $E = \mathbb{C}$, we will simply write ℓ_p and c_0 .

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For $p \in (1, \infty)$, the p -convex hull of a sequence $(x_n) \in \ell_p(E)$ is defined by

$$p\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_p} \right\}.$$

Similarly, we set

$$1\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{c_0} \right\}, \quad (x_n) \in \ell_1(E),$$

$$\infty\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_1} \right\}, \quad (x_n) \in c_0(E).$$

Note that $\infty\text{-conv}(x_n)$ coincides with $\overline{\text{abco}}(\{x_n : n \in \mathbb{N}\})$, the norm-closed absolutely convex hull of the set $\{x_n : n \in \mathbb{N}\}$ in E .

Following [20], given $p \in [1, \infty]$, a set $K \subseteq E$ is said to be *relatively p -compact* if there is a sequence $(x_n) \in \ell_p(E)$ ($(x_n) \in c_0(E)$ if $p = \infty$) such that $K \subseteq p\text{-conv}(x_n)$. Such a sequence is not unique but a *measure of the size of the p -compact set K* is introduced in [10, p. 297] (see also [14]) defining

$$m_p(K, E) = \begin{cases} \inf\{\|(x_n)\|_p : (x_n) \in \ell_p(E), K \subseteq p\text{-conv}(x_n)\} & \text{if } 1 \leq p < \infty, \\ \inf\{\|(x_n)\|_p : (x_n) \in c_0(E), K \subseteq p\text{-conv}(x_n)\} & \text{if } p = \infty, \end{cases}$$

and $m_p(K, E) = \infty$ if K is not p -compact. We frequently will write $m_p(K)$ instead of $m_p(K, E)$.

A linear operator between Banach spaces $T: E \rightarrow F$ is said to be *p -compact* if $T(B_E)$ is a relatively p -compact subset of F . The space of all p -compact linear operators from E to F is denoted by $\mathcal{K}_p(E, F)$, and \mathcal{K}_p is a Banach operator ideal endowed with the norm $k_p(T) = m_p(T(B_E))$ (see [20, Theorem 4.2] and [10, Proposition 3.15]).

From the cited Grothendieck’s result [12], a set $K \subseteq E$ is relatively compact if and only if for every $\varepsilon > 0$, there is a sequence $(x_n) \in c_0(E)$ with $\|(x_n)\|_{\infty} \leq \sup_{x \in K} \|x\| + \varepsilon$ such that $K \subseteq \infty\text{-conv}(x_n)$. Hence we can consider compact sets as ∞ -compact sets and, in this way, compact operators can be viewed as ∞ -compact operators with k_{∞} being the usual operator norm.

The work of Sinha and Karn [20] motivated many papers on p -compactness in operator spaces (see [3, 8–11, 14, 19], among others) and also in Lipschitz spaces [1, 2].

The extension of this theory to the polynomial and holomorphic settings was addressed in [5, 6]. In these environments, the property of p -compactness was studied from the following local point of view: a mapping $f: U \rightarrow F$ is said to be *locally p -compact* if every point $x \in U$ has a neighborhood $U_x \subseteq U$ such that $f(U_x)$ is relatively p -compact in F .

The aim of this note is to study a subclass of locally p -compact holomorphic mappings, namely, *holomorphic mappings with relatively p -compact range*. Notice that every such mapping is locally p -compact but the converse is not true. For instance, if \mathbb{D} denotes the open complex unit disc, the holomorphic mapping $f: \mathbb{D} \rightarrow c_0$ defined by $f(z) = \sum_{n=1}^{\infty} z^n e_n$, where (e_n) is the canonical basis of ℓ_1 , is locally 1-compact but it has not relatively compact range (see [15, Example 3.2]) and, consequently, neither relatively p -compact range for any $p \geq 1$.

Our motivation to deal with this class of mappings also arises from the study (initiated in [15] and continued in [7, 13]) on the Banach space $\mathcal{H}_{\mathcal{K}}^{\infty}(U, F)$ formed by all holomorphic mappings from U to F with relatively compact range, equipped with the supremum norm.

We now briefly describe the content of this paper. Let E and F be complex Banach spaces, U an open subset of E and $p \in [1, \infty]$. Let $\mathcal{H}^{\infty}(U, F)$ denote the Banach space of all bounded holomorphic mappings from U into F , endowed with the supremum norm. In particular, $\mathcal{H}^{\infty}(U)$ stands for $\mathcal{H}^{\infty}(U, \mathbb{C})$.

In [15], Mujica provided a linearisation method of the members of $\mathcal{H}^{\infty}(U, F)$, which will be an essential tool in our analysis of the subject. If $\mathcal{G}^{\infty}(U)$ is the canonical predual of $\mathcal{H}^{\infty}(U)$ obtained by Mujica [15] via an

identification denoted g_U , we will establish that a bounded holomorphic mapping $f: U \rightarrow F$ has relatively p -compact range if and only if Mujica's linearisation $T_f: \mathcal{G}^\infty(U) \rightarrow F$ is a p -compact operator. This fact has some interesting applications.

If $\mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$ denotes the space of all holomorphic mappings with relatively p -compact range $f: U \rightarrow F$ with the natural norm $k_p^{\mathcal{H}^\infty}(f) = m_p(f(U))$, we will prove that $\mathcal{H}_{\mathcal{K}_p}^\infty$ is a surjective Banach ideal of bounded holomorphic mappings which is generated by composition with the ideal \mathcal{K}_p of p -compact operators. This means that each mapping $f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$ admits a factorization $f = T \circ g$, where G is a complex Banach space, $g \in \mathcal{H}^\infty(U, G)$ and $T \in \mathcal{K}_p(G, F)$. Moreover, $k_p^{\mathcal{H}^\infty}(f)$ coincides with $\inf\{k_p(T) \|g\|_\infty\}$, where the infimum is extended over all such factorizations of f and, curiously, this infimum is attained at the factorization $f = T_f \circ g_U$ due to Mujica [15].

In parallelism with the linear case, we introduce the notion of right p -nuclear holomorphic mapping from U to F , study its linearisation on $\mathcal{G}^\infty(U)$, analyse its ideal property and show that every right p -nuclear holomorphic mapping has relatively p -compact range.

Moreover, we characterize the members of $\mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$ as those bounded holomorphic mappings from U to F which factorize through a quotient space of ℓ_p , and also as those whose transposes are quasi p -nuclear operators (respectively, factor through a closed subspace of ℓ_p).

2. The results

From now on, unless otherwise stated, E and F will denote complex Banach spaces, U will be an open subset of E and $p \in [1, \infty]$.

As usual, $\mathcal{L}(E, F)$ denotes the Banach space of all bounded linear operators from E to F endowed with the operator canonical norm and E^* stands for the dual space of E . The subspaces of $\mathcal{L}(E, F)$ consisting of all compact operators and finite-rank bounded operators from E to F will be denoted by $\mathcal{K}(E, F)$ and $\mathcal{F}(E, F)$, respectively.

In this section, we will study holomorphic mappings $f: U \rightarrow F$ so that $f(U)$ is a relatively p -compact subset of F . We denote by $\mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$ the set formed by such mappings and, for each $f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$, we define $k_p^{\mathcal{H}^\infty}(f) = m_p(f(U))$.

The space of all holomorphic mappings with relatively compact range from U to F , denoted $\mathcal{H}_{\mathcal{K}}^\infty(U, F)$, is a Banach space equipped with the supremum norm (see [13, Corollary 2.11]). On account of the following result, we will only study in this paper the case $1 \leq p < \infty$.

Proposition 2.1. $\mathcal{H}_{\mathcal{K}_\infty}^\infty(U, F) = \mathcal{H}_{\mathcal{K}}^\infty(U, F)$ and $k_\infty^{\mathcal{H}^\infty}(f) = \|f\|_\infty$ for all $\mathcal{H}_{\mathcal{K}_\infty}^\infty(U, F)$.

Proof. Let $f \in \mathcal{H}_{\mathcal{K}_\infty}^\infty(U, F)$ and let $(y_n) \in c_0(F)$ be such that $f(U) \subseteq \infty\text{-conv}(y_n)$. Since $\infty\text{-conv}(y_n)$ is relatively compact in F , it follows that $f \in \mathcal{H}_{\mathcal{K}}^\infty(U, F)$ with $\|f\|_\infty \leq \|(y_n)\|_\infty$, and taking infimum over all such sequences (y_n) , we have $\|f\|_\infty \leq k_\infty^{\mathcal{H}^\infty}(f)$.

Conversely, let $f \in \mathcal{H}_{\mathcal{K}}^\infty(U, F)$. By classical Grothendieck's result, for every $\varepsilon > 0$, there exists $(y_n) \in c_0(F)$ with $\|(y_n)\|_\infty \leq \|f\|_\infty + \varepsilon$ such that $f(U) \subseteq \infty\text{-conv}(y_n)$. Hence $f \in \mathcal{H}_{\mathcal{K}_\infty}^\infty(U, F)$ and $k_\infty^{\mathcal{H}^\infty}(f) \leq \|f\|_\infty$. \square

2.1. Linearisation

Our first aim is to characterize holomorphic mappings with relatively p -compact range in terms of the p -compactness of their linearisations on the canonical preduel of $\mathcal{H}^\infty(U)$.

Towards this end, we first recall the following result due to Mujica [15] concerning linearisation of holomorphic mappings on Banach spaces.

Theorem 2.2. [15] Let E be a complex Banach space and U be an open set in E . Let $\mathcal{G}^\infty(U)$ denote the norm-closed linear subspace of $\mathcal{H}^\infty(U)^*$ generated by the functionals $\delta(x) \in \mathcal{H}^\infty(U)^*$ with $x \in U$, defined by $\delta(x)(f) = f(x)$ for all $f \in \mathcal{H}^\infty(U)$.

- (i). The mapping $g_U: U \rightarrow \mathcal{G}^\infty(U)$ defined by $g_U(x) = \delta(x)$ is holomorphic with $\|\delta(x)\| = 1$ for all $x \in U$.
- (ii). For every complex Banach space F and every mapping $f \in \mathcal{H}^\infty(U, F)$, there exists a unique operator $T_f \in \mathcal{L}(\mathcal{G}^\infty(U), F)$ such that $T_f \circ g_U = f$. Furthermore, $\|T_f\| = \|f\|_\infty$.
- (iii). For every complex Banach space F , the mapping $f \mapsto T_f$ is an isometric isomorphism from $\mathcal{H}^\infty(U, F)$ onto $\mathcal{L}(\mathcal{G}^\infty(U), F)$.
- (iv). $\mathcal{H}^\infty(U)$ is isometrically isomorphic to $\mathcal{G}^\infty(U)^*$, via the mapping $J_U: \mathcal{H}^\infty(U) \rightarrow \mathcal{G}^\infty(U)^*$ given by $J_U(f)(g_U(x)) = f(x)$ for all $f \in \mathcal{H}^\infty(U)$ and $x \in U$.
- (v). $B_{\mathcal{G}^\infty(U)}$ coincides with $\overline{\text{abco}}(g_U(U))$. □

We are now ready to state the aforementioned characterization.

Theorem 2.3. Let $p \in [1, \infty)$ and $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent:

- (i). f has relatively p -compact range.
- (ii). $T_f: \mathcal{G}^\infty(U) \rightarrow F$ is a p -compact linear operator.

In this case, $k_p^{\mathcal{H}^\infty}(f) = k_p(T_f)$. Furthermore, $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{H}_{\mathcal{K}_p}^\infty(U, F), k_p^{\mathcal{H}^\infty})$ onto $(\mathcal{K}_p(\mathcal{G}^\infty(U), F), k_p)$.

Proof. Using Theorem 2.2, we have the following inclusions:

$$\begin{aligned} f(U) &= T_f \circ g_U(U) \subseteq T_f(\overline{\text{abco}}(g_U(U))) = T_f(B_{\mathcal{G}^\infty(U)}) \\ &\subseteq \overline{\text{abco}}(T_f \circ g_U(U)) = \overline{\text{abco}}(f(U)). \end{aligned}$$

(ii) \Rightarrow (i): If $T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U), F)$, then $f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$ with

$$k_p^{\mathcal{H}^\infty}(f) = m_p(f(U)) \leq m_p(T_f(B_{\mathcal{G}^\infty(U)})) = k_p(T_f),$$

by the first inclusion above.

(i) \Rightarrow (ii): If $f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$, then $T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U), F)$ with

$$k_p(T_f) = m_p(T_f(B_{\mathcal{G}^\infty(U)})) \leq m_p(\overline{\text{abco}}(f(U))) = m_p(f(U)) = k_p^{\mathcal{H}^\infty}(f),$$

by the second inclusion above and the known fact that a set $K \subseteq F$ is p -compact if and only if $\overline{\text{abco}}(K)$ is p -compact, in whose case $m_p(K) = m_p(\overline{\text{abco}}(K))$.

The last assertion of the statement follows immediately from part (iii) of Theorem 2.2 and from the above proof. □

2.2. Banach ideal property

Our next goal is to study the Banach ideal structure of $(\mathcal{H}_{\mathcal{K}_p}^\infty, k_p^{\mathcal{H}^\infty})$. Inspired by the notion of Banach operator ideal [18], the following type of ideals was considered in [7].

An ideal of bounded holomorphic mappings (or simply, a bounded-holomorphic ideal) is a subclass $\mathcal{I}^{\mathcal{H}^\infty}$ of the class of bounded holomorphic mappings \mathcal{H}^∞ such that for each complex Banach space E , each open subset U of E and each complex Banach space F , the components

$$\mathcal{I}^{\mathcal{H}^\infty}(U, F) := \mathcal{I}^{\mathcal{H}^\infty} \cap \mathcal{H}^\infty(U, F)$$

satisfy the following three conditions:

- (I1) $\mathcal{I}^{\mathcal{H}^\infty}(U, F)$ is a linear subspace of $\mathcal{H}^\infty(U, F)$,
- (I2) For any $g \in \mathcal{H}^\infty(U)$ and $y \in F$, the mapping $g \cdot y: x \mapsto g(x)y$ from U to F is in $\mathcal{I}^{\mathcal{H}^\infty}(U, F)$,

(I3) *The ideal property:* if H, G are complex Banach spaces, V is an open subset of H , $h \in \mathcal{H}(V, U)$, $f \in \mathcal{I}^{\mathcal{H}^\infty}(U, F)$ and $S \in \mathcal{L}(F, G)$, then $S \circ f \circ h \in \mathcal{I}^{\mathcal{H}^\infty}(V, G)$.

Suppose that a function $\|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty}} : \mathcal{I}^{\mathcal{H}^\infty} \rightarrow \mathbb{R}_0^+$ satisfies the following three properties:

- (N1) $(\mathcal{I}^{\mathcal{H}^\infty}(U, F), \|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty}})$ is a normed (Banach) space with $\|f\|_\infty \leq \|f\|_{\mathcal{I}^{\mathcal{H}^\infty}}$ for all $f \in \mathcal{I}^{\mathcal{H}^\infty}(U, F)$,
- (N2) $\|g \cdot y\|_{\mathcal{I}^{\mathcal{H}^\infty}} = \|g\|_\infty \|y\|$ for all $g \in \mathcal{H}^\infty(U)$ and $y \in F$,
- (N3) If H, G are complex Banach spaces, V is an open subset of H , $h \in \mathcal{H}(V, U)$, $f \in \mathcal{I}^{\mathcal{H}^\infty}(U, F)$ and $S \in \mathcal{L}(F, G)$, then $\|S \circ f \circ h\|_{\mathcal{I}^{\mathcal{H}^\infty}} \leq \|S\| \|f\|_{\mathcal{I}^{\mathcal{H}^\infty}}$.

Then $(\mathcal{I}^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty}})$ is called a *normed (Banach) bounded-holomorphic ideal*.

A normed bounded-holomorphic ideal $\mathcal{I}^{\mathcal{H}^\infty}$ is said to be:

- (R) *regular* if for any $f \in \mathcal{H}^\infty(U, F)$, we have that $f \in \mathcal{I}^{\mathcal{H}^\infty}(U, F)$ with $\|f\|_{\mathcal{I}^{\mathcal{H}^\infty}} = \|\kappa_F \circ f\|_{\mathcal{I}^{\mathcal{H}^\infty}}$ whenever $\kappa_F \circ f \in \mathcal{I}^{\mathcal{H}^\infty}(U, F^{**})$, where κ_F denotes the isometric linear embedding from F into F^{**} .
- (S) *surjective* if for any mapping $f \in \mathcal{H}^\infty(U, F)$, any open subset V of a complex Banach space G and any surjective mapping $\pi \in \mathcal{H}(V, U)$, we have that $f \in \mathcal{I}^{\mathcal{H}^\infty}(U, F)$ with $\|f\|_{\mathcal{I}^{\mathcal{H}^\infty}} = \|f \circ \pi\|_{\mathcal{I}^{\mathcal{H}^\infty}}$ whenever $f \circ \pi \in \mathcal{I}^{\mathcal{H}^\infty}(V, F)$.

Bearing in mind Theorem 2.3, Theorem 3.2 in [4] (see also Theorem 2.4 in [7]) shows that $\mathcal{H}_{\mathcal{K}_p}^\infty$ is generated by composition with the operator ideal \mathcal{K}_p (see [7, Definition 2.3]).

Corollary 2.4. *Let $p \in [1, \infty)$ and $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent:*

- (i). $f : U \rightarrow F$ has relatively p -compact range.
- (ii). $f = T \circ g$ for some complex Banach space G , $g \in \mathcal{H}^\infty(U, G)$ and $T \in \mathcal{K}_p(G, F)$.

In this case, we have

$$k_p^{\mathcal{H}^\infty}(f) = \|f\|_{\mathcal{K}_p \circ \mathcal{H}^\infty} := \inf\{k_p(T) \|g\|_\infty\},$$

where the infimum runs over all factorizations of f as in (ii), and this infimum is attained at $T_f \circ g_U$ (Mujica’s factorization of f [15]).

Furthermore, $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{K}_p \circ \mathcal{H}^\infty(U, F), \|\cdot\|_{\mathcal{K}_p \circ \mathcal{H}^\infty})$ onto $(\mathcal{K}_p(\mathcal{G}^\infty(U), F), k_p)$. \square

The following result gathers some Banach ideal properties of $\mathcal{H}_{\mathcal{K}_p}^\infty$.

Theorem 2.5. *For each $p \in [1, \infty)$, $(\mathcal{H}_{\mathcal{K}_p}^\infty, k_p^{\mathcal{H}^\infty})$ is a surjective Banach bounded-holomorphic ideal. Furthermore, the ideal $(\mathcal{H}_{\mathcal{K}_p}^\infty(U, F), k_p^{\mathcal{H}^\infty})$ is regular whenever F is reflexive.*

Proof. In view of Corollary 2.4, Corollary 2.5 in [7] yields that $(\mathcal{H}_{\mathcal{K}_p}^\infty, k_p^{\mathcal{H}^\infty})$ is a Banach bounded-holomorphic ideal. Then we only have to study its surjectivity and its regularity.

(S) Let $f \in \mathcal{H}^\infty(U, F)$ and assume that $f \circ \pi \in \mathcal{H}_{\mathcal{K}_p}^\infty(V, F)$, where V is an open subset of a complex Banach space G and $\pi \in \mathcal{H}(V, U)$ is surjective. Since $f(U) = (f \circ \pi)(V)$, it is immediate that $f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$ with $k_p^{\mathcal{H}^\infty}(f) = k_p^{\mathcal{H}^\infty}(f \circ \pi)$. Hence $(\mathcal{H}_{\mathcal{K}_p}^\infty, k_p^{\mathcal{H}^\infty})$ is surjective.

(R) Suppose now that F is reflexive. Let $f \in \mathcal{H}^\infty(U, F)$ and assume that $\kappa_F \circ f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F^{**})$. We can take a sequence (y_n) in $\ell_p(F)$ (in $c_0(F)$ if $p = 1$) such that $(\kappa_F \circ f)(U) \subseteq p\text{-conv}(\kappa_F(y_n))$, that is, $\kappa_F(f(U)) \subseteq \kappa_F(p\text{-conv}(y_n))$ which yields $f(U) \subseteq p\text{-conv}(y_n)$ by the injectivity of κ_F . Hence $f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$ with $k_p^{\mathcal{H}^\infty}(f) \leq \|(y_n)\|_p = \|(\kappa_F(y_n))\|_p$ and so $k_p^{\mathcal{H}^\infty}(f) \leq k_p^{\mathcal{H}^\infty}(\kappa_F \circ f)$ by extending this infimum over all such sequences $(\kappa_F(y_n))$. The converse inequality follows from the condition (N3) satisfied by $(\mathcal{H}_{\mathcal{K}_p}^\infty, k_p^{\mathcal{H}^\infty})$ and this completes the proof. \square

2.3. Factorization

We now present a factorization result for holomorphic mappings with relatively p -compact range which should be compared with [11, Proposition 2.9].

Corollary 2.6. *Let $p \in [1, \infty)$ and $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent:*

- (i). $f: U \rightarrow F$ has relatively p -compact range.
- (ii). There exist a closed subspace M in ℓ_{p^*} (c_0 instead of ℓ_{p^*} if $p = 1$), a separable Banach space G , an operator T in $\mathcal{K}_p(\ell_{p^*}/M, G)$, a mapping g in $\mathcal{H}_{\mathcal{K}}^\infty(U, \ell_{p^*}/M)$ and an operator S in $\mathcal{K}(G, F)$ such that $f = S \circ T \circ g$.

In this case, $k_p^{\mathcal{H}^\infty}(f) = \inf\{\|S\| k_p(T) \|g\|_\infty\}$, where the infimum is extended over all factorizations of f as in (ii).

Proof. We will only prove it for $p \in (1, \infty)$. The case $p = 1$ is similarly obtained.

(i) \Rightarrow (ii): Suppose that $f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$. By Theorem 2.3, $T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U), F)$ with $k_p(T_f) = k_p^{\mathcal{H}^\infty}(f)$. Applying [11, Proposition 2.9], for each $\varepsilon > 0$, there exist a closed subspace $M \subseteq \ell_{p^*}$, a separable Banach space G , an operator $T \in \mathcal{K}_p(\ell_{p^*}/M, G)$, an operator $S \in \mathcal{K}(G, F)$ and an operator $R \in \mathcal{K}(\mathcal{G}^\infty(U), \ell_{p^*}/M)$ such that $T_f = S \circ T \circ R$ with $\|S\| k_p(T) \|R\| \leq k_p(T_f) + \varepsilon$. Moreover, $R = T_g$ with $\|g\|_\infty = \|R\|$ for some $g \in \mathcal{H}_{\mathcal{K}}^\infty(U, \ell_{p^*}/M)$ by [13, Corollary 2.11]. Thus we obtain

$$f = T_f \circ g_U = S \circ T \circ R \circ g_U = S \circ T \circ T_g \circ g_U = S \circ T \circ g,$$

with

$$\|S\| k_p(T) \|g\|_\infty = \|S\| k_p(T) \|R\| \leq k_p(T_f) + \varepsilon = k_p^{\mathcal{H}^\infty}(f) + \varepsilon.$$

Since ε was arbitrary, we deduce that $\|S\| k_p(T) \|g\|_\infty \leq k_p^{\mathcal{H}^\infty}(f)$.

(ii) \Rightarrow (i): Assume that $f = S \circ T \circ g$ is a factorization as in (ii). Since $S \circ T \in \mathcal{K}_p(\ell_{p^*}/M, F)$ by the ideal property of \mathcal{K}_p , Corollary 2.4 yields that $f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$ with

$$k_p^{\mathcal{H}^\infty}(f) \leq k_p(S \circ T) \|g\|_\infty \leq \|S\| k_p(T) \|g\|_\infty,$$

and taking infimum over all such factorizations of f , we have $k_p^{\mathcal{H}^\infty}(f) \leq \inf\{\|S\| k_p(T) \|g\|_\infty\}$. \square

2.4. Transposition

Let us recall that the *transpose* of a mapping $f \in \mathcal{H}^\infty(U, F)$ is the bounded linear operator $f^t: F^* \rightarrow \mathcal{H}^\infty(U)$ defined by

$$f^t(y^*) = y^* \circ f \quad (y^* \in F^*).$$

Moreover, $\|f^t\| = \|f\|_\infty$ and $f^t = J_U^{-1} \circ (T_f)^*$, where $J_U: \mathcal{H}^\infty(U) \rightarrow \mathcal{G}^\infty(U)^*$ is the isometric isomorphism defined in Theorem 2.2.

The Banach ideal \mathcal{K}_p is associated by duality with the ideal of quasi- p -nuclear operators. According to [17], for every $p \in [1, \infty)$, an operator $T \in \mathcal{L}(E, F)$ is said to be *quasi p -nuclear* if there is a sequence $(x_n^*) \in \ell_p(E^*)$ such that

$$\|T(x)\| \leq \left(\sum_{n=1}^{\infty} |x_n^*(x)|^p \right)^{1/p} \quad (x \in E).$$

If $\mathcal{QN}_p(E, F)$ denotes the set formed by such operators, then \mathcal{QN}_p is a Banach operator ideal equipped with the norm

$$v_p^{\mathcal{Q}}(T) = \inf \left\{ \|(x_n^*)\|_p : \|T(x)\| \leq \left(\sum_{n=1}^{\infty} |x_n^*(x)|^p \right)^{1/p}, \forall x \in E \right\}.$$

By [10, Proposition 3.8], an operator $T \in \mathcal{K}_p(E, F)$ if and only if its adjoint $T^* \in \mathcal{QN}_p(F^*, E^*)$. Moreover, $k_p(T) = v_p^{\mathcal{Q}}(T^*)$ by [11, Corollary 2.7]. A holomorphic version of this result can be stated as follows.

Theorem 2.7. Let $p \in [1, \infty)$ and $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent:

- (i). $f: U \rightarrow F$ has relatively p -compact range.
- (ii). $f^t: F^* \rightarrow \mathcal{H}^\infty(U)$ is a quasi p -nuclear operator.

In this case, $k_p^{\mathcal{H}^\infty}(f) = v_p^Q(f^t)$.

Proof. Applying Theorem 2.3, [11, Corollary 2.7] and [17, p. 32], respectively, we have

$$\begin{aligned} f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F) &\Leftrightarrow T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U), F) \\ &\Leftrightarrow (T_f)^* \in \mathcal{QN}_p(F^*, \mathcal{G}^\infty(U)^*) \\ &\Leftrightarrow f^t = J_U^{-1} \circ (T_f)^* \in \mathcal{QN}_p(F^*, \mathcal{H}^\infty(U)). \end{aligned}$$

In this case, $k_p^{\mathcal{H}^\infty}(f) = k_p(T_f) = v_p^Q((T_f)^*) = v_p^Q(f^t)$. \square

Given $p \in [1, \infty)$, let us recall (see [18]) that an operator $T \in \mathcal{L}(E, F)$ is p -summing if there exists a constant $C \geq 0$ such that, regardless of the natural number n and regardless of the choice of vectors x_1, \dots, x_n in E , we have

$$\left(\sum_{i=1}^n \|T(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^n |x^*(x_i)|^p \right)^{\frac{1}{p}}.$$

The infimum of such constants C is denoted by $\pi_p(T)$ and the linear space of all p -summing operators from E into F by $\Pi_p(E, F)$.

The following result could be compared with [10, Proposition 3.13] stated in the linear setting.

Proposition 2.8. Let $f \in \mathcal{H}^\infty(U, F)$ and $g \in \mathcal{H}_{\mathcal{K}}^\infty(U, F^*)$. Assume that $T_f \in \Pi_p(\mathcal{G}^\infty(U), F)$ with $p \in [1, \infty)$. Then $f^t \circ g \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, \mathcal{H}^\infty(U))$ with $k_p^{\mathcal{H}^\infty}(f^t \circ g) \leq \pi_p(T_f) \|g\|_\infty$.

Proof. By Theorem 2.3 and Proposition 2.1, $T_g \in \mathcal{K}(\mathcal{G}^\infty(U), F^*)$ with $\|T_g\| = \|g\|_\infty$. Consequently, by [10, Proposition 3.13], the linear operator $(T_f)^* \circ T_g \in \mathcal{K}_p(\mathcal{G}^\infty(U), \mathcal{G}^\infty(U)^*)$ with $k_p((T_f)^* \circ T_g) \leq \pi_p(T_f) \|T_g\|$. From the equality $f^t \circ T_g = J_U^{-1} \circ (T_f)^* \circ T_g$, we infer that $f^t \circ T_g \in \mathcal{K}_p(\mathcal{G}^\infty(U), \mathcal{H}^\infty(U))$ with $k_p(f^t \circ T_g) = k_p((T_f)^* \circ T_g)$ by the ideal property of \mathcal{K}_p . Applying Theorem 2.3, there exists $h \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, \mathcal{H}^\infty(U))$ with $k_p^{\mathcal{H}^\infty}(h) = k_p(T_h)$ such that $f^t \circ T_g = T_h$. Hence $f^t \circ g = h$ and thus $f^t \circ g \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, \mathcal{H}^\infty(U))$ with $k_p^{\mathcal{H}^\infty}(f^t \circ g) = k_p(T_h) = k_p((T_f)^* \circ T_g) \leq \pi_p(T_f) \|T_g\| = \pi_p(T_f) \|g\|_\infty$. \square

p -Compact operators were characterized as those operators whose adjoints factor through a subspace of ℓ_p [20, Theorem 3.2]. We now obtain a similar factorization for the transpose of a holomorphic mapping with relatively p -compact range (compare also to [10, Proposition 3.10]).

Corollary 2.9. Let $p \in [1, \infty)$ and $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent:

- (i). $f: U \rightarrow F$ has relatively p -compact range.
- (ii). There exist a closed subspace $M \subseteq \ell_p$ and $R \in \mathcal{QN}_p(F^*, M)$, $S \in \mathcal{L}(M, \mathcal{H}^\infty(U))$ such that $f^t = S \circ R$.

Proof. (i) \Rightarrow (ii): If $f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$, then $T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U), F)$ by Theorem 2.3. By [10, Proposition 3.10], there exist a closed subspace $M \subseteq \ell_p$ and operators $R \in \mathcal{QN}_p(F^*, M)$ and $S_0 \in \mathcal{L}(M, \mathcal{G}^\infty(U)^*)$ such that $(T_f)^* = S_0 \circ R$. Taking $S = J_U^{-1} \circ S_0 \in \mathcal{L}(M, \mathcal{H}^\infty(U))$, we have $f^t = S \circ R$.

(ii) \Rightarrow (i): Assume that $f^t = S \circ R$ being S and R as in the statement. It follows that $(T_f)^* = J_U \circ f^t = J_U \circ S \circ R$, and thus $T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U), F)$ by [10, Proposition 3.10]. Hence $f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$ by Theorem 2.3 \square

2.5. Inclusions

We will study the inclusion relations of holomorphic mappings with relatively p -compact range between them and with other classes of bounded holomorphic mappings.

Our first result follows immediately by applying Theorem 2.3 and the fact stated in [20, Proposition 4.3] that $\mathcal{K}_p \subseteq \mathcal{K}_q$ whenever $1 \leq p \leq q \leq \infty$.

Corollary 2.10. *If $1 \leq p \leq q < \infty$ and $f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$, then $f \in \mathcal{H}_{\mathcal{K}_q}^\infty(U, F)$ and $k_q^{\mathcal{H}^\infty}(f) \leq k_p^{\mathcal{H}^\infty}(f)$. □*

Let us recall that a mapping $f \in \mathcal{H}^\infty(U, F)$ has *finite dimensional rank* if the linear hull of its range is a finite dimensional subspace of F . We denote by $\mathcal{H}_{\mathcal{F}}^\infty(U, F)$ the set of all finite-rank bounded holomorphic mappings from U to F . In the light of Theorem 2.5, it is clear that $\mathcal{H}_{\mathcal{F}}^\infty(U, F)$ is a linear subspace of $\mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$.

In similarity with the linear case, it seems natural to introduce the following class of holomorphic mappings.

Definition 2.11. *Let $p \in [1, \infty)$. A mapping $f \in \mathcal{H}^\infty(U, F)$ is said to be p -approximable if there exists a sequence (f_n) in $\mathcal{H}_{\mathcal{F}}^\infty(U, F)$ such that $k_p^{\mathcal{H}^\infty}(f_n - f) \rightarrow 0$ as $n \rightarrow \infty$. We denote by $\mathcal{H}_{\mathcal{F}_p}^\infty(U, F)$ the space of all p -approximable holomorphic mappings from U to F .*

Proposition 2.12. *For $p \in [1, \infty)$, every p -approximable holomorphic mapping from U to F has relatively p -compact range.*

Proof. Let $f \in \mathcal{H}_{\mathcal{F}_p}^\infty(U, F)$. Hence there is a sequence (f_n) in $\mathcal{H}_{\mathcal{F}}^\infty(U, F)$ such that $k_p^{\mathcal{H}^\infty}(f_n - f) \rightarrow 0$ as $n \rightarrow \infty$. Since $T_{f_n} \in \mathcal{F}(\mathcal{G}^\infty(U), F)$ by [15, Proposition 3.1], $\mathcal{F}(\mathcal{G}^\infty(U), F) \subseteq \mathcal{K}_p(\mathcal{G}^\infty(U), F)$ by [20, Theorem 4.2] and $k_p(T_{f_n} - T_f) = k_p(T_{f_n - f}) = k_p^{\mathcal{H}^\infty}(f_n - f)$ for all $n \in \mathbb{N}$ by Theorems 2.2 and 2.3, we deduce that $T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U), F)$ by [20, Theorem 4.2], and so $f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$ by Theorem 2.3. □

Given $p \in [1, \infty)$, $\ell_p^{\text{weak}}(E)$ denotes the Banach space of all weakly p -summable sequences (x_n) in E , endowed with the norm

$$\|(x_n)\|_p^{\text{weak}} = \sup \left\{ \left(\sum_{n=1}^{\infty} |f(x_n)|^p \right)^{1/p} : f \in B_{E^*} \right\}.$$

Let us recall (see [16]) that an operator $T \in \mathcal{L}(E, F)$ is said to be *right p -nuclear* if there are sequences $(x_n^*) \in \ell_{p^*}^{\text{weak}}(E^*)$ and $(y_n) \in \ell_p(F)$ such that

$$T(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n \quad (x \in E),$$

and the series converges in $\mathcal{L}(E, F)$. The set of such operators, denoted $\mathcal{N}^p(E, F)$, is a Banach space with the norm

$$\nu^p(T) = \inf \left\{ \|(x_n^*)\|_{p^*}^{\text{weak}} \|(y_n)\|_p \right\},$$

where the infimum is taken over all representations of T as above.

A holomorphic variant of this class of operators can be introduced as follows.

Definition 2.13. *Given $p \in [1, \infty)$, a holomorphic mapping $f: U \rightarrow F$ is said to be *right p -nuclear* if there exist sequences (g_n) in $\ell_{p^*}^{\text{weak}}(\mathcal{H}^\infty(U))$ and (y_n) in $\ell_p(F)$ such that $f = \sum_{n=1}^{\infty} g_n \cdot y_n$ in $(\mathcal{H}^\infty(U, F), \|\cdot\|_\infty)$. We set*

$$\nu^{p\mathcal{H}^\infty}(f) = \inf \left\{ \|(g_n)\|_{p^*}^{\text{weak}} \|(y_n)\|_p \right\},$$

with the infimum taken over all right p -nuclear holomorphic representations of f as above. Let $\mathcal{H}_{\mathcal{N}^p}^\infty(U, F)$ denote the set of all right p -nuclear holomorphic mappings from U into F .

We now establish the relationships of a right p -nuclear holomorphic mapping $f: U \rightarrow F$ with its linearisation $T_f: \mathcal{G}^\infty(U) \rightarrow F$.

Theorem 2.14. *Let $p \in [1, \infty)$ and $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent:*

- (i). $f: U \rightarrow F$ is right p -nuclear.
- (ii). $T_f: \mathcal{G}^\infty(U) \rightarrow F$ is a right p -nuclear operator.

In this case, $v^{p\mathcal{H}^\infty}(f) = v^p(T_f)$. Furthermore, $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{H}_{\mathcal{N}^p}^\infty(U, F), v^{p\mathcal{H}^\infty})$ onto $(\mathcal{N}^p(\mathcal{G}^\infty(U), F), v^p)$.

Proof. (i) \Rightarrow (ii): Assume that $f \in \mathcal{H}_{\mathcal{N}^p}^\infty(U, F)$ and let $\sum_{n \geq 1} g_n \cdot y_n$ be a right p -nuclear holomorphic representation of f . By Theorem 2.2, there is a unique operator $T_f \in \mathcal{L}(\mathcal{G}^\infty(U, F))$ such that $T_f \circ g_U = f$. Similarly, for each $n \in \mathbb{N}$, there is a functional $T_{g_n} \in \mathcal{G}^\infty(U)^*$ with $\|T_{g_n}\| = \|g_n\|_\infty$ such that $T_{g_n} \circ g_U = g_n$. Notice that $\sum_{n=1}^{+\infty} T_{g_n} \cdot y_n \in \mathcal{L}(\mathcal{G}^\infty(U), F)$ since

$$\sum_{k=1}^m \|T_{g_k} \cdot y_k\| = \sum_{k=1}^m \|T_{g_k}\| \|y_k\| = \sum_{k=1}^m \|g_k\|_\infty \|y_k\| \leq \|(g_n)\|_{p^*}^{\text{weak}} \|(y_n)\|_p,$$

for all $m \in \mathbb{N}$. We can write

$$f = \sum_{n=1}^{\infty} g_n \cdot y_n = \sum_{n=1}^{\infty} (T_{g_n} \circ g_U) \cdot y_n = \left(\sum_{n=1}^{\infty} T_{g_n} \cdot y_n \right) \circ g_U,$$

in $(\mathcal{H}^\infty(U, F), \|\cdot\|_\infty)$. Hence $T_f = \sum_{n=1}^{\infty} T_{g_n} \cdot y_n$ by Theorem 2.2, where $(T_{g_n}) \in \ell_{p^*}^{\text{weak}}(\mathcal{G}^\infty(U)^*)$ with $\|(T_{g_n})\|_{p^*}^{\text{weak}} \leq \|(g_n)\|_{p^*}^{\text{weak}}$. Therefore $T_f \in \mathcal{N}^p(\mathcal{G}^\infty(U), F)$ with $v^p(T_f) \leq \|(g_n)\|_{p^*}^{\text{weak}} \|(y_n)\|_p$. Taking infimum over all right p -nuclear holomorphic representation of f , we deduce that $v^p(T_f) \leq v^{p\mathcal{H}^\infty}(f)$.

(ii) \Rightarrow (i): Suppose that $T_f \in \mathcal{N}^p(\mathcal{G}^\infty(U), F)$ and let $\sum_{n \geq 1} \phi_n \cdot y_n$ be a right p -nuclear representation of T_f . By Theorem 2.2, for each $n \in \mathbb{N}$, there is a $g_n \in \mathcal{H}^\infty(U)$ such that $J_U(g_n) = \phi_n$ with $\|g_n\|_\infty = \|\phi_n\|$. We have

$$\begin{aligned} \left\| \left(f - \sum_{k=1}^n g_k \cdot y_k \right) (x) \right\| &= \left\| f(x) - \sum_{k=1}^n g_k(x) y_k \right\| = \left\| T_f(g_U(x)) - \sum_{k=1}^n J_U(g_k)(g_U(x)) y_k \right\| \\ &= \left\| \left(T_f - \sum_{k=1}^n \phi_k \cdot y_k \right) (g_U(x)) \right\| \leq \left\| T_f - \sum_{k=1}^n \phi_k \cdot y_k \right\| \|g_U(x)\| \\ &= \left\| T_f - \sum_{k=1}^n \phi_k \cdot y_k \right\|, \end{aligned}$$

for all $x \in U$ and $n \in \mathbb{N}$. Taking supremum over all $x \in U$, we obtain

$$\left\| f - \sum_{k=1}^n g_k \cdot y_k \right\|_\infty \leq \left\| T_f - \sum_{k=1}^n \phi_k \cdot y_k \right\|,$$

for all $n \in \mathbb{N}$. Hence $f = \sum_{n=1}^{\infty} g_n \cdot y_n$ in $(\mathcal{H}^\infty(U, F), \|\cdot\|_\infty)$, where $(g_n) \in \ell_{p^*}^{\text{weak}}(\mathcal{H}^\infty(U))$ with $\|(g_n)\|_{p^*}^{\text{weak}} \leq \|(\phi_n)\|_{p^*}^{\text{weak}}$. So $f \in \mathcal{H}_{\mathcal{N}^p}^\infty(U, F)$ with $v^{p\mathcal{H}^\infty}(f) \leq \|(\phi_n)\|_{p^*}^{\text{weak}} \|(y_n)\|_p$, and this implies that $v^{p\mathcal{H}^\infty}(f) \leq v^p(T_f)$.

The last assertion in the statement follows easily from what was proved above and from Theorem 2.2. \square

Combining Theorem 2.14, firstly with [4, Theorem 3.2] and secondly with [7, Corollary 2.5], we derive the following two results.

Corollary 2.15. *Let $p \in [1, \infty)$ and $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent:*

- (i). $f: U \rightarrow F$ is right p -nuclear.
- (ii). $f = T \circ g$ for some complex Banach space G , $g \in \mathcal{H}^\infty(U, G)$ and $T \in \mathcal{N}^p(G, F)$.

In this case, we have

$$v^p \mathcal{H}^\infty(f) = \|f\|_{\mathcal{N}^p \circ \mathcal{H}^\infty} := \inf\{v^p(T) \|g\|_\infty\},$$

where the infimum is taken over all factorizations of f as in (ii) and this infimum is attained at $T_f \circ g_U$.

As a consequence, the correspondence $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{N}^p \circ \mathcal{H}^\infty(U, F), \|\cdot\|_{\mathcal{N}^p \circ \mathcal{H}^\infty})$ onto $(\mathcal{N}^p(\mathcal{G}^\infty(U), F), v^p)$. □

Corollary 2.16. *For each $p \in [1, \infty)$, $(\mathcal{H}_{\mathcal{N}^p}^\infty, v^p \mathcal{H}^\infty)$ is a Banach bounded-holomorphic ideal.* □

The following relation is readily obtained.

Corollary 2.17. *Let $p \in [1, \infty)$ and $f \in \mathcal{H}_{\mathcal{N}^p}^\infty(U, F)$. Then $f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$ with $k_p^{\mathcal{H}^\infty}(f) \leq v^p \mathcal{H}^\infty(f)$.*

Proof. By Proposition 2.14, we have $T_f \in \mathcal{N}^p(\mathcal{G}^\infty(U), F)$ with $v^p(T_f) = v^p \mathcal{H}^\infty(f)$. It follows that $T_f \in \mathcal{K}_p(\mathcal{G}^\infty(U), F)$ with $k_p(T_f) \leq v^p(T_f)$ (see [10, p. 295]). Hence $f \in \mathcal{H}_{\mathcal{K}_p}^\infty(U, F)$ with $k_p^{\mathcal{H}^\infty}(f) \leq v^p \mathcal{H}^\infty(f)$ by Theorem 2.3. □

Given Banach spaces E, F, G , let us recall that a normed operator ideal \mathcal{I} is *surjective* if for every surjection $Q \in \mathcal{L}(G, E)$ and every $T \in \mathcal{L}(E, F)$, it follows from $T \circ Q \in \mathcal{I}(G, F)$ that $T \in \mathcal{I}(E, F)$ with $\|T\|_{\mathcal{I}} = \|T \circ Q\|_{\mathcal{I}}$. The smallest surjective ideal which contains \mathcal{I} , denoted by \mathcal{I}^{sur} , is called the *surjective hull* of \mathcal{I} .

We now introduce the analogue concept in the holomorphic setting.

Definition 2.18. *The surjective hull of a bounded-holomorphic ideal $\mathcal{I}^{\mathcal{H}^\infty}$ is the smallest surjective ideal which contains $\mathcal{I}^{\mathcal{H}^\infty}$ and it is denoted by $(\mathcal{I}^{\mathcal{H}^\infty})^{\text{sur}}$.*

We have seen that $\mathcal{H}_{\mathcal{K}_p}^\infty$ is a surjective bounded-holomorphic ideal which contains $\mathcal{H}_{\mathcal{N}^p}^\infty$, and therefore $(\mathcal{H}_{\mathcal{N}^p}^\infty)^{\text{sur}} \subseteq \mathcal{H}_{\mathcal{K}_p}^\infty$, but we do not know if both sets are equal as it happens in the linear case (see [10, Proposition 3.11]).

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