Filomat 37:24 (2023), 8041–8055
https://doi.org/10.2298/FIL2324041L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Extensions of Fejér type inequalities for GA-convex functions and related results

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Abstract. In this paper, new Fejér-type inequalities for GA-convex functions are obtained. Some mappings related to the Fejér-type inequalities for GA-convex are defined. The properties of these mappings are explored, and as a result, certain known results are refined.

1. Introduction

For convex functions the following double inequality has great significance in literature and is known as Hermite-Hadamard's inequality [20, 21]:

$$\lambda\left(\frac{\nu_1+\nu_2}{2}\right) \leq \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\lambda(\kappa)d\kappa \leq \frac{\lambda(\nu_1)+\lambda(\nu_2)}{2}, \tag{1}$$

where $\lambda : X \longrightarrow \mathbb{R}$, $\emptyset \neq X \subseteq \mathbb{R}$, $\nu_1, \nu_2 \in X$ with $\nu_1 < \nu_2$, is a convex function. If λ is concave, the inequality holds in the other direction.Dragomir [12] defined the following mappings $\mathbb{H}, \mathbb{F} : [0,1] \to \mathbb{R}$

$$\mathbb{H}(\varkappa) = \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\lambda\left(\varkappa\kappa + (1-\varkappa)\left(\frac{\nu_1+\nu_2}{2}\right)\right)d\kappa$$

and

$$\mathbb{F}(\varkappa) = \frac{1}{(\nu_2-\nu_1)^2}\int_{\nu_1}^{\nu_2}\int_{\nu_1}^{\nu_2}\lambda\left(\varkappa\kappa + (1-\varkappa)\sigma\right)d\kappa d\sigma,$$

for a convex function $\lambda : [\nu_1, \nu_2] \to \mathbb{R}$ and the first inequality in (1) have been refined in [12].

Theorem 1.1. *[12] Let $\lambda : [\nu_1, \nu_2] \to \mathbb{R}$ be a convex function on $[\nu_1, \nu_2]$. Then*

(i) $\mathbb{H}$ *is convex on* $[0, 1]$.

2020 *Mathematics Subject Classification*. Primary 05C38, 15A15; Secondary 05A15, 15A18.
Keywords. Hermite-Hadamard Inequality, convex function, GA-convex function, Fejér inequality.
Received: 24 March 2023; Revised: 06 April 2023; Accepted: 15 April 2023
Communicated by Dragan S. Djordjević
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(ii) *The following hold:*

$$\inf_{\varkappa\in[0,1]} \mathbb{H}(\varkappa) = \mathbb{H}(0) = \lambda\left(\frac{\nu_1+\nu_2}{2}\right)$$

$$\sup_{\varkappa\in[0,1]} \mathbb{H}(\varkappa) = \mathbb{H}(1) = \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} \lambda(\kappa)\,d\kappa.$$

(iii) $\mathbb{H}$ *is monotonically increasing on* $[0,1]$.

Theorem 1.2. *[12] Let* $\lambda:[\nu_1,\nu_2]\to\mathbb{R}$ *be a convex function on* $[\nu_1,\nu_2]$. *Then*

(i) $\mathbb{F}\left(\varkappa+\frac{1}{2}\right)=\mathbb{F}\left(\frac{1}{2}-\varkappa\right)$ *for all* $\varkappa\in\left[0,\frac{1}{2}\right]$

(ii) $\mathbb{F}$ *is convex convex on* $[\nu_1,\nu_2]$.

(iii) *The following hold:*

$$\sup_{\varkappa\in[0,1]} \mathbb{F}(\varkappa) = \mathbb{F}(1) = \mathbb{F}(0) = \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} \lambda(\kappa)\,d\kappa$$

$$\inf_{\varkappa\in[0,1]} \mathbb{F}(\varkappa) = \mathbb{F}\left(\frac{1}{2}\right) = \frac{1}{(\nu_2-\nu_1)^2}\int_{\nu_1}^{\nu_2}\int_{\nu_1}^{\nu_2} \lambda\left(\frac{\kappa+\sigma}{2}\right)d\kappa d\sigma.$$

(iv) *The inequality*

$$\lambda\left(\frac{\nu_1+\nu_2}{2}\right) \le \mathbb{F}\left(\frac{1}{2}\right)$$

holds.

(v) $\mathbb{F}$ *is increasing monotonically on* $\left[\frac{1}{2},1\right]$.

(vi) $\mathbb{F}$ *is decreasing monotonically on* $\left[0,\frac{1}{2}\right]$.

(vii) *We have the inequality* $\mathbb{H}(\varkappa)\le\mathbb{F}(\varkappa)$ *for all* $\varkappa\in[0,1]$.

Yang and Hong [45] improved the relationship between the middle and rightmost terms in (1) by constructing the following mapping $\mathbb{P}:[0,1]\to\mathbb{R}$

$$\mathbb{P}(\varkappa) = \frac{1}{2(\nu_2-\nu_1)}\int_{\nu_1}^{\nu_2}\left[\lambda\left(\left(\frac{1+\varkappa}{2}\right)\nu_2+\left(\frac{1-\varkappa}{2}\right)\kappa\right)+\lambda\left(\left(\frac{1+\varkappa}{2}\right)\nu_1+\left(\frac{1-\varkappa}{2}\right)\kappa\right)\right]d\kappa,$$

for a convex function $\lambda:[\nu_1,\nu_2]\to\mathbb{R}$.

Theorem 1.3. *[45] Let* $\lambda:[\nu_1,\nu_2]\to\mathbb{R}$ *be a convex function on* $[\nu_1,\nu_2]$. *Then*

(i) $\mathbb{P}$ *is convex on* $[0,1]$.

(ii) $\mathbb{P}$ *increases monotonically on* $[0,1]$.

(iii) *The following hold*

$$\inf_{\varkappa\in[0,1]} \mathbb{P}(\varkappa) = \mathbb{P}(0) = \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} \lambda(\kappa)\,d\kappa$$

and

$$\sup_{\varkappa\in[0,1]} \mathbb{P}(\varkappa) = \mathbb{P}(1) = \frac{\lambda(\nu_1)+\lambda(\nu_2)}{2}.$$

As a weighted generalization of (1), Fejér [19] established the following double inequality:

Let $\lambda : X \longrightarrow \mathbb{R}$, $\emptyset \neq X \subseteq \mathbb{R}$, $\nu_1, \nu_2 \in X$ with $\nu_1 < \nu_2$ be a convex function and $\zeta : [\nu_1, \nu_2] \to \mathbb{R}$ is non-negative integrable symmetric about $\kappa = \frac{\nu_1+\nu_2}{2}$

$$\lambda\left(\frac{\nu_1+\nu_2}{2}\right)\int_{\nu_1}^{\nu_2}\zeta(\kappa)d\kappa \leq \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}\lambda(\kappa)\zeta(\kappa)d\kappa \leq \frac{\lambda(\nu_1)+\lambda(\nu_2)}{2}\int_{\nu_1}^{\nu_2}\zeta(\kappa)d\kappa. \tag{2}$$

These inequalities have many extensions and generalizations, see [14]-[22] and [23]-[43].

Teseng et al. [40] refined inequalities (2) by defining the following mappings on $[0, 1]$:

$$\mathbb{I}(\varkappa) = \frac{1}{2}\int_{\nu_1}^{\nu_2}\left[\lambda\left(\varkappa\frac{\kappa+\nu_1}{2}+(1-\varkappa)\frac{\nu_1+\nu_2}{2}\right)+\lambda\left(\varkappa\frac{\kappa+\nu_2}{2}+(1-\varkappa)\frac{\nu_1+\nu_2}{2}\right)\right]\zeta(\kappa)d\kappa,$$

$$\mathbb{J}(\varkappa) = \frac{1}{2}\int_{\nu_1}^{\nu_2}\left[\lambda\left(\varkappa\frac{\kappa+\nu_1}{2}+(1-\varkappa)\frac{3\nu_1+\nu_2}{4}\right)+\lambda\left(\varkappa\frac{\kappa+\nu_2}{2}+(1-\varkappa)\frac{\nu_1+3\nu_2}{4}\right)\right]\zeta(\kappa)d\kappa,$$

$$\begin{aligned}\mathbb{M}(\varkappa) = &\frac{1}{2}\int_{\nu_1}^{\frac{\nu_1+\nu_2}{2}}\left[\lambda\left(\varkappa\nu_1+(1-\varkappa)\frac{\kappa+\nu_1}{2}\right)+\lambda\left(\varkappa\frac{\nu_1+\nu_2}{2}+(1-\varkappa)\frac{\kappa+\nu_2}{2}\right)\right]\zeta(\kappa)d\kappa\\ &+\frac{1}{2}\int_{\frac{\nu_1+\nu_2}{2}}^{\nu_2}\left[\lambda\left(\varkappa\frac{\nu_1+\nu_2}{2}+(1-\varkappa)\frac{\kappa+\nu_1}{2}\right)+\lambda\left(\varkappa\nu_2+(1-\varkappa)\frac{\kappa+\nu_2}{2}\right)\right]\zeta(\kappa)d\kappa\end{aligned}$$

and

$$\mathbb{N}(\varkappa) = \frac{1}{2}\int_{\nu_1}^{\nu_2}\left[\lambda\left(\varkappa\nu_1+(1-\varkappa)\frac{\kappa+\nu_1}{2}\right)+\lambda\left(\varkappa\nu_2+(1-\varkappa)\frac{\kappa+\nu_2}{2}\right)\right]\zeta(\kappa)d\kappa,$$

for a convex function $\lambda : [\nu_1, \nu_2] \to \mathbb{R}$ a and non-negative integrable symmetric function $\zeta : [\nu_1, \nu_2] \to \mathbb{R}$ about $\kappa = \frac{\nu_1+\nu_2}{2}$.

By applying the result given below:

Lemma 1.4. *[40] Let $\lambda : [\nu_1, \nu_2] \to \mathbb{R}$ be a convex function and let $\nu_1 \leq \sigma_1 \leq \kappa_1 \leq \kappa_2 \leq \sigma_2 \leq \nu_2$ with $\kappa_1 + \kappa_2 = \sigma_1 + \sigma_2$. Then*

$$\lambda(\kappa_1) + \lambda(\kappa_2) \leq \lambda(\sigma_1) + \lambda(\sigma_2).$$

Teseng et al. obtained the following important refinement inequalities.

Theorem 1.5. *[40] For a non-negative integrable symmetric function $\zeta : [\nu_1, \nu_2] \to \mathbb{R}$ about $\kappa = \frac{\nu_1+\nu_2}{2}$ and a convex function $\lambda : [\nu_1, \nu_2] \to \mathbb{R}$, the mapping $\mathbb{I}$ is convex, increasing on $[0, 1]$ and the Fejér-type inequalities*

$$\lambda\left(\frac{\nu_1+\nu_2}{2}\right)\int_{\nu_1}^{\nu_2}\zeta(\kappa)d\kappa = \mathbb{I}(0) \leq \mathbb{I}(\varkappa) \leq \mathbb{I}(1) = \frac{1}{2}\int_{\nu_1}^{\nu_2}\left[\lambda\left(\frac{\nu_1+\kappa}{2}\right)+\lambda\left(\frac{\kappa+\nu_2}{2}\right)\right]\zeta(\kappa)d\kappa$$

hold for all $\varkappa \in [0, 1]$

Theorem 1.6. *[40] The mapping $\mathbb{J}$ is convex, increasing on $[0,1]$ and the Fejér-type inequalities*

$$\frac{\lambda\left(\frac{3\nu_1+\nu_2}{4}\right)+\lambda\left(\frac{\nu_1+3\nu_2}{4}\right)}{2}\int_{\nu_1}^{\nu_2}\zeta(\kappa)d\kappa=\mathbb{J}(0)\leq\mathbb{J}(\varkappa)\leq\mathbb{J}(1)=\frac{1}{2}\int_{\nu_1}^{\nu_2}\left[\lambda\left(\frac{\nu_1+\kappa}{2}\right)+\lambda\left(\frac{\kappa+\nu_2}{2}\right)\right]\zeta(\kappa)d\kappa$$

hold for all $\varkappa\in[0,1]$, where $\lambda:[\nu_1,\nu_2]\to\mathbb{R}$ is convex and $\zeta:[\nu_1,\nu_2]\to\mathbb{R}$ is non-negative integrable symmetric about $\kappa=\frac{\nu_1+\nu_2}{2}$.

Theorem 1.7. *[40] Let $\lambda:[\nu_1,\nu_2]\to\mathbb{R}$ and $\zeta:[\nu_1,\nu_2]\to\mathbb{R}$ be defined as in Theorem 1.5, then $\mathbb{I}(\varkappa)\leq\mathbb{J}(\varkappa)$ for all* $[0,1]$.

Theorem 1.8. *[40] The mapping $\mathbb{M}$ is convex, increasing on $[0,1]$ and the Fejér-type inequalities*

$$\frac{1}{2}\int_{\nu_1}^{\nu_2}\left[\lambda\left(\frac{\nu_1+\kappa}{2}\right)+\lambda\left(\frac{\kappa+\nu_2}{2}\right)\right]\zeta(\kappa)d\kappa=\mathbb{M}(0)\leq\mathbb{M}(\varkappa)\leq\mathbb{M}(1)=\frac{1}{2}\int_{\nu_1}^{\nu_2}\left[\lambda\left(\frac{\nu_1+\nu_2}{2}\right)+\frac{\lambda(\nu_1)+\lambda(\nu_2)}{2}\right]\zeta(\kappa)d\kappa.$$

hold for all $\varkappa\in[0,1]$ for a convex function $\lambda:[\nu_1,\nu_2]\to\mathbb{R}$ and a non-negative integrable symmetric mapping $\zeta:[\nu_1,\nu_2]\to\mathbb{R}$ about $\kappa=\frac{\nu_1+\nu_2}{2}$.

Theorem 1.9. *[40] The mapping $\mathbb{N}$ is convex, increasing on $[0,1]$ and the Fejér-type inequalities*

$$\frac{1}{2}\int_{\nu_1}^{\nu_2}\left[\lambda\left(\frac{\nu_1+\kappa}{2}\right)+\lambda\left(\frac{\kappa+\nu_2}{2}\right)\right]\zeta(\kappa)d\kappa=\mathbb{N}(0)\leq\mathbb{N}(\varkappa)\leq\mathbb{N}(1)\leq\frac{\lambda(\nu_1)+\lambda(\nu_2)}{2}\int_{\nu_1}^{\nu_2}\zeta(\kappa)d\kappa.$$

hold for all $\varkappa\in[0,1]$, where $\lambda:[\nu_1,\nu_2]\to\mathbb{R}$ is a convex function and $\zeta:[\nu_1,\nu_2]\to\mathbb{R}$ is non-negative integrable symmetric about $\kappa=\frac{\nu_1+\nu_2}{2}$.

Theorem 1.10. *[40] Let $\lambda:[\nu_1,\nu_2]\to\mathbb{R}$ and $\zeta:[\nu_1,\nu_2]\to\mathbb{R}$ be defined as in Theorem 1.5, then $\mathbb{M}(\varkappa)\leq\mathbb{N}(\varkappa)$ for all $\varkappa\in[0,1]$.*

One of the generalizations of the convex functions is *GA*-convex functions:

Definition 1.11. *[24] Let $X\subseteq(0,\infty)$ as an interval. A function $\lambda:X\to\mathbb{R}$ is considered to be GA-convex (concave), if*

$$\lambda\left(\kappa^{1-\varkappa}\sigma^{\varkappa}\right)\leq(\geq)(1-\varkappa)\lambda(\kappa)+\varkappa\lambda(\sigma) \tag{3}$$

for all $\kappa,\sigma\in X$ and $\varkappa\in[0,1]$.
Since the condition (3) can be written as

$$\lambda\circ\exp((1-\varkappa)\ln\kappa+\varkappa\ln\sigma)\leq(\geq)(1-\varkappa)\lambda\circ\exp(\ln\kappa)+\varkappa\lambda\circ\exp(\ln\sigma),$$

then we observe that $\lambda:X\to\mathbb{R}$ is GA-convex (concave) on X if and only if $\lambda\circ\exp$ is convex (concave) on $\ln X=\{\ln\kappa:\kappa\in X\}$. We note that if $X=[\nu_1,\nu_2]$, then $\ln X=[\ln\nu_1,\ln\nu_2]$.

Remark 1.12. *If function $g:[\ln\nu_1,\ln\nu_2]$ is convex (concave) on $[\ln\nu_1,\ln\nu_2]$, then $\lambda:[\nu_1,\nu_2]\subseteq(0,\infty)\to\mathbb{R}$, $\lambda(\varkappa)=g(\ln\varkappa)$ is GA-convex (concave) on $[\nu_1,\nu_2]$.*

Using *GA*-convexity, the Hermite-Hadamard type were obtained by İşcan in the following result.

Theorem 1.13. *[24] Let $\lambda:X\subseteq(0,\infty)\to\mathbb{R}$ be a GA-convex function and $\nu_1,\nu_2\in X$ with $\nu_1<\nu_2$. If $\lambda\in L([\nu_1,\nu_2])$ then the following inequalities hold:*

$$\lambda\left(\sqrt{\nu_1\nu_2}\right)\leq\frac{1}{\ln\nu_2-\ln\nu_1}\int_{\nu_2}^{\nu_1}\frac{\lambda(\kappa)}{\kappa}d\kappa\leq\frac{\lambda(\nu_1)+\lambda(\nu_2)}{2}. \tag{4}$$

Let $\lambda : [\nu_1, \nu_2] \subseteq (0, \infty) \to \mathbb{R}$ be a GA-convex mapping and let $S, \mathbb{U}, \mathbb{V} : [0, 1] \to \mathbb{R}$ be defined by

$$S(\varkappa) = \frac{1}{\ln \nu_2 - \ln \nu_1} \int_{\nu_1}^{\nu_2} \frac{1}{\kappa} \lambda \left(\kappa^{\varkappa} \left(\sqrt{\nu_1 \nu_2} \right)^{1-\varkappa} \right) d\kappa, \tag{5}$$

$$\mathbb{U}(\varkappa) = \frac{1}{(\ln \nu_2 - \ln \nu_1)^2} \int_{\nu_1}^{\nu_2} \int_{\nu_1}^{\nu_2} \frac{1}{\kappa\sigma} \lambda \left(\kappa^{\varkappa} \sigma^{1-\varkappa} \right) d\kappa d\sigma \tag{6}$$

and

$$\mathbb{V}(\varkappa) = \frac{1}{2(\ln \nu_2 - \ln \nu_1)} \int_{\nu_1}^{\nu_2} \frac{1}{\kappa} \left[\lambda \left(\nu_2^{\frac{1+\varkappa}{2}} \kappa^{\frac{1-\varkappa}{2}} \right) + \lambda \left(\nu_1^{\frac{1+\varkappa}{2}} \kappa^{\frac{1-\varkappa}{2}} \right) \right] d\kappa. \tag{7}$$

The following refinement inequalities for (4) mappings were obtained by the author:

Theorem 1.14. *[26] Let $\lambda : [\nu_1, \nu_2] \subseteq (0, \infty) \to \mathbb{R}$ be a GA-convex function on $[\nu_1, \nu_2]$. Then*

(i) S is GA-convex on $[0, 1]$.
(ii) The following hold:

$$\inf_{\varkappa \in [0,1]} S(\varkappa) = S(0) = \lambda \left(\sqrt{\nu_1 \nu_2} \right)$$

$$\sup_{\varkappa \in [0,1]} S(\varkappa) = S(1) = \frac{1}{\ln \nu_2 - \ln \nu_1} \int_{\nu_1}^{\nu_2} \frac{\lambda(\kappa)}{\kappa} d\kappa.$$

(iii) S monotonically increasing on $[0, 1]$.

Theorem 1.15. *[26] Let $\lambda : [\nu_1, \nu_2] \subseteq (0, \infty) \to \mathbb{R}$ be a GA-convex function on $[\nu_1, \nu_2]$. Then*

(i) The identity

$$\mathbb{U}\left(\varkappa + \frac{1}{2} \right) = \mathbb{U}\left(\frac{1}{2} - \varkappa \right) \text{ holds for all } \varkappa \in \left[0, \frac{1}{2} \right].$$

(ii) $\mathbb{U}$ is GA-convex on $[\nu_1, \nu_2]$.
(iii) The identities

$$\inf_{\varkappa \in [0,1]} \mathbb{U}(\varkappa) = \mathbb{U}\left(\frac{1}{2} \right) = \frac{1}{(\ln \nu_2 - \ln \nu_1)^2} \int_{\nu_1}^{\nu_2} \int_{\nu_1}^{\nu_2} \frac{1}{\kappa\sigma} \lambda \left(\sqrt{\kappa\sigma} \right) d\kappa d\sigma$$

and

$$\sup_{\varkappa \in [0,1]} \mathbb{U}(\varkappa) = \mathbb{U}(0) = \mathbb{U}(1) = \frac{1}{\ln \nu_2 - \ln \nu_1} \int_{\nu_1}^{\nu_2} \frac{\lambda(\kappa)}{\kappa} d\kappa$$

hold.
(iv) The inequality

$$\lambda \left(\sqrt{\kappa\sigma} \right) \leq \mathbb{U}\left(\frac{1}{2} \right)$$

is valid.
(v) $\mathbb{U}$ is monotonically increasing on $\left[\frac{1}{2}, 1 \right]$ and monotonically decreasing on $\left[0, \frac{1}{2} \right]$.
(vi) $S(\varkappa) \leq \mathbb{U}(\varkappa)$ for all $\varkappa \in [0, 1]$.

Theorem 1.16. *[26] Let $\lambda : [\nu_1, \nu_2] \subseteq (0, \infty) \to \mathbb{R}$ be a GA-convex function on $[\nu_1, \nu_2]$. Then* *(i)* $\mathbb{V}$ *is GA-convex on* $[\nu_1, \nu_2]$.

(ii) *The equalities*

$$\inf_{\varkappa\in[0,1]} \mathbb{V}(\varkappa) = \mathbb{V}(0) = \frac{1}{\ln \nu_2 - \ln \nu_1} \int_{\nu_1}^{\nu_2} \frac{\lambda(\kappa)}{\kappa} d\kappa$$

and

$$\sup_{\varkappa\in[0,1]} \mathbb{V}(\varkappa) = \mathbb{V}(1) = \frac{\lambda(\nu_1) + \lambda(\nu_2)}{2}$$

hold.

(iii) $\mathbb{V}$ *is monotonically increasing on* $[0, 1]$.

Geometrically symmetric functions are defined in the definition below.

Definition 1.17. *[27] A function* $\zeta : [\nu_1, \nu_2] \subseteq (0, \infty) \to \mathbb{R}$ *is geometrically symmetric with respect to* $\sqrt{\nu_1\nu_2}$ *if*

$$\zeta(\kappa) = \zeta\left(\frac{\nu_1\nu_2}{\kappa}\right)$$

holds for all $\kappa \in [\nu_1, \nu_2]$.

Fejér type inequalities using *GA*-convexity and the notion of geometrical symmetricity were proven by the author in [27].

Theorem 1.18. *[27] If* $\lambda \in L([\nu_1, \nu_2])$ *and* $\zeta : [\nu_1, \nu_2] \to \mathbb{R}$ *is non-negative, integrable geometrically symmetric with respect to* $\sqrt{\nu_1\nu_2}$*, then*

$$\lambda\left(\sqrt{\nu_1\nu_2}\right) \int_{\nu_2}^{\nu_1} \frac{\zeta(\kappa)}{\kappa} d\kappa \le \int_{\nu_2}^{\nu_1} \frac{\lambda(\kappa)\zeta(\kappa)}{\kappa} d\kappa \le \frac{\lambda(\nu_1) + \lambda(\nu_2)}{2} \int_{\nu_2}^{\nu_1} \frac{\zeta(\kappa)}{\kappa} d\kappa, \tag{8}$$

where $\lambda : X \subseteq (0, \infty) \to \mathbb{R}$ *is a GA-convex function for* $\nu_1, \nu_2 \in X$ *with* $\nu_1 < \nu_2$.

Inspired by the research in [14, 26, 40, 45], we establish some novel mappings in relation to (8) and show new Féjer type inequalities that offer refinement inequalities.

2. Main Results

We start this section by stating the Jensen's inequality for *GA*-convex and convex functions.

Theorem 2.1. *[10, 11] Let* $\lambda : X \subset (0, \infty) \to \mathbb{R}$ *be a GA-convex function and* $[\nu_1, \nu_2] \subset X^\circ$. *If* $\zeta(\varkappa) \ge 0$ *a.e. on* $[\nu_1, \nu_2]$ *with* $\int_{\nu_1}^{\nu_2} \zeta(\varkappa) d\varkappa > 0$*, then*

$$\frac{\int_{\nu_1}^{\nu_2} \lambda(\varkappa)\zeta(\varkappa) d\varkappa}{\int_{\nu_1}^{\nu_2} \zeta(\varkappa) d\varkappa} \ge \lambda \circ \exp\left(\frac{\int_{\nu_1}^{\nu_2} \zeta(\varkappa) \ln \varkappa d\varkappa}{\int_{\nu_1}^{\nu_2} \zeta(\varkappa) d\varkappa}\right).$$

Let us now define some mappings on $[0, 1]$ related to (8) and prove some refinement inequalities.

$$\mathbb{I}_1(\varkappa) = \frac{1}{2}\int_{\nu_1}^{\nu_2} \left[\lambda\left(\kappa^{\frac{\varkappa}{2}} \nu_2^{-\frac{\varkappa}{2}} \sqrt{\nu_1\nu_2}\right) + \lambda\left(\kappa^{\frac{\varkappa}{2}} \nu_1^{-\frac{\varkappa}{2}} \sqrt{\nu_1\nu_2}\right)\right] \frac{\zeta(\kappa)}{\kappa} d\kappa,$$

$$\mathbb{J}_1(\varkappa) = \frac{1}{2}\int_{\nu_1}^{\nu_2}\left[\lambda\left(\kappa^{\frac{\varkappa}{2}}\nu_1^{\frac{3-\varkappa}{4}}\nu_2^{\frac{1-\varkappa}{4}}\right) + \lambda\left(\kappa^{\frac{\varkappa}{2}}\nu_1^{\frac{1-\varkappa}{4}}\nu_2^{\frac{3-\varkappa}{4}}\right)\right]\frac{\zeta(\kappa)}{\kappa}d\kappa,$$

$$\mathbb{M}_1(\varkappa) = \frac{1}{2}\int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\left[\lambda\left(\nu_1^{\frac{1+\varkappa}{2}}\kappa^{\frac{1-\varkappa}{2}}\right) + \lambda\left(\nu_1^{\frac{\varkappa}{2}}\nu_2^{\frac{1}{2}}\kappa^{\frac{1-\varkappa}{2}}\right)\right]\zeta(\kappa)d\kappa + \frac{1}{2}\int_{\sqrt{\nu_1\nu_2}}^{\nu_2}\left[\lambda\left(\nu_1^{\frac{1}{2}}\nu_2^{\frac{\varkappa}{2}}\kappa^{\frac{1-\varkappa}{2}}\right) + \lambda\left(\nu_2^{\frac{1+\varkappa}{2}}\kappa^{\frac{1-\varkappa}{2}}\right)\right]\zeta(\kappa)d\kappa$$

and

$$\mathbb{N}_1(\varkappa) = \frac{1}{2}\int_{\nu_1}^{\nu_2}\left[\lambda\left(\nu_1^{\frac{1+\varkappa}{2}}\kappa^{\frac{1-\varkappa}{2}}\right) + \lambda\left(\nu_2^{\frac{1+\varkappa}{2}}\kappa^{\frac{1-\varkappa}{2}}\right)\right]\frac{\zeta(\kappa)}{\kappa}d\kappa,$$

where $\lambda : [\nu_1, \nu_2] \subseteq (0, \infty) \to \mathbb{R}$ is a *GA*-convex function and $\zeta : [\nu_1, \nu_2] \to \mathbb{R}$ is non-negative integrable and symmetric about $\kappa = \sqrt{\nu_1\nu_2}$.

Lemma 2.2. *Let $\lambda : [\nu_1, \nu_2] \to \mathbb{R}$ be a GA-convex function and let $\nu_1 \leq \sigma_1 \leq \kappa_1 \leq \kappa_2 \leq \sigma_2 \leq \nu_2$ with $\kappa_1\kappa_2 = \sigma_1\sigma_2$. Then*

$$\lambda(\kappa_1) + \lambda(\kappa_2) \leq \lambda(\sigma_1) + \lambda(\sigma_2).$$

Proof. For $\sigma_1 = \sigma_2$, the result is obvious. We observe that

$$\kappa_1 = (\sigma_1)^{\frac{\ln\sigma_2 - \ln\kappa_1}{\ln\sigma_2 - \ln\sigma_1}}(\sigma_2)^{\frac{\ln\kappa_1 - \ln\sigma_1}{\ln\sigma_2 - \ln\sigma_1}}$$

and

$$\kappa_2 = (\sigma_1)^{\frac{\ln\sigma_2 - \ln\kappa_2}{\ln\sigma_2 - \ln\sigma_1}}(\sigma_2)^{\frac{\ln\kappa_2 - \ln\sigma_1}{\ln\sigma_2 - \ln\sigma_1}}.$$

are in the interval $[\nu_1, \nu_2]$, and $\kappa_1\kappa_2 = \sigma_1\sigma_2$.
By applying the *GA*-convexity, we obtain

$$\begin{aligned}\lambda(\kappa_1) + \lambda(\kappa_2) &\leq \left(\frac{\ln\sigma_2 - \ln\kappa_1}{\ln\sigma_2 - \ln\sigma_1}\right)\lambda(\sigma_1) + \left(\frac{\ln\kappa_1 - \ln\sigma_1}{\ln\sigma_2 - \ln\sigma_1}\right)\lambda(\sigma_2) + \left(\frac{\ln\sigma_2 - \ln\kappa_2}{\ln\sigma_2 - \ln\sigma_1}\right)\lambda(\sigma_1) + \left(\frac{\ln\kappa_2 - \ln\sigma_1}{\ln\sigma_2 - \ln\sigma_1}\right)\lambda(\sigma_2)\\ &= \lambda(\sigma_1) + \lambda(\sigma_2).\end{aligned}$$

□

Theorem 2.3. *Let $\lambda, \zeta, \mathbb{I}_1$ be defined as above. Then $\mathbb{I}_1$ is GA-convex, increasing on $[0, 1]$ and Fejér-type inequalities*

$$\lambda\left(\sqrt{\nu_1\nu_2}\right)\int_{\nu_1}^{\nu_2}\frac{\zeta(\kappa)}{\kappa}d\kappa = \mathbb{I}_1(0) \leq \mathbb{I}_1(\varkappa) \leq \mathbb{I}_1(1) = \frac{1}{2}\int_{\nu_1}^{\nu_2}\left[\lambda\left(\sqrt{\nu_1\kappa}\right) + \lambda\left(\sqrt{\kappa\nu_2}\right)\right]\frac{\zeta(\kappa)}{\kappa}d\kappa \tag{9}$$

hold for all $\varkappa \in [0, 1]$.

Proof. The mapping $\mathbb{I}_1 : [0, 1] \to \mathbb{R}$ is *GA*-convex if and only of the mapping $\bar{I}_1 : [0, 1] \to \mathbb{R}$ defined by

$$\begin{aligned}\bar{I}_1(\varkappa) = \frac{1}{2}\int_{\ln\nu_2}^{\ln\nu_1}&\left[g \circ \exp\left(\varkappa\ln\left(\sqrt{\frac{\kappa}{\nu_2}}\right) + \ln\left(\sqrt{\nu_1\nu_2}\right)\right)\right.\\ &\left.+ g \circ \exp\left(\varkappa\ln\left(\sqrt{\frac{\kappa}{\nu_1}}\right) + \ln\left(\sqrt{\nu_1\nu_2}\right)\right)\right]\zeta \circ \exp(\kappa)\,d\kappa\end{aligned}$$

is convex for a convex mapping $g : [\ln \nu_1, \ln \nu_2] \to \mathbb{R}$. Let $\varkappa_1, \varkappa_2 \in [0,1]$, $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, then

$$\begin{aligned}
\bar{I}_1(\varkappa_1\alpha + \varkappa_2\beta) &= \frac{1}{2}\int_{\ln \nu_2}^{\ln \nu_1}\Bigg[g \circ \exp\left((\varkappa_1\alpha + \varkappa_2\beta)\ln\left(\sqrt{\frac{\kappa}{\nu_2}}\right) + (\alpha+\beta)\ln\left(\sqrt{\nu_1\nu_2}\right)\right)\\
&\quad + g \circ \exp\left((\varkappa_1\alpha + \varkappa_2\beta)\ln\left(\sqrt{\frac{\kappa}{\nu_1}}\right) + (\alpha+\beta)\ln\left(\sqrt{\nu_1\nu_2}\right)\right)\Bigg]\zeta \circ \exp(\kappa)\, d\kappa\\
&= \frac{1}{2}\int_{\ln \nu_2}^{\ln \nu_1}\Bigg[g \circ \exp\left(\alpha\left(\varkappa_1 \ln\left(\sqrt{\frac{\kappa}{\nu_2}}\right) + \ln\left(\sqrt{\nu_1\nu_2}\right)\right) + \beta\left(\varkappa_2 \ln\left(\sqrt{\frac{\kappa}{\nu_2}}\right) + \ln\left(\sqrt{\nu_1\nu_2}\right)\right)\right)\\
&\quad + g \circ \exp\Bigg(\alpha\left(\varkappa_1 \ln\left(\sqrt{\frac{\kappa}{\nu_1}}\right) + \ln\left(\sqrt{\nu_1\nu_2}\right)\right)\\
&\quad + \beta\left(\varkappa_2 \ln\left(\sqrt{\frac{\kappa}{\nu_1}}\right) + \ln\left(\sqrt{\nu_1\nu_2}\right)\right)\Bigg)\Bigg]\zeta \circ \exp(\kappa)\, d\kappa\\
&\le \alpha\frac{1}{2}\int_{\ln \nu_2}^{\ln \nu_1}\Bigg[g \circ \exp\left(\varkappa_1 \ln\left(\sqrt{\frac{\kappa}{\nu_2}}\right) + \ln\left(\sqrt{\nu_1\nu_2}\right)\right)\\
&\quad + g \circ \exp\left(\varkappa_1 \ln\left(\sqrt{\frac{\kappa}{\nu_1}}\right) + \ln\left(\sqrt{\nu_1\nu_2}\right)\right)\Bigg]\zeta \circ \exp(\kappa)\, d\kappa\\
&\quad + \beta\frac{1}{2}\int_{\ln \nu_2}^{\ln \nu_1}\Bigg[g \circ \exp\left(\varkappa_2 \ln\left(\sqrt{\frac{\kappa}{\nu_2}}\right) + \ln\left(\sqrt{\nu_1\nu_2}\right)\right)\\
&\quad + g \circ \exp\left(\varkappa_2 \ln\left(\sqrt{\frac{\kappa}{\nu_1}}\right) + \ln\left(\sqrt{\nu_1\nu_2}\right)\right)\Bigg]\zeta \circ \exp(\kappa)\, d\kappa = \alpha\bar{I}_1(\varkappa_1) + \beta\bar{I}_1(\varkappa_2).
\end{aligned}$$

This proves the GA-convexity of $\mathbb{I}_1 : [0,1] \to \mathbb{R}$.
Using integration techniques and under the assumptions on ζ, the following identity holds on $[0,1]$:

$$\mathbb{I}_1(\varkappa) = \int_{\nu_1}^{\sqrt{\nu_1\nu_2}} \left[\lambda\left(\kappa^{\varkappa}(\nu_1\nu_2)^{\frac{1-\varkappa}{2}}\right) + \lambda\left(\kappa^{-\varkappa}(\nu_1\nu_2)^{\frac{1+\varkappa}{2}}\right)\right]\zeta\left(\frac{\kappa^2}{\nu_1}\right)d\kappa. \tag{10}$$

Let $\varkappa_1$, $\varkappa_2 \in [0,1]$ with $\varkappa_1 < \varkappa_2$. Choosing

$$\begin{aligned}
\kappa_1 &= \kappa^{\varkappa_2}(\nu_1\nu_2)^{\frac{1-\varkappa_2}{2}},\\
\kappa_2 &= \kappa^{-\varkappa_2}(\nu_1\nu_2)^{\frac{1+\varkappa_2}{2}},\\
\sigma_1 &= \kappa^{\varkappa_1}(\nu_1\nu_2)^{\frac{1-\varkappa_1}{2}}
\end{aligned}$$

and

$$\sigma_2 = \kappa^{-\varkappa_1}(\nu_1\nu_2)^{\frac{1+\varkappa_1}{2}}.$$

We observe that

$$\left(\kappa^{\varkappa_2}(\nu_1\nu_2)^{\frac{1-\varkappa_2}{2}}\right)\left(\kappa^{-\varkappa_2}(\nu_1\nu_2)^{\frac{1+\varkappa_2}{2}}\right) = \left(\kappa^{\varkappa_1}(\nu_1\nu_2)^{\frac{1-\varkappa_1}{2}}\right)\left(\kappa^{-\varkappa_1}(\nu_1\nu_2)^{\frac{1+\varkappa_1}{2}}\right) = \nu_1\nu_2.$$

Applying Lemma 2.2 to get the following inequality holds for all $\kappa \in \left[\nu_1, \sqrt{\nu_1\nu_2}\right]$:

$$\lambda\left(\kappa^{\varkappa_1}(\nu_1\nu_2)^{\frac{1-\varkappa_1}{2}}\right)+\lambda\left(\kappa^{-\varkappa_1}(\nu_1\nu_2)^{\frac{1+\varkappa_1}{2}}\right)\le\lambda\left(\kappa^{\varkappa_2}(\nu_1\nu_2)^{\frac{1-\varkappa_2}{2}}\right)+\lambda\left(\kappa^{\varkappa_2}(\nu_1\nu_2)^{\frac{1+\varkappa_2}{2}}\right).$$

Multiplying the inequality (10) by $\zeta\left(\frac{\kappa^2}{\nu_1}\right)$, integrating both sides over κ on $\left[\nu_1, \sqrt{\nu_1\nu_2}\right]$ and using identity (10), we derive $\mathbb{I}_1(\varkappa_1) \le \mathbb{I}_1(\varkappa_2)$. Thus $\mathbb{I}_1$ is increasing on $[0,1]$ and then the inequality (9) holds. □

Remark 2.4. *Let $\zeta(\kappa) = \frac{1}{\ln\nu_2-\ln\nu_1}$, $\kappa \in [\nu_1,\nu_2]$ in Theorem 2.3. Then $\mathbb{I}_1(\varkappa) = S(\varkappa)$, $\varkappa \in [0,1]$ and the inequality (9) reduces to the inequality*

$$\lambda\left(\sqrt{\nu_1\nu_2}\right)=S(0)\le S(\varkappa)\le S(1)=\frac{1}{\ln\nu_2-\ln\nu_1}\int_{\nu_1}^{\nu_2}\frac{\lambda(\kappa)}{\kappa}d\kappa,$$

where S is defined by (5).

Theorem 2.5. *Let $\lambda, \zeta, \mathbb{J}_1$ be defined as above. Then $\mathbb{J}_1$ is GA-convex, increasing on $[0,1]$ and Fejér-type inequalities*

$$\frac{\lambda\left(\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}\right)+\lambda\left(\nu_1^{\frac{1}{4}}\nu_2^{\frac{3}{4}}\right)}{2}\int_{\nu_1}^{\nu_2}\frac{\zeta(\kappa)}{\kappa}d\kappa=\mathbb{J}_1(0)\le\mathbb{J}_1(\varkappa)\le\mathbb{J}_1(1)=\frac{1}{2}\int_{\nu_1}^{\nu_2}\left[\lambda\left(\sqrt{\nu_1\kappa}\right)+\lambda\left(\sqrt{\kappa\nu_2}\right)\right]\frac{\zeta(\kappa)}{\kappa}d\kappa \quad (11)$$

hold for all $\varkappa \in [0,1]$

Proof. The GA-convexity of $\mathbb{J}_1$ on $[0,1]$ can be proved similarly as in proving the GA-convexity of $\mathbb{I}_1$ on $[0,1]$. The following identity holds on $[0,1]$:

$$\mathbb{J}_1(\varkappa)=\int_{\nu_1}^{\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}}\left[\lambda\left(\kappa^{\varkappa}\nu_1^{\frac{3(1-\varkappa)}{4}}\nu_2^{\frac{1-\varkappa}{4}}\right)+\lambda\left(\kappa^{-\varkappa}\nu_1^{\frac{3(1+\varkappa)}{4}}\nu_2^{\frac{1+\varkappa}{4}}\right)+\lambda\left(\kappa^{\varkappa}\nu_1^{\frac{1-3\varkappa}{4}}\nu_2^{\frac{3-\varkappa}{4}}\right)+\lambda\left(\kappa^{-\varkappa}\nu_1^{\frac{3\varkappa+1}{4}}\nu_2^{\frac{\varkappa+3}{4}}\right)\right]\frac{\zeta\left(\frac{\kappa^2}{\nu_1}\right)}{\kappa}d\kappa. \quad (12)$$

Let $\varkappa_1, \varkappa_2 \in [0,1]$ with $\varkappa_1 < \varkappa_2$. Choosing

$$\kappa_1=\kappa^{\varkappa_2}\nu_1^{\frac{3(1-\varkappa_2)}{4}}\nu_2^{\frac{1-\varkappa_2}{4}},$$
$$\kappa_2=\kappa^{-\varkappa_2}\nu_1^{\frac{3(1+\varkappa_2)}{4}}\nu_2^{\frac{1+\varkappa_2}{4}},$$
$$\sigma_1=\kappa^{\varkappa_1}\nu_1^{\frac{3(1-\varkappa_1)}{4}}\nu_2^{\frac{1-\varkappa_1}{4}}$$

and

$$\sigma_2=\kappa^{-\varkappa_1}\nu_1^{\frac{3(1+\varkappa_1)}{4}}\nu_2^{\frac{1+\varkappa_1}{4}}.$$

Application of Lemma 2.2 leads to the following inequality:

$$\lambda\left(\kappa^{\varkappa_1}\nu_1^{\frac{3(1-\varkappa_1)}{4}}\nu_2^{\frac{1-\varkappa_1}{4}}\right)+\lambda\left(\kappa^{-\varkappa_1}\nu_1^{\frac{3(1+\varkappa_1)}{4}}\nu_2^{\frac{1+\varkappa_1}{4}}\right)\le\lambda\left(\kappa^{\varkappa_2}\nu_1^{\frac{3(1-\varkappa_2)}{4}}\nu_2^{\frac{1-\varkappa_2}{4}}\right)+\lambda\left(\kappa^{-\varkappa_2}\nu_1^{\frac{3(1+\varkappa_2)}{4}}\nu_2^{\frac{1+\varkappa_2}{4}}\right) \quad (13)$$

for all $\kappa \in \left[\nu_1, \nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}\right]$.
In a similar way, with the choices

$$\kappa_1=\kappa^{\varkappa_2}\nu_1^{\frac{1-3\varkappa_2}{4}}\nu_2^{\frac{3-\varkappa_2}{4}},$$
$$\kappa_2=\kappa^{-\varkappa_2}\nu_1^{\frac{3\varkappa_2+1}{4}}\nu_2^{\frac{\varkappa_2+3}{4}},$$
$$\sigma_1=\kappa^{\varkappa_1}\nu_1^{\frac{1-3\varkappa_1}{4}}\nu_2^{\frac{3-\varkappa_1}{4}}$$

and

$$\sigma_2 = \kappa^{-\varkappa_1}\nu_1^{\frac{3\varkappa_1+1}{4}}\nu_2^{\frac{\varkappa_1+3}{4}}$$

for $\varkappa_1, \varkappa_2 \in [0,1]$, where $\varkappa_1 < \varkappa_2$ and using Lemma 2.2, we obtain

$$\lambda\left(\kappa^{\varkappa_1}\nu_1^{\frac{1-3\varkappa_1}{4}}\nu_2^{\frac{3-\varkappa_1}{4}}\right)+\lambda\left(\kappa^{-\varkappa_1}\nu_1^{\frac{3\varkappa_1+1}{4}}\nu_2^{\frac{\varkappa_1+3}{4}}\right)\le\lambda\left(\kappa^{\varkappa_2}\nu_1^{\frac{1-3\varkappa_2}{4}}\nu_2^{\frac{3-\varkappa_2}{4}}\right)+\lambda\left(\kappa^{-\varkappa_2}\nu_1^{\frac{3\varkappa_2+1}{4}}\nu_2^{\frac{\varkappa_2+3}{4}}\right), \tag{14}$$

where for all $\kappa \in \left[\nu_1, \nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}\right]$.

Adding (13) and (14), multiplying both sides by $\frac{\zeta\left(\frac{\kappa^2}{\nu_1}\right)}{\kappa^2}$ and then integrating over $\left[\nu_1, \nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}\right]$, we get that $\mathbb{J}_1(\varkappa_1) \le \mathbb{J}_1(\varkappa_2)$ for $\varkappa_1, \varkappa_2 \in [0,1]$, where $\varkappa_1 < \varkappa_2$. It is proved that $\mathbb{J}_1$ is increasing on $[0,1]$ and hence the inequality (11) is proved because of the fact that $\mathbb{J}_1(0) \le \mathbb{J}_1(\varkappa) \le \mathbb{J}_1(1)$. □

A comparison between $\mathbb{I}_1$ and $\mathbb{J}_1$ is given in the theorem below:

Theorem 2.6. *Let λ, ζ, $\mathbb{I}_1$, $\mathbb{J}_1$ be defined as above. Then $\mathbb{I}_1(\varkappa) \le \mathbb{J}_1(\varkappa)$ on $[0,1]$.*

Proof. We observe that the following identities hold for all $\varkappa \in [0,1]$ and $\kappa \in \left[\nu_1, \sqrt{\nu_1\nu_2}\right]$:

$$\mathbb{J}_1(\varkappa) = \int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\left[\lambda\left(\kappa^{\varkappa}\nu_1^{\frac{3(1-\varkappa)}{4}}\nu_2^{\frac{1-\varkappa}{4}}\right)+\lambda\left(\kappa^{-\varkappa}\nu_1^{\frac{3\varkappa+1}{4}}\nu_2^{\frac{\varkappa+3}{4}}\right)\right]\frac{\zeta\left(\frac{\kappa^2}{\nu_1}\right)}{\kappa}d\kappa \tag{15}$$

and

$$\mathbb{I}_1(\varkappa) = \int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\mathbb{I}_1(\varkappa) = \int_{\nu_1}^{\sqrt{\nu_1\nu_2}}\left[\lambda\left(\kappa^{\varkappa}(\nu_1\nu_2)^{\frac{1-\varkappa}{2}}\right)+\lambda\left(\kappa^{-\varkappa}(\nu_1\nu_2)^{\frac{1+\varkappa}{2}}\right)\right]\zeta\left(\frac{\kappa^2}{\nu_1}\right)d\kappa. \tag{16}$$

Let

$$\kappa_1 = \kappa^{\varkappa}(\nu_1\nu_2)^{\frac{1-\varkappa}{2}},$$
$$\kappa_2 = \kappa^{-\varkappa}(\nu_1\nu_2)^{\frac{1+\varkappa}{2}},$$
$$\sigma_1 = \kappa^{\varkappa}\nu_1^{\frac{3(1-\varkappa)}{4}}\nu_2^{\frac{1-\varkappa}{4}}$$

and

$$\sigma_2 = \kappa^{-\varkappa}\nu_1^{\frac{3\varkappa+1}{4}}\nu_2^{\frac{\varkappa+3}{4}}$$

for all $\varkappa \in [0,1]$ and $\kappa \in \left[\nu_1, \sqrt{\nu_1\nu_2}\right]$. Then

$$\kappa_1\kappa_2 = \sigma_1\sigma_2 = \nu_1\nu_2.$$

Lemma 2.2 leads to the inequality

$$\lambda\left(\kappa^{\varkappa}(\nu_1\nu_2)^{\frac{1-\varkappa}{2}}\right)+\lambda\left(\kappa^{-\varkappa}(\nu_1\nu_2)^{\frac{1+\varkappa}{2}}\right)\le\lambda\left(\kappa^{\varkappa}\nu_1^{\frac{3(1-\varkappa)}{4}}\nu_2^{\frac{1-\varkappa}{4}}\right)+\lambda\left(\kappa^{-\varkappa}\nu_1^{\frac{3\varkappa+1}{4}}\nu_2^{\frac{\varkappa+3}{4}}\right)$$

for all $\varkappa \in [0,1]$ and $\kappa \in \left[\nu_1, \sqrt{\nu_1\nu_2}\right]$.

Multiplying both sides by $\frac{\zeta\left(\frac{\kappa^2}{\nu_1}\right)}{\kappa}$ and then integrating over $\left[\nu_1, \sqrt{\nu_1\nu_2}\right]$, we get that $\mathbb{I}_1(\varkappa) \le \mathbb{J}_1(\varkappa)$ for $\varkappa \in [0,1]$. □

The following result demonstrates how the function attributes of $\mathbb{M}_1$ are incorporated:

Theorem 2.7. *Let $\lambda, \zeta, \mathbb{M}_1$ be defined as above. Then $\mathbb{M}_1$ is GA-convex, increasing on $[0,1]$, and for all $\varkappa \in [0,1]$, we have the following Fejér-type inequality*

$$\frac{1}{2}\int_{\nu_1}^{\nu_2}\left[\lambda\left(\sqrt{\nu_1\kappa}\right)+\lambda\left(\sqrt{\kappa\nu_2}\right)\right]\frac{\zeta(\kappa)}{\kappa}d\kappa=\mathbb{M}_1(0)$$
$$\leq \mathbb{M}_1(\varkappa)\leq \mathbb{M}_1(1)=\frac{1}{2}\left[\lambda\left(\sqrt{\nu_1\nu_2}\right)+\frac{\lambda(\nu_1)+\lambda(\nu_2)}{2}\right]\int_{\nu_1}^{\nu_2}\frac{\zeta(\kappa)}{\kappa}d\kappa. \tag{17}$$

Proof. We can prove the *GA*-convexity of $\mathbb{M}_1$ on $[0,1]$ by following the same method as that of proving the *GA*-convexity of $\mathbb{I}_1$ on $[0,1]$ in Theorem 2.3.
The identity

$$\mathbb{M}_1(\varkappa)=\int_{\nu_1}^{\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}}\left[\lambda\left(\kappa^{1-\varkappa}\nu_1^{\varkappa}\right)+\lambda\left(\kappa^{-(1-\varkappa)}\nu_1^{\frac{3}{2}-\varkappa}\nu_2^{\frac{1}{2}}\right)+\lambda\left(\kappa^{1-\varkappa}\nu_1^{\varkappa-\frac{1}{2}}\nu_2^{\frac{1}{2}}\right)+\lambda\left(\kappa^{-(1-\varkappa)}\nu_1^{1-\varkappa}\nu_2\right)\right]\frac{\zeta\left(\frac{\kappa^2}{\nu_1}\right)}{\kappa}d\kappa$$

holds for all $\varkappa \in [0,1]$ and $\kappa \in \left[\nu_1, \nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}\right]$.
According to Lemma 2.2, the following inequalities are valid for all $\varkappa_1, \varkappa_2 \in [0,1]$ with $\varkappa_1 < \varkappa_2$ and $\kappa \in \left[\nu_1, \nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}\right]$:

$$\lambda\left(\kappa^{1-\varkappa_1}\nu_1^{\varkappa_1}\right)+\lambda\left(\kappa^{-(1-\varkappa_1)}\nu_1^{\frac{3}{2}-\varkappa_1}\nu_2^{\frac{1}{2}}\right)\leq\lambda\left(\kappa^{1-\varkappa_2}\nu_1^{\varkappa_2}\right)+\lambda\left(\kappa^{1-\varkappa_2}\nu_1^{\frac{3}{2}-\varkappa_2}\nu_2^{\frac{1}{2}}\right) \tag{18}$$

$$\lambda\left(\kappa^{1-\varkappa_1}\nu_1^{\varkappa_1-\frac{1}{2}}\nu_2^{\frac{1}{2}}\right)+\lambda\left(\kappa^{-(1-\varkappa_1)}\nu_1^{1-\varkappa_1}\nu_2\right)\leq\lambda\left(\kappa^{1-\varkappa_2}\nu_1^{\varkappa_2-\frac{1}{2}}\nu_2^{\frac{1}{2}}\right)+\lambda\left(\kappa^{-(1-\varkappa_2)}\nu_1^{1-\varkappa_2}\nu_2\right) \tag{19}$$

since

$$\left(\kappa^{1-\varkappa_1}\nu_1^{\varkappa_1}\right)\left(\kappa^{-(1-\varkappa_1)}\nu_1^{\frac{3}{2}-\varkappa_1}\nu_2^{\frac{1}{2}}\right)=\left(\kappa^{1-\varkappa_2}\nu_1^{\varkappa_2-\frac{1}{2}}\nu_2^{\frac{1}{2}}\right)\left(\kappa^{-(1-\varkappa_2)}\nu_1^{1-\varkappa_2}\nu_2\right)=\nu_1^{\frac{3}{2}}\nu_2^{\frac{1}{2}}$$

and

$$\left(\kappa^{1-\varkappa_1}\nu_1^{\varkappa_1-\frac{1}{2}}\nu_2^{\frac{1}{2}}\right)\left(\kappa^{-(1-\varkappa_1)}\nu_1^{1-\varkappa_1}\nu_2\right)=\left(\kappa^{1-\varkappa_2}\nu_1^{\varkappa_2-\frac{1}{2}}\nu_2^{\frac{1}{2}}\right)\left(\kappa^{-(1-\varkappa_2)}\nu_1^{1-\varkappa_2}\nu_2\right)=\nu_1^{\frac{1}{2}}\nu_2^{\frac{3}{2}}.$$

Adding (18) and (19) and multiplying both sides of the resulting inequality by $\frac{\zeta\left(\frac{\kappa^2}{\nu_1}\right)}{\kappa}$ and then integrating over $\left[\nu_1, \nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}\right]$, we get that $\mathbb{M}_1(\varkappa_1) \leq \mathbb{M}_1(\varkappa_2)$ for $\varkappa_1, \varkappa_2 \in [0,1]$ with $\varkappa_1 < \varkappa_2$. Hence $\mathbb{M}_1$ is increasing on $[0,1]$ and thus the inequalities (17) follow. □

The properties of the mapping $\mathbb{N}_1$ are presented in the given result:

Theorem 2.8. *Let $\lambda, \zeta, \mathbb{N}_1$ be defined as above. Then $\mathbb{N}_1$ is GA-convex, increasing on $[0,1]$, and for all $\varkappa \in [0,1]$, then Fejér-type inequalities*

$$\frac{1}{2}\int_{\nu_1}^{\nu_2}\left[\lambda\left(\sqrt{\nu_1\kappa}\right)+\lambda\left(\sqrt{\kappa\nu_2}\right)\right]\frac{\zeta(\kappa)}{\kappa}d\kappa=\mathbb{N}_1(0)\leq\mathbb{N}_1(\varkappa)\leq\mathbb{N}_1(1)=\frac{\lambda(\nu_1)+\lambda(\nu_2)}{2}\int_{\nu_1}^{\nu_2}\frac{\zeta(\kappa)}{\kappa}d\kappa \tag{20}$$

holds.

Proof. We can prove the *GA*-convexity of $\mathbb{N}_1$ on $[0,1]$ by following the same method as that of proving the *GA*-convexity of $\mathbb{I}_1$ on $[0,1]$ in Theorem 2.3.
The identity

$$\mathbb{N}_1(\varkappa) = \int_{\nu_1}^{\sqrt{\nu_1\nu_2}} \left[\lambda\left(\nu_1^{\varkappa}\kappa^{1-\varkappa}\right) + \lambda\left(\nu_1^{1-\varkappa}\nu_2\kappa^{-(1-\varkappa)}\right)\right] \frac{\zeta\left(\frac{\kappa^2}{\nu_1}\right)}{\kappa} d\kappa$$

holds for all $\varkappa \in [0,1]$ and $\kappa \in \left[\nu_1, \sqrt{\nu_1\nu_2}\right]$.
As an application of Lemma 2.2, the following inequality holds:

$$\lambda\left(\nu_1^{\varkappa_1}\kappa^{1-\varkappa_1}\right) + \lambda\left(\nu_1^{1-\varkappa_1}\nu_2\kappa^{-(1-\varkappa_1)}\right) \leq \lambda\left(\nu_1^{\varkappa_2}\kappa^{1-\varkappa_2}\right) + \lambda\left(\nu_1^{1-\varkappa_2}\nu_2\kappa^{-(1-\varkappa_2)}\right) \tag{21}$$

for all $\varkappa_1, \varkappa_2 \in [0,1]$ with $\varkappa_1 < \varkappa_2$ and $\kappa \in \left[\nu_1, \sqrt{\nu_1\nu_2}\right]$
since

$$\left(\nu_1^{\varkappa_1}\kappa^{1-\varkappa_1}\right)\left(\nu_1^{1-\varkappa_1}\nu_2\kappa^{-(1-\varkappa_1)}\right) = \left(\nu_1^{\varkappa_2}\kappa^{1-\varkappa_2}\right)\left(\nu_1^{1-\varkappa_2}\nu_2\kappa^{-(1-\varkappa_2)}\right) = \nu_1\nu_2.$$

Multiplying both sides of (21) by $\frac{\zeta\left(\frac{\kappa^2}{\nu_1}\right)}{\kappa}$ and then integrating over $\left[\nu_1, \sqrt{\nu_1\nu_2}\right]$, we get that $\mathbb{N}_1(\varkappa_1) \leq \mathbb{N}_1(\varkappa_2)$ for $\varkappa_1, \varkappa_2 \in [0,1]$ with $\varkappa_1 < \varkappa_2$. Hence $\mathbb{N}_1$ is increasing on $[0,1]$ and thus the inequalities (20) are proved. □

Remark 2.9. *Let $\zeta(\kappa) = \frac{1}{\ln\nu_2 - \ln\nu_1}$, $\kappa \in [\nu_1, \nu_2]$ in Theorem 2.3. Then $\mathbb{N}_1(\varkappa) = \mathbb{V}(\varkappa)$, $\varkappa \in [0,1]$ and the inequality (9) reduces to the inequality*

$$\frac{1}{\ln\nu_2 - \ln\nu_1}\int_{\nu_1}^{\nu_2} \frac{\lambda(\kappa)}{\kappa} d\kappa = \mathbb{V}(0) \leq \mathbb{V}(\varkappa) \leq \mathbb{V}(1) = \frac{\lambda(\nu_1) + \lambda(\nu_2)}{2},$$

where S is defined by (7).

Theorem 2.10. *Let λ, ζ, $\mathbb{M}_1$, $\mathbb{N}_1$ be defined as above. Then $\mathbb{M}_1(\varkappa) \leq \mathbb{N}_1(\varkappa)$ on $[0,1]$.*

Proof. The identities:

$$\mathbb{N}_1(\varkappa) = \int_{\nu_1}^{\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}} \left[\lambda\left(\nu_1^{\varkappa}\kappa^{1-\varkappa}\right) + \lambda\left(\nu_1^{\frac{3-\varkappa}{2}}\nu_2^{\frac{1-\varkappa}{2}}\kappa^{-(1-\varkappa)}\right) + \lambda\left(\nu_1^{1-\varkappa}\nu_2\kappa^{-(1-\varkappa)}\right) + \lambda\left(\nu_1^{-\frac{1-\varkappa}{2}}\nu_2^{\frac{1}{2}+\varkappa}\kappa^{1-\varkappa}\right)\right] \frac{\zeta\left(\frac{\kappa^2}{\nu_1}\right)}{\kappa} d\kappa. \tag{22}$$

and

$$\mathbb{M}_1(\varkappa) = \int_{\nu_1}^{\nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}} \left[\lambda\left(\nu_1^{\varkappa}\kappa^{1-\varkappa}\right) + \lambda\left(\nu_1^{\frac{3}{2}-\varkappa}\nu_2^{\frac{1}{2}}\kappa^{-(1-\varkappa)}\right) + \lambda\left(\nu_1^{\varkappa-\frac{1}{2}}\nu_2^{\frac{1}{2}}\kappa^{1-\varkappa}\right) + \lambda\left(\nu_1^{1-\varkappa}\nu_2\kappa^{-(1-\varkappa)}\right)\right] \frac{\zeta\left(\frac{\kappa^2}{\nu_1}\right)}{\kappa} d\kappa \tag{23}$$

hold for all $\varkappa \in [0,1]$ and $\kappa \in \left[\nu_1, \nu_1^{\frac{3}{4}}\nu_2^{\frac{1}{4}}\right]$.
Let

$$\kappa_1 = \nu_1^{\varkappa}\kappa^{1-\varkappa},$$
$$\kappa_2 = \nu_1^{\frac{3}{2}-\varkappa}\nu_2^{\frac{1}{2}}\kappa^{-(1-\varkappa)},$$
$$\sigma_1 = \nu_1^{\varkappa}\kappa^{1-\varkappa}$$

and

$$\sigma_2 = \nu_1^{\frac{3-\varkappa}{2}} \nu_2^{\frac{1-\varkappa}{2}} \kappa^{-(1-\varkappa)}$$

for all $\varkappa \in [0,1]$ and $\kappa \in \left[\nu_1, \nu_1^{\frac{3}{4}} \nu_2^{\frac{1}{4}}\right]$. Then

$$\kappa_1 \kappa_2 = \sigma_1 \sigma_2 = \nu_1^{\frac{3}{2}} \sqrt{\nu_2}.$$

Lemma 2.2 gives the inequality:

$$\lambda\left(\nu_1^{\varkappa} \kappa^{1-\varkappa}\right) + \lambda\left(\nu_1^{\frac{3}{2}-\varkappa} \nu_2^{\frac{1}{2}} \kappa^{-(1-\varkappa)}\right) \leq \lambda\left(\nu_1^{\varkappa} \kappa^{1-\varkappa}\right) + \lambda\left(\nu_1^{\frac{3-\varkappa}{2}} \nu_2^{\frac{1-\varkappa}{2}} \kappa^{-(1-\varkappa)}\right) \tag{24}$$

for all $\varkappa \in [0,1]$ and $\kappa \in \left[\nu_1, \nu_1^{\frac{3}{4}} \nu_2^{\frac{1}{4}}\right]$.
Similarly with the choices

$$\begin{aligned} \kappa_1 &= \nu_1^{\varkappa-\frac{1}{2}} \nu_2^{\frac{1}{2}} \kappa^{1-\varkappa}, \\ \kappa_2 &= \nu_1^{1-\varkappa} \nu_2 \kappa^{-(1-\varkappa)}, \\ \sigma_1 &= \nu_1^{1-\varkappa} \nu_2 \kappa^{-(1-\varkappa)} \end{aligned}$$

and

$$\sigma_2 = \nu_1^{-\frac{1-\varkappa}{2}} \nu_2^{\frac{1}{2}+\varkappa} \kappa^{1-\varkappa}$$

and using Lemma 2.2, we get

$$\lambda\left(\nu_1^{\varkappa-\frac{1}{2}} \nu_2^{\frac{1}{2}} \kappa^{1-\varkappa}\right) + \lambda\left(\nu_1^{1-\varkappa} \nu_2 \kappa^{-(1-\varkappa)}\right) \leq \lambda\left(\nu_1^{1-\varkappa} \nu_2 \kappa^{-(1-\varkappa)}\right) + \lambda\left(\nu_1^{-\frac{1-\varkappa}{2}} \nu_2^{\frac{1}{2}+\varkappa} \kappa^{1-\varkappa}\right). \tag{25}$$

Adding (24) and (25) and multiplying both sides of the resulting inequality by $\frac{\zeta\left(\frac{\kappa^2}{\nu_1}\right)}{\kappa}$ and then integrating over $\left[\nu_1, \nu_1^{\frac{3}{4}} \nu_2^{\frac{1}{4}}\right]$, we get that $\mathbb{M}_1(\varkappa) \leq \mathbb{N}_1(\varkappa)$ for $\varkappa \in [0,1]$. □

Theorems 2.5-2.10 naturally lead to the following Fejér-type inequalities.

Corollary 2.11. *Let λ, ζ be defined as above. Then we have*

$$\begin{aligned} \lambda\left(\sqrt{\nu_1 \nu_2}\right) \int_{\nu_1}^{\nu_2} \frac{\zeta(\kappa)}{\kappa} d\kappa &\leq \frac{\lambda\left(\nu_1^{\frac{3}{4}} \nu_2^{\frac{1}{4}}\right) + \lambda\left(\nu_1^{\frac{1}{4}} \nu_2^{\frac{3}{4}}\right)}{2} \int_{\nu_1}^{\nu_2} \frac{\zeta(\kappa)}{\kappa} d\kappa \leq \frac{1}{2} \int_{\nu_1}^{\nu_2} \left[\lambda\left(\sqrt{\nu_1 \kappa}\right) + \lambda\left(\sqrt{\kappa \nu_2}\right)\right] \frac{\zeta(\kappa)}{\kappa} d\kappa \\ &\leq \frac{1}{2}\left[\lambda\left(\sqrt{\nu_1 \nu_2}\right) + \frac{\lambda(\nu_1) + \lambda(\nu_2)}{2}\right] \int_{\nu_1}^{\nu_2} \frac{\zeta(\kappa)}{\kappa} d\kappa \leq \frac{\lambda(\nu_1) + \lambda(\nu_2)}{2} \int_{\nu_1}^{\nu_2} \frac{\zeta(\kappa)}{\kappa} d\kappa. \end{aligned} \tag{26}$$

Corollary 2.12. *Let $\zeta(\kappa) = \frac{1}{\ln \nu_2 - \ln \nu_1}$, $\kappa \in [\nu_1, \nu_2]$ in Corollary 2.11. Then the inequality (26) reduces to*

$$\begin{aligned} \lambda\left(\sqrt{\nu_1 \nu_2}\right) &\leq \frac{\lambda\left(\nu_1^{\frac{3}{4}} \nu_2^{\frac{1}{4}}\right) + \lambda\left(\nu_1^{\frac{1}{4}} \nu_2^{\frac{3}{4}}\right)}{2} \leq \frac{1}{2}\left(\frac{1}{\ln \nu_2 - \ln \nu_1}\right) \int_{\nu_1}^{\nu_2} \frac{1}{\kappa}\left[\lambda\left(\sqrt{\nu_1 \kappa}\right) + \lambda\left(\sqrt{\kappa \nu_2}\right)\right] d\kappa \\ &\leq \frac{1}{2}\left[\lambda\left(\sqrt{\nu_1 \nu_2}\right) + \frac{\lambda(\nu_1) + \lambda(\nu_2)}{2}\right] \leq \frac{\lambda(\nu_1) + \lambda(\nu_2)}{2}. \end{aligned} \tag{27}$$

3. Conclusions

The topic of mathematical inequalities has been an emerging topic since the last century and considerable research has been conducted by a number of mathematicians. Towards the development of this topic, a number of researchers have tried to generalized the concept of convex sets and convex functions. One of the generalizations of convex functions is *GA*-convex functions. The research in this paper discusses some new Fejér-type inequalities for *GA*-convex functions. In this study, we considered some mappings related to the Fejér-type inequalities for *GA*-convex and discussed properties of these mappings. As a result of discussions of the properties, we get refinements of some known results. The research of this paper could be a source of inspiration for young researchers to explore the topic of generalization of convex functions especially related to *GA*-convex functions and to prove new Hermite-Hadamard and Fejér-type inequalities for *GA*-convex functions.

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