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The Bishop's property (β) for class A operators

Atsushi Uchiyama[a], Kotaro Tanahashi[a]

[a]*Tohoku Medical and Pharmaceutical University, Komatsushima 4-4-1, Aoba-ku, Sendai 981-8558, Japan*

Abstract. We say that an operator T on a Hilbert space $\mathcal{H}$ has the Bishop's property (β) if for an arbitrary open set $\mathcal{U} \subset \mathbb{C}$ and analytic functions $f_n : \mathcal{U} \to \mathcal{H}$ with $\|(T-z)f_n(z)\|$ converges to 0 uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$ then $\|f_n\|$ converges to 0 uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$. An operator T on $\mathcal{H}$ is called to be hyponormal if $T^*T \geq TT^*$, and T is called to be class A if $T^*T \leq |T^2|$. In this papaer, we give an elementary proof of the assertion that every hyponormal operator has the Bishop's property (β). And we show that every class A operator has the Bishop's property (β). Moreover, we also show a class A operator T is similar to a hyponormal operator if T is invertible, and hence T has the growth condition (G_1).

1. Introduction and Preliminaries

The class of hyponormal operators is one of the most important class of Hilbert space operators which has many interesting properties, the Bishop's property (β) is one of them. Many mathematicians extend this results to the classes including hyponormal operators, e.g., the class of p-hyponormal operators, w-hyponormal operators and class A operators. Some of them, it seems that there are several gaps in the proofs. We will show an elementary proof of Bishop's property (β) for hyponormal operator and also give the proofs of Bishop's property (β) for p-hyponormal, w-hyponormal, M-hyponormal and class A. Here, we say that an operator T is p-hyponormal for a $p > 0$ if $(T^*T)^p \geq (TT^*)^p$, w-hyponormal if the Aluthge transform $\widetilde{T} := |T|^{1/2}U|T|^{1/2}$ satisfies

$$|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|,$$

where $T = U|T|$ is the polar decomposition of T. T is called to be w-$A(s,t)$ operator for $s,t > 0$ if the generalized Aluthege transform $T(s,t) := |T|^sU|T|^t$ satisfies that

$$|T(s,t)|^{\frac{2t}{t+s}} \geq |T|^{2t} \text{ and } |T|^{2s} \geq |T(s,t)^*|^{\frac{2s}{t+s}}.$$

We also say that T is M-hyponormal for an $M \geq 1$ if $\|(T-\lambda)^*x\| \leq M\|(T-\lambda)x\|$ for arbitrary vector $x \in \mathcal{H}$ and complex number $\lambda \in \mathbb{C}$. We remark that if $p = 1$ for p-hyponormal or $M = 1$ for M-hyponormal then they are equal to hyponormal. These classes are invariant under scalar multiplication, i.e, if T is hyponormal, p-hyponormal, M-hyponormal or w-hyponormal then αT is also hyponormal, p-hyponormal, M-hyponormal or w-hyponormal respectively for all $\alpha \in \mathbb{C}$. However, only hyponormal and M-hyponormal

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Email addresses: uchiyama@tohoku-mpu.ac.jp (Atsushi Uchiyama), tanahasi@tohoku-mpu.ac.jp (Kotaro Tanahashi)

are translation invariant, i.e., if T is hyponormal or M-hyponormal then $T - \alpha$ is also hyponormal or M-hyponormal respectively for all $\alpha \in \mathbb{C}$. Translation invariant is the most important property to prove the Bishop's property (β). Though p-hyponormal and w-hyponormal are not translation invariant, the Aluthge transform of them or the second Aluthge transform of them are always hyponormal with the same spectrum. The following theorem is famous.

Theorem.(Aluthege) If T is p-hyponormal and $T = U|T|$ is its polar decomposition, then the Aluthge transform $\widetilde{T} = |T^{1/2}U|T|^{1/2}$ is hyponormal if $1/2 \le p \le 1$ and $(p + 1/2)$-hyponormal if $0 < p < 1/2$.

2. The Bishop's property (β) for hyponormal, M-hyponormal, p-hyponormal or w-hyponormal operators

In [6], the authors proved every hyponormal operator has the Bishop's condition (β). Here, an operator T on a Banach space X is said to have the Bishop's condition (β), if the operator

$$T_z : \mathcal{O}(\mathcal{U}, X) \to \mathcal{O}(\mathcal{U}, X), \quad f \to (z - T)f$$

is a topological monomorphism for each open set $\mathcal{U}$ in $\mathbb{C}$, where $\mathcal{O}(\mathcal{U}, X)$ denotes the Fréchet space of all X-valued analytic function on $\mathcal{U}$.

Many mathematicians extended this result to several classes of Hilbert space operators including the class of hyponormal operators, however, the definition of the Bishop's property (β) in these papers is as follows:

An operator T on a Hilbert space $\mathcal{H}$ is said to have the Bishop's property (β) if for an arbitrary open set $\mathcal{U} \subset \mathbb{C}$ and analytic functions $f_n : \mathcal{U} \to \mathcal{H}$ with $\|(T - z)f_n(z)\|$ converges to 0 uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$ then $\|f_n\|$ converges to 0 uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$.

In this section, we shall give an elementary proof of the assertion that every hyponormal operator has the Bishop's property (β) in this sense, and also extend this result to the classes of p-hyponormal, M-hyponormal and w-hyponormal operators.

Theorem 2.1. Every hyponormal operator has the Bishop's property (β).

Proof. Let T be an arbitrary hyponormal operator, $\mathcal{U} \subset \mathbb{C}$ be an arbitrary open subset and $\{f_n\}$ be a sequence of analytic functions $f_n : \mathcal{U} \to \mathcal{H}$ with $\|(T - z)f_n(z)\|$ converges to 0 uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$. We shall show that for any $z_0 \in \mathcal{U}$ there exists $\rho > 0$ such that $\|(T - z)f_n(z)\| \to 0$ uniformly on $\overline{N(z_0; \rho)}$ as $n \to \infty$, where $N(z_0; \rho) = \{z : |z - z_0| < \rho\}$ denotes the ρ-neighborhood of z_0.

Without loss of generality we may assume $z_0 = 0$, if necessary consider the hyponormal operator $S = T - z_0$, the functions $g_n(z) = f_n(z + z_0)$ and open set $\mathcal{U} - z_0 := \{z - z_0 : z \in \mathcal{U}\}$ instead of T, f_n $(n \ge 1)$ and $\mathcal{U}$ respectively.

Choose $r > 0$ such as $\mathbb{D}_r := \overline{N(0; r)} \subset \mathcal{U}$. Put $C_r := \{z \in \mathbb{C} : |z| = r\}$. Then, by the assumption

$$\|(T - z)f_n\|_{\mathbb{D}_r} = \sup\{\|(T - z)f_n(z)\| : z \in \mathbb{D}_r\} \to 0 \ (n \to \infty).$$

Since $\|(T - z)^* f_n(z)\| \le \|(T - z)f_n(z)\|$ for all $z \in \mathcal{U}$, it follows that

$$\begin{aligned}
\|(T - z)^* f_n\|_{\mathbb{D}_r} &\le \sup\{\|(T - z)f_n(z)\| \mid z \in \mathbb{D}_r\} \to 0 \ (n \to \infty) \\
\|(|T|^2 - r^2)f_n\|_{C_r} &= \|(|T|^2 - |z|^2)f_n(z)\|_{C_r} = \|T^*(T - z)f_n(z) + z(T - z)^* f_n(z)\|_{C_r} \\
&\le (\|T\| + r)\|(T - z)f_n\|_{\mathbb{D}_r} \to 0 \ (n \to \infty).
\end{aligned}$$

Let $0 < \rho < r$ be arbitrary and $z \in \mathbb{D}_\rho$. Then

$$(|T|^2 - r^2)f_n(z) = \frac{1}{2\pi i}\int_{C_r} \frac{(|T|^2 - r^2)f_n(\zeta)}{\zeta - z}\, d\zeta,$$

where integral is taken by the positive direction. And

$$\begin{aligned}
\|(|T|^2 - r^2)f_n(z)\| &\le \frac{1}{2\pi}\int_{C_r} \frac{\|(|T|^2 - r^2)f_n\|_{C_r}}{|\zeta - z|}\,|d\zeta| \\
&\le \frac{1}{2\pi}\int_{C_r} \frac{\|(|T|^2 - r^2)f_n\|_{C_r}}{|\zeta| - |z|}\,|d\zeta| \\
&\le \frac{1}{2\pi}\int_{C_r} \frac{\|(|T|^2 - r^2)f_n\|_{C_r}}{r - \rho}\,|d\zeta| \\
&= \frac{1}{2\pi}\cdot\frac{2\pi r}{r-\rho}\|(|T|^2 - r^2)f_n\|_{C_r} \\
&= \frac{r}{r-\rho}\|(|T|^2 - r^2)f_n\|_{C_r} \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

This implies that $\|(|T|^2 - r^2)f_n\|_{\mathbb{D}_\rho} \to 0$ as $n \to \infty$. Recall

$$\begin{aligned}
\|(|T|^2 - |z|^2)f_n(z)\|_{\mathbb{D}_\rho} &= \|T^*(T-z)f_n(z) + z(T-z)^* f_n(z)\|_{\mathbb{D}_\rho} \\
&\le (\|T^*\| + \rho)\|(T-z)f_n(z)\|_{\mathbb{D}_\rho} \to 0 \quad (n \to \infty),
\end{aligned}$$

it follows that $\|(r^2 - |z|^2)f_n(z)\|_{\mathbb{D}_\rho} \to 0$. Since $r^2 - |z|^2 \ge r^2 - \rho^2 > 0$, we have $\frac{1}{r^2 - |z|^2} \le \frac{1}{r^2-\rho^2}$ for all $z \in \mathbb{D}_\rho$ and

$$\|f_n\|_{\mathbb{D}_\rho} \le \frac{1}{r^2-\rho^2}\|(r^2 - |z|^2)f_n(z)\|_{\mathbb{D}_\rho} \to 0 \quad (n \to \infty).$$

This completes the proof.

Corollary 2.2. Let C be a class of operators such as

1) translation invariant, i.e., $T \in C \Longrightarrow T + z \in C$ for all $z \in \mathbb{C}$,

2) $\|(T-z)^*x\|^\alpha \le M\|(T-z)x\|$ $(x \in \mathcal{H},\ z \in \mathbb{C})$ for some $\alpha > 0, M > 0$.

Then T has the Bishop's property (β).

Hence, every M-hyponormal operator has the Bishop's property (β).

Proposion 2.3.

1) Every p-hyponormal operator has the Bishop's property (β).

2) Every w-hyponormal has the Bishop's property (β).

3) Every Class w-$A(s,t)$ operator with $s + t = 1$ has the Bishop's property (β).

Proof. 1) Let T be p-hyponormal for $1/2 \le p \le 1$. Then the Aluthge transform $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ is hyponormal, where $T = U|T|$ is the polar decomposition. Let $\mathcal{U}$ be an open subset of $\mathbb{C}$ and $f_n : \mathcal{U} \to \mathcal{H}$ be an analytic function which satisfies

$$\|(T-z)f_n(z)\| \to 0$$

uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$. Put $g_n(z) = |T|^{1/2}f_n(z)$. Then $(\widetilde{T}-z)g_n(z) = |T|^{1/2}(T-z)f_n(z)$ converges to 0 uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$ in the norm topology. Since $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ is hyponormal, it has the Bishop's property (β) and since g_n is analytic, $\|g_n(z)\| = \||T|^{1/2}f_n(z)\|$ and $\|Tf_n(z)\| = \|U|T|^{1/2}g_n(z)\|$ also converge to 0 uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$. Consequently, $zf_n(z) = Tf_n(z) - (T-z)f_n(z)$ converges to 0 uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$ in the norm topology. Let $K \subset \mathcal{U}$ be an arbitrary compact subset with $0 \notin K$. We show that $\|f_n\|_K \to 0$ uniformly as $n \to \infty$. It follows that $\delta = d(0,K) := \inf\{|z| : z \in K\} > 0$, $1/|z| \le 1/\delta$ for every $z \in K$ and

$$\|f_n\|_K \le 1/\delta\|zf_n(z)\|_K \to 0\ (n \to \infty).$$

Suppose $0 \in K$. Let $0 < r$ be arbitrary such as $\mathbb{D}_{2r} \subset \mathcal{U}$ and $K' := K \cap \{z \in \mathbb{C} \mid |z| \geq r\}$. Then $\|f_n\|_{C_{2r}}$ and $\|f_n\|_{K'}$ converge to 0 uniformly as $n \to \infty$ by the previous arguments. For $z \in \mathbb{D}_r$, $f_n(z) = \dfrac{1}{2\pi i}\displaystyle\int_{C_{2r}} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta$ and

$$\begin{aligned}
\|f_n(z)\|_{\mathbb{D}_r} &\leq \frac{1}{2\pi}\int_{C_{2r}} \frac{\|f_n\|_{C_{2r}}}{|\zeta - z|}\,|d\zeta| \leq \frac{1}{2\pi}\int_{C_{2r}} \frac{\|f_n\|_{C_{2r}}}{|\zeta| - |z|}\,|d\zeta| \\
&\leq \frac{1}{2\pi}\int_{C_{2r}} \frac{\|f_n\|_{C_{2r}}}{2r - |z|}\,|d\zeta| \\
&= \frac{1}{2\pi}\cdot\frac{2\pi\cdot 2r}{2r - r}\|f_n\|_{C_{2r}} \\
&\leq 2\|f_n\|_{C_{2r}} \to 0 \ (n \to \infty).
\end{aligned}$$

This implies that $\|f_n\|_{\mathbb{D}_r}$, $\|f_n\|_{\mathbb{D}_r \cup K'}$ and hence $\|f_n\|_K$ converge to 0 as $n \to \infty$. Hence, T has the Bishop's property (β).

Let T be p-hyponormal operator with $0 < p < 1/2$. Then Aluthge transform $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ is $(p + 1/2)$-hyponormal. Let $\mathcal{U}$ be an open subset of $\mathbb{C}$ and $f_n : \mathcal{U} \to \mathcal{H}$ be an analytic function which satisfies

$$\|(T - z)f_n(z)\| \to 0$$

uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$. And put $g_n(z) = |T|^{1/2}f_n(z)$. Then $(\widetilde{T} - z)g_n(z) = |T|^{1/2}(T - z)f_n(z)$ converges to 0 uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$ in the norm topology. Since $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ is $(p + 1/2)$-hyponormal, it has the Bishop's property (β) by the previous arguments. By using same argument as above, we have that $\|f_n\|$ converges to 0 uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$. Hence, T has the Bishop's property (β).

2) Let T be w-hyponormal. Then the Aluthge transform $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ is 1/2-hyponormal. And the proof is same as above.

3) Let T be a w-$A(s,t)$ operator with $s, t > 0$ and $s + t = 1$. Then the generalized Aluthge transform $T(s,t) = |T|^s U|T|^t$ is $\min(s,t)$-hyponormal. And the proof is almost same as the case of p-hyponormal operators, use $g_n(z) = |T|^s f_n(z)$ instead of $|T|^{1/2}f_n$.

3. Bishop's property (β) for class A operators

In this section, we give a transform from class A to hyponormal which does not change the spectrum and show that every class A operator has the Bishop's property (β). Also, we show that if T is an invertible class A operator then T is similar to a hyponormal operator.

Theorem(Douglas[5]) Let A, B be bounded linear operator on a Hilbert space $\mathcal{H}$. The followin are equivalent each other:

1) $A\mathcal{H} \subset B\mathcal{H}$

2) There exists a positive real number $k > 0$ such that

$$AA^* \leq kBB^*$$

3) $A = BC$ for some C with $\|C\| \leq \sqrt{k}$. Moreover, there uniquely exists C which satisfies the following three conditions (i), (ii), (iii).

(i) $\|C\| = \inf\{\sqrt{k} \mid AA^* \leq kBB^*\}$

(ii) $\ker C = \ker A$

(iii) $\operatorname{ran} C \subset (\ker B)^{\perp}$.

Let T be a class A operator on $\mathcal{H}$, i.e., T satisfies that $T^*T \leqq |T^2|$. Then, by Douglas's theorem there exists a unique contraction C which satisfies $T^* = |T^2|^{1/2}C$ and the above conditions (i), (ii), (iii).

We shall show that $S := |T^2|^{1/2}C^*$ is hyponormal, moreover if T is invertible then S is similar to T. We also show that T has the Bishop's property (β).

Theorem 3.1. Let T be a class A operator and $T^* = |T^2|^{1/2}C$ be same as above. Then

1) $S := |T^2|^{1/2}C^*$ satisfies $\sigma(S) = \sigma(T)$ and

$$S^*S \geq |T^2| \geq SS^* \text{ (especially, } S \text{ is hyponormal).}$$

2) T is similar to $S = |T^2|^{1/2}C^*$ if T is invertible.

3) T has the Bishop's property (β).

4) T satisfies the growth condition (G_1) if T is invertible, i.e. there exists a positive number $M > 0$ such that $\|(T-z)^{-1}\| \leq \dfrac{M}{d(z,\sigma(T))}$ $(z \in \rho(T))$.

Proof. 1) Since $|T|^2 = T^*T \leq |T^2|$, it follows that $\ker|T| = \ker|T^2|^{1/2} = \ker T$ and hence $\overline{\operatorname{ran} C} \subset (\ker|T^2|^{1/2})^{\perp} = (\ker T)^{\perp} = \overline{\operatorname{ran} T^*} = \overline{\operatorname{ran}|T|} = \overline{\operatorname{ran}|T^2|^{1/2}}$. Put P be the orthogonal projection onto $\overline{\operatorname{ran}|T^2|^{1/2}}$. Then $T^{*2} = |T^2|^{1/2}C|T^2|^{1/2}C$ and

$$|T^2|^2 = (T^{*2}T^2) = |T^2|^{1/2}C|T^2|^{1/2}CC^*|T^2|^{1/2}C^*|T^2|^{1/2}$$

imply that

$$\begin{aligned} &|T^2|^{1/2}\Big(|T^2| - C|T^2|^{1/2}CC^*|T^2|^{1/2}C^*\Big)|T^2|^{1/2} = 0, \\ &\quad P\Big(|T^2| - C|T^2|^{1/2}CC^*|T^2|^{1/2}C^*\Big)P = 0. \end{aligned}$$

Since $P|T^2| = |T^2|$ and $PC = C$, it follows that $|T^2|P = |T^2|$, $C^*P = C^*$ and

$$P\Big(|T^2| - C|T^2|^{1/2}CC^*|T^2|^{1/2}C^*\Big)P = |T^2| - C|T^2|^{1/2}CC^*|T^2|^{1/2}C^* = 0.$$

Hence,

$$|T^2| = C|T^2|^{1/2}CC^*|T^2|^{1/2}C^* \leq C|T^2|C^*.$$

Thus $S = |T^2|^{1/2}C^*$ satisfiles that

$$\begin{aligned} S^*S &= C|T^2|C^* \geq |T^2| \\ &\geq |T^2|^{1/2}C^*C|T^2|^{1/2} = SS^*. \end{aligned}$$

If T is invertible then C and C^* are invertible by $T^* = |T^2|^{1/2}C$. Thus $S = |T^2|^{1/2}C^*$ is also invertible. Conversely, if $S = |T^2|^{1/2}C^*$ is invertible then $\operatorname{ran}|T^2|^{1/2} = \mathcal{H}$ and $\ker|T^2|^{1/2} = (\operatorname{ran}|T^2|^{1/2})^{\perp} = \{0\}$. This implies that $|T^2|^{1/2}$ is one-to-one onto, so it is invertible. Hence, $C^* = |T^2|^{-1/2}S$ and $T = C^*|T^2|^{1/2}$ are also invertible.

Since $\sigma(S) \setminus \{0\} = \sigma(|T^2|^{1/2}C^*) \setminus \{0\} = \sigma(C^*|T^2|^{1/2}) \setminus \{0\} = \sigma(T) \setminus \{0\}$ and S is invertible if and only if T is invertible, $\sigma(S) = \sigma(T)$ holds.

2) If T is invertible then C and C^* are also invertible, and therefore $T = C^*|T^2|^{1/2} = C^*(|T^2|^{1/2}C^*)C^{*-1} = C^*SC^{*-1}$ is similar to S.

3) Let $\mathcal{U}$ be an open subset of $\mathbb{C}$ and $f_n : \mathcal{U} \to \mathcal{H}$ be analytic functions which satisfy

$$\|(T-z)f_n(z)\| \to 0$$

uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$. Since $S|T^2|^{1/2} = |T^2|^{1/2}\Big(C^*|T^2|^{1/2}\Big) = |T^2|^{1/2}T$, it follows that

$$\|(S-z)|T^2|^{1/2}f_n(z)\| = \|\,|T^2|^{1/2}(T-z)f_n(z)\,\| \leq \|\,|T^2|^{1/2}\| \cdot \|(T-z)f_n(z)\| \to 0 \ (n \to \infty)$$

uniformly on every compact subset of $\mathcal{U}$. Since S is hyponormal it has Bishop's property (β) and $|T^2|^{1/2}f_n(z)$ is analytic, $\|\,|T^2|^{1/2}f_n(z)\,\|$ converges to 0 uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$. This implies that $\| \, |T| f_n(z) \, \|$ also converges to 0 uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$ by the following inequality

$$\| \, |T| f_n(z) \, \|^2 = \langle T^* T f_n(z), f_n(z) \rangle \le \langle |T^2| f_n(z), f_n(z) \rangle = \| \, |T^2|^{1/2} f_n(z) \, \|^2.$$

Thus $\|z f_n(z)\| = \|T f_n(z) - (T-z) f_n(z)\| = \|U|T| f_n(z) - (T-z) f_n(z)\|$ converges to 0 uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$. By using same argument in the case of hyponormal operator, $\|f_n(z)\|$ also converges to 0 uniformly on every compact subset of $\mathcal{U}$ as $n \to \infty$. This completes the proof.

4) If T is invertible then C^* is also invertible. Since $(T-z)^{-1} = C^*(S-z)^{-1} C^{*-1}$,

$$\begin{aligned} \|(T-z)^{-1}\| &= \|C^*(S-z)^{-1} C^{*-1}\| \le \|C^*\| \|C^{*-1}\| \|(S-z)^{-1}\| \\ &\le \|C^{*-1}\| \frac{1}{d(z, \sigma(S))} \qquad (\because S \text{ is hyponormal}) \\ &= \frac{\|C^{*-1}\|}{d(z, \sigma(T))} \quad (z \in \rho(T)). \end{aligned}$$

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