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# The weak group-star matrix

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**Abstract.** In this paper, we introduce one type of matrix, called the weak group-star matrix. We investigate the characterizations, representations, and properties of the matrix. A variant of the successive matrix squaring computational iterative scheme is given for calculating the weak group-star matrix. Moreover, the Cramer's rule for the solution of a singular equation $(A^{\dagger})^{*}x = b$ is presented. Then, the perturbation is also given for the weak group-star matrix. In the final, the weak group-star matrix being used in solving appropriate systems of linear equations is established.

## 1. Introduction

Throughout this paper, we denote the set of all $m \times n$ complex matrices by $\mathbb{C}^{m\times n}$. For $A \in \mathbb{C}^{n\times n}$, the symbols $A^{*}$, $rank(A)$, $N(A)$, *and* $R(A)$ stand for the conjugate transpose, the rank, the null space and the range space of $A$, respectively. Moreover, $I_n$ will refer to the $n \times n$ identity matrix. Let $A \in \mathbb{C}^{n\times n}$, the smallest positive integer $k$ for which $rank(A^k) = rank(A^{k+1})$ is called the index of $A$ and is denoted by Ind$(A)$. Then $\mathbb{C}_k^{n\times n}$ represents all $n \times n$ complex matrices sets with index $k$. $P_{E,F}$ represents the projector on the subspace $E$ along the subspace $F$. For $A \in \mathbb{C}^{n\times n}$, $P_A$ stands for the orthogonal projection onto $R(A)$. The symbol $\mathbb{C}_n^{CM}$ represents the subset of all $n \times n$ complex matrices sets with index 1.

Next, let's review the definitions of some generalized inverses. For $A \in \mathbb{C}^{m\times n}$, the Moore-Penrose inverse $A^{\dagger}$ of $A$ is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four Penrose equations [1]:

$$AXA = A, \quad XAX = X, \quad (AX)^{*} = AX, \quad (XA)^{*} = XA.$$

The Moore-Penrose inverse can be used to represent orthogonal projectors $P_A := AA^{\dagger}$ onto $R(A)$ and $Q_A := A^{\dagger}A$ onto $R(A^{*})$, respectively. A matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the equality $AXA = A$ is called an inner inverse or $\{1\}$-inverse of $A$, and a matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the equality $XAX = X$ is called an outer inverse or $\{2\}$-inverse of $A$.

The Drazin inverse is a kind of outer inverse defined for square matrices. For $A \in \mathbb{C}^{n\times n}$ and Ind$(A) = k$, the Drazin inverse $A^{D}$ of $A$ is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying the following three equations [13]:

$$A^{k+1}X = A^{k}, \qquad XAX = X, \qquad AX = XA.$$

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In particular, if $\mathrm{Ind}(A) = 1$, $A^D = A^\#$ is the group inverse of $A$.

For $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$, the core-EP inverse $A^{\textcircled{\dagger}}$ of $A$ is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying the following conditions [12]:

$$XAX = X, \qquad R(A^k) = R(X) = R(X^*).$$

Obviously, the core-EP inverse is an outer inverse of $A$. Recall that, by [6], the core-EP inverse can be expressed as $A^{\textcircled{\dagger}} = A^D A^k (A^k)^\dagger$.

The weak group inverse is proposed by Wang and Chen [15] for square matrices of an arbitrary index as an extension of the group inverse. For $A \in \mathbb{C}^{m\times n}$, the weak group inverse $A^{\textcircled{w}}$ $of$ $A$ is the uniquely determined matrix that satisfying:

$$AX^2 = X, \qquad AX = A^{\textcircled{\dagger}}A.$$

Notice that, by [15], we have$A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2 A$. Two new generalized inverses have emerged by combining Moore-Penrose inverse and the weak group inverse, which are the weak core inverse (WCI) $A^{\textcircled{w},\dagger}$ and the dual weak core inverse (d-WCI) $A^{\dagger,\textcircled{w}}$, respectively [2]. Precisely, the weak core inverse of $A \in \mathbb{C}^{n\times n}$ presents a unique solution to the matrix system [2]:

$$XAX = X, \quad AX = CA^\dagger, \quad XA = A^D C,$$

where $C$ is the weak core part of $A$ with $C = AA^{\textcircled{w}}A$. Notice that $A^{\textcircled{w},\dagger} = A^{\textcircled{w}}AA^\dagger$ and $A^{\dagger,\textcircled{w}} = A^\dagger AA^{\textcircled{w}}$.

In [2], let $A\in\mathbb{C}^{n\times n}$ and $\mathrm{Ind}(A) = k$. The weak core part $C$ of $A$ satisfies the following equations:

$$CA^k = A^{k+1}, \qquad C = A^{\textcircled{\dagger}}A^2, \qquad (I - AA^D)C = 0, \tag{1}$$

$$(I - AA^{\textcircled{\dagger}})C = (I - AA^{\textcircled{w}})C = 0, \qquad C(I - Q_A) = 0. \tag{2}$$

The DMP-inverse of $A \in \mathbb{C}_k^{n\times n}$, written by $A^{D,\dagger}$, was defined in [8] as the unique matrix $X \in \mathbb{C}_k^{n\times n}$ satisfying

$$XAX = X, \qquad XA = A^D A, \qquad A^k X = A^k A^\dagger.$$

Moreover, it was proved that $A^{D,\dagger} = A^D AA^\dagger$. Also, the dual DMP-inverse of $A$ was introduced in [8], namely $A^{\dagger,D} = A^\dagger AA^D$.

D. Mosić in [9] introduced the Drazin-Star and the Star-Drazin matrices of a square matrix. Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$. The Drazin-Star matrix of $A$ (or Drazin-Star inverse of $(A^\dagger)^*$ ) is

$$A^{D,*} = A^D AA^*$$

which is the unique solution of the following equations:

$$X(A^\dagger)^* X = X, \qquad A^k X = A^k A^*, \qquad X(A^\dagger)^* = A^D A.$$

Recall that the Star-Drazin matrix of $A$ (or Star-Drazin inverse of $(A^\dagger)^*$) is also defined in [9] as $A^{*,D} = A^* AA^D$. Inspired by this types of matrices, we will introduce the weak group-star matrix in this article.

First of all, let us review the core-EP decomposition. Wang gave the core-EP decomposition in the document [14]. Let $A\in\mathbb{C}^{n\times n}$ with $Ind(A) = k$, $rank(A^k) = p$. Then, one has $A = A_1 + A_2$, $A_1\in\mathbb{C}_n^{CM}$, where $A_2^k = 0$, $A_1^* A_2 = A_2 A_1 = 0$. Furthermore, there exists an unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$A = U\begin{pmatrix} T & S \\ 0 & N \end{pmatrix} U^*, \ A_1 = U\begin{pmatrix} T & S \\ 0 & 0 \end{pmatrix} U^*, \ A_2 = U\begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix} U^*, \tag{3}$$

where $T\in\mathbb{C}^{p\times p}$ is nonsingular and $S\in\mathbb{C}^{p\times(n-p)}$, $N\in\mathbb{C}^{(n-p)\times(n-p)}$ is nilpotent of index $k$, i.e., $N^k = 0$.

**Lemma 1.1.** *[4, 14, 16] Let $A\in\mathbb{C}_k^{n\times n}$ as in (3). Then*

$(i)$ $A^{\dagger} = U\begin{pmatrix} T^{*}\triangle & -T^{*}\triangle SN^{\dagger} \\ (I_{n-p}-N^{\dagger}N)S^{*}\triangle & N^{\dagger}-(I_{n-p}-N^{\dagger}N)S^{*}\triangle SN^{\dagger} \end{pmatrix}U^{*}$,

$(ii)$ $A^{\textcircled{\dagger}} = U\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^{*}$,

$(iii)$ $A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^{2}A = U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^{*}$,

$(iv)$ $AA^{\dagger} = U\begin{pmatrix} I_{p} & 0 \\ 0 & NN^{\dagger} \end{pmatrix}U^{*}$,

$(v)$ $A^{\dagger}A = \begin{pmatrix} T^{*}\triangle T & T^{*}\triangle S(I-NN^{\dagger}) \\ (I-NN^{\dagger})S^{*}\triangle T & (I-NN^{\dagger})S^{*}\triangle S(I-NN^{\dagger})+N^{\dagger}N \end{pmatrix}$,

*where* $\triangle = [TT^{*}+S(I_{n-p}-N^{\dagger}N)S^{*}]^{-1}$.

**Lemma 1.2.** *[7] Let* $A\in\mathbb{C}^{n\times n}$ *with rank* $r>0$. *Then there exists a unitary matrix* $U\in\mathbb{C}^{n\times n}$ *such that*

$$A = U\begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix}U^{*}, \tag{4}$$

*where* $\Sigma = diag(\sigma_1 I_{r1}, \sigma_2 I_{r2}, \ldots, \sigma_t I_{rt})$ *is the diagonal matrix of singular values of* $A$, $\sigma_1 > \sigma_2 > \ldots, > \sigma_t > 0$, $r_1 + r_2 + \ldots + r_t = r$, *and* $K\in\mathbb{C}^{n\times n}$, $L\in\mathbb{C}^{n\times(n-r)}$ *satisfy* $KK^{*}+LL^{*} = I_r$.

**Lemma 1.3.** *Let* $A\in\mathbb{C}^{n\times n}$ *be a matrix written as in* (4). *Then,*

$(i)$*[3] the core-EP inverse of* $A$ *is*

$$A^{\textcircled{\dagger}} = U\begin{pmatrix} (\Sigma K)^{\textcircled{\dagger}} & 0 \\ 0 & 0 \end{pmatrix}U^{*}.$$

$(ii)$*[2] the weak group inverse of* $A$ *is*

$$A^{\textcircled{w}} = U\begin{pmatrix} ((\Sigma K)^{\textcircled{\dagger}})^{2}\Sigma K & ((\Sigma K)^{\textcircled{\dagger}})^{2}\Sigma L \\ 0 & 0 \end{pmatrix}U^{*} = U\begin{pmatrix} (\Sigma K)^{\textcircled{w}} & ((\Sigma K)^{\textcircled{\dagger}})^{2}\Sigma L \\ 0 & 0 \end{pmatrix}U^{*}.$$

The main structure of this paper is as follows. In Sect. 2, we introduce the weak group-star matrix. Then, we give some representations and characterizations of this type of the matrix. In Sect. 3, we develop the SMS method for finding the weak group-star matrix. In Sect. 4, the Cramer's rule for the solution of a singular equation $(A^{\dagger})^{*}x = b$ is presented. In Sect. 5, we study the perturbation of the weak group-star matrix. In Sect. 6, we give the application of the weak group-star matrix in solving linear equations.

## 2. Definition, characterizations and representations of the weak group-star Matrix

**Theorem 2.1.** *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$, $C$ is the weak core part of $A$. Then, the system of equations*

$$X(A^{\dagger})^{*}X = X,\ AX = CA^{*},\ X(A^{\dagger})^{*} = A^{D}C \tag{5}$$

*is consistent and its unique solution is $X = A^{D}CA^{*}$.*

Proof. For $X = A^{D}CA^{*}$. In fact, (1) implies $AX = AA^{D}CA^{*} = CA^{*}$. On the other hand, (2) implies $X(A^{\dagger})^{*} = A^{D}CA^{*}(A^{\dagger})^{*} = A^{D}CA^{\dagger}A = A^{D}C$. Finally,

$$X(A^{\dagger})^{*}X = A^{D}CX = A^{D}AA^{\textcircled{W}}CA^{*} = A^{D}CA^{*} = X,$$

where the last equality follows by (2). Hence, $X = A^{D}CA^{*}$ satisfies the system of (5).

In order to show that system (5) has a unique solution, assume that both two matrices $X_1$ and $X_2$ satisfy (5), then

$$AX_1 = CA^{*} = AX_2,\ X_1(A^{\dagger})^{*} = A^{D}C = X_2(A^{\dagger})^{*}.$$

Thus, we can obtain

$$\begin{aligned} X_2 &= X_2(A^{\dagger})^{*}X_2 = A^{D}CX_2 = A^{D}AA^{\textcircled{W}}AX_2 \\ &= A^{D}AA^{\textcircled{W}}AX_1 = A^{D}CX_1 = X_1(A^{\dagger})^{*}X_1 = X_1, \end{aligned}$$

which implies that system (5) has the unique solution. □

**Definition 2.2.** *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$, and $C$ be the weak core part of $A$. The weak group-star matrix of $A$ (or the weak group-star inverse of $(A^{\dagger})^{*}$) denoted as $A^{\textcircled{W},*}$, is defined to be the solution of the system* (5).

**Theorem 2.3.** *Let $A \in \mathbb{C}^{n\times n}$ and $\mathrm{Ind}(A) = k$. Then,*

$$A^{\textcircled{W},*} = A^{\textcircled{W}}AA^{*}.$$

Proof. Since $R(A^{\textcircled{W}}) = R(A^{k})$, then $A^{\textcircled{W}} = A^{k}Z$, for some $Z \in \mathbb{C}^{n\times n}$. Thus, we have

$$A^{\textcircled{W},*} = A^{D}CA^{*} = A^{D}AA^{\textcircled{W}}AA^{*} = A^{D}AA^{k}ZAA^{*} = A^{k}ZAA^{*} = A^{\textcircled{W}}AA^{*}.$$

□

**Remark 2.4.** *Obviously, the weak group-star matrix is named based on the expressions whom are defined. In general, the weak group-star matrix are not generalized inverses of a given matrix $A$, but they are outer inverses of $(A^{\dagger})^{*}$.*

We observe that the weak group-star matrix provide new classes of square matrices by the following example, because they are different from each of the Moore-Penrose inverse, the weak group inverse, the weak core inverse and the dual weak core inverse.

**Example 2.5.** *Let*

$$A = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

*Then,*

$$A^{\dagger} = \begin{pmatrix} 2/3 & -1/3 & 2/3 & 0 \\ -1/3 & 2/3 & -1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ A^{\textcircled{W}} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{\textcircled{W},\dagger} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ A^{\dagger,\textcircled{W}} = \begin{pmatrix} 2/3 & -1/3 & 1/3 & -2/3 \\ -1/3 & 2/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 2/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ A^{\textcircled{W},*} = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the next example, we show that the weak group-star inverse of $(A^{\dagger})^*$ is different from each of the Moore-Penrose inverse, the weak group inverse, the weak core inverse and the dual weak core inverse of $(A^{\dagger})^*$. Note that the weak group-star inverse present new classes of generalized inverse.

**Example 2.6.** *Let*

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

*It is easy to check that* $\mathrm{Ind}(A) = 2$. *We can obtain that the Moore-Penrose inverse, the Weak group inverse and the core EP inverse are*

$$A^{\dagger} = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ A^{ⓦ} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ A^{\textcircled{\dagger}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

*We also have*

$$(A^{\dagger})^* = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ [(A^{\dagger})^*]^{\dagger} = A^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

$$[(A^{\dagger})^*]^{ⓦ} = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ [(A^{\dagger})^*]^{\textcircled{\dagger}} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$[(A^{\dagger})^*]^{ⓦ,\dagger} = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ [(A^{\dagger})^*]^{\dagger,ⓦ} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ [(A^{\dagger})^*]^{ⓦ,*} = \begin{pmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

**Theorem 2.7.** *Let* $A \in \mathbb{C}^{n\times n}$ *be a matrix written as in* (3). *Then*

$$A^{ⓦ,*} = U\begin{pmatrix} T^* + (T^{-1}S + T^{-2}SN)S^* & (T^{-1}S + T^{-2}SN)N^* \\ 0 & 0 \end{pmatrix}U^*. \tag{6}$$

Proof. From Lemma 1.1, we can obtain

$$\begin{aligned} A^{ⓦ,*} = A^{ⓦ}AA^* &= U\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}\begin{pmatrix} T^* & 0 \\ S^* & N^* \end{pmatrix}U^* \\ &= U\begin{pmatrix} T^* + (T^{-1}S + T^{-2}SN)S^* & (T^{-1}S + T^{-2}SN)N^* \\ 0 & 0 \end{pmatrix}U^*. \end{aligned}$$

□

**Corollary 2.8.** *Let* $A \in \mathbb{C}^{n\times n}$ *be a matrix written as in* (3). *Then*

$$AA^{ⓦ,*} = U\begin{pmatrix} F_1 & F_2 \\ 0 & 0 \end{pmatrix}U^*, \tag{7}$$

*where* $F_1 = TT^* + (S + T^{-1}SN)S^*$, $F_2 = (S + T^{-1}SN)N^*$. *Besides,*

$$A^{ⓦ,*}A = U\begin{pmatrix} F_3 & F_4 \\ 0 & 0 \end{pmatrix}U^*,$$

*where* $F_3 = T^*T + (T^{-1}S + T^{-2}SN)S^*T$, $F_4 = T^*S + (T^{-1}S + T^{-2}SN)S^*S + (T^{-1}S + T^{-2}SN)N^*N$.

**Remark 2.9.** *Let $A\in\mathbb{C}^{n\times n}$ as in* (3) *and with* $\mathrm{Ind}(A) = k$. *We can obtain $A^{ⓦ} = A^{\#}$ if and only if $A \in \mathbb{C}_n^{CM}$, i.e., $N = 0$.*

**Lemma 2.10.** *If $A\in\mathbb{C}^{n\times n}$ with* $\mathrm{Ind}(A) = k$. *Then $R(A^{ⓦ,*}) = R(A^k)$.*

Proof. In fact, according to Theorem 2.1, we have

$$R(A^{ⓦ,*}) \subseteq R(A^{ⓦ,*}(A^{\dagger})^*) = R(A^D C) \subseteq R(A^D) = R(A^k).$$

On the other hand, $R(A^k) \subseteq R(A^{ⓦ,*})$. By Theorem 2.1, we can see

$$R(A^k) \subseteq R(A^D) \subseteq R(A^D C) = R(A^{ⓦ,*}(A^{\dagger})^*) \subseteq R(A^{ⓦ,*}).$$

Hence, $R(A^k) = R(A^{ⓦ,*})$. □

**Lemma 2.11.** *[5] Let $A\in\mathbb{C}^{n\times n}$. Then, the following statements hold.*

(*i*) $AA^{ⓦ} = P_{R(A^k),N((A^k)^*A)}$,

(*ii*) $A^{ⓦ}A = P_{R(A^k),N((A^k)^*A^2)}$.

According Theorem 2.1 and Lemma 2.10, we can obtain Lemma 2.12.

**Lemma 2.12.** *[11] Let $A\in\mathbb{C}^{n\times n}$ be such that* $\mathrm{Ind}(A) = k$. *Then*

(*i*) $A^{ⓦ,*} = [(A^{\dagger})^*]^{(2)}_{R(A^k),N(A^k)^*}$,

(*ii*) $(A^{\dagger})^*)A^{ⓦ,*}$ *is a projector on $R((A^{\dagger})^*)A^{ⓦ})$ along $N((A^k)^*A^2A^*)$,*

(*iii*) $A^{ⓦ,*}(A^{\dagger})^*$ *is a projector on $R(A^k)$ along $N((A^k)^*A^2)$.*

**Corollary 2.13.** *Let $A\in\mathbb{C}^{n\times n}$ with* $\mathrm{Ind}(A) = k$. *For $l \geq k$,*

$$A^{ⓦ,*} = A^l(A^{l+2})^{\dagger}A^2A^*. \tag{8}$$

Proof. According to [10], it follows $A^{ⓦ} = A^l(A^{l+2})^{\dagger}A$. By the corresponding Theorem 2.3, we get the equality (8).

**Theorem 2.14.** *Let $A\in\mathbb{C}^{n\times n}$ be a matrix written as in* (4). *Then*

$$A^{ⓦ,*} = U\begin{pmatrix} (\Sigma K)^{ⓦ}\Sigma\Sigma^* & 0 \\ 0 & 0 \end{pmatrix}U^*.$$

Proof. From Lemma 1.3, we can obtain

$$\begin{aligned} A^{ⓦ,*} = A^{ⓦ}AA^* = (A^{ⓣ})^2A^2A^* &= U\begin{pmatrix} (\Sigma K)^{ⓦ}\Sigma KK^*\Sigma^* + (\Sigma K)^{ⓦ}\Sigma LL^*\Sigma^* & 0 \\ 0 & 0 \end{pmatrix}U^* \\ &= U\begin{pmatrix} (\Sigma K)^{ⓦ}\Sigma(KK^* + LL^*)\Sigma^* & 0 \\ 0 & 0 \end{pmatrix}U^*. \end{aligned}$$

□

**Theorem 2.15.** *Let $A\in\mathbb{C}^{n\times n}$ with* $\mathrm{Ind}(A) = k$ *and $C$ is the weak core part of $A$. Then, the following statements are equivalent:*

(*i*) *$X\in\mathbb{C}^{n\times n}$ is the weak group-star matrix of $A$.*

(*ii*) *X satisfies equations*

$$X(A^{\dagger})^{*}X = X,\ AX = CA^{*},\ X(A^{\dagger})^{*} = A^{D}C.$$

(*iii*) *X satisfies equations*

$$A^{Ⓦ}AX = X,\ AX = CA^{*}.$$

(*iv*) *X satisfies equations*

$$AX(A^{\dagger})^{*} = C,\ A^{Ⓦ}AXAA^{\dagger} = X,\ (A^{\dagger})^{*}X(A^{\dagger})^{*} = (A^{\dagger})^{*}A^{Ⓦ}A.$$

(*v*) *X satisfies equations*

$$XAA^{\dagger} = X,\ X(A^{\dagger})^{*} = A^{Ⓦ}A,\ X(A^{\dagger})^{*}A^{\dagger} = A^{Ⓦ,\dagger},\ XA = A^{Ⓦ}AA^{*}A.$$

(*vi*) *X satisfies equations*

$$X(A^{\dagger})^{*}A^{Ⓦ}AA^{*} = X,\ X(A^{\dagger})^{*}A^{Ⓦ}AX = X,\ (A^{\dagger})^{*}A^{Ⓦ}AX = (A^{\dagger})^{*}A^{Ⓦ}AA^{*}.$$

(*vii*) *X satisfies equations*

$$(A^{\dagger})^{*}A^{Ⓦ}AX(A^{\dagger})^{*}A^{Ⓦ}A = (A^{\dagger})^{*}A^{Ⓦ}A,\ X(A^{\dagger})^{*}A^{Ⓦ}A = A^{Ⓦ}A.$$

Proof. (*i*) ⇒ (*ii*): By Theorem 2.1, the proof is clear.

(*ii*) ⇒ (*iii*): Using $AX = CA^{*}$, we can obtain

$$A^{Ⓦ}AX = ACA^{*} = X.$$

(*iii*) ⇒ (*i*): The hypothesis $A^{Ⓦ}AX = X$, $AX = CA^{*}$ imply

$$X = A^{Ⓦ}AX = A^{Ⓦ}CA^{*} = A^{Ⓦ}AA^{Ⓦ}AA^{*} = A^{Ⓦ}AA^{*} = X.$$

(*i*) ⇒ (*iv*): Since $X = A^{Ⓦ}AA^{*}$ and by (2), it follows that

$$AX(A^{\dagger})^{*} = AA^{Ⓦ}AA^{*}(A^{\dagger})^{*} = CAA^{\dagger} = C,$$

$$A^{Ⓦ}AXAA^{\dagger} = (A^{Ⓦ}AA^{Ⓦ})A(A^{*}AA^{\dagger}) = A^{Ⓦ}AA^{*} = X,$$

and

$$(A^{\dagger})^{*}X(A^{\dagger})^{*} = (A^{\dagger})^{*}A^{Ⓦ}AA^{*}(A^{\dagger})^{*} = (A^{\dagger})^{*}A^{Ⓦ}A(A^{\dagger}A)^{*} = (A^{\dagger})^{*}A^{Ⓦ}AA^{\dagger}A = (A^{\dagger})^{*}A^{Ⓦ}A.$$

(*iv*) ⇒ (*i*): By $A^{Ⓦ}AXAA^{\dagger} = X, AX = AA^{Ⓦ}AA^{*}$,we have

$$X = A^{Ⓦ}AXAA^{\dagger} = A^{Ⓦ}AA^{Ⓦ}AA^{*}AA^{\dagger} = A^{Ⓦ}AA^{*}AA^{\dagger} = A^{Ⓦ}AA^{*} = X.$$

The rest can be proved similarly according to the above method. □

By Lemma 2.12 and $A^{Ⓦ,*} = A^{Ⓦ}AA^{*}$, we obtain

$$(A^{\dagger})^{*}A^{Ⓦ,*} = P_{R((A^{\dagger})^{*}A^{Ⓦ}),N((A^{k})^{*}A^{2}A^{*})},\ R(A^{Ⓦ,*}) \subseteq R(A^{Ⓦ}) = R(A^{k}).$$

Then we can get Theorem 2.16.

**Theorem 2.16.** *[11] Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$. Then, the matrix equation*

$$(A^{\dagger})^{*}X = P_{R((A^{\dagger})^{*}A^{Ⓦ}),N((A^{k})^{*}A^{2}A^{*})},\ R(X) \subseteq R(A^{k}) \tag{9}$$

*is consistent and it has the unique solution $X = A^{Ⓦ,*}$.*

Lemma 2.17 can be checked by using the same method of [11]. Therefore, we omit the proof.

**Lemma 2.17.** *[11] Let* $A\in\mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A)=k$. *Then,*

$(i)$ $(A^{\dagger})^{*}A^{Ⓦ,*}(A^{\dagger})^{*}=(A^{\dagger})^{*}\Leftrightarrow A^{\dagger}AA^{Ⓦ}A=A^{\dagger}A\Leftrightarrow AA^{Ⓦ}A=A\Leftrightarrow AA^{Ⓦ}AA^{\dagger}=AA^{\dagger}$;

$(ii)$ $A^{k}A^{Ⓦ,*}A^{k}=A^{k}\Leftrightarrow A^{k}A^{*}A^{k}=A^{k}$;

$(iii)$ $AA^{Ⓦ,*}=AA^{Ⓦ}\Leftrightarrow A^{Ⓦ,*}=A^{Ⓦ}$;

$(iv)$ $A^{Ⓦ,*}A=AA^{Ⓦ}\Leftrightarrow A^{Ⓦ,*}=A^{Ⓦ,\dagger}$;

$(v)$ $A^{Ⓦ,*}A=A^{\dagger}A\Leftrightarrow A^{Ⓦ,*}=A^{\dagger}$;

$(vi)$ $AA^{Ⓦ,*}=AA^{\dagger}\Leftrightarrow AA^{Ⓦ,*}A=A$;

$(vii)$ $A^{Ⓦ,*}=A^{*}\Leftrightarrow A^{Ⓦ,\dagger}=A^{\dagger}$.

Let $A\in\mathbb{C}^{n\times n}$ with $ind(A)=k$, then

$$A^{Ⓦ,*}=A^{Ⓦ}AA^{*}=U\begin{pmatrix}G_1 & G_2\\ 0 & 0\end{pmatrix}U^{*},$$

where $G_1=T^{*}+(T^{-1}S+T^{-2}SN)S^{*}$, $G_2=(T^{-1}S+T^{-2}SN)N^{*}$.

**Theorem 2.18.** *Let* $A\in\mathbb{C}^{n\times n}$ *be a matrix with* $\mathrm{Ind}(A)=k$ *written as in* (3). *Then*

$(i)$ $A^{Ⓦ,*}A=A^{*}A\Leftrightarrow A$ *is a symmetrical and EP matrix.*

$(ii)$ $AA^{Ⓦ,*}=AA^{*,Ⓦ}\Leftrightarrow S+T^{-1}SN=(TT^{*}+SS^{*})T^{-1}S,\ NS^{*}=0$.

Proof.

(i)

$$\begin{aligned}A^{Ⓦ,*}A=A^{*}A &\Leftrightarrow \begin{pmatrix}G_1T & G_1S+G_2N\\ 0 & 0\end{pmatrix}=\begin{pmatrix}T^{*}T & T^{*}S\\ S^{*}T & S^{*}S+N^{*}N\end{pmatrix}\\ &\Leftrightarrow T^{*}T+(T^{-1}S+T^{-2}SN)S^{*}T=T^{*}T,\\ &T^{*}S+(T^{-1}S+T^{-2}SN)S^{*}S+(T^{-1}S+T^{-2}SN)N^{*}N=T^{*}S,\ S^{*}T=0,\ S^{*}S+N^{*}N=0.\\ &\Leftrightarrow S=0,\ N=0.\\ &\Leftrightarrow\ A\ \text{is a symmetrical and EP matrix}.\end{aligned}$$

(ii)

$$\begin{aligned}AA^{Ⓦ,*}=AA^{*,Ⓦ} &\Leftrightarrow \begin{pmatrix}TG_1 & TG_2\\ 0 & 0\end{pmatrix}=\begin{pmatrix}TT^{*}+SS^{*} & TT^{*}T^{-1}S+SS^{*}T^{-1}S\\ NS^{*} & NS^{*}T^{-1}S\end{pmatrix}\\ &\Leftrightarrow TT^{*}+(S+T^{-1}SN)S^{*}=TT^{*}+SS^{*},\ NS^{*}=0,\ T(T^{-1}S+T^{-2}SN)=TT^{*}T^{-1}S+SS^{*}T^{-1}S.\\ &\Leftrightarrow S+T^{-1}SN=(TT^{*}+SS^{*})T^{-1}S,\ NS^{*}=0.\ \square\end{aligned}$$

**Theorem 2.19.** *Let* $A\in\mathbb{C}^{n\times n}$ *be a matrix with* $\mathrm{Ind}(A)=k$ *written as in* (3). *Then*

$(i)$ $A^{Ⓦ,*}=A\Leftrightarrow A$ *is a symmetrical and EP matrix.*

$(ii)$ $A^{Ⓦ,*}=A^{*}\Leftrightarrow A$ *is an EP matrix.*

$(iii)$ $A^{Ⓦ,*}=AA^{\dagger}\Leftrightarrow TT^{*}+SS^{*}=T,\ N=0$.

$(iv)$ $A^{Ⓦ,*}=A^{*,Ⓦ}\Leftrightarrow S=0$.

Proof.
(i)

$$\begin{aligned} A^{Ⓦ,*} = A &\Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T & S \\ 0 & N \end{pmatrix} \\ &\Leftrightarrow T^* + (T^{-1}S + T^{-2}SN)S^* = T,\ (T^{-1}S + T^{-2}SN)N^* = S \text{ and } N = 0. \\ &\Leftrightarrow T = T^*,\ S = 0,\ N = 0. \\ &\Leftrightarrow A \text{ is a symmetrical and EP matrix}. \end{aligned}$$

(ii)

$$\begin{aligned} A^{Ⓦ,*} = A^* &\Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^* & 0 \\ S^* & N^* \end{pmatrix} \\ &\Leftrightarrow T^* + (T^{-1}S + T^{-2}SN)S^* = T^*,\ (T^{-1}S + T^{-2}SN)N^* = 0,\ S^* = 0 \text{ and } N^* = 0. \\ &\Leftrightarrow S = 0,\ N = 0. \\ &\Leftrightarrow A \text{ is an EP matrix}. \end{aligned}$$

(iii)

$$\begin{aligned} A^{Ⓦ,*} = AA^\dagger &\Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & NN^\dagger \end{pmatrix} \\ &\Leftrightarrow T^* + (T^{-1}S + T^{-2}SN)S^* = I,\ (T^{-1}S + T^{-2}SN)N^* = 0 \text{ and } NN^\dagger = 0. \\ &\Leftrightarrow TT^* + SS^* = T,\ N = 0. \end{aligned}$$

(iv)

$$\begin{aligned} A^{Ⓦ,*} = A^{*,Ⓦ} &\Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^* & T^*T^{-1}S \\ S^* & S^*T^{-1}S \end{pmatrix} \\ &\Leftrightarrow T^* + (T^{-1}S + T^{-2}SN)S^* = T^*,\ (T^{-1}S + T^{-2}SN)N^* = T^*T^{-1}S,\ S^* = 0, \text{ and } S^*T^{-1}S = 0. \\ &\Leftrightarrow S = 0. \ \square \end{aligned}$$

**Theorem 2.20.** *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = 1$. Then, the following statements are equivalent:*

(*i*) *$A$ is a partial isometry and $A$ is an EP matrix.*

(*ii*) $AA^{Ⓦ,*} = AA^\dagger$.

(*iii*) $A^{Ⓦ,*}A = AA^\dagger$.

(*iv*) $AA^{Ⓦ,*} = A^\dagger A$.

(*v*) $A^{Ⓦ,*}A = A^\dagger A$.

Proof.
(*i*) ⇔ (*ii*)

$$\begin{aligned} AA^{Ⓦ,*} = AA^\dagger &\Leftrightarrow \begin{pmatrix} TG_1 & TG_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & NN^\dagger \end{pmatrix} \\ &\Leftrightarrow TT^* + (S + T^{-1}SN)S^* = I,\ (S + T^{-1}SN)N^* = S \text{ and } NN^\dagger = 0. \\ &\Leftrightarrow TT^* = I,\ N = 0,\ (S + T^{-1}SN)N^* = S = 0. \\ &\Leftrightarrow TT^* = I,\ S = 0,\ N = 0. \end{aligned} \tag{10}$$

$(i) \Leftrightarrow (iii)$

$$A^{ⓦ,*}A = AA^{\dagger} \Leftrightarrow \begin{pmatrix} G_1T & G_1S + G_2N \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & NN^{\dagger} \end{pmatrix}$$
$$\Leftrightarrow T^*T + (T^{-1}S + T^{-2}SN)S^*T = I,\ T^*S + (T^{-1}S + T^{-2}SN)S^*S + (T^{-1}S + T^{-2}SN)N^*N = 0 \text{ and } NN^{\dagger} = 0.$$
$$\Leftrightarrow N = 0,\ (TT^* + SS^*)S = 0 \text{ and } (TT^* + SS^*)T = T.$$
$$\Leftrightarrow TT^* = I,\ S = 0,\ N = 0.$$

$(i) \Leftrightarrow (iv)$

$$AA^{ⓦ,*} = A^{\dagger}A \Leftrightarrow \begin{pmatrix} TG_1 & TG_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^* \triangle T & T^* \triangle S(I - NN^{\dagger}) \\ (I - NN^{\dagger})S^* \triangle T & (I - NN^{\dagger})S^* \triangle S(I - NN^{\dagger}) + N^{\dagger}N \end{pmatrix}.$$
$$\Leftrightarrow T^* \triangle T = TT^* + (S + T^{-1}SN)S^*,\ T^* \triangle S(I - NN^{\dagger}) = (S + T^{-1}SN)N^*,\ S = SNN^{\dagger},\ N^{\dagger}N = 0.$$
$$\Leftrightarrow T^* \triangle T = TT^*,\ S = 0, \text{ and } N = 0.$$
$$\Leftrightarrow TT^* = I,\ S = 0 \text{ and } N = 0.$$

$(i) \Leftrightarrow (v)$

$$A^{ⓦ,*}A = A^{\dagger}A \Leftrightarrow \begin{pmatrix} G_1T & G_1S + G_2N \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T^* \triangle T & T^* \triangle S(I - NN^{\dagger}) \\ (I - NN^{\dagger})S^* \triangle T & (I - NN^{\dagger})S^* \triangle S(I - NN^{\dagger}) + N^{\dagger}N \end{pmatrix}$$
$$\Leftrightarrow T^* \triangle T = T^*T + (T^{-1}S + T^{-2}SN)S^*T,\ S = SNN^{\dagger},\ N^{\dagger}N = 0,$$
$$T^* \triangle S(I - NN^{\dagger}) = T^*S + (T^{-1}S + T^{-2}SN)S^*S + (T^{-1}S + T^{-2}SN)N^*N.$$
$$\Leftrightarrow T^* \triangle T = T^*T,\ N = 0 \text{ and } S = SNN^{\dagger} = 0.$$
$$\Leftrightarrow T^*T = I,\ S = 0 \text{ and } N = 0.$$

Therefore, the above conditions are equivalent. □

**Definition 2.21.** *Let $A, B \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$. We call $A$ is below $B$ under the relation $\leqslant^{ⓦ,*}$ if*

$$AA^{ⓦ,*} = BA^{ⓦ,*} \text{ and } A^{ⓦ,*}A = A^{ⓦ,*}B.$$

Naturally, we will consider whether this binary relationship can become a partial order. The answer to this question is No. A binary relation is called a partial order if it is reflexive, transitive, and anti-symmetric on a non-empty set. Next, we give a concrete example to prove that this relationship is not satisfied antisymmetry.

**Example 2.22.** *Consider the matrices*

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

*Since*

$$A^{ⓦ} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ B^{ⓦ} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

*we can get*

$$A^{ⓦ,*}A = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ A^{ⓦ,*}B = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$AA^{\textcircled{W},*} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ BA^{\textcircled{W},*} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$BB^{\textcircled{W},*} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ AB^{\textcircled{W},*} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B^{\textcircled{W},*}B = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B^{\textcircled{W},*}A = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

*Thus,*

$$AA^{\textcircled{W},*} = BA^{\textcircled{W},*}, A^{\textcircled{W},*}A = A^{\textcircled{W},*}B,$$
$$AB^{\textcircled{W},*} = BB^{\textcircled{W},*}, B^{\textcircled{W},*}B = B^{\textcircled{W},*}A.$$

*Clearly, $A \leqslant^{\textcircled{W},*} B$ and $B \leqslant^{\textcircled{W},*} A$ hold, but $A \neq B$. Hence, the weak group-star relation can not be a partial order.*

## 3. Successive matrix squaring algorithm for the weak group-star matrix

In this section, we give successive matrix squaring algorithms for computing the weak group-star matrix. The development of the SMS iterations start from the transformations.

Since

$$\begin{aligned}(A^{k+2})^{\dagger}A(AA^{\textcircled{W},*}) &= (A^{k+2})^{\dagger}A^2A^k(A^{k+2})^{\dagger}A^2A^* \\ &= (A^{k+2})^{\dagger}A^{k+2}(A^{k+2})^{\dagger}A^2A^* = (A^{k+2})^{\dagger}A^2A^*,\end{aligned}$$

we have

$$\begin{aligned}A^{\textcircled{W},*} &= A^{\textcircled{W},*} - \beta((A^{k+2})^{\dagger}A(AA^{\textcircled{W},*}) - (A^{k+2})^{\dagger}A^2A^*) \\ &= (I - \beta(A^{k+2})^{\dagger}A^2)A^{\textcircled{W},*} + \beta(A^{k+2})^{\dagger}A^2A^*.\end{aligned}$$

Observe the following matrices

$$P = I - \beta(A^{k+2})^{\dagger}A^2, \ \ Q = \beta(A^{k+2})^{\dagger}A^2A^*, \ \ \beta > 0.$$

It is obvious that $A^{\textcircled{W},*}$ is the unique solution of $X = PX + Q$. Then an iterative procedure for computing the weak group-star matrix $A^{\textcircled{W},*}$ can be defined as follows

$$X_1 = Q, \ X_{m+1} = PX_m + Q. \tag{11}$$

This algorithm can be implemented in parallel by considering the block matrix

$$T = \begin{pmatrix} P & Q \\ 0 & I \end{pmatrix}, \quad T^m = \begin{pmatrix} P^m & \Sigma_{i=0}^{m-1}P^iQ \\ 0 & I \end{pmatrix}.$$

The top right block of $T^m$ is $X^m$, the $m$th approximation to $A^{\textcircled{W},*}$. The matrix power $T^m$ can be computed by the successive squaring, i.e.

$$T_0 = T, \ T_{i+1} = T_i^2, \ i = 0, 1, \ldots, j,$$

where the integer $j$ is such that $2^j \geq m$. The following theorem gives the sufficient condition for the convergence of the iterative process (11).

**Theorem 3.1.** *Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A)=k$ and $rank(A^k)=r$. Then the approximation*

$$X_{2^m}=\sum_{i=0}^{2^m-1}(I-\beta(A^{k+2})^{\dagger}A^2)^i\beta(A^{k+2})^{\dagger}A^2A^*,$$

*defined by the iterative process* (11) *converges to the weak group-star matrix $A^{Ⓦ,*}$ if the spectral radius $\rho(I-X_1(A^{\dagger})^*)\leq 1$. Moreover, the following error estimation holds:*

$$\left\|A^{Ⓦ,*}-X_{2^m}\right\|\leq\left\|(I-X_1(A^{\dagger})^*)^{2^m}\right\|.$$

*As a result,*

$$\lim_{m\to\infty} sup\sqrt[2^m]{\left\|A^{Ⓦ,*}-X_{2^m}\right\|}\leq(I-X_1(A^{\dagger})^*).$$

Proof. We know that

$$A^{Ⓦ,*}(A^{\dagger})^*A^{Ⓦ,*}=A^{Ⓦ,*},\quad X_{2^m}(A^{\dagger})^*A^{Ⓦ,*}=X_{2^m}.$$

By the mathematical induction, we can get

$$I-X_{2^m}(A^{\dagger})^*=(I-X_1(A^{\dagger})^*)^{2^m}.$$

Therefore,

$$\begin{aligned}\left\|A^{Ⓦ,*}-X_{2^m}\right\| &= \left\|A^{Ⓦ,*}-X_{2^m}(A^{\dagger})^*A^{Ⓦ,*}\right\|\\ &= \left\|(I-X_{2^m}(A^{\dagger})^*)A^{Ⓦ,*}\right\|\\ &\leq \left\|A^{Ⓦ,*}\right\|\left\|I-X_{2^m}(A^{\dagger})^*\right\|\\ &= \left\|A^{Ⓦ,*}\right\|\left\|(I-X_1(A^{\dagger})^*)^{2^m}\right\|,\end{aligned}$$

and

$$\begin{aligned}\lim_{m\to\infty} sup\sqrt[2^m]{\left\|A^{Ⓦ,*}-X_{2^m}\right\|} &\leq \lim_{m\to\infty} sup\sqrt[2^m]{\left\|A^{Ⓦ,*}\right\|\left\|(I-X_1(A^{\dagger})^*)^{2^m}\right\|}\\ &= \rho(I-X_1(A^{\dagger})^*).\end{aligned}$$

In the last equality, we use the fact that $\lim_{m\to\infty}\|B^n\|^{1/n}=\rho(B)$, for any square matrix $B$.

If $\beta$ is a real parameter such that $\underset{1\leq i\leq t}{max}\left|1-\beta\lambda_i\right|<1$, where $\lambda_i$ $(i=1,2,\ldots,s)$ are the nonzero eigenvalues of $(A^{k+2})^{\dagger}A^2A^*$, then

$$\rho(I-X_1(A^{\dagger})^*)=\rho(I-\beta(A^{k+2})^{\dagger}A^2)\leq 1.$$

It completes the proof. □

**Example 3.2.** *Consider the following matrix:*

$$A=\begin{pmatrix}0 & 4/3 & -1/3\\ -1/3 & 1 & -1/3\\ -2/3 & -2/3 & 0\end{pmatrix},\mathrm{Ind}(A)=2.$$

*Let*

$$P=I-\beta(A^4)^{\dagger}A^2,\;\; Q=\beta(A^4)^{\dagger}A^2A^*,\beta=0.6.$$

*The eigenvalues $\lambda_i$ of $QA$ are included in the set $\{0,0,0.5\}$. The nonzero eigenvalues $\lambda_i$ satisfy*

$$\underset{i}{max}|1-\lambda_i|=|1-0.5|=0.5<1.$$

*Then we obtain the satisfactory approximation for $A^{\textcircled{W},*}$ after the 6th iteration of the successive matrix squaring algorithm.*

$$(T^2)^6 \approx \begin{pmatrix} 0.982 & 0.130 & -0.037 & -0.185 & -0.148 & 0.074 \\ 0.130 & 0.093 & 0.026 & 1.300 & 1.037 & -0.519 \\ -0.031 & 0.218 & 0.938 & -0.311 & -0.249 & 0.125 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

*The upper right corner of $(T^2)^6$ is an approximation of the weak group-star matrix, that is*

$$A^{\textcircled{W},*} = \begin{pmatrix} -0.185 & -0.148 & 0.074 \\ 1.300 & 1.037 & -0.519 \\ -0.311 & -0.249 & 0.125 \end{pmatrix}.$$

## 4. The Cramer's rule for the solution of a singular equation $(A^\dagger)^* x = b$

Since $R(A^{\textcircled{W},*}) = R(A^k) \subseteq N(V)$, we obtain $VA^{\textcircled{W},*} = 0$. By $R(I - AA^{\textcircled{W},*}) \subseteq R(U) = R(UU^\dagger) = N(I - UU^\dagger)$, we can obtain $I - AA^{\textcircled{W},*} = UU^\dagger(I - AA^{\textcircled{W},*})$. Then, we get Theorem 4.1.

**Theorem 4.1.** *[11] Let $A\in\mathbb{C}^{n\times n}$ with $\operatorname{Ind}(A) = k$. Suppose $U\in\mathbb{C}^{n\times r}$ and $V^*\in\mathbb{C}^{n\times r}$ having full column rank such that*

$$R(I - AA^{\textcircled{W},*}) \subseteq R(U) \subseteq N(A^{\textcircled{W},*}), \text{ and } R(A^k) \subseteq N(V).$$

*Then, the bordered matrix*

$$X = \begin{pmatrix} A & U \\ V & 0 \end{pmatrix}$$

*is nonsingular and*

$$X^{-1} = \begin{pmatrix} A^{\textcircled{W},*} & (I - A^{\textcircled{W},*}A)V^\dagger \\ U^\dagger(I - AA^{\textcircled{W},*}) & -U^\dagger(A - AA^{\textcircled{W},*}A)V^\dagger \end{pmatrix}. \tag{12}$$

Similarly, we can get the following result.

**Theorem 4.2.** *Let $A\in\mathbb{C}^{n\times n}$ with $\operatorname{Ind}(A) = k$. Suppose $U\in\mathbb{C}^{n\times r}$ and $V^*\in\mathbb{C}^{n\times r}$ having full column rank such that*

$$R(A^k) = N(V),\ R(U) = N(A^k)^*.$$

*Then the bordered matrix*

$$X = \begin{pmatrix} (A^\dagger)^* & U \\ V & 0 \end{pmatrix}$$

*is nonsingular and*

$$X^{-1} = \begin{pmatrix} A^{\textcircled{W},*} & (I - A^{\textcircled{W}}A)V^\dagger \\ U^\dagger(I - (A^\dagger)^*A^{\textcircled{W},*}) & -U^\dagger((A^\dagger)^* - (A^\dagger)^*A^{\textcircled{W}}A)V^\dagger \end{pmatrix}. \tag{13}$$

Since $B \in R((A^\dagger)^*A^{\textcircled{W}})$, we have $B = (A^\dagger)^*A^{\textcircled{W}}Z$, for some $Z\in\mathbb{C}^{n\times n}$. If $X = A^{\textcircled{W},*}B$, we obtain

$$(A^\dagger)^*X = (A^\dagger)^*A^{\textcircled{W},*}B = (A^\dagger)^*A^{\textcircled{W}}AA^*(A^\dagger)^*A^{\textcircled{W}}Z = (A^\dagger)^*A^{\textcircled{W}}Z = B.$$

Then we can get the following theorem.

**Theorem 4.3.** *[11] Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$, *and* $B \in R((A^{\dagger})^{*}A^{\textcircled{W}})$. *Then*

$$(A^{\dagger})^{*}X = B \tag{14}$$

*in* $R(A^{k})$ *has the unique solution* $X = A^{\textcircled{W},*}B$.

Similar to the Theorem 4.3, we can prove the following theorem.

**Theorem 4.4.** *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$ *and* $B \in R(AA^{\textcircled{W}})$. *Then* $A^{*}B$ *is the unique solution in* $R(A^{*}(A^{k})^{*}A^{2})$ *of* $(A^{\dagger})^{*}X = B$.

Using the relationship between the weak group-star inverse of $(A^{\dagger})^{*}$ and a nonsingular bordered matrix, we give the Cramer's rule for solving a singular linear equation $(A^{\dagger})^{*}x = B$. $(A^{\dagger})^{*}(ij \to b_j)$ denotes the matrix obtained by replacing $i$th column of $(A^{\dagger})^{*}$ with $b_j$, where $b_j$ is the $j$th column of $B$.

**Theorem 4.5.** *Let* $A, B \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$. *Suppose* $U \in \mathbb{C}^{n\times r}$ *and* $V^{*} \in \mathbb{C}^{n\times r}$ *having full column rank such that*

$$R(A^{\textcircled{W},*}) = R(A^{k}) = N(V), \text{ and } R(U) = N(A^{\textcircled{W},*}).$$

*If* $R(B) \subseteq R((A^{\dagger})^{*}A^{\textcircled{W}})$, *then the unique solution* $X = A^{\textcircled{W},*}B$ *of the singular linear equation* (14) *is given by*

$$x_{ij} = \frac{det\begin{pmatrix}(A^{\dagger})^{*}(i\to b_j) & U\\ V(i\to 0) & 0\end{pmatrix}}{det\begin{pmatrix}(A^{\dagger})^{*} & U\\ V & 0\end{pmatrix}},\ i = 1,2,\ldots n,\ j = 1,2,\ldots n. \tag{15}$$

Proof. Since $X = A^{\textcircled{W},*}B \in R(A^{k}) = N(V)$ and $B \in R((A^{\dagger})^{*}A^{\textcircled{W}}) = AR(A^{k})$, we have

$$VX = 0,\ (I - AA^{\textcircled{W},*})B = 0. \tag{16}$$

It follows from (16) that the solution of $(A^{\dagger})^{*}X = B$ satisfies

$$\begin{pmatrix}(A^{\dagger})^{*} & U\\ V & 0\end{pmatrix}\begin{pmatrix}X\\ 0\end{pmatrix} = \begin{pmatrix}B\\ 0\end{pmatrix}. \tag{17}$$

By Theorem 4.2, the coefficient matrix of (17) is nonsingular. Using (13) and (16), we can obtain

$$\begin{pmatrix}X\\ 0\end{pmatrix} = \begin{pmatrix}A^{\textcircled{W},*} & (I - A^{\textcircled{W}}A)V^{\dagger}\\ U^{\dagger}(I - (A^{\dagger})^{*}A^{\textcircled{W}}) & -U^{\dagger}((A^{\dagger})^{*} - (A^{\dagger})^{*}A^{\textcircled{W}}A)V^{\dagger}\end{pmatrix}\begin{pmatrix}B\\ 0\end{pmatrix} = \begin{pmatrix}A^{\textcircled{W},*}B\\ 0\end{pmatrix}.$$

Therefore, $x = A^{\textcircled{W},*}B$ and (15) follows from the classical Cramer's rule [13]. □

## 5. Perturbations of the weak group-star matrix

Using the form of the core-EP decomposition of $A^{\textcircled{W},*}$, we can calculate the perturbation of $A^{\textcircled{W},*}$.

**Theorem 5.1.** *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$, $B = A + E \in \mathbb{C}^{n\times n}$. *If*

$$EAA^{\textcircled{W}} = E, AA^{\textcircled{W}}E = E, \text{ and } \| A^{\textcircled{W}}E \| < 1,$$

*then*

$$B^{\textcircled{W},*} = (I_n + A^{\textcircled{W}}E)^{-1}A^{\textcircled{W}}(A + E)(A + E)^{*} = A^{\textcircled{W}}(I_n + EA^{\textcircled{W}})^{-1}(A + E)(A + E)^{*}.$$

Proof. Let $A$ have the form of (3), and $E = U\begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}U^*$, where $E_1 \in \mathbb{C}^{r\times r}$. Since $AA^{Ⓦ}E = E$, we get

$$\begin{aligned} AA^{Ⓦ}E &= U\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}\begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}U^* \\ &= U\begin{pmatrix} E_1 + T^{-1}SE_3 & E_2 + T^{-1}SE_4 \\ 0 & 0 \end{pmatrix}U^* \\ &= U\begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}U^*. \end{aligned} \tag{18}$$

Thus, we can get $E_3 = 0, E_4 = 0$. And applying $EAA^{Ⓦ} = E$, we have

$$\begin{aligned} EAA^{Ⓦ} &= U\begin{pmatrix} E_1 & E_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}\begin{pmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{pmatrix}U^* \\ &= U\begin{pmatrix} E_1 & E_1T^{-1}S \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} E_1 & E_2 \\ 0 & 0 \end{pmatrix}U^*. \end{aligned}$$

Hence, $E_2 = E_1T^{-1}S$.

Owing to $\rho(EA^{Ⓦ}) = \rho(A^{Ⓦ}E) \leq \| A^{Ⓦ}E \| < 1$, we can get $I + A^{Ⓦ}E$ is reversible and $T + E_1$ is nonsingular. Furthermore, notice that

$$E = U\begin{pmatrix} E_1 & E_2 \\ 0 & 0 \end{pmatrix}U^*, \ B = A + E = U\begin{pmatrix} T + E_1 & S + E_2 \\ 0 & N \end{pmatrix}U^*,$$

we can get

$$B^{Ⓦ} = U\begin{pmatrix} (T + E_1)^{-1} & (T + E_1)^{-2}(S + E_2) \\ 0 & 0 \end{pmatrix}U^*.$$

Therefore,

$$B^{Ⓦ,*} = U\begin{pmatrix} (T + E_1)^* + \triangle_1(S + E_2)^* & \triangle_1 N^* \\ 0 & 0 \end{pmatrix}U^*,$$

where $\triangle_1 = [(T + E_1)^{-1}(S + E_2) + (T + E_1)^{-2}(S + E_2)N]$. Thus,

$$B^{Ⓦ,*} = (I_n + A^{Ⓦ}E)^{-1}A^{Ⓦ}(A + E)(A + E)^* = A^{Ⓦ}(I_n + EA^{Ⓦ})^{-1}(A + E)(A + E)^*. \ \square$$

Furthermore, we have the following result.

**Theorem 5.2.** *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$, $B = A + E \in \mathbb{C}^{n\times n}$. *If*

$$AA^{Ⓦ,*}E = E, \ \textit{and} \ \| A^{Ⓦ}E \| < 1,$$

*then*

$$\begin{aligned} B^{Ⓦ,*} &= ((I_n + A^{Ⓦ}E)^{-1}A^{Ⓦ})^2AA^{Ⓣ}(A + E)^2(A + E)^* \\ &= (I_n + A^{Ⓦ}E)^{-1}A^{Ⓦ}(I_n + A^{Ⓦ}E)^{-1}A^{Ⓣ}(A + E)^2(A + E)^*. \end{aligned}$$

## 6. Applications

In this section, we will give the application of the weak group-star matrix in solving linear equations.

**Theorem 6.1.** *Let $A\in\mathbb{C}^{n\times n}$ and $\mathrm{Ind}(A) = k$, the equation*

$$(A^{k+2})^*A^2x = (A^{k+2})^*A^2A^*b, \quad b\in\mathbb{C}^n, \tag{19}$$

*is consistent and its general solution is*

$$x = A^{\textcircled{w},*}b + (I_n - A^{\textcircled{w}}A)y, \tag{20}$$

*for arbitrary $y\in\mathbb{C}^m$.*

Proof. Suppose that $x$ has the form (20). Applying $A^{\textcircled{w},*} = A^k(A^{k+2})^\dagger A^2A^*$, we have

$$\begin{aligned}(A^{k+2})^*A^2A^{\textcircled{w},*} &= (A^{k+2})^*A^2A^k(A^{k+2})^\dagger A^2A^* \\ &= (A^{k+2})^*A^{k+2}(A^{k+2})^\dagger A^2A^* \\ &= (A^{k+2})^*A^2A^*.\end{aligned}$$

Therefore $(A^{k+2})^*A^2A^{\textcircled{w},*}b = (A^{k+2})^*A^2A^*b$, which implies that (19) holds for $x$.

For a solution $x$ to (19), we obtain

$$\begin{aligned}A^{\textcircled{w},*}b &= A^k(A^{k+2})^\dagger A^2A^*b \\ &= A^k(A^{k+2})^\dagger((A^{k+2})^\dagger)^*(A^{k+2})^*A^2A^*b \\ &= A^k(A^{k+2})^\dagger((A^{k+2})^\dagger)^*(A^{k+2})^*A^2x \\ &= A^{\textcircled{w}}Ax.\end{aligned}$$

Now, we get

$$x = A^{\textcircled{w},*}b + x - A^{\textcircled{w},*}Ax = A^{\textcircled{w},*}b + (I_n - A^{\textcircled{w}}A)x.$$

i.e., $x$ possesses the form (20). □

Since $A^{\textcircled{w}}AX = A^{\textcircled{w}}AA^{\textcircled{w},*}b = A^{\textcircled{w},*}b$, we have $A^{\textcircled{w}}AX = A^{\textcircled{w}}AA^{\textcircled{w},*}b = A^{\textcircled{w},*}b$. Then we can obtain Theorem 6.2.

**Theorem 6.2.** *[11] Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$, then the equation*

$$A^{\textcircled{w}}AX = A^{\textcircled{w},*}b \tag{21}$$

*is consistent and its general solution is*

$$x = A^{\textcircled{w},*}b + (I - A^{\textcircled{w}}A)y, \tag{22}$$

*for arbitrary $y\in\mathbb{C}^{n\times n}$.*

Similarly, the following theorem can be proved.

**Theorem 6.3.** *Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$. Then the equation*

$$(A^\dagger)^*x = AA^{\textcircled{w}}b$$

*is consistent and its general solution is*

$$x = A^{*,\textcircled{w}}b + (I - A^\dagger A)y,$$

*for arbitrary $y\in\mathbb{C}^{n\times n}$.*

Now, we can get the following consequence by the result of Theorem 6.3 in the case that $b \in R(A^k)$.

**Corollary 6.4.** *Let $A\in\mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$. Then the equation*

$$(A^\dagger)^*x = b, b \in R(A^k)$$

*is consistent and its general solution is*

$$x = A^*b + (I - A^\dagger A)y,$$

*for arbitrary $y\in\mathbb{C}^{n\times n}$.*

## 7. Conclusion

In this paper, the definition and characterizations of the weak group-star matrix are given. The equivalence between various matrices and the weak group-star matrix are established. For Cramer's rule and the perturbation, we also give relevant theorems. Moreover, the weak group-star matrix can be applied to solving equations.

Moreover, dual weak group-star matrix can be called star-weak group matrix. Let $A \in \mathbb{C}^{n\times n}$ and $\mathrm{Ind}(A) = k$, $C$ is the weak core part of $A$. Then

$$X(A^{\dagger})^{*}X = X,\ (A^{\dagger})^{*}X = CA^{D},\ XA = A^{*}C,$$

is consistent and its unique solution is $X = A^{*}CA^{D}$. The matrix satisfying the above equations is defined as $A^{*,\text{Ⓦ}} = A^{*}AA^{\text{Ⓦ}}$ and named the star-weak group matrix.

The star-weak group matrix also possesses similar properties of the weak group-star matrix.

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