



Solvability of (P, Q) -functional integral equations of fractional order using generalized Darbo's fixed point theorem

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Abstract.

In this article, we establish a generalized version of Darbo's fixed point theorem via some newly defined condensing operators and we define a new fractional integral using (P, Q) -calculus and study its properties. Finally, we apply this generalized Darbo's fixed point theorem to check the existence of a solution of (P, Q) -functional integral equations of fractional order in a Banach space. We explain the results with the help of simple examples.

1. Introduction

The measure of non-compactness which was first introduced by Kuratowski [14] plays a very important role in many branches of mathematics. There are several types of non-compactness measures in metric and topological spaces. For more information on the subject of measure of non-compactness, see [7]. Non-compactness measures are used in various types of integral and differential equations, see [7]. Arab et al. [6] proved the existence of solutions for infinite systems of integral equations that generate via two variables. In [10], the existence of solutions for singular integral equations was discussed using a measure of non-compactness.

The idea of Q -calculus was introduced by Jackson [11, 12]. Fractional q -difference concept was introduced by Agarwal [2] and Al-Salam [4]. In [13], the existence of solution of Q -integral equations of fractional order have been discussed.

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In this article, we apply a generalized version of Darbo’s theorem to study the solvability of the equation:

$$X(\theta) = \eta \left(\theta, X(\bar{a}(\theta)), \frac{\mathcal{F}(\theta, X(\bar{b}(\theta)))}{\Gamma_{P,Q}(\alpha)} \int_0^\theta (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_1, X(\theta_1)) d_{P,Q}\theta_1 \right), \tag{1}$$

where $\theta \in I = [0, 1]$, $0 < Q < P \leq 1$, $\mathcal{F}, \mathcal{U} : I \times \mathbb{R} \rightarrow \mathbb{R}$, $\bar{a}, \bar{b} : I \rightarrow I$, $\eta : I \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\alpha > 1$.

2. Preliminaries

At first, we recall some facts about Q -calculus. For more details, we refer to [2, 5, 17]. Let $Q \in [1, \infty)$. For arbitrary $\mathbb{L} \in \mathbb{R}$, the Q -real number $[\mathbb{L}]_Q$ is defined by

$$[\mathbb{L}]_Q = \frac{1 - Q^\mathbb{L}}{1 - Q}.$$

The Q -shifted factorial of real number \mathbb{L}' is defined by

$$(\mathbb{L}', Q)_0 = 1, (\mathbb{L}', Q)_\ell = \prod_{i=0}^{\ell-1} (1 - \mathbb{L}'Q^i), \ell = 1, 2, \dots, \infty.$$

For $(\mathbb{L}, \mathbb{L}') \in \mathbb{R}^2$, the Q -analog of $(\mathbb{L} - \mathbb{L}')^\ell$ is defined by

$$(\mathbb{L} - \mathbb{L}')^{(0)} = 1, (\mathbb{L} - \mathbb{L}')^{(\ell)} = \prod_{i=0}^{\ell-1} (\mathbb{L} - \mathbb{L}'Q^i), \ell = 1, 2, \dots, \infty.$$

For arbitrary $\beta \in \mathbb{R}$, $(\mathbb{L}, \mathbb{L}') \in \mathbb{R}^2$ and $\mathbb{L} \geq 0$,

$$(\mathbb{L} - \mathbb{L}')^{(\beta)} = \mathbb{L}^\beta \prod_{i=0}^{\infty} \left(\frac{\mathbb{L} - \mathbb{L}'Q^i}{\mathbb{L} - \mathbb{L}'Q^{\beta+i}} \right).$$

For $\mathbb{L}' = 0$, we have $\mathbb{L}^{(\beta)} = \mathbb{L}^\beta$.

The Q -gamma function is given by

$$\Gamma_Q(\mathbb{L}) = \frac{(1 - Q)^{(\mathbb{L}-1)}}{(1 - Q)^{\mathbb{L}-1}}, \mathbb{L} \notin \{0, -1, -2, \dots\}.$$

The (P, Q) -bracket or twin-basic number is defined by Sadjang [16] as follows. For arbitrary $\mathbb{L} \in \mathbb{R}$, we have

$$[\mathbb{L}]_{P,Q} = \frac{P^\mathbb{L} - Q^\mathbb{L}}{P - Q}.$$

For arbitrary $\mathbb{L}, \mathbb{L}' \in \mathbb{R}$, we define the (P, Q) -analog of $(\mathbb{L} - \mathbb{L}')^\ell$ as follows:

$$(\mathbb{L} - \mathbb{L}')_{P,Q}^{(0)} = 1, (\mathbb{L} - \mathbb{L}')_{P,Q}^{(\ell)} = \prod_{i=0}^{\ell-1} (\mathbb{L}P^i - \mathbb{L}'Q^i), \ell = 1, 2, 3, \dots$$

and for arbitrary $\beta \in \mathbb{R}$ and for arbitrary $\mathbb{L} \geq 0$,

$$(\mathbb{L} - \mathbb{L}')_{P,Q}^{(\beta)} = \mathbb{L}^\beta \prod_{i=0}^{\infty} \left(\frac{\mathbb{L}P^i - \mathbb{L}'Q^i}{\mathbb{L}P^i - \mathbb{L}'Q^{\beta+i}} \right).$$

For $\mathbb{L}' = 0$, we have $(\mathbb{L} - \mathbb{L}')_{P,Q}^{(\beta)} = \mathbb{L}^\beta$.

Lemma 2.1. If $\beta > 0$ and $A \leq B \leq T$ then $(T - A)_{P,Q}^{(\beta)} \geq (T - B)_{P,Q}^{(\beta)}$.

Proof. We have to know that

$$T^\beta \prod_{i=0}^{\infty} \left(\frac{TP^i - AQ^i}{TP^i - AQ^{\beta+i}} \right) \geq T^\beta \prod_{i=0}^{\infty} \left(\frac{TP^i - BQ^i}{TP^i - BQ^{\beta+i}} \right).$$

For each $i \in \mathbb{N}_0$, we show that

$$\begin{aligned} (TP^i - AQ^i)(TP^i - BQ^{\beta+i}) &\geq (TP^i - BQ^i)(TP^i - AQ^{\beta+i}) \\ \Leftrightarrow BP^iQ^{\beta+i} + AP^iQ^i &\geq AP^iQ^{\beta+i} + BP^iQ^i \\ \Leftrightarrow A + BQ^\beta &\geq B + AQ^\beta \\ \Leftrightarrow B - A &\leq Q^\beta(B - A). \end{aligned}$$

For $A = B$, we have $B - A = Q^\beta(B - A)$ and for $A \neq B$, we have $Q^\beta \leq 1$. \square

We define the (P, Q) -analogue of the Gamma function as follows:

$$\Gamma_{P,Q}(\mathbb{L}) = \frac{(P - Q)_{P,Q}^{(\mathbb{L}-1)}}{(P - Q)^{\mathbb{L}-1}}, \quad \mathbb{L} \notin \{0, -1, -2, \dots\}.$$

For $P = 1$, we can see that $\Gamma_{P,Q}$ reduces to Γ_Q . Clearly, we can see that

$$\Gamma_{P,Q}(\mathbb{L} + 1) \neq [\mathbb{L}]_{P,Q} \Gamma_{P,Q}(\mathbb{L}).$$

Only for $P = 1$ the equality holds.

Let $f : [0, \bar{a}] \rightarrow \mathbb{R}$ be a function where \bar{a} is a nonnegative real number. Sadjang [16], defined the (P, Q) -integral of the function f as follows:

$$\int_0^\theta f(\mathbb{L}) d_{P,Q} \mathbb{L} = (P - Q)\theta \sum_{\ell=0}^{\infty} \frac{Q^\ell}{P^{\ell+1}} f\left(\frac{Q^\ell}{P^{\ell+1}}\theta\right),$$

where $\left|\frac{P}{Q}\right| > 1$ and $\theta \in [0, \bar{a}]$, provided that the sum converges absolutely. For $P = 1$, we get $\int_0^\theta f(\mathbb{L}) d_{P,Q} \mathbb{L} = \int_0^\theta f(\mathbb{L}) d_Q \mathbb{L}$.

Lemma 2.2. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Then

$$\left| \int_0^\theta f(\mathbb{L}) d_{P,Q} \mathbb{L} \right| \leq \int_0^\theta |f(\mathbb{L})| d_{P,Q} \mathbb{L}$$

for all $\theta \in [0, 1]$.

Proof. We have

$$\begin{aligned} \left| \int_0^\theta \mathbf{f}(\mathbb{L}) d_{P,Q}\mathbb{L} \right| &= \left| (P - Q)\theta \sum_{\ell=0}^\infty \frac{Q^\ell}{P^{\ell+1}} \mathbf{f}\left(\frac{Q^\ell}{P^{\ell+1}}\theta\right) \right|, \left| \frac{P}{Q} \right| > 1 \\ &\leq (P - Q)\theta \sum_{\ell=0}^\infty \frac{Q^\ell}{P^{\ell+1}} \left| \mathbf{f}\left(\frac{Q^\ell}{P^{\ell+1}}\theta\right) \right| \\ &= \int_0^\theta |\mathbf{f}(\mathbb{L})| d_{P,Q}\mathbb{L}. \end{aligned}$$

□

Remark 2.3. If $\mathbf{f}(\mathbb{L}) = 1$ for all $\mathbb{L} \in I = [0, 1]$, then for any $\theta \in I$, we have

$$\begin{aligned} \int_0^\theta \mathbf{f}(\mathbb{L}) d_{P,Q}\mathbb{L} &= \int_0^\theta d_{P,Q}\mathbb{L} \\ &= (P - Q)\theta \sum_{\ell=0}^\infty \frac{Q^\ell}{P^{\ell+1}}, \left| \frac{P}{Q} \right| > 1 \\ &= \left(\frac{P - Q}{P}\right)\theta \sum_{\ell=0}^\infty \left(\frac{Q}{P}\right)^\ell \\ &= \left(\frac{P - Q}{P}\right)\theta \left(\frac{P}{P - Q}\right) \\ &= \theta. \end{aligned}$$

We introduce the fractional (P, Q) -integral of order $\alpha \geq 0$ of the function \mathbf{f} which is given by

$$I_{P,Q}^0 \mathbf{f}(\theta) = \mathbf{f}(\theta)$$

and

$$I_{P,Q}^\alpha \mathbf{f}(\theta) = \frac{1}{\Gamma_{P,Q}(\alpha)} \int_0^\theta (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \mathbf{f}(\theta_1) d_{P,Q}\theta_1,$$

where $\theta \in [0, 1]$ and $\alpha > 1$. For $P = 1$, we get $I_{P,Q}^\alpha \mathbf{f}(\theta) = I_Q^\alpha \mathbf{f}(\theta)$.

Definition 2.4. [8] A strongly continuous semigroup on E is a mapping $S : [0, \infty) \rightarrow \mathcal{L}(E)$ so that:

- (1) $S(0) = I_i$ and $S(t + s) = S(t)S(s)$ for all $t, s \geq 0$ where I_i is the identity mapping.
- (2) $S(x)$ is continuous on $[0, \infty)$ for all $x \in E$ where E is a complex Banach space and $\mathcal{L}(E)$ is the Banach algebra of all continuous linear mappings defined on \mathbb{B} .

Let $f_1, f_2 \in C[0, 1]$ and $k_1, k_2 \in \mathbb{R}$. Therefore

$$\begin{aligned} &I_{P,Q}^\alpha [k_1 f_1(\theta) + k_2 f_2(\theta)] \\ &= \frac{1}{\Gamma_{P,Q}(\alpha)} \int_0^\theta (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} [k_1 f_1(\theta_1) + k_2 f_2(\theta_1)] d_{P,Q}\theta_1 \\ &= k_1 I_{P,Q}^\alpha f_1(\theta) + k_2 I_{P,Q}^\alpha f_2(\theta). \end{aligned}$$

Hence, the operator $I_{p,Q}^\alpha$ is linear.

Again for $f_1(\theta), f_2(\theta) \geq 0$ we observe that

$$I_{p,Q}^\alpha [f_1(\theta) + f_2(\theta)] = I_{p,Q}^\alpha [f_1(\theta)] + I_{p,Q}^\alpha [f_2(\theta)] \neq I_{p,Q}^\alpha [f_1(\theta)] I_{p,Q}^\alpha [f_2(\theta)]$$

and $I_{p,Q}^\alpha [0] = 0 \neq I_i$.

Hence, we conclude that the operator $I_{p,Q}^\alpha$ is not an strongly continuous semigroup on $C([0, 1])$.

Suppose that E be a real Banach space. Let $\bar{B}(y_0, d)$ be the closed ball in E with center y_0 and radius d . By \bar{L} and $\text{Conv}L$ we denote the closure and the convex closure of L . Moreover, let \mathbf{NB}_E be the family of all nonempty and bounded subsets of E and \mathbf{RC}_E be its subfamily consisting of all relatively compact sets.

The following definition of a measure of noncompactness has been presented in [7].

Definition 2.5. $\mu : \mathbf{NB}_E \rightarrow [0, \infty)$ is called a measure of noncompactness if:

- (i) $\mu(L) = 0$ implies that L is precompact for all $L \in \mathbf{NB}_E$,
- (ii) the family $\ker \mu = \{L \in \mathbf{NB}_E : \mu(L) = 0\}$ is nonempty and $\ker \mu \subset \mathbf{RC}_E$,
- (iii) $L \subset L' \implies \mu(L) \leq \mu(L')$,
- (iv) $\mu(\bar{L}) = \mu(L)$,
- (v) $\mu(\text{Conv}L) = \mu(L)$,
- (vi) $\mu(\lambda L + (1 - \lambda)L') \leq \lambda\mu(L) + (1 - \lambda)\mu(L')$ for all $\lambda \in [0, 1]$,
- (vii) $\bigcap_{n=1}^\infty L_n \neq \emptyset$ whenever $L_n \in \mathbf{NB}_E$, $L_n = \bar{L}_n$, $L_{n+1} \subseteq L_n$ for all $n = 1, 2, 3, \dots$ and $\lim_{n \rightarrow \infty} \mu(L_n) = 0$.

The family $\ker \mu$ is said to be the *kernel of measure* μ .

A measure μ is called sublinear if:

- (1) $\mu(\lambda L) = |\lambda| \mu(L)$ for all $\lambda \in \mathbb{R}$,
- (2) $\mu(L + L') \leq \mu(L) + \mu(L')$.

A sublinear measure of noncompactness μ so that

$$\mu(L \cup L') = \max\{\mu(L), \mu(L')\}$$

and $\ker \mu = \mathbf{RC}_E$ is said to be regular.

For a bounded subset Q of a metric space X ,

$$\alpha(Q) = \inf \left\{ \delta > 0 : Q = \bigcup_{i=1}^n Q_i, \text{diam}(Q_i) \leq \delta \text{ for } 1 \leq i \leq n \leq \infty \right\},$$

is the Kuratowski measure of noncompactness of Q where $\text{diam}(Q_i)$ denotes the diameter of the set Q_i , that is,

$$\text{diam}(Q_i) = \sup \{d(x, y) : x, y \in Q_i\},$$

and

$$\chi(Q) = \inf \{\epsilon > 0 : Q \text{ has a finite } \epsilon - \text{net in } X\},$$

is the Hausdorff measure of noncompactness for Q .

Recall the following fixed point theorems:

Theorem 2.6. [1, Schauder fixed-point theorem] Let E be a Banach space and $\mathfrak{N}(\neq \emptyset) \subseteq E$ be closed and convex. Then any $\Delta : \mathfrak{N} \rightarrow \mathfrak{N}$ which is continuous and compact, admits at least one fixed point.

Theorem 2.7. [9, Darbo fixed-point theorem] Let \mathbb{E} be a Banach space and $\mathfrak{N} \subseteq \mathbb{E}$ be nonempty, bounded, closed and convex (NBCC) and μ is a measure of noncompactness defined in \mathbb{E} . Also, let $\Delta : \mathfrak{N} \rightarrow \mathfrak{N}$ be continuous and there exists a constant $0 \leq \tau < 1$ with

$$\mu(\Delta\Pi) \leq \tau \cdot \mu(\Pi), \Pi \subseteq \mathfrak{N}.$$

Then Δ has a fixed point.

In this section, we establish a generalization of Darbo’s fixed point theorem with the help of following concepts:

Definition 2.8. [15] Let functions $\wp_1, \wp_2 : \mathbb{R}_+ \rightarrow \mathbb{R}$ be given. The pair (\wp_1, \wp_2) is called a pair of shifting distance functions(SDF) if:

- (1) $\wp_1(l) \leq \wp_2(m)$, then $l \leq m$, for all $l, m \in \mathbb{R}_+$,
- (2) for all $l_k, m_k \in \mathbb{R}_+$ with $\lim_{k \rightarrow \infty} l_k = \lim_{k \rightarrow \infty} m_k = w$, if $\wp_1(l_k) \leq \wp_2(m_k)$ for all k , then $w = 0$.

Following examples of \wp represents a pair (\wp_1, \wp_2) of a SDF.

- (1) $\wp_1(\xi) = \ln\left(\frac{1+2\xi}{2}\right)$ and $\wp_2(\xi) = \ln\left(\frac{1+\xi}{2}\right)$.
- (2) $\wp_1(\xi) = \xi$ and $\wp_2(\xi) = \lambda\xi$, $\lambda \in [0, 1)$.

Definition 2.9. Let \mathbb{F} be the family of all continuous and nondecreasing maps $F : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ with:

- (1) $\max\{a, b, c\} \leq F(a, b, c)$ for all $a, b, c \geq 0$,
- (2) $F(a, 0, 0) = a$ for all $a \geq 0$.

For example, $F : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ defined by $F(a, b, c) = a + b + c$ is an element of \mathbb{F} .

3. New results

From now on, let $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be nondecreasing and continuous with $\phi(t) = 0$ iff $t = 0$ and $\phi(t) < t$ for all $t > 0$.

Theorem 3.1. Let \mathbb{E} be a Banach space, $\mathbb{C} \subseteq \mathbb{E}$ be a NBCC and $T : \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function with

$$\wp_1 [F(\mu(T\mathbb{L}), \gamma_1(\mu(T\mathbb{L})), \gamma_2(\mu(T\mathbb{L})))] \leq \wp_2 [\phi \{F(\mu(\mathbb{L}), \gamma_1(\mu(\mathbb{L})), \gamma_2(\mu(\mathbb{L})))\}], \tag{2}$$

for all $\mathbb{L}(\neq \emptyset) \subseteq \mathbb{C}$; $F \in \mathbb{F}$; $\wp_1, \wp_2 \in \wp$ and $\gamma_1, \gamma_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous nondecreasing functions where μ is an arbitrary MNC. Then T admits a fixed point in \mathbb{C} .

Proof. Define a sequence (\mathbb{C}_s) , where $\mathbb{C}_1 = \mathbb{C}$ and $\mathbb{C}_{s+1} = \overline{\text{Conv}(T\mathbb{C}_s)}$, for all $s \geq 1$. Also, $T\mathbb{C}_1 = T\mathbb{C} \subseteq \mathbb{C} = \mathbb{C}_1$, $\mathbb{C}_2 = \overline{\text{Conv}(T\mathbb{C}_1)} \subseteq \mathbb{C} = \mathbb{C}_1$. Similarly, $\mathbb{C}_1 \supseteq \mathbb{C}_2 \supseteq \mathbb{C}_3 \supseteq \dots \supseteq \mathbb{C}_s \supseteq \mathbb{C}_{s+1} \supseteq \dots$.

If $s_0 \in \mathbb{N}$ with $\mu(\mathbb{C}_{s_0}) = 0$, then \mathbb{C}_{s_0} is compact. So, applying Theorem 2.6 we observed that T admits a fixed point.

Let $\mu(\mathbb{C}_s) > 0$ for all $s \geq 0$. By (2) we have

$$\begin{aligned} & \wp_1 [F(\mu(\mathbb{C}_{s+1}), \gamma_1(\mu(\mathbb{C}_{s+1})), \gamma_2(\mu(\mathbb{C}_{s+1})))] \\ &= \wp_1 [F(\mu(\overline{\text{Conv}(T\mathbb{C}_s)}), \gamma_1(\mu(\overline{\text{Conv}(T\mathbb{C}_s)})), \gamma_2(\mu(\overline{\text{Conv}(T\mathbb{C}_s)})))] \\ &= \wp_1 [F(\mu(T\mathbb{C}_s), \gamma_1(\mu(T\mathbb{C}_s)), \gamma_2(\mu(T\mathbb{C}_s)))] \\ &\leq \wp_2 [\phi \{F(\mu(\mathbb{C}_s), \gamma_1(\mu(\mathbb{C}_s)), \gamma_2(\mu(\mathbb{C}_s)))\}], \end{aligned}$$

which gives

$$\begin{aligned} & F(\mu(\mathbf{C}_{s+1}), \gamma_1(\mu(\mathbf{C}_{s+1})), \gamma_2(\mu(\mathbf{C}_{s+1}))) \\ & \leq \phi \{F(\mu(\mathbf{C}_s), \gamma_1(\mu(\mathbf{C}_s)), \gamma_2(\mu(\mathbf{C}_s)))\} \\ & < F(\mu(\mathbf{C}_s), \gamma_1(\mu(\mathbf{C}_s)), \gamma_2(\mu(\mathbf{C}_s))). \end{aligned}$$

Clearly, the sequence $\{F(\mu(T\mathbf{C}_s), \gamma_1(\mu(T\mathbf{C}_s)), \gamma_2(\mu(T\mathbf{C}_s)))\}_{s=1}^{\infty}$ is positive and decreasing. So, we can find a $d \geq 0$ such that

$$\lim_{s \rightarrow \infty} F(\mu(T\mathbf{C}_s), \gamma_1(\mu(\mathbf{C}_s)), \gamma_2(\mu(T\mathbf{C}_s))) = d.$$

If $d = 0$, then the result is obvious.

If possible, assume that $d > 0$.

As $s \rightarrow \infty$, then we get $d < \phi(d)$ which is a contradiction. Hence, $\lim_{s \rightarrow \infty} F(\mu(\mathbf{C}_s), \gamma_1(\mu(\mathbf{C}_s)), \gamma_2(\mu(\mathbf{C}_s))) = 0$, i.e., $d = 0$ which gives

$$F\left(\lim_{s \rightarrow \infty} \mu(\mathbf{C}_s), \lim_{s \rightarrow \infty} \gamma_1(\mu(\mathbf{C}_s)), \lim_{s \rightarrow \infty} \gamma_2(\mu(\mathbf{C}_s))\right) = 0.$$

By using the property of F we get $\lim_{s \rightarrow \infty} \mu(\mathbf{C}_s) = 0$.

We know that $\mathbf{C}_s \supseteq \mathbf{C}_{s+1}$ and by Definition 2.5 we get $\mathbf{C}_\infty = \bigcap_{s=1}^{\infty} \mathbf{C}_s \subseteq \mathbf{C}$ is nonempty, closed and convex. Also, \mathbf{C}_∞ is invariant under F . Thus, Theorem 2.6 implies that F has a fixed point in $\mathbf{C}_\infty \subseteq \mathbf{C}$. \square

Theorem 3.2. Let \mathbb{E} be a Banach space, $\mathbf{C} \subseteq \mathbb{E}$ be a NBCC and $T : \mathbf{C} \rightarrow \mathbf{C}$ be a continuous function such that

$$\wp_1 [\mu(T\mathbb{L}) + \gamma_1(\mu(T\mathbb{L})) + \gamma_2(\mu(T\mathbb{L}))] \leq \wp_2 [\phi \{ \mu(\mathbb{L}) + \gamma_1(\mu(\mathbb{L})) + \gamma_2(\mu(\mathbb{L})) \}], \quad (3)$$

for all $\mathbb{L} (\neq \emptyset) \subseteq \mathbf{C}$; $\wp_1, \wp_2 \in \wp$ where $\gamma_1, \gamma_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous nondecreasing functions and μ is an arbitrary MNC. Then T admits a fixed point in \mathbf{C} .

Proof. The result can be obtained by taking $F(a, b, c) = a + b + c$ in Theorem 3.1. \square

Corollary 3.3. Let \mathbb{E} be a Banach space, $\mathbf{C} \subseteq \mathbb{E}$ be a NBCC and $T : \mathbf{C} \rightarrow \mathbf{C}$ be a continuous function with

$$\wp_1 [\mu(T\mathbb{L})] \leq \wp_2 [\phi \{ \mu(\mathbb{L}) \}], \quad (4)$$

for all $\mathbb{L} (\neq \emptyset) \subseteq \mathbf{C}$ and $\wp_1, \wp_2 \in \wp$ where μ is an arbitrary MNC. Then T admits a fixed point in \mathbf{C} .

Proof. Let $\gamma_1(t) = \gamma_2(t) = 0$ for all $t \geq 0$ in Theorem 3.2. \square

Corollary 3.4. Let \mathbb{E} be a Banach space, $\mathbf{C} \subseteq \mathbb{E}$ be a NBCC and $T : \mathbf{C} \rightarrow \mathbf{C}$ be a continuous function with

$$\mu(T\mathbb{L}) \leq \phi \{ \mu(\mathbb{L}) \}, \quad (5)$$

for all $\mathbb{L} (\neq \emptyset) \subseteq \mathbf{C}$ where μ is an arbitrary MNC. Then T admits a fixed point in \mathbf{C} .

Proof. Let $\wp_1(t) = \wp_2(t) = t$ for all $t \geq 0$ in Corollary 3.3. \square

Remark 3.5. For $\phi(t) = kt$ where $k \in [0, 1)$ and $t \in \mathbb{R}_+$ in Corollary 3.4 we obtain the Darbo's fixed point theorem.

4. Application

In this section, we establish the existence of solution of the equation (1) in the space $E = C(I)$, where $C(I)$ is the set of real and continuous functions defined on the compact set I . We also know that E is a Banach space with respect to the norm

$$\| \mathbb{L} \| = \max \{ |\mathbb{L}(\theta)| : \theta \in I \}, \mathbb{L} \in E.$$

Let $\mathbf{M} \in \mathbf{NB}_E$. For $(\mathbb{L}, \mathbf{r}) \in \mathbf{M} \times (0, \infty)$, we denote by $\omega(\mathbb{L}, \mathbf{r})$ the modulus of continuity of \mathbb{L} , i.e.,

$$\omega(\mathbb{L}, \mathbf{r}) = \sup \{ |\mathbb{L}(\theta) - \mathbb{L}(\theta_1)| : \theta, \theta_1 \in I, |\theta - \theta_1| \leq \mathbf{r} \}.$$

Further we define

$$\omega(\mathbf{M}, \mathbf{r}) = \sup \{ \omega(\mathbb{L}, \mathbf{r}) : \mathbb{L} \in \mathbf{M} \}.$$

Define the mapping $\mu : \mathbf{NB}_E \rightarrow [0, \infty)$ by

$$\mu(\mathbf{M}) = \lim_{\mathbf{r} \rightarrow 0^+} \omega(\mathbf{M}, \mathbf{r}), \mathbf{M} \in \mathbf{NB}_E.$$

Then μ is a measure of non-compactness in E (see[7]).

Let us define the operator \mathcal{T} on E by

$$(\mathcal{T}\mathbb{L})(\theta) = \eta \left(\theta, \mathbb{L}(\bar{a}(\theta)), \frac{\mathcal{F}(\theta, \mathbb{L}(\bar{b}(\theta)))}{\Gamma_{P,Q}(\alpha)} \int_0^\theta (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_1, \mathbb{L}(\theta_1)) d_{P,Q}\theta_1 \right),$$

where $\mathbb{L} \in E$ and $\theta \in I$.

We consider the following assumptions:

- (1) The functions $\mathcal{F}, \mathcal{U} : I \times \mathbb{R} \rightarrow \mathbb{R}; \bar{a}, \bar{b} : I \rightarrow I$ and $\eta : I \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous.
- (2) There exists a constant $\mathcal{D}_\eta > 0$ and non-decreasing function $\psi_\eta : [0, \infty) \rightarrow [0, \infty)$ such that

$$|\eta(\theta, \mathbb{L}, \mathbb{L}') - \eta(\theta, Z, W)| \leq \psi_\eta(|\mathbb{L} - Z|) + \mathcal{D}_\eta |\mathbb{L}' - W|$$

for all $\theta \in I$ and for all $\mathbb{L}, \mathbb{L}', Z, W \in \mathbb{R}$.

- (3) There exists a constant $\mathcal{D}_\mathcal{F} > 0$ such that

$$|\mathcal{F}(\theta, \mathbb{L}) - \mathcal{F}(\theta, \mathbb{L}')| \leq \mathcal{D}_\mathcal{F} |\mathbb{L} - \mathbb{L}'|$$

for all $\theta \in I$ and for all $\mathbb{L}, \mathbb{L}' \in \mathbb{R}$.

- (4) There exists a non-decreasing and continuous function $\psi_\mathcal{U} : [0, \infty) \rightarrow [0, \infty)$ such that

$$|\mathcal{U}(\theta, \mathbb{L}) - \mathcal{U}(\theta, \mathbb{L}')| \leq \psi_\mathcal{U}(|\mathbb{L} - \mathbb{L}'|),$$

where $\theta \in I$ and $\mathbb{L}, \mathbb{L}' \in \mathbb{R}$. Also, $\psi_\mathcal{U}(\theta) < \theta, \theta > 0$ and $\mathcal{U}(\theta, 0) = 0$ for all $\theta \in I$.

- (5) There exists $\mathbf{r}_0 > 0$ such that

$$\psi_\eta(\mathbf{r}_0) + \frac{\mathcal{D}_\eta \psi_\mathcal{U}(\mathbf{r}_0)}{|\Gamma_{P,Q}(\alpha)|} (\mathcal{D}_\mathcal{F} \mathbf{r}_0 + \hat{\mathcal{F}}) + \hat{\eta} \leq \mathbf{r}_0,$$

where $\hat{\eta} = \max \{ |\eta(\theta, 0, 0)| : \theta \in I \}$ and $\hat{\mathcal{F}} = \max \{ |\mathcal{F}(\theta, 0)| : \theta \in I \}$.

- (6) The function $\psi_\eta : [0, \infty) \rightarrow [0, \infty)$ is continuous so that $\psi_\eta(\theta) < \hat{L}\theta$ for all $\theta > 0$ where $\hat{L} > 0$ is a constant.

(7) The function $\bar{a} : I \rightarrow I$ satisfies

$$|\bar{a}(\theta) - \bar{a}(\theta_1)| \leq \psi_{\bar{a}}(|\theta - \theta_1|)$$

for all $\theta, \theta_1 \in I$ and $\psi_{\bar{a}} : [0, \infty) \rightarrow [0, \infty)$ is non-decreasing and $\lim_{\theta \rightarrow 0^+} \psi_{\bar{a}}(\theta) = 0$.

(8) The function $\bar{b} : I \rightarrow I$ satisfies

$$|\bar{b}(\theta) - \bar{b}(\theta_1)| \leq \psi_{\bar{b}}(|\theta - \theta_1|)$$

for all $\theta, \theta_1 \in I$ and $\psi_{\bar{b}} : [0, \infty) \rightarrow [0, \infty)$ is non-decreasing and $\lim_{\theta \rightarrow 0^+} \psi_{\bar{b}}(\theta) = 0$.

(9) We suppose that $0 < \psi_{\mathcal{U}}(\mathbf{r}_0) < \frac{|\Gamma_{P,Q}(\alpha)|}{\mathcal{D}_{\mathcal{F}} \mathcal{D}_{\eta}}$ and $\frac{\mathcal{D}_{\eta}}{|\Gamma_{P,Q}(\alpha)|} (\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}) < 1$. Also,

$$\hat{\mathcal{L}} + \mathcal{L} + \mathcal{N} < 1,$$

$$\text{where } \mathcal{L} = \frac{\mathcal{D}_{\eta}(\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}})}{|\Gamma_{P,Q}(\alpha)|} \text{ and } \mathcal{N} = \frac{\mathcal{D}_{\eta} \mathcal{D}_{\mathcal{F}} \psi_{\mathcal{U}}(\mathbf{r}_0)}{|\Gamma_{P,Q}(\alpha)|}.$$

Let the closed ball with center 0 and radius \mathbf{r}_0 be denoted by $\bar{\mathbf{B}}(0, \mathbf{r}_0) = \{\mathbf{L} \in \mathbf{E} : \|\mathbf{L}\| \leq \mathbf{r}_0\}$.

Theorem 4.1. Under the hypothesis (1)-(9), equation (1) has at least one solution in $\mathbf{E} = C(I)$.

Proof. As $\theta_1 \in [0, 1] = I$ so $\theta_1 \geq 1$. Also $\alpha - 1 > 0$. By applying Lemma 2.1 we have

$$(P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \leq (P\theta - 0)_{P,Q}^{(\alpha-1)} = (P\theta)_{P,Q}^{(\alpha-1)}$$

i.e

$$(P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \leq P^{\alpha-1} \theta^{\alpha-1}.$$

Since $P \leq 1$ therefore $(P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \leq \theta^{\alpha-1}$.

Let $\mathbf{L} \in \bar{\mathbf{B}}(0, \mathbf{r}_0)$. By using assumptions (1)-(9), for all $\theta \in I$, we have

$$\begin{aligned} & |(\mathcal{F}\mathbf{L})(\theta)| \\ & \leq \left| \eta \left(\theta, \mathbf{L}(\bar{a}(\theta)), \frac{\mathcal{F}(\theta, \mathbf{L}(\bar{b}(\theta)))}{|\Gamma_{P,Q}(\alpha)|} \int_0^{\theta} (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_1, \mathbf{L}(\theta_1)) d_{P,Q} \theta_1 \right) - \eta(\theta, 0, 0) \right| + |\eta(\theta, 0, 0)| \\ & \leq \psi_{\eta} (\|\mathbf{L}(\bar{a}(\theta))\|) + \mathcal{D}_{\eta} \frac{|\mathcal{F}(\theta, \mathbf{L}(\bar{b}(\theta)))|}{|\Gamma_{P,Q}(\alpha)|} \int_0^{\theta} |(P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)}| |\mathcal{U}(\theta_1, \mathbf{L}(\theta_1))| d_{P,Q} \theta_1 + \hat{\eta} \\ & \leq \psi_{\eta} (\|\mathbf{L}\|) + \frac{\mathcal{D}_{\eta}}{|\Gamma_{P,Q}(\alpha)|} \{ |\mathcal{F}(\theta, \mathbf{L}(\bar{b}(\theta))) - \mathcal{F}(\theta, 0)| + |\mathcal{F}(\theta, 0)| \} \int_0^{\theta} \theta^{\alpha-1} |\mathcal{U}(\theta_1, \mathbf{L}(\theta_1))| d_{P,Q} \theta_1 + \hat{\eta} \\ & \leq \psi_{\eta} (\mathbf{r}_0) + \frac{\mathcal{D}_{\eta}}{|\Gamma_{P,Q}(\alpha)|} (\mathcal{D}_{\mathcal{F}} \|\mathbf{L}\| + \hat{\mathcal{F}}) \int_0^{\theta} \theta^{\alpha-1} \psi_{\mathcal{U}}(\|\mathbf{L}\|) d_{P,Q} \theta_1 + \hat{\eta} \\ & \leq \psi_{\eta} (\mathbf{r}_0) + \frac{\mathcal{D}_{\eta} \psi_{\mathcal{U}}(\mathbf{r}_0)}{|\Gamma_{P,Q}(\alpha)|} (\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}) \int_0^{\theta} \theta^{\alpha-1} d_{P,Q} \theta_1 + \hat{\eta} \\ & = \psi_{\eta} (\mathbf{r}_0) + \frac{\mathcal{D}_{\eta} \psi_{\mathcal{U}}(\mathbf{r}_0)}{|\Gamma_{P,Q}(\alpha)|} (\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}) \theta^{\alpha} + \hat{\eta} \\ & \leq \mathbf{r}_0, \end{aligned}$$

i.e., $(\mathcal{T}\mathbb{L})(\theta) \in \bar{\mathbf{B}}(0, \mathbf{r}_0)$. Thus, \mathcal{T} maps $\bar{\mathbf{B}}(0, \mathbf{r}_0)$ into itself.

We have to show that \mathcal{T} is continuous on $\bar{\mathbf{B}}(0, \mathbf{r}_0)$. Let us define the operators λ_1, λ_2 and λ_3 on \mathbf{E} by

$$(\lambda_1\mathbb{L})(\theta) = \theta$$

$$(\lambda_2\mathbb{L})(\theta) = \mathbb{L}(\bar{a}(\theta))$$

and

$$(\lambda_3\mathbb{L})(\theta) = \mathcal{F}(\theta, \mathbb{L}(\bar{b}(\theta)))$$

for all $\theta \in I$ and $\mathbb{L} \in \mathbf{E}$. It is obvious that λ_1 is continuous.

For all $\mathbb{L}, \mathbb{L}' \in \mathbf{E}$ we have

$$|(\lambda_2\mathbb{L})(\theta) - (\lambda_2\mathbb{L}')(\theta)| = |\mathbb{L}(\bar{a}(\theta)) - \mathbb{L}'(\bar{a}(\theta))| \leq \|\mathbb{L} - \mathbb{L}'\|,$$

for all $\theta \in I$ which gives $\|\lambda_2\mathbb{L} - \lambda_2\mathbb{L}'\| \leq \|\mathbb{L} - \mathbb{L}'\|$. Therefore, λ_2 is uniformly continuous on \mathbf{E} .

Similarly, we can show that $\|\lambda_3\mathbb{L} - \lambda_3\mathbb{L}'\| \leq \mathcal{D}_{\mathcal{F}} \|\mathbb{L} - \mathbb{L}'\|$ for all $\mathbb{L}, \mathbb{L}' \in \mathbf{E}$. Therefore, λ_3 is also uniformly continuous on \mathbf{E} .

To prove that \mathcal{T} is continuous on $\bar{\mathbf{B}}(0, \mathbf{r}_0)$, for this we show that

$$(\mathcal{H}\mathbb{L})(\theta) = \int_0^\theta (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_1, \mathbb{L}(\theta_1)) d_{P,Q}\theta_1$$

is continuous on $\bar{\mathbf{B}}(0, \mathbf{r}_0)$. Let $\epsilon > 0$ and $\mathbb{L}, \mathbb{L}' \in \bar{\mathbf{B}}(0, \mathbf{r}_0)$ such that $\|\mathbb{L} - \mathbb{L}'\| < \epsilon$. For all $\theta \in I$ we have

$$(\mathcal{H}\mathbb{L})(\theta) - (\mathcal{H}\mathbb{L}')(\theta) = \int_0^\theta (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \{\mathcal{U}(\theta_1, \mathbb{L}(\theta_1)) - \mathcal{U}(\theta_1, \mathbb{L}'(\theta_1))\} d_{P,Q}\theta_1.$$

Let $\mathcal{U}_{\mathbf{r}_0}(\epsilon) = \sup \{|\mathcal{U}(\theta, \mathbb{L}) - \mathcal{U}(\theta, \mathbb{L}')| : \theta \in I; \mathbb{L}, \mathbb{L}' \in \bar{\mathbf{B}}(0, \mathbf{r}_0); \|\mathbb{L} - \mathbb{L}'\| < \epsilon\}$.

Therefore,

$$\begin{aligned} (\mathcal{H}\mathbb{L})(\theta) - (\mathcal{H}\mathbb{L}')(\theta) &\leq \mathcal{U}_{\mathbf{r}_0}(\epsilon) \int_0^\theta (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} d_{P,Q}\theta_1 \\ &\leq \mathcal{U}_{\mathbf{r}_0}(\epsilon) \int_0^\theta \theta^{\alpha-1} d_{P,Q}\theta_1 \\ &= \theta^\alpha \mathcal{U}_{\mathbf{r}_0}(\epsilon) \\ &\leq \mathcal{U}_{\mathbf{r}_0}(\epsilon). \end{aligned}$$

So, we have

$$\|\mathcal{H}\mathbb{L} - \mathcal{H}\mathbb{L}'\| \leq \mathcal{U}_{\mathbf{r}_0}(\epsilon).$$

Using the uniform continuity of \mathcal{U} on the compact set $I \times [\mathbf{r}_0, \mathbf{r}_0]$ we get

$$\lim_{\epsilon \rightarrow 0^+} \mathcal{U}_{\mathbf{r}_0}(\epsilon) = 0.$$

Thus, \mathcal{H} is continuous. So, we can conclude that \mathcal{T} is also continuous. Let $\mathbf{d} > 0$ and $\theta_1, \theta_2 \in I$ such that $|\theta_1 - \theta_2| \leq \mathbf{d}$. We also assume that $\theta_1 \geq \theta_2$. Now

$$\begin{aligned} & |(\mathcal{T}\mathbb{L})(\theta_1) - (\mathcal{T}\mathbb{L})(\theta_2)| \\ &= \left| \eta \left(\theta_1, \mathbb{L}(\bar{a}(\theta_1)), \frac{\mathcal{F}(\theta_1, \mathbb{L}(\bar{b}(\theta_1)))}{\Gamma_{P,Q}(\alpha)} \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{u}(\theta_3, \mathbb{L}(\theta_3)) d_{P,Q}\theta_3 \right) \right. \\ &\quad \left. - \eta \left(\theta_2, \mathbb{L}(\bar{a}(\theta_2)), \frac{\mathcal{F}(\theta_2, \mathbb{L}(\bar{b}(\theta_2)))}{\Gamma_{P,Q}(\alpha)} \int_0^{\theta_2} (P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{u}(\theta_3, \mathbb{L}(\theta_3)) d_{P,Q}\theta_3 \right) \right| \\ &\leq \left| \eta \left(\theta_1, \mathbb{L}(\bar{a}(\theta_1)), \frac{\mathcal{F}(\theta_1, \mathbb{L}(\bar{b}(\theta_1)))}{\Gamma_{P,Q}(\alpha)} \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{u}(\theta_3, \mathbb{L}(\theta_3)) d_{P,Q}\theta_3 \right) \right. \\ &\quad \left. - \eta \left(\theta_2, \mathbb{L}(\bar{a}(\theta_1)), \frac{\mathcal{F}(\theta_1, \mathbb{L}(\bar{b}(\theta_1)))}{\Gamma_{P,Q}(\alpha)} \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{u}(\theta_3, \mathbb{L}(\theta_3)) d_{P,Q}\theta_3 \right) \right| \\ &\quad + \left| \eta \left(\theta_2, \mathbb{L}(\bar{a}(\theta_1)), \frac{\mathcal{F}(\theta_1, \mathbb{L}(\bar{b}(\theta_1)))}{\Gamma_{P,Q}(\alpha)} \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{u}(\theta_3, \mathbb{L}(\theta_3)) d_{P,Q}\theta_3 \right) \right. \\ &\quad \left. - \eta \left(\theta_2, \mathbb{L}(\bar{a}(\theta_2)), \frac{\mathcal{F}(\theta_2, \mathbb{L}(\bar{b}(\theta_2)))}{\Gamma_{P,Q}(\alpha)} \int_0^{\theta_2} (P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{u}(\theta_3, \mathbb{L}(\theta_3)) d_{P,Q}\theta_3 \right) \right| \\ &= I_1 + I_2. \end{aligned}$$

Also,

$$\begin{aligned} & \left| \frac{\mathcal{F}(\theta_1, \mathbb{L}(\bar{b}(\theta_1)))}{\Gamma_{P,Q}(\alpha)} \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{u}(\theta_3, \mathbb{L}(\theta_3)) d_{P,Q}\theta_3 \right| \\ &\leq \frac{|\mathcal{F}(\theta_1, \mathbb{L}(\bar{b}(\theta_1)))|}{|\Gamma_{P,Q}(\alpha)|} \int_0^{\theta_1} |(P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)}| |\mathbf{u}(\theta_3, \mathbb{L}(\theta_3))| d_{P,Q}\theta_3 \\ &\leq \frac{|\mathcal{F}(\theta_1, \mathbb{L}(\bar{b}(\theta_1))) - \mathcal{F}(\theta_1, 0)| + |\mathcal{F}(\theta_1, 0)|}{|\Gamma_{P,Q}(\alpha)|} \int_0^{\theta_1} |(P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)}| \psi_{\mathbf{u}}(\|\mathbb{L}(\theta_3)\|) d_{P,Q}\theta_3 \\ &\leq \frac{(\mathcal{D}_{\mathcal{F}} \|\mathbb{L}(\bar{b}(\theta_1))\| + \hat{\mathcal{F}}) \psi_{\mathbf{u}}(\|\mathbb{L}\|)}{|\Gamma_{P,Q}(\alpha)|} \int_0^{\theta_1} |(P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)}| d_{P,Q}\theta_3 \\ &\leq \frac{(\mathcal{D}_{\mathcal{F}} \|\mathbb{L}\| + \hat{\mathcal{F}}) \psi_{\mathbf{u}}(\|\mathbb{L}\|)}{|\Gamma_{P,Q}(\alpha)|} \theta_1^\alpha \\ &\leq \frac{(\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}) \psi_{\mathbf{u}}(\mathbf{r}_0)}{|\Gamma_{P,Q}(\alpha)|} = \hat{\mathcal{D}}. \end{aligned}$$

Set

$$\mathcal{D}(\eta, \mathbf{d}) = \sup \left\{ |\eta(\theta, \mathbb{L}, \mathbb{L}') - \eta(\theta_1, \mathbb{L}, \mathbb{L}')| : \theta, \theta_1 \in I; |\theta - \theta_1| < \mathbf{d}, \mathbb{L} \in [-\mathbf{r}_0, \mathbf{r}_0], \mathbb{L}' \in [-\hat{\mathcal{D}}, \hat{\mathcal{D}}] \right\}.$$

Therefore, $I_1 \leq \mathcal{D}(\eta, \mathbf{d})$. Again

$$\begin{aligned}
 I_2 &\leq \psi_\eta \left(\left| \mathfrak{L}(\bar{a}(\theta_1)) - \mathfrak{L}(\bar{a}(\theta_2)) \right| \right) \\
 &+ \frac{\mathcal{D}_\eta}{|\Gamma_{P,Q}(\alpha)|} \left| \mathcal{F}(\theta_1, \mathfrak{L}(\bar{b}(\theta_1))) \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{U}(\theta_3, \mathfrak{L}(\theta_3)) d_{P,Q}\theta_3 \right. \\
 &\left. - \mathcal{F}(\theta_2, \mathfrak{L}(\bar{b}(\theta_2))) \int_0^{\theta_2} (P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{U}(\theta_3, \mathfrak{L}(\theta_3)) d_{P,Q}\theta_3 \right|.
 \end{aligned}$$

We have

$$\left| \mathfrak{L}(\bar{a}(\theta_1)) - \mathfrak{L}(\bar{a}(\theta_2)) \right| \leq \omega(\mathfrak{L} \ 0 \ \bar{a}, \mathbf{d})$$

which gives

$$\psi_\eta \left(\left| \mathfrak{L}(\bar{a}(\theta_1)) - \mathfrak{L}(\bar{a}(\theta_2)) \right| \right) \leq \psi_\eta \left(\omega(\mathfrak{L} \ 0 \ \bar{a}, \mathbf{d}) \right).$$

Now, we have

$$\begin{aligned}
 &\left| \mathcal{F}(\theta_1, \mathfrak{L}(\bar{b}(\theta_1))) \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{U}(\theta_3, \mathfrak{L}(\theta_3)) d_{P,Q}\theta_3 \right. \\
 &\left. - \mathcal{F}(\theta_2, \mathfrak{L}(\bar{b}(\theta_2))) \int_0^{\theta_2} (P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{U}(\theta_3, \mathfrak{L}(\theta_3)) d_{P,Q}\theta_3 \right| \\
 &\leq \left| \mathcal{F}(\theta_1, \mathfrak{L}(\bar{b}(\theta_1))) \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{U}(\theta_3, \mathfrak{L}(\theta_3)) d_{P,Q}\theta_3 \right. \\
 &\left. - \mathcal{F}(\theta_2, \mathfrak{L}(\bar{b}(\theta_2))) \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{U}(\theta_3, \mathfrak{L}(\theta_3)) d_{P,Q}\theta_3 \right| \\
 &+ \left| \mathcal{F}(\theta_2, \mathfrak{L}(\bar{b}(\theta_2))) \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{U}(\theta_3, \mathfrak{L}(\theta_3)) d_{P,Q}\theta_3 \right. \\
 &\left. - \mathcal{F}(\theta_2, \mathfrak{L}(\bar{b}(\theta_2))) \int_0^{\theta_2} (P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{U}(\theta_3, \mathfrak{L}(\theta_3)) d_{P,Q}\theta_3 \right| \\
 &\leq \left| \mathcal{F}(\theta_1, \mathfrak{L}(\bar{b}(\theta_1))) - \mathcal{F}(\theta_2, \mathfrak{L}(\bar{b}(\theta_2))) \right| \int_0^{\theta_1} \left| (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \right| \left| \mathbf{U}(\theta_3, \mathfrak{L}(\theta_3)) \right| d_{p,q}\theta_3 \\
 &+ \left| \mathcal{F}(\theta_2, \mathfrak{L}(\bar{b}(\theta_2))) \right| \left| \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{U}(\theta_3, \mathfrak{L}(\theta_3)) d_{P,Q}\theta_3 - \int_0^{\theta_2} (P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathbf{U}(\theta_3, \mathfrak{L}(\theta_3)) d_{P,Q}\theta_3 \right|
 \end{aligned}$$

$$\begin{aligned} &\leq \left| \mathcal{F}(\theta_1, \mathbb{L}(\bar{b}(\theta_1))) - \mathcal{F}(\theta_2, \mathbb{L}(\bar{b}(\theta_2))) \right| \psi_{\mathcal{U}}(\|\mathbb{L}\|) \\ &+ \left| \mathcal{F}(\theta_2, \mathbb{L}(\bar{b}(\theta_2))) \right| \left| \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, \mathbb{L}(\theta_3)) d_{P,Q}\theta_3 - \int_0^{\theta_2} (P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, \mathbb{L}(\theta_3)) d_{P,Q}\theta_3 \right| \\ &= I_3 + I_4. \end{aligned}$$

We define

$$\omega_{\mathcal{F}}(\mathbf{r}_0, \mathbf{d}) = \sup \{ |\mathcal{F}(\theta_1, \mathbb{L}) - \mathcal{F}(\theta_2, \mathbb{L})| : \theta_1, \theta_2 \in I, |\theta_1 - \theta_2| \leq \mathbf{d}, \mathbb{L} \in [-\mathbf{r}_0, \mathbf{r}_0] \}.$$

Then

$$\begin{aligned} I_3 &\leq \psi_{\mathcal{U}}(\|\mathbb{L}\|) \left| \mathcal{F}(\theta_1, \mathbb{L}(\bar{b}(\theta_1))) - \mathcal{F}(\theta_1, \mathbb{L}(\bar{b}(\theta_2))) \right| \\ &+ \psi_{\mathcal{U}}(\|\mathbb{L}\|) \left| \mathcal{F}(\theta_1, \mathbb{L}(\bar{b}(\theta_2))) - \mathcal{F}(\theta_2, \mathbb{L}(\bar{b}(\theta_2))) \right| \\ &\leq \psi_{\mathcal{U}}(\mathbf{r}_0) \left[\mathcal{D}_{\mathcal{F}} \left| \mathbb{L}(\bar{b}(\theta_1)) - \mathbb{L}(\bar{b}(\theta_2)) \right| \right] + \psi_{\mathcal{U}}(\mathbf{r}_0) \omega_{\mathcal{F}}(\mathbf{r}_0, \mathbf{d}) \\ &\leq \psi_{\mathcal{U}}(\mathbf{r}_0) \left[\mathcal{D}_{\mathcal{F}} \omega(\mathbb{L} \circ \bar{b}, \mathbf{d}) + \omega_{\mathcal{F}}(\mathbf{r}_0, \mathbf{d}) \right]. \end{aligned}$$

We have

$$\begin{aligned} \left| \mathcal{F}(\theta_2, \mathbb{L}(\bar{b}(\theta_2))) \right| &\leq \left| \mathcal{F}(\theta_2, \mathbb{L}(\bar{b}(\theta_2))) - \mathcal{F}(\theta_2, 0) \right| + \left| \mathcal{F}(\theta_2, 0) \right| \\ &\leq \mathcal{D}_{\mathcal{F}} \left| \mathbb{L}(\bar{b}(\theta_2)) \right| + \hat{\mathcal{F}} \\ &\leq \mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}. \end{aligned}$$

Also, we have

$$\begin{aligned} &\int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, \mathbb{L}(\theta_3)) d_{P,Q}\theta_3 \\ &= (P - Q)\theta_1 \sum_{n=0}^{\infty} \frac{Q^n}{P^{n+1}} \left(P\theta_1 - \frac{Q^{n+1}}{P^{n+1}}\theta_1 \right)_{P,Q}^{(\alpha-1)} \mathcal{U} \left(\frac{Q^n\theta_1}{P^{n+1}}, \mathbb{L} \left(\frac{Q^n\theta_1}{P^{n+1}} \right) \right) \end{aligned}$$

and

$$\left(P\theta_1 - \frac{Q^{n+1}}{P^{n+1}}\theta_1 \right)_{P,Q}^{(\alpha-1)} = \theta_1^{\alpha-1} \left(P - \frac{Q^{n+1}}{P^{n+1}} \right)_{P,Q}^{(\alpha-1)}.$$

Therefore,

$$\begin{aligned} &\int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, \mathbb{L}(\theta_3)) d_{P,Q}\theta_3 \\ &= (P - Q)\theta_1^{\alpha} \sum_{n=0}^{\infty} \frac{Q^n}{P^{n+1}} \left(P - \frac{Q^{n+1}}{P^{n+1}} \right)_{P,Q}^{(\alpha-1)} \mathcal{U} \left(\frac{Q^n\theta_1}{P^{n+1}}, \mathbb{L} \left(\frac{Q^n\theta_1}{P^{n+1}} \right) \right) \end{aligned}$$

and

$$\left| \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, \mathbb{L}(\theta_3)) d_{P,Q}\theta_3 - \int_0^{\theta_2} (P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, \mathbb{L}(\theta_3)) d_{P,Q}\theta_3 \right|$$

$$\leq (P - Q) \sum_{n=0}^{\infty} \frac{Q^n}{P^{n+1}} \left| \left(P - \frac{Q^{n+1}}{P^{n+1}} \right)_{P,Q}^{(\alpha-1)} \left| \theta_1^\alpha \mathcal{U} \left(\frac{Q^n \theta_1}{P^{n+1}}, \mathbb{L} \left(\frac{Q^n \theta_1}{P^{n+1}} \right) \right) - \theta_2^\alpha \mathcal{U} \left(\frac{Q^n \theta_2}{P^{n+1}}, \mathbb{L} \left(\frac{Q^n \theta_2}{P^{n+1}} \right) \right) \right| \right|.$$

We have

$$\left| \left(P - \frac{Q^{n+1}}{P^{n+1}} \right)_{P,Q}^{(\alpha-1)} \right| \leq 1$$

and since $\theta_1 \leq 1$ therefore

$$\left| \theta_1^\alpha \mathcal{U} \left(\frac{Q^n \theta_1}{P^{n+1}}, \mathbb{L} \left(\frac{Q^n \theta_1}{P^{n+1}} \right) \right) - \theta_2^\alpha \mathcal{U} \left(\frac{Q^n \theta_2}{P^{n+1}}, \mathbb{L} \left(\frac{Q^n \theta_2}{P^{n+1}} \right) \right) \right|$$

$$\leq \theta_1^\alpha \left| \mathcal{U} \left(\frac{Q^n \theta_1}{P^{n+1}}, \mathbb{L} \left(\frac{Q^n \theta_1}{P^{n+1}} \right) \right) - \mathcal{U} \left(\frac{Q^n \theta_1}{P^{n+1}}, \mathbb{L} \left(\frac{Q^n \theta_2}{P^{n+1}} \right) \right) \right|$$

$$+ \left| \theta_1^\alpha \mathcal{U} \left(\frac{Q^n \theta_1}{P^{n+1}}, \mathbb{L} \left(\frac{Q^n \theta_2}{P^{n+1}} \right) \right) - \theta_2^\alpha \mathcal{U} \left(\frac{Q^n \theta_2}{P^{n+1}}, \mathbb{L} \left(\frac{Q^n \theta_2}{P^{n+1}} \right) \right) \right|$$

$$\leq \psi_{\mathcal{U}} \left(\left| \mathbb{L} \left(\frac{Q^n \theta_1}{P^{n+1}} \right) - \mathbb{L} \left(\frac{Q^n \theta_2}{P^{n+1}} \right) \right| \right) + \mathcal{A}_{\mathbf{d}}$$

$$\leq \psi_{\mathcal{U}} (\omega(\mathbb{L}, \mathbf{d})) + \mathcal{A}_{\mathbf{d}},$$

where

$$\mathcal{A}_{\mathbf{d}} = \sup \left\{ \left| \theta_1^\alpha \mathcal{U}(\theta_4, \mathbb{L}) - \theta_2^\alpha \mathcal{U}(\theta_5, \mathbb{L}) \right| : \theta_1, \theta_2, \theta_4, \theta_5 \in I, |\theta_1 - \theta_2| \leq \mathbf{d}, |\theta_4 - \theta_5| \leq \mathbf{d}, \mathbb{L} \in [-\mathbf{r}_0, \mathbf{r}_0] \right\}.$$

Again

$$\left| \int_0^{\theta_1} (P\theta_1 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, \mathbb{L}(\theta_3)) d_{P,Q}\theta_3 - \int_0^{\theta_2} (P\theta_2 - Q\theta_3)_{P,Q}^{(\alpha-1)} \mathcal{U}(\theta_3, \mathbb{L}(\theta_3)) d_{P,Q}\theta_3 \right|$$

$$\leq (\psi_{\mathcal{U}} (\omega(\mathbb{L}, \mathbf{d})) + \mathcal{A}_{\mathbf{d}}) (P - Q) \sum_{n=0}^{\infty} \frac{Q^n}{P^{n+1}}$$

$$\leq \psi_{\mathcal{U}} (\omega(\mathbb{L}, \mathbf{d})) + \mathcal{A}_{\mathbf{d}}.$$

Therefore

$$I_4 \leq (\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}) (\psi_{\mathcal{U}} (\omega(\mathbb{L}, \mathbf{d})) + \mathcal{A}_{\mathbf{d}}).$$

Using the above inequalities we get

$$I_2 \leq \left\{ \psi_{\mathcal{U}}(\mathbf{r}_0) (\mathcal{D}_{\mathcal{F}} \omega(\mathbb{L} \ 0 \ \bar{b}, \mathbf{d}) + \omega_{\mathcal{F}}(\mathbf{r}_0, \mathbf{d})) + (\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}) (\psi_{\mathcal{U}} (\omega(\mathbb{L}, \mathbf{d})) + \mathcal{A}_{\mathbf{d}}) \right\} \frac{\mathcal{D}_{\eta}}{|\Gamma_{P,Q}(\alpha)|} + \psi_{\eta} (\omega(\mathbb{L} \ 0 \ \bar{a}, \mathbf{d})).$$

Now, from assumption (7) we have

$$\omega(\mathbb{L} \ 0 \ \bar{a}, \mathbf{d}) = \sup \{ |\mathbb{L}(\bar{a}(\theta)) - \mathbb{L}(\bar{a}(\theta_1))| : \theta, \theta_1 \in I, |\theta - \theta_1| \leq \mathbf{d} \}$$

$$\leq \sup \{ |\mathbb{L}(\hat{\theta}) - \mathbb{L}(\hat{\theta}_1)| : \hat{\theta}, \hat{\theta}_1 \in I, |\hat{\theta} - \hat{\theta}_1| \leq \psi_{\bar{a}}(\mathbf{d}) \}$$

$$= \omega(\mathbb{L}, \psi_{\bar{a}}(\mathbf{d})).$$

Similarly, from assumption (8) we have

$$\omega(\mathbb{L}, \mathbf{0}, \bar{\mathbf{b}}, \mathbf{d}) \leq \omega(\mathbb{L}, \psi_{\bar{\mathbf{b}}}(\mathbf{d})).$$

Then

$$I_2 \leq \left\{ \psi_{\mathcal{U}}(\mathbf{r}_0) (\mathcal{D}_{\mathcal{F}} \omega(\mathbb{L}, \psi_{\bar{\mathbf{b}}}(\mathbf{d})) + \omega_{\mathcal{F}}(\mathbf{r}_0, \mathbf{d})) + (\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}) (\psi_{\mathcal{U}}(\omega(\mathbb{L}, \mathbf{d})) + \mathcal{A}_{\mathbf{d}}) \right\} \frac{\mathcal{D}_{\eta}}{|\Gamma_{P,Q}(\alpha)|} + \psi_{\eta}(\omega(\mathbb{L}, \psi_{\bar{\mathbf{a}}}(\mathbf{d}))).$$

Therefore

$$\begin{aligned} \omega(\mathcal{T}\mathbb{L}, \mathbf{d}) &\leq \mathcal{D}(\eta, \mathbf{d}) + \psi_{\eta}(\omega(\mathbb{L}, \psi_{\bar{\mathbf{a}}}(\mathbf{d}))) \\ &\quad + \frac{\mathcal{D}_{\eta}}{|\Gamma_{P,Q}(\alpha)|} \left\{ \psi_{\mathcal{U}}(\mathbf{r}_0) (\mathcal{D}_{\mathcal{F}} \omega(\mathbb{L}, \psi_{\bar{\mathbf{b}}}(\mathbf{d})) + \omega_{\mathcal{F}}(\mathbf{r}_0, \mathbf{d})) + (\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}) (\psi_{\mathcal{U}}(\omega(\mathbb{L}, \mathbf{d})) + \mathcal{A}_{\mathbf{d}}) \right\}. \end{aligned}$$

Let \mathbf{M} be a non-empty subset of $\bar{\mathbf{B}}(0, \mathbf{r}_0)$. Then we have

$$\begin{aligned} \omega(\mathcal{T}\mathbf{M}, \mathbf{d}) &\leq \mathcal{D}(\eta, \mathbf{d}) + \psi_{\eta}(\omega(\mathbf{M}, \psi_{\bar{\mathbf{a}}}(\mathbf{d}))) \\ &\quad + \frac{\mathcal{D}_{\eta}}{|\Gamma_{P,Q}(\alpha)|} \left\{ \psi_{\mathcal{U}}(\mathbf{r}_0) (\mathcal{D}_{\mathcal{F}} \omega(\mathbf{M}, \psi_{\bar{\mathbf{b}}}(\mathbf{d})) + \omega_{\mathcal{F}}(\mathbf{r}_0, \mathbf{d})) + (\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}) (\psi_{\mathcal{U}}(\omega(\mathbf{M}, \mathbf{d})) + \mathcal{A}_{\mathbf{d}}) \right\}. \end{aligned}$$

We have

$$\lim_{\theta \rightarrow 0^+} \psi_{\bar{\mathbf{a}}}(\theta) = \lim_{\theta \rightarrow 0^+} \psi_{\bar{\mathbf{b}}}(\theta) = 0.$$

As $\mathbf{d} \rightarrow 0$, we get

$$\begin{aligned} \mu(\mathcal{T}\mathbf{M}) &\leq 0 + \psi_{\eta}(\omega(\mathbf{M})) \\ &\quad + \frac{\mathcal{D}_{\eta}}{|\Gamma_{P,Q}(\alpha)|} \left\{ \psi_{\mathcal{U}}(\mathbf{r}_0) (\mathcal{D}_{\mathcal{F}} \mu(\mathbf{M}) + 0) + (\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}) (\psi_{\mathcal{U}}(\mu(\mathbf{M})) + 0) \right\} \\ &= \psi_{\eta}(\mu(\mathbf{M})) + \frac{\mathcal{D}_{\eta}}{|\Gamma_{P,Q}(\alpha)|} \left\{ \psi_{\mathcal{U}}(\mathbf{r}_0) \mathcal{D}_{\mathcal{F}} \mu(\mathbf{M}) + (\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}) \psi_{\mathcal{U}}(\mu(\mathbf{M})) \right\} \\ &< \hat{\mathcal{L}}\mu(\mathbf{M}) + \mathcal{L}\mu(\mathbf{M}) + \mathcal{N}\mu(\mathbf{M}) \end{aligned}$$

where $\mathcal{L} = \frac{\mathcal{D}_{\eta}(\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}})}{|\Gamma_{P,Q}(\alpha)|}$ and $\mathcal{N} = \frac{\mathcal{D}_{\eta} \mathcal{D}_{\mathcal{F}} \psi_{\mathcal{U}}(\mathbf{r}_0)}{|\Gamma_{P,Q}(\alpha)|}$.

Therefore

$$\mu(\mathcal{T}\mathbf{M}) \leq \hat{\ell}(\mu(\mathbf{M})),$$

where $\hat{\ell}(\theta) = \hat{\mathcal{L}}\theta + \mathcal{L}\theta + \mathcal{N}\theta$ for all $\theta \geq 0$ and it can be observed that $\hat{\ell}(\theta) < \theta$ for all $\theta > 0$ and $\hat{\ell}(0) = 0$ by assumption (9).

Therefore, by Corollary 3.4 there exists at least one fixed point of \mathcal{T} in $\bar{\mathbf{B}}(0, \mathbf{r}_0)$, which is a solution for (1). \square

5. Illustrative example

Example 5.1. Consider the following (P, Q) -integral equation:

$$\mathbb{L}(\theta) = \frac{\theta^2}{36} + \frac{\mathbb{L}(\theta)}{3} + \left(\frac{\theta}{3} + \frac{\mathbb{L}(\theta)}{6} \right) \int_0^{\theta} (P\theta - Q\theta_1)_{P,Q}^{(\alpha-1)} \cdot \frac{\mathbb{L}(\theta_1)}{3 + \theta_1^2} d_{P,Q}\theta_1 \tag{6}$$

where $\theta \in I = [0, 1], \alpha > 1$ and $0 < Q < P \leq 1$.

Here,

$$\bar{a}(\theta) = \bar{b}(\theta) = \theta,$$

$$\eta(\theta, \mathbb{L}, \mathbb{L}') = \frac{\theta^2}{36} + \frac{\mathbb{L}}{3} + \mathbb{L}'\Gamma_{P,Q}(\alpha),$$

$$\mathcal{F}(\theta, \mathbb{L}) = \frac{\theta}{3} + \frac{\mathbb{L}}{6}$$

and

$$\mathcal{U}(\theta, \mathbb{L}) = \frac{\mathbb{L}}{3 + \theta^2},$$

where $\theta \in I$ and $\mathbb{L} \in C(I)$.

Assumption (1) is trivial. For all $\theta \in I$ and $\mathbb{L}, \mathbb{L}', Z, W \in \mathbb{R}$ we have

$$\begin{aligned} & |\eta(\theta, \mathbb{L}, \mathbb{L}') - \eta(\theta, Z, W)| \\ &= \left| \frac{\mathbb{L} - Z}{3} + (\mathbb{L}' - W)\Gamma_{P,Q}(\alpha) \right| \\ &\leq \frac{|\mathbb{L} - Z|}{3} + |\mathbb{L}' - W| |\Gamma_{P,Q}(\alpha)|. \end{aligned}$$

If we choose $\psi_\eta(\theta) = \frac{\theta}{3}$ where $\theta \geq 0$ and $\mathcal{D}_\eta = |\Gamma_{P,Q}(\alpha)|$, then

$$|\eta(\theta, \mathbb{L}, \mathbb{L}') - \eta(\theta, Z, W)| \leq \psi_\eta(|\mathbb{L} - Z|) + \mathcal{D}_\eta |\mathbb{L}' - W|.$$

Thus, assumption (2) is satisfied. Again, for all $\mathbb{L}, \mathbb{L}' \in C(I)$ we have

$$|\mathcal{U}(\theta, \mathbb{L}) - \mathcal{U}(\theta, \mathbb{L}')| = \left| \frac{\mathbb{L}}{3 + \theta^2} - \frac{\mathbb{L}'}{3 + \theta^2} \right| \leq \frac{1}{3 + \theta^2} |\mathbb{L} - \mathbb{L}'| \leq \frac{1}{3} |\mathbb{L} - \mathbb{L}'| = \psi_{\mathcal{U}}(|\mathbb{L} - \mathbb{L}'|),$$

where $\psi_{\mathcal{U}}(\theta) = \frac{\theta}{3}$ for all $\theta \geq 0$. Also, $\mathcal{U}(\theta, 0) = 0$ for all $\theta \in I$ and $\psi_{\mathcal{U}}(\theta) < \theta$ for all $\theta > 0$. Thus, assumption (4) is satisfied. For this example we have $\hat{\eta} = \frac{1}{36}$ and $\hat{\mathcal{F}} = \frac{1}{3}$. Now

$$\psi_\eta(\mathbf{r}_0) + \frac{\mathcal{D}_\eta \psi_{\mathcal{U}}(\mathbf{r}_0)}{|\Gamma_{P,Q}(\alpha)|} (\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}) + \hat{\eta} \leq \mathbf{r}_0$$

is equivalent to

$$\frac{\mathbf{r}_0}{3} + \frac{\mathbf{r}_0}{3} \left(\frac{\mathbf{r}_0}{6} + \frac{1}{3} \right) + \frac{1}{36} \leq \mathbf{r}_0,$$

i.e.,

$$2\mathbf{r}_0^2 - 20\mathbf{r}_0 + 1 \leq 0.$$

The inequality holds for $\mathbf{r}_0 \in \left[\frac{5-7\sqrt{2}}{2}, \frac{5+7\sqrt{2}}{2} \right]$. Since $\psi_\eta(\theta) = \frac{\theta}{3} < \theta$ for all $\theta > 0$ and ψ_η is continuous, therefore assumption (6) is satisfied.

Again since

$$|\bar{a}(\theta) - \bar{a}(\theta_1)| = |\theta - \theta_1| = \psi_{\bar{a}}(|\theta - \theta_1|),$$

where $\psi_{\hat{a}}(\theta) = \theta$ which is non-decreasing and $\lim_{\theta \rightarrow 0^+} \psi_{\hat{a}}(\theta) = 0$, therefore, assumption (7) is satisfied.

From

$$0 < \psi_{\mathcal{U}}(\mathbf{r}_0) < \frac{|\Gamma_{P,Q}(\alpha)|}{\mathcal{D}_{\mathcal{F}} \mathcal{D}_{\eta}},$$

we have $0 < \frac{\mathbf{r}_0}{3} < 6$, i.e., $0 < \mathbf{r}_0 < 18$, where $\mathcal{D}_{\mathcal{F}} = \frac{1}{6}$.

Again, from

$$\frac{\mathcal{D}_{\eta}}{|\Gamma_{P,Q}(\alpha)|} (\mathcal{D}_{\mathcal{F}} \mathbf{r}_0 + \hat{\mathcal{F}}) < 1$$

we get $\mathbf{r}_0 < 4$.

Since

$$\left[\frac{5 - 7\sqrt{2}}{2}, \frac{5 + 7\sqrt{2}}{2} \right] \cap (0, 4) \neq \emptyset.$$

Also, $\hat{\mathcal{L}} = \frac{1}{3}$, $\mathcal{L} = \frac{\mathbf{r}_0}{6} + \frac{1}{3}$, $\mathcal{N} = \frac{\mathbf{r}_0}{18}$ and

$$\hat{\mathcal{L}} + \mathcal{L} + \mathcal{N} = \frac{2}{3} + \frac{2\mathbf{r}_0}{9} < 1$$

for $\mathbf{r}_0 = \frac{1}{2}$. Therefore, by Theorem 4.1 the equation (6) has at least one solution in $C(I)$.

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