Filomat 37:23 (2023), 7955–7971
https://doi.org/10.2298/FIL2323955T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fixed point theorem for a new $\mathcal{S}_{\mathcal{F}}$-$\mathcal{G}_{\mathcal{F}}$-contraction mappings in metric space with supportive applications

Abdellah Taqbibt[a,*], Mohamed Chaib[a], M'hamed Elomari[a], Said Melliani[a]

[a]*Laboratory of Applied Mathematics and Scientific Calculus, Sultan Moulay Slimane University, PO Box 523, FST, Beni Mellal, 23000, Morocco*

Abstract. This manuscript has for goal to present a new type of contraction, namely $\mathcal{S}_{\mathcal{F}}$-$\mathcal{G}_{\mathcal{F}}$-contraction, based on the simulation function and the Geraghty function. Some existence and uniqueness results of fixed point in complete metric spaces have been shown with this contraction. Our results generalize and unify several existing results in the literature. Moreover, we illustrated these results with some examples and applications.

1. Introduction

In recent years, fixed point theory has become one of the most important tools for solving certain problems in various fields like in nonlinear analysis, physics, biology and game theory. Banach provided the first fixed point theorem in complete metric spaces, which was generalized in different fields, and one of these generalizations was given by Berinde [1], where he introduced the concept of quasicontraction as a generalization of the weak contraction, in the sense of Berinde. Subsequently, several results were obtained, for example, see [2, 2–5, 18, 22, 24]. Recently, Babu et al.[6] introduced a contractive condition, namely "condition (B)", and they proved important results of a fixed point for this contractive condition mappings, similarly, Ćirić et al.[7] introduced the concept of generalized quasi-contraction condition and shown some results of existence of fixed point in ordered metric spaces. Samet et al. [10] introduced a class of functions, namely α-admissible and they proved some results of existence of fixed point for α-ψ-contractive maps. We refer the reader to [11–13] for more results about this class of functions. In 2016, by combining the ideas of [25] and [14], Karapinar [8] introduced the concept of α-admissible $\mathcal{Z}$-contraction to obtain certain fixed point results in metric spaces.

Motivated by the above works, we present a new type of contraction, namely type $\mathcal{S}_{\mathcal{F}}$-$\mathcal{G}_{\mathcal{F}}$-contraction. In addition, we show some results concerning the existence and uniqueness of fixed points for this $\mathcal{S}_{\mathcal{F}}$-$\mathcal{G}_{\mathcal{F}}$-contraction mappings, Our results unify several well-known type of contractions and generalize several existing results in the literature. Moreover, we can describe these results with some exemples and applictions. In 2012, B. Samet and Erdal Karapinar [23] originated the concept of α-admissibility presented in [14].

2020 *Mathematics Subject Classification*. 37C25 , 47H10, 47H09.
Keywords. Simulation function, Admissible mapping, Geraghty functions, Fixed point theory, Contraction mapping, Fractional differential equations.
Received: 04 March 2023; Revised: 01 April 2023; Accepted: 02 April 2023
Communicated by Maria Alessandra Ragusa
* Corresponding author: Abdellah Taqbibt
Email addresses: taqbibt.gmi@gmail.com (Abdellah Taqbibt), mohamed15lmd@gmail.com (Mohamed Chaib), m.elomari@usms.ma (M'hamed Elomari), s.melliani@usms.ma (Said Melliani)

Definition 1.1. *[14] Let* $\alpha : \Sigma \times \Sigma \longrightarrow \mathbb{R}^+$ *be a function and* $Q : \Sigma \longrightarrow \Sigma$ *be a self-mapping. We say that* Q *is a* α*-admissible if*

$$\alpha(\vartheta, \eta) \geq 1 \Longrightarrow \alpha(Q\vartheta, Q\eta) \geq 1 \text{ for all } \vartheta, \eta \in \Sigma. \quad (1)$$

Example 1.2. *[14] Let* $\Sigma = \mathbb{R}^*_+$. *Define* $Q : \Sigma \longrightarrow \Sigma$ *and* $\alpha : \Sigma \times \Sigma \longrightarrow \mathbb{R}^+$, *as follows* $Q\vartheta = \ln(\vartheta)$ *for all* $\vartheta \in \Sigma$ *and*

$$\alpha(\vartheta, \eta) = \begin{cases} 0 & \text{if } \vartheta < \eta, \\ 2 & \text{if } \vartheta \geq \eta. \end{cases}$$

Let $\vartheta, \eta \in \mathbb{R}^+$ *such that* $\alpha(\vartheta, \eta) \geq 1$, *then* $\vartheta \geq \eta$ *and thus* $\ln(\vartheta) \geq \ln(\eta)$, *implies that* $Q\vartheta \geq Q\eta$. *So,* $\alpha(Q\vartheta, Q\eta) = 2 \geq 1$. *Then,* Q *is* α*-admissible.*

Example 1.3. *[14] We define*

$$Q : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$$
$$\vartheta \longmapsto Q\vartheta = \sqrt{\vartheta}$$

and

$$\alpha : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^+$$
$$(\vartheta, \eta) \longmapsto \alpha(\vartheta, \eta) = \begin{cases} 0 & \text{if } \vartheta < \eta, \\ e^{\vartheta - \eta} & \text{if } \vartheta \geq \eta. \end{cases}$$

Let $\vartheta, \eta \in \mathbb{R}^+$ *such that* $\alpha(\vartheta, \eta) \geq 1$, *then* $\vartheta \geq \eta$ *and thus* $\sqrt{\vartheta} \geq \sqrt{\eta}$, *implies that* $\mathcal{P}\vartheta \geq \mathcal{P}\eta$. *So,* $\alpha(\mathcal{P}\vartheta, \mathcal{P}\eta) = e^{\vartheta - \eta} \geq 1$. *Then,* $\mathcal{P}$ *is* α*-admissible.*

Definition 1.4. *[15] Let* $\alpha : \Sigma \times \Sigma \longrightarrow \mathbb{R}^+$ *be a function and* $Q : \Sigma \to \Sigma$ *be a mapping. We say that* Q *is an extended-*α*-admissible if*

$$\alpha(\eta, Q\eta) \geq 1 \Longrightarrow \alpha(Q\eta, Q^2\eta) \geq 1.$$

Definition 1.5. *[16] An* α*-admissible map* Q *is called triangular* α*-admissible if* $\alpha(\vartheta, \sigma) \geq 1$ *and* $\alpha(\sigma, \eta) \geq 1$ *implies* $\alpha(\vartheta, \eta) \geq 1$, *for all* $\vartheta, \eta, \sigma \in \Upsilon$.

A new concept of contractions with the simulation functions has been introduced by Khojasteh et al[25].

Definition 1.6. *[25] The function* $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *is said to be a simulation function, if the following conditions hold*

(Ψ1) $\Psi(\tau, \theta) = 0$ *if and only if* $\tau = \theta = 0$;
(Ψ2) $\Psi(\tau, \theta) < \theta - \tau$ *for all* $\tau, \theta > 0$;
(Ψ3) *if* $\{\theta_n\}, \{\tau_n\}$ *two real positive sequences such that* $\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \theta_n > 0$, *then* $\limsup_{n\to\infty} \Psi(\tau_n, \theta_n) < 0$.

For more details the reader can see [19, 25]. We denote by $\mathcal{S}_{\mathcal{F}}$ the set of all simulation functions.

Definition 1.7. *[21] Let* $\beta : \mathbb{R}^+ \to]0, 1[$ *such that*

$$\text{for every } \{b_m\} \subset \mathbb{R}^+ \text{ and } \lim_{m\to\infty} \beta(b_m) = 1, \text{ implies } \lim_{m\to\infty} b_m = 0^+$$

such a function is said a Geraghty function.

The set of all Geraghty functions will be denoted by $\mathcal{G}_{\mathcal{F}}$.

Definition 1.8. *[25] Let* $Q : \Sigma \to \Sigma$ *be a self-mapping over a metric space* (Σ, δ) *and let* $\Psi \in \mathcal{S}_{\mathcal{F}}$. *We say that* Q *is a* $\mathcal{S}_{\mathcal{F}}$*-contraction with respect to* Ψ, *if the following condition holds*

$$\Psi(\delta(Q\eta, Q\theta), \delta(\eta, \theta)) \geq 0 \qquad \text{for all } \eta, \theta \in \Sigma. \quad (2)$$

Theorem 1.9. *[25] Let* (Σ, δ) *be a complete metric space and* $Q : \Sigma \to \Sigma$ *be a* $\mathcal{S}_{\mathcal{F}}$*-contraction with respect to* $\Psi \in \mathcal{S}_{\mathcal{F}}$. *Then* Q *has a unique fixed point in* Σ *and for every* $\sigma_0 \in \Sigma$ *the Picard sequence* $\{\sigma_n\}$, *where* $\sigma_n = Q\sigma_{n-1}$ *for all* $n \in \mathbb{N}$, *converges to the fixed point of* Q.

2. Main results

In this section we present our main results, to do this we start with the following definition.

Definition 2.1. *Let (Σ, δ) be a metric space and $\alpha : \Sigma \times \Sigma \longrightarrow \mathbb{R}^+$ be a function. We say that $Q : \Sigma \longrightarrow \Sigma$ is a $\mathcal{S}_{\mathcal{F}}$-$\mathcal{G}_{\mathcal{F}}$-contraction if there are $\Psi \in \mathcal{S}_{\mathcal{F}}$, $\beta \in \mathcal{G}_{\mathcal{F}}$ and $L \geq 0$ such that*

$$\alpha(\eta, \theta) \geq 1 \Rightarrow \Psi(\delta(Q\eta, Q\theta), \beta(M(\eta, \theta))M(\eta, \theta) + LN(\eta, \theta)) \geq 0, \tag{3}$$

where

$$N(\eta, \theta) = \min\{\delta(\eta, Q\eta), \delta(\theta, Q\theta), \delta(\eta, Q\theta), \delta(y, Qx)\};$$
$$M(\eta, \theta) = \max\{\delta(\eta, \theta), \frac{\delta(\eta, Q\eta) + \delta(\theta, Q\theta)}{2}, \frac{\delta(\eta, Q\theta) + \delta(\theta, Q\eta)}{2}\}$$

for all $\eta, \theta \in \Sigma$.

According to this definition, we have

Theorem 2.2. *Let (Σ, δ) be a complete metric space and let $Q : \Sigma \longrightarrow \Sigma$ be a $\mathcal{S}_{\mathcal{F}}$-$\mathcal{G}_{\mathcal{F}}$-contraction. In addition we assume that the following conditions*

(**C1**) *Q is a triangular α-admissible;*
(**C2**) *there is $\sigma_0 \in \Sigma$ with $\alpha(\sigma_0, Q\sigma_0) \geq 1$;*
(**C3**) *Q is continuous;*

hold. Then there is $\gamma \in \Sigma$ with $Q\gamma = \gamma$.

Proof. Condition (**C2**) ensure the existence of $\sigma_0 \in \Sigma$, with $\alpha(\sigma_0, Q\sigma_0) \geq 1$. Now let $\{\sigma_n\}$ be an iterative sequence in Σ given by

$$\sigma_{n+1} = Q\sigma_n \text{ for all } n \in \mathbb{N}.$$

If there is $m \in \mathbb{N}$ such that $\sigma_m = Q\sigma_m$ the proof is obvious. Assume that $\sigma_{n+1} \neq \sigma_n$ for all $n \in \mathbb{N}$. Combine both conditions (**C1**) and (**C2**), we have

$$\alpha(\sigma_0, \sigma_1) = \alpha(\sigma_0, Q\sigma_0) \geq 1 \text{ implies that } \alpha(Q\sigma_0, Q\sigma_1) = \alpha(\sigma_1, \sigma_2) \geq 1,$$

and so on by recurrence, we find

$$\alpha(\sigma_n, \sigma_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}. \tag{4}$$

From **(3)** and **(4)** with $p = \sigma_n$ and $q = \sigma_{n-1}$, we obtain

$$\begin{aligned} 0 &\leq \Psi(\delta(Q\sigma_n, Q\sigma_{n-1}), \beta(M(\sigma_n, \sigma_{n-1}))M(\sigma_n, \sigma_{n-1}) + LN(\sigma_n, \sigma_{n-1})) \\ &= \Psi(\delta(\sigma_{n+1}, \sigma_n), \beta(M(\sigma_n, \sigma_{n-1}))M(\sigma_n, \sigma_{n-1}) + LN(\sigma_n, \sigma_{n-1})) \\ &< M(\sigma_n, \sigma_{n-1}) + LN(\sigma_n, \sigma_{n-1}) - \delta(\sigma_n, \sigma_{n+1}), \end{aligned} \tag{5}$$

where

$$\begin{aligned} N(\sigma_n, \sigma_{n-1}) &= \min\{\delta(\sigma_n, Q\sigma_n), \delta(\sigma_{n-1}, Q\sigma_{n-1}), \delta(\sigma_n, Q\sigma_{n-1}), \delta(\sigma_{n-1}, Q\sigma_n)\} \\ &= \min\{\delta(\sigma_{n-1}, \sigma_n), \delta(\sigma_n, \sigma_{n+1}), \delta(\sigma_{n-1}, \sigma_{n+1}, \delta(\sigma_n, \sigma_n))\} \\ &= 0 \end{aligned} \tag{6}$$

and

$$\begin{aligned}M(\sigma_{n-1},\sigma_n) &= \max\{\delta(\sigma_n,\sigma_{n-1}), \frac{\delta(\sigma_n,Q\sigma_n)+\delta(\sigma_{n-1},Q\sigma_{n-1})}{2},\\ &\qquad \frac{\delta(\sigma_n,Q\sigma_{n-1})+\delta(\sigma_{n-1},Q\sigma_n)}{2}\}\\ &= \max\{\delta(\sigma_n,\sigma_{n-1}), \frac{\delta(\sigma_n,\sigma_{n+1})+\delta(\sigma_{n-1},\sigma_n)}{2}, \frac{\delta(\sigma_{n-1},\sigma_{n+1})}{2}\}\\ &\le \max\{\delta(\sigma_n,\sigma_{n-1}), \frac{\delta(\sigma_n,\sigma_{n+1})+\delta(\sigma_{n-1},\sigma_n)}{2}\}\\ &\le \max\{\delta(\sigma_n,\sigma_{n-1}),\delta(\sigma_n,\sigma_{n+1})\}. \end{aligned} \tag{7}$$

By **(5)**, **(4)**, **(6)** and **(7)**, we get

$$\delta(\sigma_n,\sigma_{n+1}) < \max\{\delta(\sigma_n,\sigma_{n-1}),\delta(\sigma_n,\sigma_{n+1})\}$$

for all $n \ge 1$.

Now, if $\max\{\delta(\sigma_n,\sigma_{n-1}),\delta(\sigma_n,\sigma_{n+1})\} = \delta(\sigma_n,\sigma_{n+1})$ for some $n \ge 1$. Then from the above inequality, we get

$$\delta(\sigma_n,\sigma_{n+1}) < \delta(\sigma_n,\sigma_{n+1}),$$

which constitutes a contradiction. Therefore

$$\max\{\delta(\sigma_{n-1},\sigma_n),\delta(\sigma_n,\sigma_{n+1})\} = \delta(\sigma_n,\sigma_{n-1}) \text{ for all } n \ge 1. \tag{8}$$

Hence

$$\delta(\sigma_n,\sigma_{n+1}) < \delta(\sigma_n,\sigma_{n-1}) \text{ for all } n \ge 1. \tag{9}$$

Consequently, we conclude that $\{\delta(\sigma_n,\sigma_{n-1})\}$ is a decreasing sequence of positive real numbers. Thus, there is $r \ge 0$ such that $\lim_{n\to\infty} \delta(\sigma_n,\sigma_{n-1}) = r \ge 0$. We claim that

$$\lim_{n\to\infty} \delta(\sigma_n,\sigma_{n-1}) = 0. \tag{10}$$

On contrary if $r > 0$. From **(9)** it follows that

$$\lim_{n\to\infty} \delta(\sigma_n,\sigma_{n+1}) = r. \tag{11}$$

Now, if we take the sequences $\{\delta_n = \delta(\sigma_n,\sigma_{n+1})\}$ and $\{\tau_n = \delta(\sigma_n,\sigma_{n-1})\}$ and considering **(11)**, then $\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \tau_n = r$ therefore by (**Ψ3**), we get that

$$0 \le \lim_{n\to\infty} \sup \Psi(\delta(\sigma_n,\sigma_{n+1}),\delta(\sigma_n,\sigma_{n-1})) < 0 \tag{12}$$

which constitutes a contradiction, we conclude that $r = 0$.

In the next step, let us prove that $\{\sigma_n\}$ is a Cauchy sequence in Σ. On contrary, assume that $\{\sigma_n\}$ is not a Cauchy sequence. So, there exists $\epsilon > 0$, for every $N \in \mathbb{N}$, there are $n, m \in \mathbb{N}$ such that $N < m < n$ and $\delta(\sigma_m,\sigma_n) > \epsilon$. In account of **(10)**, there exists $n_0 \in \mathbb{N}$ with

$$\delta(\sigma_n,\sigma_{n+1}) < \epsilon \text{ for every } n > n_0. \tag{13}$$

We can find subsequences $\{\sigma_{m_k}\}$ and $\{\sigma_{n_k}\}$ of $\{\sigma_n\}$ such that $m_k > n_k \ge n_0$ and

$$\delta(\sigma_{m_k},\sigma_{n_k}) \quad > \epsilon, \text{ for all } k. \tag{14}$$

where m_k is the smallest integer that satisfies the formula **(14)**. Then

$$\delta(\sigma_{m_{k-1}}, \sigma_{n_k}) \leq \epsilon \text{ for every } k, \tag{15}$$

Now, using **(14)**, **(15)** and the triangular inequality, we obtain

$$\begin{aligned} \epsilon < \delta(\sigma_{m_k}, \sigma_{n_k}) &\leq \delta(\sigma_{m_k}, \sigma_{m_k-1}) + \delta(\sigma_{m_k-1}, \sigma_{n_k}) \\ &\leq \delta(\sigma_{m_k}, \sigma_{m_k-1}) + \epsilon \end{aligned} \tag{16}$$

Letting $k \to \infty$ and using Equation **(10)**, we derive that

$$\lim_{n\to\infty} \delta(\sigma_{m_k}, \sigma_{n_k}) = \epsilon. \tag{17}$$

Again, by the triangular inequality, we have

$$\delta(\sigma_{m_k}, \sigma_{n_k}) \leq \delta(\sigma_{m_k}, \sigma_{m_k+1}) + \delta(\sigma_{m_k+1}, \sigma_{n_k+1}) + \delta(\sigma_{n_k+1}, \sigma_{n_k}) \text{ for all } k. \tag{18}$$

Also, we have

$$\delta(\sigma_{m_k+1}, \sigma_{n_k+1}) \leq \delta(\sigma_{m_k+1}, \sigma_{m_k}) + \delta(\sigma_{m_k}, \sigma_{n_k}) + \delta(\sigma_{n_k}, \sigma_{n_k+1}) \text{ for all } k. \tag{19}$$

By passing to the limit as $k \to \infty$ in **(19)**, **(18)** and **(10)** we conclude that

$$\lim_{n\to\infty} \delta(\sigma_{m_k+1}, \sigma_{n_k+1}) = \epsilon. \tag{20}$$

Based on the same reasoning as above, we can write

$$\lim_{n\to\infty} \delta(\sigma_{m_k}, \sigma_{n_k+1}) = \lim_{n\to\infty} \delta(\sigma_{m_k+1}, \sigma_{n_k}) = \epsilon. \tag{21}$$

As Q is a triangular α-admissible, we have

$$\alpha(\sigma_{m_k}, \sigma_{n_k}) \geq 1. \tag{22}$$

As well, since Q is a $\mathcal{S}_{\mathcal{F}}$-$\mathcal{G}_{\mathcal{F}}$-contraction, we get

$$\begin{aligned} 0 &\leq \Psi(\delta(Q\sigma_{m_k}, Q\sigma_{n_k}), \beta(M(\sigma_{m_k}, \sigma_{n_k}))M(\sigma_{m_k}, \sigma_{n_k}) + LN(\sigma_{m_k}, \sigma_{n_k})) \\ &= \Psi(\delta(\sigma_{m_k+1}, \sigma_{n_k+1}), \beta(M(\sigma_{m_k}, \sigma_{n_k}))M(\sigma_{m_k}, \sigma_{n_k}) + LN(\sigma_{m_k}, \sigma_{n_k})) \\ &< \beta(M(\sigma_{m_k}, \sigma_{n_k}))M(\sigma_{m_k}, \sigma_{n_k}) + LN(\sigma_{m_k}, \sigma_{n_k}) - \delta(\sigma_{m_k+1}, \sigma_{n_k+1}). \end{aligned} \tag{23}$$

Hence

$$\begin{aligned} 0 < \delta(\sigma_{m_k+1}, \sigma_{n_k+1}) &< \beta(M(\sigma_{m_k}, \sigma_{n_k}))M(\sigma_{m_k}, \sigma_{n_k}) + LN(\sigma_{m_k}, \sigma_{n_k}) \\ &< M(\sigma_{m_k}, \sigma_{n_k}) + LN(\sigma_{m_k}, \sigma_{n_k}). \end{aligned} \tag{24}$$

for all $k \geq n_1$. Where

$$\begin{aligned} M(\sigma_{m_k}, \sigma_{n_k}) &= \max\{\delta(\sigma_{m_k}, \sigma_{n_k}), \frac{\delta(\sigma_{m_k}, Q\sigma_{m_k}) + \delta(\sigma_{n_k}, Q\sigma_{n_k})}{2}, \\ &\qquad \frac{\delta(\sigma_{m_k}, Q\sigma_{n_k}) + \delta(\sigma_{n_k}, Q\sigma_{m_k})}{2}\} \\ &= \max\{\delta(\sigma_{m_k}, \sigma_{n_k}), \frac{\delta(\sigma_{m_k}, \sigma_{m_k+1}) + \delta(\sigma_{n_k}, \sigma_{n_k+1})}{2}, \\ &\qquad \frac{\delta(\sigma_{m_k}, \sigma_{n_k+1}) + \delta(\sigma_{n_k}, \sigma_{m_k+1})}{2}\} \end{aligned} \tag{25}$$

and

$$\begin{aligned} &N(\sigma_{m_k}, \sigma_{n_k}) \\ &\quad = \min\{\delta(\sigma_{m_k}, Q\sigma_{m_k}), \delta(\sigma_{n_k}, Q\sigma_{n_k}), \delta(\sigma_{m_k}, Q\sigma_{n_k}), \delta(\sigma_{n_k}, Q\sigma_{m_k})\} \\ &\quad = \min\{\delta(\sigma_{m_k}, \sigma_{m_k+1}), \delta(\sigma_{n_k}, \sigma_{n_k+1}), \delta(\sigma_{m_k}, \sigma_{n_k+1}), \delta(\sigma_{n_k}, \sigma_{m_k+1})\} \end{aligned} \tag{26}$$

Taking the limit as k tends to $+\infty$ in **(26)** and **(25)**, by **(17)**, **(10)**, **(21)** and **(20)** we get

$$\lim_{k\to\infty} M(\sigma_{m_k}, \sigma_{n_k}) = \epsilon \tag{27}$$

and

$$\lim_{k\to\infty} N(\sigma_{m_k}, \sigma_{n_k}) = 0. \tag{28}$$

From **(28)**, **(24)** and **(27)**, we derive that

$\mu_{n_k} = \delta(\sigma_{m_k+1}, \sigma_{n_k+1})$ and $\nu_{n_k} = M(\sigma_{m_k}, \sigma_{n_k}) + LN(\sigma_{m_k}, \sigma_{n_k})$, we have $\lim_{k\to\infty} \mu_{n_k} = \lim_{k\to\infty} \nu_{n_k} = \epsilon$. Therefore by (**Ψ3**), we get

$$0 \le \limsup_{k\to\infty} \Psi(\delta(\sigma_{m_k+1}, \sigma_{n_k+1}), M(\sigma_{m_k}, \sigma_{n_k}) + LN(\sigma_{m_k}, \sigma_{n_k}) < 0,$$

which gives a contradiction. So $\{\sigma_n\}$ is a Cauchy sequence in (Σ, δ). Therefore, there is $\gamma \in \Sigma$ satisfies

$$\lim_{n\to\infty} \delta(\sigma_n, \gamma) = 0, \tag{29}$$

the continuity of Q implies that

$$\lim_{n\to\infty} \delta(\sigma_{n+1}, Q\gamma) = \lim_{n\to\infty} \delta(Q\sigma_n, Q\gamma) = 0. \tag{30}$$

By using **(30)**, **(29)**, we conclude that $Q\gamma = \gamma$.
□

Theorem 2.3. *Let (Σ, δ) be a complete metric space and let $Q : \Sigma \to \Sigma$ be a $\mathcal{S}_{\mathcal{F}}$-$\mathcal{G}_{\mathcal{F}}$-contraction mappings verifies the following assumptions*

C1) *Q is a triangular α-admissible;*
(**C2**) *there exists $\sigma_0 \in \Sigma$ with $\alpha(\sigma_0, Q\sigma_0) \ge 1$;*
(**C3**) *if $\{\sigma_n\}$ is a sequence in Σ with*

$$\alpha(\sigma_n, \sigma_{n+1}) \ge 1 \text{ for every } n \text{ and } \lim_{n\to\infty} \sigma_n = p \in \Sigma,$$

then there is a subsequence $\{\sigma_{n(k)}\}$ of $\{\sigma_n\}$ such that $\alpha(\sigma_{n(k)}, p) \ge 1$ for every k.

Then, Q posses a fixed point.

Proof. We adopt the same reasoning as in the proof of the previous theorem, we can show that $\sigma_{n+1} = Q\sigma_n$ for all $n \ge 0$ is a Cauchy sequence in Σ. Since (Σ, δ) is complete, there is $\gamma \in \Sigma$ satisfies $\lim_{n\to\infty} \sigma_n = \gamma$. By **(4)** and the assumption (**C3**), there is a subsequence $\{\sigma_{n(k)}\}$ of $\{\sigma_n\}$ such that $\alpha(\sigma_{n(k)}, \gamma) \ge 1$ for every $k \in \mathbb{N}$. According to the contration given in **(3)**, we obtain

$$\begin{aligned} 0 &\le \Psi(\delta(Q\sigma_{n(k)}, Q\gamma), \beta(M(\sigma_{n(k)}, \gamma))M(\sigma_{n(k)}, \gamma) + LN(\sigma_{n(k)}, \gamma)) \\ &= \Psi(\delta(\sigma_{n(k)+1}, Q\gamma), \beta(M(\sigma_{n(k)}, \gamma))M(\sigma_{n(k)}, \gamma) + LN(\sigma_{n(k)}, \gamma)) \\ &< \beta(M(\sigma_{n(k)}, \gamma))M(\sigma_{n(k)}, \gamma) + LN(\sigma_{n(k)}, \gamma) - \delta(\sigma_{n(k)+1}, Q\gamma). \end{aligned}$$

Hence

$$\begin{aligned}\delta(\sigma_{n(k)+1}, Q\gamma) &< \beta(M(\sigma_{n(k)}, \gamma))M(\sigma_{n(k)}, \gamma) + LN(\sigma_{n(k)}, \gamma)\\ &< M(\sigma_{n(k)}, \gamma) + LN(\sigma_{n(k)}, \gamma)\end{aligned} \tag{31}$$

where

$$\begin{aligned}M(\sigma_{n(k)}, \gamma) &= \max\{(\delta(\sigma_{n(k)}, \gamma), \frac{\delta(\sigma_{n(k)}, \sigma_{n(k)+1}) + \delta(\gamma, Q\gamma)}{2},\\ &\qquad \frac{\delta(\sigma_{n(k)}, Q\gamma) + \delta(\gamma, \sigma_{n(k)+1})}{2}\}\\ N(\sigma_{n(k)}, \gamma) &= \min\{(\delta(\sigma_{n(k)}, \sigma_{n(k)+1}), \delta(\gamma, Q\gamma), \delta(\sigma_{n(k)}, Q\gamma), \delta(\gamma, \sigma_{n(k)+1})\}.\end{aligned}$$

By passing to the limite as $k \to \infty$, we get

$$\lim_{k\to\infty} M(\sigma_{n(k)}, \gamma) = \frac{\delta(z, Q\gamma)}{2} \tag{32}$$

$$\lim_{k\to\infty} N(\sigma_{n(k)}, \gamma) = 0. \tag{33}$$

Suppose that $\delta(\gamma, Q\gamma) > 0$. By **(31)**, it yields

$$\delta(\sigma_{n(k)+1}, Q\gamma) < M(\sigma_{n(k)}, \gamma) + LN(\sigma_{n(k)}, \gamma)$$

Now, by passing to the limite as $k \to \infty$, thank's to **(33)** and **(32)**, we obtain

$$\delta(\gamma, Q\gamma) \leq \frac{\delta(\gamma, Q\gamma)}{2}$$

a contradiction and hence $\delta(\gamma, Q\gamma) = 0$, that is $\gamma = Q\gamma$. □

To ensure the uniqueness we need in addition to the following condition
(**C**) $\alpha(\eta, \theta) \geq 1$ for every $\eta, \theta \in \mathbf{Fix}(Q)$, where $\mathbf{Fix}(Q)$ denotes the set of all fixed points of Q.

Theorem 2.4. *Suppose that the assumptions of Theorem* **2.2** *(resp. Theorem* **2.3***) satisfied, if in addition* (**C**) *holds, then* Q *posses a unique fixed point.*

Proof. Assume that there are two distinct points $\gamma, \gamma^* \in \Sigma$ with $\gamma = Q\gamma$ and $\gamma^* = Q\gamma^*$. According to (**C**), we can write

$$\alpha(\gamma, \gamma^*) \geq 1.$$

And thus, from **(3)** and (**Ψ2**), it follows that

$$\begin{aligned}0 &\leq \Psi(\delta(Q\gamma, Q\gamma^*), \beta(M(\gamma, \gamma^*))M(\gamma, \gamma^*) + LN(\gamma, \gamma^*))\\ &= \Psi(\delta(\gamma, \gamma^*), \beta(M(\gamma, \gamma^*))M(\gamma, \gamma^*) + LN(\gamma, \gamma^*))\\ &< \beta(M(\gamma, \gamma^*))M(\gamma, \gamma^*) + LN(\gamma, \gamma^*) - \delta(\gamma, \gamma^*)\\ &< M(\gamma, \gamma^*) + LN(\gamma, \gamma^*) - \delta(\gamma, \gamma^*).\end{aligned} \tag{34}$$

Where

$$\begin{aligned} M(\gamma,\gamma^*) &= \max\{\delta(\gamma,\gamma^*), \frac{\delta(\gamma,Q\gamma)+\delta(\gamma^*,Q\gamma^*)}{2}, \frac{\delta(\gamma,Q\gamma^*)+\delta(\gamma^*,Q\gamma)}{2}\} \\ &= \max\{\delta(\gamma,\gamma^*), \frac{\delta(\gamma,\gamma)+\delta(\gamma^*,\gamma^*)}{2}, \frac{\delta(\gamma,\gamma^*)+\delta(\gamma^*,\gamma)}{2}\} \\ &= \delta(\gamma,\gamma^*) \end{aligned} \tag{35}$$

and

$$\begin{aligned} N(\gamma,\gamma^*) &= \min\{\delta(\gamma,Q\gamma),\delta(\gamma^*,Q\gamma^*),\delta(\gamma,Q\gamma^*),\delta(\gamma^*,Q\gamma)\} \\ &= \min\{\delta(\gamma,\gamma),\delta(\gamma^*,\gamma^*),\delta(\gamma,\gamma^*),\delta(\gamma^*,\gamma)\} \\ &= 0 \end{aligned} \tag{36}$$

From **(34)**, **(36)** and **(35)** we deduce that

$$0 < \delta(\gamma,\gamma^*) - \delta(\gamma,\gamma^*).$$

So,

$$\delta(\gamma,\gamma^*) < \delta(\gamma,\gamma^*)$$

which is a contradiction. Hence $\gamma = \gamma^*$.

□

Example 2.5. *Let $\Sigma = [0, \frac{1}{4}]$ endowed with the metric induced by absolute value. Consider $\Psi_1 : [0,\infty)\times[0,\infty) \longrightarrow \mathbb{R}$ be defined by $\Psi_1(\tau,s) = \frac{s}{2} - \tau$, and let $\beta : \mathbb{R}^+ \to [0,1)$, $\beta(\tau) = \frac{1}{1+\tau}$, $\forall\tau > 0, \beta(0) = \frac{1}{2}$ and L be a positive real number.*
We define the mappings $Q : \Sigma \longrightarrow \Sigma$ and $\alpha : \Sigma\times\Sigma \longrightarrow [0,\infty)$, as follows

$$Q\eta = \frac{\eta}{3}, \quad \textit{for every } \eta \in \Sigma,$$

and

$$\alpha(\eta,\theta) = \begin{cases} 1 & \textit{if } \eta,\theta \in [0,\frac{1}{4}], \\ 0 & \textit{otherwise.} \end{cases}$$

We have

$$\begin{aligned} \Psi_1\Big(\delta(Q\eta,Q\theta), \beta(M(\eta,\theta))M(\eta,\theta) &+ LN(\eta,\theta)\Big) \\ &= \frac{\beta(M(\eta,\theta))M(\eta,\theta)+LN(\eta,\theta)}{2} \\ &- \delta(Q\eta,Q\theta) \\ &= \frac{M(\eta,\theta)}{2(1+M(\eta,\theta))} + \frac{LN(\eta,\theta)}{2} \\ &- \delta(Q\eta,Q\theta). \end{aligned}$$

Let $p,q \in \Sigma$, we have

$$\begin{aligned} \delta(Q\eta,Q\theta) &= |\, Q\eta - Q\theta \,| \\ &\le \frac{1}{3}\max\{\delta(\eta,\theta), \frac{\delta(\eta,Q\eta)+\delta(\theta,Q\theta)}{2}, \frac{\delta(\eta,Q\theta)+\delta(\theta,Q\eta)}{2}\} \\ &= \frac{1}{3}M(\eta,\theta). \end{aligned}$$

Then,

$$\begin{aligned}
&\Psi_1\Big(\delta(Q\eta, Q\theta), \beta(M(\eta,\theta))M(\eta,\theta) + LN(\eta,\theta)\Big)\\
&\qquad = \frac{\beta(M(\eta,\theta))M(\eta,\theta) + LN(\eta,\theta)}{2}\\
&\qquad - \delta(Q\eta, Q\theta)\\
&\qquad \geq \frac{M(\eta,\theta)}{2(1+M(\eta,\theta))} + \frac{LN(\eta,\theta)}{2}\\
&\qquad - \frac{1}{3}M(\eta,\theta)\\
&\qquad = \frac{M(\eta,\theta) - 2M^2(\eta,\theta)}{6(1+M(\eta,\theta))}\\
&\qquad + \frac{LN(\eta,\theta)}{2}.
\end{aligned}$$

On the other hand, we have $\forall \eta, \theta \in \Sigma,\ \ M(\eta,\theta) \in [0, \frac{1}{4}]$. *So*

$$M(\eta,\theta) - 2M^2(\eta,\theta) \geq 0.$$

Finally

$$\Psi_1\Big(\delta(Q\eta, Q\theta), \beta(M(\eta,\theta))M(\eta,\theta) + LN(\eta,\theta)\Big) \geq 0.$$

Example 2.6. *Let* $\Sigma = [0,3]$ *endowed with the metric induced by absolute value. We define the mapping* $g : \Sigma \longrightarrow \Sigma$ *by*

$$g\eta = \tfrac{\eta}{7} + \tfrac{1}{9}, \quad \forall \eta \in \Sigma.$$

and the simulation function defined by

$$\Psi_2(\tau, s) = s - \frac{\tau+2}{\tau+1}\tau, \quad \forall s, \tau \geq 0.$$

Define the mapping $\alpha : \Sigma \times \Sigma \longrightarrow [0,\infty)$ *by*

$$\alpha(\eta,\theta) = \begin{cases} 1 & \text{if } \eta \neq \theta, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\eta, \theta \in \Sigma$ *with* $\eta \neq \theta$, *we have*

$$\begin{aligned}
&\Psi_2\Big(\delta(g\eta, g\theta), \beta(M(\eta,\theta))M(\eta,\theta) + LN(\eta,\theta)\Big)\\
&\qquad = \beta(M(\eta,\theta))M(\eta,\theta) + LN(\eta,\theta)\\
&\qquad - \delta(g\eta, g\theta)\frac{\delta(g\eta, g\theta) + 2}{\delta(g\eta, g\theta) + 1}\\
&\qquad = \frac{M(\eta,\theta)}{1+M(\eta,\theta)} + LN(\eta,\theta)\\
&\qquad - \delta(g\eta, g\theta)\frac{\delta(g\eta, g\theta) + 2}{\delta(g\eta, g\theta) + 1}
\end{aligned}$$

and

$$\begin{aligned}\delta(g\eta, g\theta) &= \mid g\eta - g\theta \mid \\ &\leq \frac{1}{7}\max\{\delta(\eta,\theta), \frac{\delta(\eta, g\theta)+\delta(\theta, g\theta)}{2}, \frac{\delta(\eta, g\theta)+\delta(\theta, g\theta)}{2}\} \\ &= \frac{1}{7}M(\eta,\theta).\end{aligned}$$

Then,

$$\begin{aligned}&\Psi_2\Big(\delta(g\eta, g\theta), \beta(M(\eta,\theta))M(\eta,\theta)+LN(\eta,\theta)\Big) \\ &= \beta(M(\eta,\theta))M(\eta,\theta)+LN(\eta,\theta) \\ &- \delta(g\eta, g\theta)\frac{\delta(g\eta, g\theta)+2}{\delta(g\eta, g\theta)+1} \\ &\geq \frac{M(\eta,\theta)}{1+M(\eta,\theta)}+LN(\eta,\theta) \\ &- \frac{1}{7}M(\eta,\theta) \\ &\geq \frac{M(\eta,\theta)}{1+M(\eta,\theta)}+LN(\eta,\theta) \\ &- \frac{1}{7}M(\eta,\theta) \\ &= \frac{6M(\eta,\theta)-M^2(\eta,\theta)}{7(1+M(\eta,\theta))}+LN(\eta,\theta).\end{aligned}$$

On the other hand, we have $\forall \eta, \theta \in \Sigma, \quad M(\eta,\theta) \in [0,3]$. *So*

$$6M(\eta,\theta) - M^2(\eta,\theta) \geq 0.$$

Finally

$$\Psi_2\Big(\delta(g\eta, g\theta), \beta(M(\eta,\theta))M(\eta,\theta)+LN(\eta,\theta)\Big) \geq 0.$$

3. Consequences

In this section, several classical fixed point results can be easily deduced using our main results.

Corollary 3.1. (*See E. Karapinar and V. M. L. Hima Bindu[26]) Let* (Σ, δ) *be a complete metric space,* $\mathcal{Q} : \Sigma \to \Sigma$ *be a mapping and*

$$\alpha : \Sigma \times \Sigma \longrightarrow [0, \infty)$$

be an admissible function. Assume that there exist $\Psi \in \mathcal{S}_{\mathcal{F}}$, $\beta \in \mathcal{G}_{\mathcal{F}}$ *and a constant* $L \geq 0$ *such that*

$$\alpha(\eta,\theta) \geq 1 \Rightarrow \Psi(\delta(\mathcal{Q}\eta, \mathcal{Q}\theta), \beta(\delta(\eta,\theta))\delta(\eta,\theta) + LN(\eta,\theta)) \geq 0, \quad \forall \eta, \theta \in \Sigma,$$

where

$$N(\eta,\theta) = \min\{\delta(\eta, \mathcal{Q}\eta), \delta(\theta, \mathcal{Q}\theta), \delta(\eta, \mathcal{Q}\theta), \delta(\theta, \mathcal{Q}\eta)\}.$$

Furthermore, we suppose that

(*i*) $\mathcal{Q}$ *is a triangular* α*-admissible;*

(ii) *there is* $\sigma_0 \in \Sigma$ *with* $\alpha(\sigma_0, Q\sigma_0) \geq 1$;

(iii) *either,* Q *is continuous, or*

(iii)′ *if* $\{\sigma_n\}$ *is a sequence in* Σ *with* $\alpha(\sigma_n, \sigma_{n+1}) \geq 1$ *for every* n *and*

$$\lim_{n\to\infty} \sigma_n = p \in \Sigma,$$

then is a subsequence $\{\sigma_{n(k)}\}$ *of* $\{\sigma_n\}$ *such that* $\alpha(\sigma_{n(k)}, \eta) \geq 1$ *for every* k;

(iv) $\forall \eta, \theta \in \boldsymbol{Fix}(Q)$ *we have* $\alpha(\eta, \theta) \geq 1$.

Then there exists $\gamma \in \Sigma$ *such that* $Q\gamma = \gamma$.

Proof. We choose the mapping $M(\eta, \theta) = \delta(\eta, \theta)$ in Theorems **2.2**, **2.3** and **2.4**. □

Corollary 3.2. *We define over the complete metric space* (Σ, δ) *a self-mapping* Q. *Assume that there are* $\Psi \in \mathcal{S}_{\mathcal{F}}$, $\beta \in \mathcal{G}_{\mathcal{F}}$, *such that*

$$\Psi(\delta(Q\eta, Q\theta), \beta(M(\eta, \theta))M(\eta, \theta) + LN(\eta, \theta)) \geq 0, \quad \text{for every } \eta, \theta \in \Sigma,$$

where

$$N(\eta, \theta) = \min\{\delta(\eta, Q\eta), \delta(\theta, Q\theta), \delta(\eta, Q\theta), \delta(y, Qx)\};$$

$$M(\eta, \theta) = \max\{\delta(\eta, \theta), \frac{\delta(\eta, Q\eta) + \delta(\theta, Q\theta)}{2}, \frac{\delta(\eta, Q\theta) + \delta(\theta, Q\eta)}{2}\}.$$

Then there is $\gamma \in \Sigma$ *with* $Q\gamma = \gamma$.

Proof. Let us consider the mapping $\alpha : \Sigma \times \Sigma \longrightarrow [0, \infty)$ with $\alpha(\eta, \theta) = 1$, for every $\eta, \theta \in \Sigma$ in Theorems **2.2**, **2.3** and **2.4**. □

Corollary 3.3. *Let* $Q : \Sigma \longrightarrow \Sigma$ *be a given mapping over a complete metric space. Assume that there are* $\Psi \in \mathcal{S}_{\mathcal{F}}$, $\beta \in \mathcal{G}_{\mathcal{F}}$ *such that*

$$\Psi(\delta(Q\eta, Q\theta), \beta(M(\eta, \theta))M(\eta, \theta))) \geq 0, \quad \text{for every } \eta, \theta \in \Sigma,$$

where

$$M(\eta, \theta) = \max\{\delta(\eta, \theta), \frac{\delta(\eta, Q\eta) + \delta(\theta, Q\theta)}{2}, \frac{\delta(\eta, Q\theta) + \delta(\theta, Q\eta)}{2}\}.$$

Then there is $\gamma \in \Sigma$ *with* $Q\gamma = \gamma$.

Proof. Taking $\alpha(\eta, \theta) = 1$, for every $\eta, \theta \in \Sigma$ and $L = 0$ in Theorems **2.2**, **2.3** and **2.4**. □

Theorem 3.4. *Let* Q *be a self-mapping defined on a complete metric space* (Σ, δ). *Assume that there is* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *be an upper semi-continuous mapping with*

$$\begin{cases} \psi(\tau) < \tau, & \text{if } \tau > 0; \\ \psi(\tau) = 0, & \text{if } \tau = 0, \end{cases} \tag{37}$$

and $\beta \in \mathcal{G}_{\mathcal{F}}$, *such that*

$$\delta(Q\eta, Q\theta) \leq \psi(K(\eta, \theta)), \quad \text{for every } \eta, \theta \in \Sigma. \tag{38}$$

where

$$K(\eta, \theta) = \beta(M(\eta, \theta))M(\eta, \theta) + LN(\eta, \theta);$$

$$N(\eta, \theta) = \min\{\delta(\eta, Q\eta), \delta(\theta, Q\theta), \delta(\eta, Q\theta), \delta(y, Qx)\};$$

$$M(\eta, \theta) = \max\{\delta(\eta, \theta), \frac{\delta(\eta, Q\eta) + \delta(\theta, Q\theta)}{2}, \frac{\delta(\eta, Q\theta) + \delta(\theta, Q\eta)}{2}\}.$$

Then there is $\gamma \in \Sigma$ *with* $Q\gamma = \gamma$.

Proof. Taking $\alpha(\eta, \theta) = 1$, for every $\eta, \theta \in \Sigma$, and let

$$\Psi(\tau, s) = \psi(s) - \tau, \ \forall \tau, s \geq 0.$$

It is clear $\Psi \in \mathcal{S}_{\mathcal{F}}$. What needed to be shown from Theorems **2.2**, **2.3** and **2.4**. □

4. Applications

I) We consider the Caputo-Fabrizio fractional differential equation

$$\begin{cases} ({}^{CF}\mathcal{D}_0^r x)(t) & = f\Big(\tau, x(\tau), ({}^{CF}\mathcal{D}_0^r x)(\tau)\Big); \quad \tau \in J = [0, T], \\ x(0) = \sigma_0 & , \end{cases} \tag{39}$$

where $T > 0, f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, ${}^{CF}\mathcal{D}_0^r$ is the Caputo-Fabrizio fractional derivative at order $r \in (0, 1)$, and $u_0 \in \mathbb{R}$.

Lemma 4.1. *[27]The porblem* **(39)** *is equivalent to the following equation*

$$x(\tau) = \theta + \eta_r h(\tau) + \delta_r \int_0^\tau h(s)ds,$$

where $h \in C(J)$ *with* $h(\tau) = f(\tau, x(\tau), h(\tau))$ *and*

$$\begin{aligned} \theta &= \sigma_0 + \eta_r h(0), \\ \eta_r &= \frac{2(1 - r}{(2 - r)M(r)}, \\ \delta_r &= \frac{2r}{(2 - r)M(r)}. \end{aligned}$$

Consider the complete metric space $(C(J), \delta)$, such that δ is given by

$$\begin{aligned} \delta : C(J) \times C(J) &\to \mathbb{R}_+ \\ (x, y) &\mapsto \delta(z, y) = \sup_{\tau \in J} \mid z(\tau) - y(\tau) \mid . \end{aligned}$$

Let Φ denote the class of all continuous and increasing functions $\varphi : \mathbb{R}^+ \to [0, 1)$ satisfying

$$\varphi(b\tau) \leq b\varphi(\tau) \quad \text{for } b \geq 1 \text{ and } \tau \geq 0.$$

The following hypotheses will be used in the sequel.

(A_1) There exist $\varphi \in \Phi$ and a function $\nu : C(J) \times C(J) \to \mathbb{R}$ and $u_0 \in C(J)$ with

$$\nu\Big(u_0(\tau), \theta + \eta_r h(\tau) + \delta_r \int_0^\tau h(s)ds\Big) \geq 0,$$

$h \in C(J)$, with $h(\tau) = f(\tau, u_0(\tau), h(\tau))$.

(A_2) There exist $\vartheta : C(J) \times C(J) \to]0, \infty[$ and $\gamma : J \to (0, 1)$ such that for each $z, y, \sigma_1, y_1 \in C(J)$ and $\tau \in J$

$$\mid f(\tau, z, y) - f(t, \sigma_1, y_1) \mid \leq \vartheta(z, y) \mid z - \sigma_1 \mid + \gamma(t) \mid y - y_1 \mid,$$

with

$$\| 1 + 2\eta_r \frac{\vartheta(z, y)}{1 - \gamma_s} + \delta_r \int_0^t \frac{\vartheta(z, y)}{1 - \gamma_s} \|_\infty \leq \frac{1}{4}\varphi(\| z - y \|_\infty),$$

where $\gamma_s = \sup_{\tau \in J} \mid \gamma(\tau) \mid$.

(A_3**)** For each $\tau \in J$ and $z, y \in C(J)$, we have

$$\begin{aligned} &\nu\Big(z(\tau) \quad , \quad y(\tau)\Big) \geq 0 \Rightarrow \\ &\qquad \nu\Big(\theta_g + \eta_r g(\tau) + \delta_r \int_0^\tau g(s)ds, \theta_h + \eta_r h(\tau) + \delta_r \int_0^\tau h(s)ds\Big) \geq 0, \end{aligned}$$

where $h, g \in C(J)$, with $h(\tau) = f(\tau, y(\tau), h(\tau), \ g(\tau) = f(\tau, z(\tau), h(\tau)))$ and $\theta_h = y_0 + \eta_r h(0), \ \theta_g = \sigma_0 + \eta_r g(0)$.

(A_4**)** If $(p_n)_{n\in\mathbb{N}} \subset C(J)$ such that $\lim_{n\to\infty} p_n = p$ and $\nu(p_n, p_{n+1}) \geq 0$, then $\nu(p_n, p) \geq 0$.

(A_5**)** If u, v two fiexd solutions of problem **(39)**, either

$$\nu(u, v) \geq 0 \quad \text{or} \quad \nu(v, u) \geq 0.$$

Theorem 4.2. *Under assumptions (*A_1*)-(*A_5*), the problem* **(39)** *has a unique solution.*

Proof. Consider the maping $Q : C(J) \to C(J)$ with

$$\begin{aligned} Q : C(J) \quad &\to \quad C(J) \\ x \quad &\mapsto \quad Qx(t) = \theta + \eta_r h(t) + \delta_r \int_0^t h(s)ds, \end{aligned}$$

where $h \in C(J)$, such that $h(\tau) = f(\tau, x(\tau), h(\tau))$ and $\theta = \sigma_0 + \eta_r h(0)$.

Using Lemma **4.1**, it is therefore sufficient to show that Q has a fixed point.

Let a function $\alpha : C(J) \times C(J) \to [0, \infty)$ defined by

$$\alpha(z, y) = \begin{cases} 1 & \text{if } \ \nu(z(\tau), y(\tau)) \geq 0 \quad \tau \in J, \\ 0 & \qquad\qquad \textit{otherwise}. \end{cases}$$

We have to prove that Q is a $\mathcal{S}_{\mathcal{F}}$-$\mathcal{G}_{\mathcal{F}}$-contraction
Lets $z, y \in C(J)$ and $\tau \in J$, we have

$$| Qz(\tau) - Qy(\tau) | \leq | \theta_g - \theta_h | + \eta_r | g(\tau) - h(\tau) | + \delta_r \int_0^t | g(s) - h(s) | ds,$$

where $h, g \in C(J)$, such that $h(\tau) = f(\tau, y(\tau), h(\tau)), \ g(\tau) = f(\tau, z(\tau), g(\tau))$ and $\theta_g = \sigma_0 + \eta_r g(0), \ \theta_h = y_0 + \eta_r h(0)$.
By **(**A_1**)**, we get

$$\begin{aligned} | g(\tau) - h(\tau) | \quad &= \quad | f(\tau, x(\tau), g(\tau)) - f(\tau, y(\tau), h(\tau)) | \\ &\leq \quad \vartheta(z, y) | z(\tau) - y(\tau) | + \gamma(\tau) | g(\tau) - h(\tau) | . \end{aligned}$$

Thus,

$$\| g - h \|_\infty \leq \frac{\vartheta(z, y)}{1 - \gamma_s} \| z - y \|_\infty .$$

Next, let us write

$$\begin{aligned}
| Qz(\tau) - Qy(\tau) | &\leq \| z - y \|_\infty + 2\eta_r \frac{\vartheta(\eta,\theta)}{1-\gamma_s} \| z - y \|_\infty \\
&+ \delta_r \int_0^\tau \frac{\vartheta(z,y)}{1-\gamma_s} \| z - y \|_\infty \, ds \\
&\leq \| z - y \|_\infty \left[1 + 2\eta_r \frac{\vartheta(z,y)}{1-\gamma_s} + \delta_r \int_0^t \frac{\vartheta(\eta,\theta)}{1-\gamma_s} ds \right] \\
&\leq \| z - y \|_\infty \frac{1}{4} \varphi(\| z - y \|_\infty) \\
&\leq \frac{1}{4} \delta(x,y) \varphi(\delta(z,y)).
\end{aligned}$$

Then

$$\delta(Qz, Qy) \leq \frac{1}{4} \delta(z,y) \varphi(\delta(z,y)).$$

And thus

$$\begin{aligned}
\alpha(z,y)\delta(Qz(\tau) - Qy(\tau)) &\leq \alpha(z,y)\frac{1}{4}\delta(z,y)\varphi(\delta(z,y)) \\
&\leq \frac{1}{4} M(z,y)\varphi(M(z,y)) + LN(z,y), \text{ where } L \geq 0.
\end{aligned}$$

Hence

$$\Psi\Big(\delta\big(Qz(\tau) - Qy(\tau)\big), M(z,y)\varphi(M(z,y)) + LN(z,y)\Big) \geq 0 \quad \text{where } L \geq 0,$$

where $\Psi(\tau, s) = \frac{s}{4} - \tau$ and $\beta(\tau) = \varphi(\tau)$.
So, Q is $\mathcal{S}_{\mathcal{F}}$-$\mathcal{G}_{\mathcal{F}}$-contraction.

Lets $z, y \in C(J)$ such that $\alpha(a, Qy) \geq 1$. Thus, for each $\tau \in J$, we get

$$\nu(a(\tau), Qy(t)) \geq 0.$$

By **(**A_3**)**, this implies that

$$\nu(Qz(\tau), Q^2 y(\tau)) \geq 0,$$

then

$$\alpha(Qz, Q^2 y) \geq 1.$$

Hence, Q is an extended-α-admissible.

From A_2, ther exists $u_0 \in C(J)$ such that

$$\alpha(u_0, Qu_0) \geq 1.$$

Finally, if $(p_n)_{n\in\mathbb{N}} \subset C(J)$ such that $\lim_{n\to\infty} p_n = p$ and $\alpha(p_n, p_{n+1}) \geq 1$, then by**(**$A_4$**)**, we have $\alpha(p_n, p) \geq 1$.

So, from Theorem **2.3**, we conclude that Q has a fixed point in $C(J)$ which is a solution of problem **(39)**. From (A_5)**)**, if v and u two fixed points of Q, then either $\nu(u,v) \geq 0$ or $\nu(v,u) \geq 0$. This implies that either $\alpha(u,v) \geq 1$ or $\alpha(v,u) \geq 1$. So, by Theorem **2.4**, the problem **(39)** has the uniqueness solution. □

II) Consider the folllowing integral equation

$$u(\tau) = g(\tau) + \int_0^1 h(\tau, s) f(\lambda, u(\lambda)) d\lambda \quad \text{for all } \tau \in [0, 1]. \tag{40}$$

Let Ψ denote the set of all non-decreasing functions $\varphi : \mathbb{R}^+ \to [0, 1)$ satisfying

$$\varphi(b\tau) \leq bt\varphi(\tau) \quad \text{for all } b \geq 1 \quad \text{and } \tau \geq 0.$$

To evaluate the intergal equation **(40)** we will take into consideration the following assumptions

(H_1) $g : [0, 1] \to \mathbb{R}$ is a continuous function.

(H_2) $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $f(\tau, u) \geq 0$ and there exists $\varphi \in \Psi$ such that

$$| f(\tau, u) - f(\tau, v) | \leq \frac{1}{3}\varphi(| u - v |), \text{ with } \lim_{n\to\infty} \varphi(\tau_n) = 1 \Rightarrow \lim_{n\to\infty} \tau_n = 0.$$

(H_3) $h : [0, 1]^2 \to \mathbb{R}$ is a continuous function, $h(\tau, s) \geq 0$ and

$$\int_0^1 h(\tau, s) ds \leq 1.$$

Let $E = C([0, 1])$, with the usual metric given by

$$\varrho(u, v) = \sup_{\tau\in[0,1]} |u(\tau) - v(\tau)| \quad \text{for every } u, v \in E.$$

Next, we define

$$\delta(u, v) = \frac{1}{2}\varrho(u, v) \quad \text{for every } u, v \in E.$$

We know that ϱ is a usual metric on E and that (E, ϱ) is a complete space. Since $\delta = \frac{1}{2}\varrho$, therefore (E, δ) is a complete metric space.

Theorem 4.3. *Under assumptions (H_1)-(H_3), equation* **(40)** *has a unique solution in E.*

Proof. Consider the maping $P : E \to E$ defined by

$$Pu(\tau) = g(\tau) + \int_0^1 h(\tau, \lambda) f(\lambda, u(\lambda)) d\lambda \quad \text{for all } \tau \in [0, 1]$$

It is clair P is well defined (this means that if $u \in E$, then $Pu \in E$). Also, for $u, v \in E$, we have

$$\begin{aligned} | Pu(\tau) - Pv(\tau) | &= | g(\tau) + \int_0^1 h(\tau, \lambda) f(\lambda, u(\lambda)) d\lambda - g(\tau) \\ &+ \int_0^1 h(\tau, \lambda) f(\lambda, v(\lambda)) ds | \\ &= | \int_0^1 [h(\tau, \lambda) f(\lambda, u(\lambda)) - h(\tau, \lambda) f(\lambda, v(\lambda))] ds | \\ &\leq \int_0^1 h(\tau, \lambda) | f(\lambda, u(\lambda)) - f(\lambda, v(\lambda)) | ds \\ &\leq \int_0^1 h(\tau, \lambda) \frac{1}{3}\varphi\Big(| u(\lambda) - v(\lambda) | \Big) ds. \end{aligned}$$

Since φ is non-decreasing, we obtain

$$\begin{aligned}\varphi\Big(\mid u(\lambda)-v(\lambda)\mid\Big) &\leq \varphi\Big(\sup_{\lambda\in[0,1]}\mid u(\lambda)-v(\lambda)\mid\Big)\\ &= \varphi(\varrho(u,v)).\end{aligned}$$

Therefore,

$$\mid Pu(\tau)-Pv(\tau)\mid \leq \frac{1}{3}\varphi(\varrho(u,v)).$$

On the other hand, we have

$$\begin{aligned}\delta(Pu,Pv) &= \frac{1}{2}\varrho(Pu,Pv)\\ &= \frac{1}{2}\sup_{t\in[0,1]}\mid Pu(s)-Pv(s)\mid\\ &\leq \frac{1}{2}\frac{1}{3}\varphi(\varrho(u,v))\\ &= \frac{1}{2}\frac{1}{3}\varphi(2\delta(u,v))\\ &\leq \frac{1}{3}\delta(u,v)\varphi(\delta(u,v))\\ &\leq \frac{1}{3}\Big[M(u,v)\varphi(M(u,v))+LN(u,v)\Big] \quad \text{where } L\geq 0.\end{aligned}$$

Then

$$\Psi\Big(\delta(Pu,Pv),\beta(M(u,v))M(u,v)+LN(u,v)\Big)\geq 0,$$

where $\beta(\tau)=\varphi(\tau)$ and $\Psi(\tau,s)=\frac{s}{3}-\tau$. By corollary **3.2**, equation **(40)** has a solution in E and the proof is finished. □

Acknowledgements

The authors are thankful to the referee for her/his valuable suggestions towards the improvement of the paper.

Conflict of interest

The authors declare that they have no conflict of interest.

Data Availability

The data used to support the findings of this study are included in the references within the article.

References

[1] Berinde, V., Approximating fixed points of weak Φ-contractions using the Picard iteration, Fixed Point Theory., **4**, 131-142 (2003).
[2] Ahmed, H. M., Ragusa M. A., Nonlocal controllability of Sobolev-type conformable fractional stochastic evolution inclusions with Clarke subdifferential, Bull. of the Malaysian Mathematical Sciences Society, **45** (6), 3239-3253 (2022).
[3] Beloul, S., Common fixed point theorems for multi-valued contractions satisfying generalized condition(B) on partial metric spaces, Facta Univ. Ser. Math. Inform., **30**, 555-566 (2015).
[4] Beloul, S., A common fixed point theorem For generalized almost contractions in metriclike spaces, Appl. Math. E-Notes., **18** , 127-139 (2018).
[5] Shatanawi, W., Postolache, M., Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces, Fixed Point Theory Appl., **1**, 1-17 (2013).
[6] Babu G. V. R., Sandhy, M. L., Kameshwari, M. V. R., A note on a fixed point theorem of Berinde on weak contractions, Carpathian J. Math., **24** , 8-12 (2008).

[7] Ćirić, B. L., Multi-valued nonlinear contraction mappings, Nonlinear Anal., **71**, 2716-2723 (2009)
[8] Karapinar, E., Fixed points results via simulation functions, Filomat., **30**, 2343-2350 (2016)
[9] Khojasteh, F., Shukla, S., Radenovi'c, S., A new approach to the study of fixed point theory for simulation functions, Filomat., **29**, 1189-1194 (2015)
[10] Samet, B., Vetro, C., Vetro V., Fixed point theorems for α-ψ contractive type mappings., Nonlinear Analysis. Theory, Methods and Applications, **75**, 2154-2165 (2012)
[11] Karapinar, E., Discussion on (α-ψ)-contractions on generalized metric spaces, Abstr. Appl. Anal, **2014**, 1–8 (2014).
[12] Sistani, T., Kazemipour, M., Fixed point theorems for α-ψ-contractions on metric spaces with a graph, J. Adv. Math. Stud., **39**, 65-79 (2014).
[13] Badehiana, Z., Asgaria, M. S., Integral type Fixed point Theorems for α-ψ-admissible Mappings Satisfying α-ψ-contractive Inequality, Filomat., **30**, 3227-3234 (2016)
[14] Samet, B., Vetro, C., Vetro, P., Fixed point for $\alpha - \psi$-contractive type mappings. Nonlinear Anal., **75**, 2145-2165 (2012).
[15] Popescu, O., Some new fixed point theorems for α-Geraghty contractive type maps in metric spaces, Fixed Point Theory Appl., **2014**, 1-12 (2014).
[16] Karapinar, E., Kumam, P., Salimi, P., On α-ψ-Meri-Keeler contractive mappings, Fixed Point Theory Appl. **2013**, 1-12 (2013).
[17] Karapinar, E., Fulga, A., Shahzad, N., Roldán López de Hierro, A. F., Solving integral equations by means of fixed point theory, Journal of Function Spaces, **2022**, art.n.7667499 (2022).
[18] Lu N., He F., Li S.F., Fixed point theorem for question of set-valued quasicontraction, Filomat, **36** (19), 6777-6790, (2022).
[19] Nastasi, A., Vetro P., Fixed point results on metric and partial metrric spaces via simulation functions, J. Nonlinear Sci. Appl., **8**, 1059-1069 (2015).
[20] Romaguera, S, Valero, O., A quantitative computational model for complete partial metric spaces via formal balls, Mathematical Structures in Computer Science., **19**, 541-563 (2009).
[21] Geraghty, M., On contractive mappings, Proc. Amer. Math. Soc., **40**, 604-608 (1973).
[22] Tomar, A., Beloul, S., Sharma, R., Upadhyay, S., Common fixed point theorems via generalized condition (B) in quasi-partial metric space and applications, Demonstr. Math., **50**, 278-298 (2017).
[23] Karapinar, E., Samet., Generalized α-ψ-contractive type mappings and related fixed point theorems with applications. Abstr. Appl. Anal., **2012**, 1-18 (2012).
[24] Taqbibt, A., Elomari, M., Melliani, S., Nonlocal semilinear Φ-Caputo fractional evolution equation with a measure of noncompactness in Banach space, Filomat, **37**(20), 6877–6890 .(2023).
[25] Khojasteh, F., Shukla, S., Radenović, S., A new approach to the study of fixed point theory for simulation functions, Filomat, 29 , 1189-1194 (2015).
[26] Karapinar, E., Hima Bindu, V. M. L., Discussions on the almost $\mathcal{Z}$-contraction. Open Mathematics, **18**, 448-457 (2020).
[27] Abbas, S., Benchohra, M., Krim, S., initial value problems for caputo-fabrizio implicit fractional differntial equations in b-metrice spaces. Bulletin of the transilvania university of Brasoc., **63**, 1-12 (2021).