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The average behaviour of a hybrid arithmetic function associated to cusp form coefficients over certain sparse sequence

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Abstract. Let f be a normalized primitive holomorphic cusp form of even integral weight for the full modular group $\Gamma = SL(2,\mathbb{Z})$, and let $\lambda_f(n)$, $\sigma(n)$ and $\varphi(n)$ be the nth normalized Fourier coefficient of the cusp form f, the sum-of-divisors function and the Euler totient function, respectively. In this paper, we investigate the asymptotic behaviour of the following summatory function

$$S_{j,b,c}(x) := \sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2\leq x\\ (a_1,a_2,a_3,a_4)\in\mathbb{Z}^4}} \lambda_f^j(n)\sigma^b(n)\varphi^c(n),$$

where $j \geq 2$ is any given integer. In a similar manner, we also establish other similar results related to normalized coefficients of the symmetric power L-functions associated to holomorphic cusp form f.

1. Introduction

The Fourier coefficients of automorphic forms are interesting and important research objects in modern number theory. Let H_k^* be the set of normalized primitive holomorphic cusp forms of even integral weight k for the full modular group $\Gamma = SL(2,\mathbb{Z})$, which consists of the eigenfunctions for the all Hecke operators T_n. The cusp form $f \in H_k^*$ at the cusp infinity admits the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z},$$

where we normalize $\lambda_f(1) = 1$ and $\lambda_f(n) \in \mathbb{R}$ is the nth normalized Fourier coefficient (Hecke eigenvalue) of f. It is well-known that the Hecke eigenvalue $\lambda_f(n)$ satisfies the Hecke relation

$$\lambda_f(n)\lambda_f(m) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right) \tag{1}$$

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for all integers $m, n \geq 1$. In 1974, Deligne [6] proved the celebrated Ramanujan-Petersson conjecture which asserts that

$$|\lambda_f(n)| \leq d(n), \tag{2}$$

where $d(n)$ is the classical divisor function.

Then the result (2) implies that for any prime number p, there exist two complex numbers $\alpha_f(p), \beta_f(p)$ such that

$$\lambda_f(p) = \alpha_f(p) + \beta_f(p), \qquad |\alpha_f(p)| = |\beta_f(p)| = \alpha_f(p)\beta_f(p) = 1. \tag{3}$$

The average behaviour of Hecke eigenvalues of normalized cuspidal Hecke eigenforms is an important topic in modern number theory. In 1927, Hecke [11] proved that

$$\sum_{n\leq x} \lambda_f(n) \ll x^{\frac{1}{2}}. \tag{4}$$

Later, the upper bound in (4) was improved by several authors (see e.g. [6, 13, 36]). In particular, Wu [44] has shown that

$$\sum_{n\leq x} \lambda_f(n) \ll x^{\frac{1}{3}} \log^{\rho} x,$$

where

$$\rho = \frac{102 + 7\sqrt{21}}{210}\Big(\frac{6 - \sqrt{21}}{5}\Big)^{\frac{1}{2}} + \frac{102 - 7\sqrt{21}}{210}\Big(\frac{6 + \sqrt{21}}{5}\Big)^{\frac{1}{2}} - \frac{33}{35} = -0.118\cdots.$$

In 1930s, Rankin [35] and Selberg [37] independently proved the following asymptotic formula

$$\sum_{n\leq x} \lambda_f^2(n) = c_f x + O(x^{3/5}) \tag{5}$$

for any $\varepsilon > 0$, where $c_f > 0$ is a constant depending on f. Very recently, the exponent in (5) has been improved to $\frac{3}{5} - \delta$ in place of $\frac{3}{5}$ by Huang [16], where $\delta \leq 1/560$. This remain the best possible result to date.

In 2015, Manski, Mayle and Zbacnik [31] considered the average behaviour of a hybrid arithmetic function and proved that

$$\sum_{n\leq x} d^a(n)\sigma^b(n)\varphi^c(n) = x^{b+c+1}P^*_{2^a-1}(\log x) + O\Big(x^{b+c+r_a+\varepsilon}\Big)$$

where $a, b, c \in \mathbb{R}$ and $\frac{1}{2} \leq r_a < 1$, here $P^*_l(t)$ denote the polynomial in t with degree l. Later, Li [29], Cui [5] investigated the average behaviour of the sum

$$\tilde{S}_{j,b,c}(x) := \sum_{n\leq x} \lambda_f^j(n)\sigma^b(n)\varphi^c(n) \tag{6}$$

for $1 \leq j \leq 6$. Very recently, Wei and Lao [45] refined the results of $\tilde{S}_{j,b,c}(x)$ for $j = 2, 4, 6$ and gave the asymptotic behaviour of $\tilde{S}_{j,b,c}(x)$ for $j = 7, 8$.

Let $\lambda_{\mathrm{sym}^j f}(n)$ denote the nth normalized coefficient of the Dirichlet expansion of the jth symmetric power L-function $L(\mathrm{sym}^j f, s)$. Fomenko [8] proved that

$$\sum_{n\leq x} \lambda_{\mathrm{sym}^2 f}(n) \ll x^{\frac{1}{2}}(\log x)^2.$$

Later, this sum has been studied by many authors (see e.g. [20, 25, 38]). The analogous cases for symmetric power lifting $\mathrm{sym}^j\pi_f$ for large j were considered by Lau and Lü [27], and Tang and Wu [43]. On the other hand, Fomenko [9] studied the sum of $\lambda^2_{\mathrm{sym}^2 f}(n)$. Later, this result was improved and generalized by some authors (see e.g. [14, 28, 39, 42]).

In [40], Sharma and Sankaranarayanan considered the asymptotic behaviour of the sum

$$U_{f,j}(x) := \sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2\leq x\\(a_1,a_2,a_3,a_4)\in\mathbb{Z}^4}} \lambda^j_{\mathrm{sym}^2 f}(n) \tag{7}$$

for $j = 2$ for $x \geq x_0$, where x_0 is sufficiently large. In fact, the authors established the following formula

$$U_{f,2}(x) = c_f x^2 + O_f\Big(x^{\frac{9}{5}+\varepsilon}\Big)$$

for any $\varepsilon > 0$, where $c_f > 0$ is some suitable constant depending on f. Later, Sharma and Sankaranarayanan [41] established the asymptotic formulae for $U_{f,j}(x)$ with $j = 3, 4$. In fact, they proved that

$$\begin{aligned} U_{f,3}(x) &= c_1 x^2 + O_f\Big(x^{\frac{27}{14}+\varepsilon}\Big),\\ U_{f,4}(x) &= c_2 x^2 \log x + O_f\Big(x^{\frac{160}{81}+\varepsilon}\Big), \end{aligned}$$

where c_1, c_2 are suitable effective constants depending on f. Very recently, the author [17] improved and generalized the above results by showing that

$$U_{f,j}(x) = c_j x^2 + O_f\Big(x^{2-\frac{60}{30(j+1)^2-13}+\varepsilon}\Big)$$

for $j \geq 2$, where c_j is some suitable constant which can be determined explicitly, and the author in the same paper also established some other similar results.

Inspired by the above results, in this paper the author firstly consider the summatory function

$$S_{j,b,c}(x) := \sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2\leq x\\(a_1,a_2,a_3,a_4)\in\mathbb{Z}^4}} \lambda^j_f(n)\sigma^b(n)\varphi^c(n), \tag{8}$$

where $j \geq 2$ is any given integer. More precisely, we establish the following result.

Theorem 1.1. *Let $b, c \in \mathbb{R}$ and $f \in H_k^*$, then for any $\varepsilon > 0$,*
(i) For $j = 2$, we have

$$S_{2,b,c}(x) = \widetilde{c}_f x^{b+c+2} + O\Big(x^{b+c+\frac{1264}{737}+\varepsilon}\Big),$$

where $\widetilde{c}_f$ is an effective constant given by

$$\widetilde{c}_f = \Big(-\frac{4}{b+c+2}\Big)\zeta(2)L(sym^2 f, 2)L(sym^2 f\otimes\tilde{\chi}_0, 1)\widetilde{U}(b+c+2),$$

here $\tilde{U}(b+c+2) \neq 0$ and $\tilde{\chi}_0$ is a nonprincipal Dirichlet character modulo 4.

(ii) Let $j = 2m \geq 4$ be an even integer, we have

$$S_{j,b,c}(x) = x^{b+c+2}P_{A_m-1}(\log x) + O\Big(x^{b+c+2-2^{-j+1}+\varepsilon}\Big),$$

where $P_{A_m-1}(t)$ is a polynomial of t which takes the form

$$\begin{aligned} P_{A_m-1}(t) &= \Big(\frac{8}{a+b+2}\Big)\frac{(-1/2)^{A_m}}{(A_m-1)!}\zeta(2)^{A_m}L(sym^{2m}f, 2)L(sym^{2m}f\otimes\tilde{\chi}_0, 1)\\ &\quad\times\prod_{1\leq r\leq m-1}L(sym^{2r}f, 2)^{C_m(r)}L(sym^{2r}f\otimes\tilde{\chi}_0, 1)^{C_m(r)}U_{j,b,c}(2)t^{A_m-1} + \ldots + c_f^*, \end{aligned}$$

and c_f^* is some suitable constant depending on f, and the constants A_m, $C_m(r)$ are given by (15), and $U_{j,b,c}(b+c+2) \neq 0$ and $\tilde{\chi}_0$ is a nonprincipal Dirichlet character modulo 4.

(iii) Let $j = 2m+1 \geq 3$ be an odd integer, we have

$$S_{j,b,c}(x) \ll x^{b+c+2-2^{-j+1}+\varepsilon}.$$

By using the similar argument, we also investigate the asymptotic behaviour of the following sum

$$S^*_{j,b,c}(x) := \sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2\leq x\\ (a_1,a_2,a_3,a_4)\in\mathbb{Z}^4}} \lambda^2_{\mathrm{sym}^j}(n)\sigma^b(n)\varphi^c(n), \tag{9}$$

where $j \geq 2$ is any given integer. We have the following theorem.

Theorem 1.2. *Let $b, c \in \mathbb{R}$ and $f \in H_k^*$, then for any $\varepsilon > 0$,*

$$S^*_{j,b,c}(x) = c_{f,j}x^{b+c+2} + O\Big(x^{b+c+2-\frac{60}{30(j+1)^2-13}+\varepsilon}\Big),$$

where $c_{f,j}$ is the constant given by

$$c_{f,j} = \Big(\frac{-4}{b+c+2}\Big)\zeta(2)\prod_{n=1}^{j} L(sym^{2n}f,2)L(sym^{2n}f\otimes\tilde{\chi}_0,1)H_{j,b,c}(b+c+2),$$

$H_{j,b,c}(b+c+2) \neq 0$ and $\tilde{\chi}_0$ is a nonprincipal Dirichlet character modulo 4.

The proofs are mainly based on the recent breakthrough of Newton and Thorne [32, 33] that $\mathrm{sym}^j f$ corresponds to a cuspidal automorphic representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ for all $j \geq 1$, along with some nice analytic properties of the associated L-functions, via classical Perron's formula applying for the generating L-functions.

Throughout the paper, for the sake of simplicity, we always work on the finite dimensional vector space H_k^*. And we also assume that $f \in H_k^*$ be a normalized cuspidal Hecke eigenform. Let $\varepsilon > 0$ denotes an arbitrarily small constant which may vary in different occurrence. The constant in O terms and $\ll$ terms depend at most on f, ε.

2. Auxiliary results

In this section, we review some relevant facts about the automorphic L-functions, and also collect some important lemmas which play an important role in the proof of the main results in this paper.

Let $f \in H_k^*$ be a Hecke eigenform. The jth symmetric power L-function attached to f is given by

$$L(\mathrm{sym}^j f, s) := \prod_p \prod_{m=0}^{j}\Big(1 - \frac{\alpha_f(p)^{j-m}\beta_f(p)^m}{p^s}\Big)^{-1} \tag{10}$$

for $\Re(s) > 1$. We can rewrite it as a Dirichlet series

$$\begin{aligned} L(\mathrm{sym}^j f, s) &= \prod_p\Big(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \ldots + \frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}} + \ldots\Big) \\ &:= \sum_{n=1}^{\infty}\frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s},\ \Re(s) > 1. \end{aligned} \tag{11}$$

It is well-known that $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function. In particular, for $j = 1$, we have $L(\mathrm{sym}^1 f, s) = L(f, s)$. And from (3), (10), (11) and the Hecke operator theory, we have

$$\lambda_f(p^j) = \sum_{m=0}^{j}\alpha_f(p)^{j-2m} = \lambda_{\mathrm{sym}^j f}(p), \qquad j \geq 1. \tag{12}$$

It is not hard to find that

$$|\lambda_{\mathrm{sym}^j f}(n)| \le d_{j+1}(n)$$

for all $j \ge 1$, where $d_\nu(n)$ denotes the ν-dimensional divisor function, which is defined as the number of ordered representations $n = n_1 \dots n_\nu$ with integers $n_1, \dots, n_\nu \ge 1$.

Let χ be a Dirichlet character modulo q. In a similar manner, we can also define the twisted jth symmetric power L-function by the Euler product representation with degree $j+1$

$$L(\mathrm{sym}^j f \otimes \chi, s) = \prod_p \prod_{m=0}^{j} (1 - \alpha_f(p)^{j-m} \beta_f(p)^m \chi(p) p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)\chi(n)}{n^s}$$

for $\Re(s) > 1$.

Let π_f be a automorphic cuspidal automorphic representations of $GL_2(\mathbb{A}_{\mathbb{Q}})$. It is well-known that an automorphic cuspidal representation π of $GL_2(\mathbb{A}_{\mathbb{Q}})$ is associated to a primitive form f, and hence an automorphic function $L(\pi_f, s)$ coincides with $L(f, s)$. Denote by $\mathrm{sym}^j \pi_f$ the jth symmetric power lift of π_f. For $2 \le j \le 8$, the automorphy of $\mathrm{sym}^j \pi_f$, was proved by a series of important works of Gelbart and Jacquet [10], Kim and Shahidi [22–24], Dieulefait [7], and Clozel and Thorne [2–4]. Very recently, Newton and Thorne [32, 33] showed that there exists a cuspidal automorphy representation of $GL_{j+1}(\mathbb{A}_{\mathbb{Q}})$ whose L-function equals $L(\mathrm{sym}^j f, s)$ for all $j \ge 1$. Hence for $j \ge 1$, the L-function $L(\mathrm{sym}^j f, s)$ is an entire function and satisfies a functional equation of certain Riemann-type with degree $j+1$.

We firstly state some basic definitions and analytic properties of general L-functions. Let $L(\phi, s)$ be a Dirichlet series (associated with the object ϕ) that admits an Euler product of degree $m \ge 1$, namely

$$L(\phi, s) = \sum_{n=1}^{\infty} \frac{\lambda_\phi(n)}{n^s} = \prod_{p<\infty} \prod_{j=1}^{m} \Big(1 - \frac{\alpha_\phi(p, j)}{p^s}\Big)^{-1},$$

where $\alpha_\phi(p, j), j = 1, 2, \cdots, m$ are the local parameters of $L(\phi, s)$ at a finite prime p. Suppose that this series and its Euler product are absolutely convergent for $\Re(s) > 1$. We denote the gamma factor by

$$L_\infty(\phi, s) = \prod_{j=1}^{m} \pi^{-\frac{s+\mu_\phi(j)}{2}} \Gamma\Big(\frac{s + \mu_\phi(j)}{2}\Big)$$

with local parameters $\mu_\phi(j), j = 1, 2, \cdots, m$ of $L(\phi, s)$ at ∞. The complete L-function $\Lambda(\phi, s)$ is defined as

$$\Lambda(\phi, s) = q(\phi)^{\frac{s}{2}} L_\infty(\phi, s) L(\phi, s),$$

where $q(\phi)$ is the conductor of $L(\phi, s)$. We assume that $\Lambda(\phi, s)$ admits an analytic continuation to the whole complex plane $\mathbb{C}$ and is holomorphic everywhere except for possible poles of finite order at $s = 0, 1$. Furthermore, we assume that it satisfies a functional equation of the Riemann-type

$$\Lambda(\phi, s) = \epsilon_\phi \Lambda(\tilde{\phi}, 1 - s),$$

where ϵ_ϕ is the root number with $|\epsilon_\phi| = 1$ and $\tilde{\phi}$ is the dual of ϕ such that $\lambda_{\tilde{\phi}}(n) = \overline{\lambda_\phi(n)}, L_\infty(\tilde{\phi}, s) = L_\infty(\phi, s)$ and $q(\tilde{\phi}) = q(\phi)$. We write $\phi \in S_e^\#$ if it is endowed with the above conditions. We say the L-function $L(\phi, s)$ satisfies the Ramanujan conjecture if $\lambda_\phi(n) \ll n^\varepsilon$ for any ε.

Here we state a very general theorem due to Lau and Lü [27].

Lemma 2.1. ([27, Lemma 2.4]) *Let $L(f, s)$ be a product of two L-functions $L_1, L_2 \in S_e^\#$ with $\deg L_i \ge 2, i = 1, 2$ and suppose that $L(f, s)$ satisfies the Ramanujan conjecture. Then for any $\varepsilon > 0$, we have*

$$\sum_{n \le x} \lambda_f(n) = M(x) + O\Big(x^{1 - \frac{2}{m} + \varepsilon}\Big),$$

where $M(x) = Res_{s=1}\{L(f,s)x^s/s\}$ *and* $m = \deg L$.

Now we introduce the truncated Perron's formula, which is given in Karatsuba and Voronin [21], pp. 334-336.

Lemma 2.2. *Suppose that the series* $f(s) = \sum_{n\geq 1} a_n n^{-s}$ *converges absolutely in* $\Re(s) > 1$, *and* $|a(n)| \leq A(n)$, *where* $A(n)$ *is a positive monotonously increasing function and*

$$\sum_{n\geq 1} |a_n| n^{-\sigma} = O\Big((\sigma-1)^{-\alpha}\Big)$$

for some $\alpha > 1$ *as* $\sigma \to 1^+$. *Then*

$$\sum_{n\leq x} a_n = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s)\frac{x^s}{s}ds + O\Big(\frac{x^b}{T(b-1)^{\alpha}}\Big) + O\Big(\frac{xA(2x)\log x}{T}\Big)$$

holds for any $1 < b \leq b_0, T \geq 2, x = N + \frac{1}{2}$ *(the constants in O-terms depend on* b_0*).*

Let

$$r_4(n) := \#\Big\{(n_1,n_2,n_3,n_4) \in \mathbb{Z}^4 \ : \ n_1^2+n_2^2+n_3^2+n_4^2 = n\Big\}.$$

We learn from [40, Sec.2] that $r_4(n) = 8r(n)$, where $r(n) = \sum_{d|n}\tilde{\chi}_0(d)d$ is multiplicative, and $\tilde{\chi}_0$ is a character modulo 4 given by

$$\tilde{\chi}_0(p^\nu) = \begin{cases} \chi_0(p^\nu), & \text{if } p > 2, \\ 3, & \text{if } p = 2, \end{cases}$$

and χ_0 is the principal character modulo 4. In particular, for any prime p, we have

$$r(p) = \sum_{d|p}\tilde{\chi}_0(d)d = 1 + p\tilde{\chi}_0(p)$$

and

$$r(p^2) = \sum_{d|p^2}\tilde{\chi}_0(d)d = 1 + p\tilde{\chi}_0(p) + p^2\tilde{\chi}_0(p^2).$$

It is well-known that $r(n) \ll n^{1+\varepsilon}$ for any $\varepsilon > 0$ (cf. [15, (1.1)]).

Let $j \geq 2$ be any fixed positive integer. Note that

$$\begin{aligned} S_{j,b,c}(x) &= \sum_{n\leq x}\lambda_f^j(n)\sigma^b(n)\varphi^c(n) \sum_{\substack{n=a_1^2+a_2^2+a_3^2+a_4^2 \\ (a_1,a_2,a_3,a_4)\in\mathbb{Z}^4}} 1 \\ &= \sum_{n\leq x}\lambda_f^j(n)\sigma^b(n)\varphi^c(n)r_4(n) = 8\sum_{n\leq x}\lambda_f^j(n)\sigma^b(n)\varphi^c(n)r(n), \end{aligned}$$

where $S_{j,b,c}(x)$ is defined as (8). In the similar manner, we also have

$$S_{j,b,c}^*(x) = 8\sum_{n\leq x}\lambda_{\mathrm{sym}^j f}^2(n)\sigma^b(n)\varphi^c(n)r(n),$$

where $S_{j,b,c}^*(x)$ is given by (9).

In order to give the asymptotic behaviour of sums via Perron's formula considered in this paper, we need the decompositions of the associated generating L-functions, which are illustrated as follows.

Lemma 2.3. *Let $b, c \in \mathbb{R}$ and $j \geq 2$ be any fixed integer, and let $f \in H_k^*$ be a Hecke eigenform. Define*

$$L_{j,b,c}(s) = \sum_{n=1}^{\infty} \frac{\lambda^2_{sym^j f}(n)\sigma^b(n)\varphi^c(n)r(n)}{n^s}.$$

Then

$$L_{j,b,c}(s) = G_{j,b,c}(s)H_{j,b,c}(s),$$

where

$$\begin{aligned} G_{j,b,c}(s) \quad &:= \quad \zeta(s-b-c)L(s-b-c-1, \tilde{\chi}_0) \\ &\times \prod_{n=1}^{j} L(sym^{2n} f, s-b-c)L(sym^{2n} f \otimes \tilde{\chi}_0, s-b-c-1), \end{aligned}$$

and $\tilde{\chi}_0$ is a Dirichlet character modulo 4. The function $H_{j,b,c}(s)$ admits a Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) \geq b+c+\frac{3}{2}+\varepsilon$ and $H_{j,b,c}(s) \neq 0$ for $\Re(s) = b+c+2$.

Proof The result follows from the similar argument as that of [45, Lemma 2.4] with some modifications. Since $\lambda^2_{\mathrm{sym}^j f}(n)\sigma^b(n)\varphi^c(n)r(n)$ is a multiplicative function, then for $\Re(s) \gg 1$, we have the Euler product

$$\begin{aligned} L_{j,b,c}(s) &= \prod_p f_{1,p}(s) = \prod_p \Big(1 + \sum_{k\geq 1} \frac{\lambda^2_{\mathrm{sym}^j f}(p^k)\sigma^b(p^k)\varphi^c(p^k)r(p^k)}{p^{ks}}\Big) \\ &= \prod_p \Big(1 + \frac{\lambda^2_{\mathrm{sym}^j f}(p)\sigma^b(p)\varphi^c(p)r(p)}{p^s} + \frac{\lambda^2_{\mathrm{sym}^j f}(p^2)\sigma^b(p^2)\varphi^c(p^2)r(p^2)}{p^{2s}} + \ldots\Big). \end{aligned}$$

In the half-plane $\Re(s) > b+c+2$, the p-th coefficient of the L-function determine the analytic properties of $L_{j,b,c}(s)$.

On taking $m = n = p^j$ in the Hecke relation (1),

$$\lambda_f^2(p^j) = \sum_{d|p^j} \lambda_f\Big(\frac{p^{2j}}{d^2}\Big) = 1 + \sum_{l=1}^{j} \lambda_f(p^{2l}).$$

Therefore,

$$\begin{aligned} \lambda^2_{\mathrm{sym}^j f}(p)r(p) &= \lambda_f^2(p^j)r(p) = \Big(1 + \sum_{l=1}^{j} \lambda_f(p^{2l})\Big)(1 + p\tilde{\chi}_0(p)) \\ &= \Big(1 + \sum_{l=1}^{j} \lambda_{\mathrm{sym}^{2l} f}(p)\Big)(1 + p\tilde{\chi}_0(p)). \end{aligned}$$

Let $s = \sigma + it$. Therefore,

$$\begin{aligned} f_{1,p}(s) &= 1 + \frac{\Big(1 + \sum_{l=1}^{j} \lambda_{\mathrm{sym}^{2l} f}(p)\Big)(p+1)^b(p-1)^c(1 + p\tilde{\chi}_0(p))}{p^s} \\ &+ \frac{\lambda^2_{\mathrm{sym}^j f}(p^2)(p^2+p+1)^b(p^2-p)^c(1 + p\tilde{\chi}_0(p) + p^2\tilde{\chi}_0(p^2))}{p^{2s}} + \cdots \\ &= 1 + \frac{\Big(1 + \sum_{l=1}^{j} \lambda_{\mathrm{sym}^{2l} f}(p)\Big)}{p^{s-b-c}} + \frac{\Big(1 + \sum_{l=1}^{j} \lambda_{\mathrm{sym}^{2l} f}(p)\Big)\tilde{\chi}_0(p)}{p^{s-b-c-1}} \\ &+ O\Big(p^{2(b+c+1-\sigma)} + p^{(b+c-\sigma)}\Big). \end{aligned}$$

Hence,

$$
\begin{aligned}
L_{j,b,c}(s) &= \prod_p f_{1,p}(s) = \prod_p \Big(1 + \frac{\big(1+\sum_{l=1}^{j}\lambda_{\mathrm{sym}^{2l}f}(p)\big)}{p^{s-b-c}} + \frac{\big(1+\sum_{l=1}^{j}\lambda_{\mathrm{sym}^{2l}f}(p)\big)\tilde{\chi}_0(p)}{p^{s-b-c-1}} \\
&\quad + O\Big(p^{2(b+c+1-\sigma)} + p^{(b+c-\sigma)}\Big)\Big) \\
&= \prod_p \Big(1 + \frac{1}{p^{s-b-c}} + \sum_{l=1}^{j}\frac{\lambda_{\mathrm{sym}^{2l}f}(p)}{p^{s-b-c}} + \frac{\tilde{\chi}_0(p)}{p^{s-b-c-1}} + \sum_{l=1}^{j}\frac{\lambda_{\mathrm{sym}^{2l}f}(p)\tilde{\chi}_0(p)}{p^{s-b-c-1}} \\
&\quad + O\Big(p^{2(b+c+1-\sigma)} + p^{(b+c-\sigma)}\Big)\Big) \\
&:= \zeta(s-b-c)L(s-b-c-1,\tilde{\chi}_0) \\
&\quad \times \prod_{n=1}^{j} L(\mathrm{sym}^{2n}f, s-b-c)L(\mathrm{sym}^{2n}f\otimes\tilde{\chi}_0, s-b-c-1)H_{j,b,c}(s),
\end{aligned}
$$

where the Dirichlet series $H_{j,b,c}(s)$ converges absolutely and uniformly in the half-plane $\Re(s) \geq b + c + \frac{3}{2} + \varepsilon$ and $H_{j,b,c}(s) \neq 0$ with $\Re(s) = b + c + 2$. □

Lemma 2.4. *Let $b, c \in \mathbb{R}$ and $j \geq 2$ be any fixed integer, and let $f \in H_k^*$ be a Hecke eigenform. Define*

$$
L^*_{2,b,c}(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^2(n)\sigma^b(n)\varphi^c(n)r(n)}{n^s}.
$$

Then

$$
\begin{aligned}
L^*_{2,b,c}(s) &= \zeta(s-b-c)L(s-b-c-1,\tilde{\chi}_0)L(sym^2 f, s-b-c) \\
&\quad \times L(sym^2 f\otimes\tilde{\chi}_0, s-b-c-1)\tilde{U}(s),
\end{aligned}
$$

where $\tilde{\chi}_0$ is a Dirichlet character modulo 4. The function $\tilde{U}(s)$ admits a Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) \geq b + c + \frac{3}{2} + \varepsilon$ and $\tilde{U}(s) \neq 0$ for $\Re(s) = b + c + 2$.

Proof This can be proved by following the similar argument as that of Lemma 2.3, since

$$
\begin{aligned}
\lambda_f(p)^2 r(p) &= (1+\lambda_f(p^2))(1+\tilde{\chi}_0(p)p) \\
&= (1+\lambda_{\mathrm{sym}^2 f}(p))(1+\tilde{\chi}_0(p)p). \qquad \square
\end{aligned}
$$

Lemma 2.5. *Let $b, c \in \mathbb{R}$ and $j \geq 3$ be any fixed integer, and let $f \in H_k^*$ be a Hecke eigenform. Define*

$$
L^*_{j,b,c}(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^j(n)\sigma^b(n)\varphi^c(n)r(n)}{n^s}.
$$

Then

$$
L^*_{j,b,c}(s) = G^*_{j,b,c}(s)U_{j,b,c}(s),
$$

where

$$
\begin{aligned}
G^*_{2m,b,c}(s) &= \zeta(s-b-c)^{A_m}L(sym^{2m}f, s-b-c)L(s-b-c-1,\tilde{\chi}_0)^{A_m} \\
&\quad \times L(sym^{2m}f\otimes\tilde{\chi}_0, s-b-c-1) \\
&\quad \times \prod_{1\leq r\leq m-1} L(sym^{2r}f, s-b-c)^{C_m(r)}L(sym^{2r}f\otimes\tilde{\chi}_0, s-b-c-1)^{C_m(r)}
\end{aligned}
\tag{13}
$$

for $j = 2m$, and

$$\begin{aligned} G^*_{2m+1,b,c}(s) &= L(f, s-b-c)^{B_m} L(sym^{2m+1} f, s-b-c) L(f \otimes \tilde{\chi}_0, s-b-c-1)^{B_m} \\ &\quad \times L(sym^{2m+1} f \otimes \tilde{\chi}_0, s-b-c-1) \\ &\quad \times \prod_{1 \le r \le m-1} L(sym^{2r+1} f, s-b-c)^{D_m(r)} L(sym^{2r+1} f \otimes \tilde{\chi}_0, s-b-c-1)^{D_m(r)} \end{aligned} \tag{14}$$

for $j = 2m+1$, and $A_m, B_m, C_m(r), D_m(r)$ are suitable constants, and

$$A_m = \frac{(2m)!}{m!(m+1)!}, \quad C_m(r) = \frac{(2m)!(2r+1)}{(m-r)!(m+r+1)!}, \quad m \ge 1. \tag{15}$$

and $\tilde{\chi}_0$ is a Dirichlet character modulo 4. The function $U_{j,b,c}(s)$ admits a Dirichlet series which converges absolutely and uniformly in the half-plane $\Re(s) \ge b + c + \frac{3}{2} + \varepsilon$ and $U_{j,b,c}(s) \ne 0$ for $\Re(s) = b + c + 2$.

Proof Since $\lambda_f^j(n)\sigma^b(n)\varphi^c(n)r(n)$ is a multiplicative function, then for $\Re(s) \gg 1$ we have the Euler product

$$L^*_{j,b,c}(s) = \prod_p f_{2,p}(s) = \prod_p \Big(1 + \sum_{k \ge 1} \frac{\lambda_f^j(p^k)\sigma^b(p^k)\varphi^c(p^k)r(p^k)}{p^{ks}}\Big).$$

We only give the proof for the case $j = 2m$, since the other case can be handled in the similar approach. From the result of Lau-Lü [27, Lemma 7.1], one has

$$\lambda_f^j(p)r(p) = \Big(A_m + \sum_{1 \le r \le m-1} C_m(r)\lambda_{\mathrm{sym}^{2r} f}(p) + \lambda_{\mathrm{sym}^{2m} f}(p)\Big)(1 + \tilde{\chi}_0(p)p).$$

where $A_m, C_m(r)$ are defined by (15).

Let $s = \sigma + it$. Therefore,

$$\begin{aligned} f_{2,p}(s) &= 1 + \frac{\lambda_f^j(p)(p+1)^b(p-1)^c(1 + p\tilde{\chi}_0(p))}{p^s} \\ &\quad + \frac{\lambda_f^j(p^2)(p^2+p+1)^b(p^2-p)^c(1 + p\tilde{\chi}_0(p) + p^2\tilde{\chi}_0(p^2))}{p^{2s}} + \cdots \\ &= 1 + \frac{\lambda_f^j(p)}{p^{s-b-c}} + \frac{\lambda_f^j(p)\tilde{\chi}_0(p)}{p^{s-b-c-1}} + O\Big(p^{2(b+c+1-\sigma)} + p^{(b+c-\sigma)}\Big). \end{aligned}$$

Then,

$$\begin{aligned} L^*_{j,b,c}(s) &= \prod_p \Big(1 + \frac{\lambda_f^j(p)}{p^{s-b-c}} + \frac{\lambda_f^j(p)\tilde{\chi}_0(p)}{p^{s-b-c-1}} + O\Big(p^{2(b+c+1-\sigma)} + p^{(b+c-\sigma)}\Big)\Big) \\ &= \prod_p \Big(1 + \frac{\Big(A_m + \sum_{1 \le r \le m-1} C_m(r)\lambda_{\mathrm{sym}^{2r} f}(p) + \lambda_{\mathrm{sym}^{2m} f}(p)\Big)(1 + \tilde{\chi}_0(p)p)}{p^{s-b-c}} \\ &\quad + O\Big(p^{2(b+c+1-\sigma)} + p^{(b+c-\sigma)}\Big)\Big) \\ &:= \zeta(s-b-c)^{A_m} L(\mathrm{sym}^{2m} f, s-b-c) L(s-b-c-1, \tilde{\chi}_0)^{A_m} \\ &\quad \times L(\mathrm{sym}^{2m} f \otimes \tilde{\chi}_0, s-b-c-1) \\ &\quad \times \prod_{1 \le r \le m-1} L(\mathrm{sym}^{2r} f, s-b-c)^{C_m(r)} L(\mathrm{sym}^{2r} f \otimes \tilde{\chi}_0, s-b-c-1)^{C_m(r)} U_{j,b,c}(s), \end{aligned}$$

where the Dirichlet series $U_{j,b,c}(s)$ converges absolutely and uniformly in the half-plane $\Re(s) \geq b+c+\frac{3}{2}+\varepsilon$ and $U_{j,b,c}(s) \neq 0$ with $\Re(s) = b+c+2$. □

Lemma 2.6. *For $\varepsilon > 0$, one has*

$$\int_1^T \left|\zeta\Big(\frac{1}{2}+it\Big)\right|^{12} dt \ll T^{2+\varepsilon} \tag{16}$$

uniformly for $T \geq 1$.

Proof This result follows from [12]. □

Lemma 2.7. *For any $\varepsilon > 0$, we have*

$$\begin{aligned} \zeta(\sigma+it) &\ll \Big(1+|t|\Big)^{\max\{\frac{13}{42}(1-\sigma),0\}+\varepsilon}, \\ L(sym^2 f, \sigma+it) &\ll \Big(1+|t|\Big)^{\max\{\frac{6}{5}(1-\sigma),0\}+\varepsilon}, \end{aligned}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 2$ and $|t| \geq 1$.

Proof The first result is the new breakthrough of Bourgain [1], and the second result follows from the recent work of Lin, Nunes and Qi [30, Corollary 1.2]. □

From above we observe that the symmetric power L-functions $L(\text{sym}^j f, s)$, $j \geq 1$ and its twisted L-functions are general L-functions in the sense of Perelli [34]. For the general L-functions, we have the following averaged or individual convexity bounds.

Lemma 2.8. *Assume that $\mathfrak{L}(s)$ is a general L-function of degree m. Then*

$$\int_T^{2T} \left|\mathfrak{L}(\sigma+it)\right|^2 dt \ll T^{m(1-\sigma)+\varepsilon}, \tag{17}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$ and $T \geq 1$, and

$$\mathfrak{L}(\sigma+it) \ll \Big(1+|t|\Big)^{\max\{\frac{m}{2}(1-\sigma),0\}+\varepsilon} \tag{18}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1+\varepsilon$ and $|t| \geq 1$.

Proof This follows the results of Perelli's mean value theorem and convexity bounds for general L-functions in [34]. □

3. Proof of Theorem 1.1

We firstly consider the case $j = 2$. By applying Lemma 2.2, we obtain

$$\sum_{n\leq x} \lambda_f^2(n)\sigma^b(n)\varphi^c(n)r_4(n) = \frac{8}{2\pi i}\int_{b+c+2+\varepsilon-iT}^{b+c+2+\varepsilon+iT} L^*_{2,b,c}(s)\frac{x^s}{s}ds + O\Big(\frac{x^{b+c+2+\varepsilon}}{T}\Big),$$

where $s = \sigma+it$ and $1 \leq T \leq x$ is some parameter to be chosen later.

By shifting the line of integration to the parallel segment with $\Re(s) = b+c+\frac{3}{2}+\varepsilon$ and invoking Cauchy's residue theorem, by Lemma 2.4 we have

$$\begin{aligned} &\sum_{n\leq x} \lambda_f^2(n)\sigma^b(n)\varphi^c(n)r_4(n) = 8\text{Res}_{s=b+c+2}\Big\{L^*_{2,b,c}(s)\frac{x^s}{s}\Big\} \\ &+\frac{8}{2\pi i}\Big\{\int_{b+c+\frac{3}{2}+\varepsilon-iT}^{b+c+\frac{3}{2}+\varepsilon+iT} + \int_{b+c+2+\varepsilon-iT}^{b+c+\frac{3}{2}+\varepsilon-iT} + \int_{b+c+\frac{3}{2}+\varepsilon+iT}^{b+c+1+\varepsilon+iT}\Big\}L^*_{2,b,c}(s)\frac{x^s}{s}ds + O\Big(\frac{x^{b+c+1+\varepsilon}}{T}\Big) \\ &:= \tilde{c}_f x^{b+c+2} + I_1^* + I_2^* + I_3^* + O\Big(\frac{x^{b+c+2+\varepsilon}}{T}\Big), \end{aligned} \tag{19}$$

where $\tilde{c}_f$ is some suitable constant depending on f and various associated L-function. The function $L^*_{2,b,c}(s)$ has only simple pole at $s = b+c+2$ coming from the factor $\zeta(s-b-c-1)$. This contributes a residue, which is $\tilde{c}_f x^{b+c+2}$ that can be determined by the following calculations.

From [40, Sec.3], we learn that

$$L(s-b-c-1,\tilde{\chi}_0) = \Big(1-\frac{3}{2^{s-b-c-1}}\Big)^{-1}\Big(1-\frac{1}{2^{s-b-c-1}}\Big)^{2}\zeta(s-b-c-1).$$

Similarly, using the similar argument, for $i \geq 1$ we have

$$\begin{aligned} L(\mathrm{sym}^i f\otimes\tilde{\chi}_0, s-b-c-1) &= \sum_{n=1}^{\infty}\frac{\lambda_{\mathrm{sym}^i f}(n)\tilde{\chi}_0(n)}{n^{s-b-c-1}} \\ &= \Big(1-\frac{\lambda_{\mathrm{sym}^i f}(2)\tilde{\chi}_0(2)}{2^{s-b-c-1}}\Big)\prod_{p>2}\Big(1-\frac{\lambda_{\mathrm{sym}^i f}(p)\tilde{\chi}_0(p)}{p^{s-b-c-1}}\Big)^{-1} \\ &= \Big(1-\frac{3\lambda_{\mathrm{sym}^i f}(2)}{2^{s-b-c-1}}\Big)^{-1}\Big(1-\frac{\lambda_{\mathrm{sym}^i f}(2)}{2^{s-b-c-1}}\Big)\prod_{p}\Big(1-\frac{\lambda_{\mathrm{sym}^i f}(p)\chi_0(p)}{p^{s-b-c-1}}\Big)^{-1} \\ &= \Big(1-\frac{3\lambda_{\mathrm{sym}^i f}(2)}{2^{s-b-c-1}}\Big)^{-1}\Big(1-\frac{\lambda_{\mathrm{sym}^i f}(2)}{2^{s-b-c-1}}\Big)L(\mathrm{sym}^i f\otimes\chi_0, s-b-c-1) \\ &= \Big(1-\frac{3\lambda_{\mathrm{sym}^i f}(2)}{2^{s-b-c-1}}\Big)^{-1}\Big(1-\frac{\lambda_{\mathrm{sym}^i f}(2)}{2^{s-b-c-1}}\Big)^{2}L(\mathrm{sym}^i f, s-b-c-1), \end{aligned} \tag{20}$$

since

$$\begin{aligned} L(\mathrm{sym}^i f\otimes\chi_0, s-b-c-1) &= \sum_{n=1}^{\infty}\frac{\lambda_{\mathrm{sym}^i f}(n)\chi_0(n)}{n^{s-b-c-1}} \\ &= \prod_{\substack{p\\(p,4)=1}}\Big(1-\frac{\lambda_{\mathrm{sym}^i f}(p)}{p^{s-b-c-1}}\Big)^{-1} \\ &= \Big(1-\frac{\lambda_{\mathrm{sym}^i f}(2)}{2^{s-b-c-1}}\Big)\prod_{p}\Big(1-\frac{\lambda_{\mathrm{sym}^i f}(p)}{p^{s-b-c-1}}\Big)^{-1} \\ &= \Big(1-\frac{\lambda_{\mathrm{sym}^i f}(2)}{2^{s-b-c-1}}\Big)L(\mathrm{sym}^i f, s-b-c-1). \end{aligned}$$

More precisely,

$$\begin{aligned} \widetilde{c_f} &= 8\lim_{s\to(b+c+2)}\Big\{(s-(b+c+2))\frac{L^*_{2,b,c}(s)}{s}\Big\} \\ &= \Big(-\frac{4}{b+c+2}\Big)\zeta(2)L(\mathrm{sym}^2 f,2)L(\mathrm{sym}^2 f\otimes\tilde{\chi}_0,1)\widetilde{U}(b+c+2). \end{aligned}$$

Now we need to handle these three terms I_1^*, I_2^* and I_3^*. For the integrals over the horizontal segments I_2^* and I_3^*, by Lemma 2.7 and (20), we have

$$\begin{aligned} I_2^*+I_3^* &\ll \int_{\frac12+\varepsilon}^{1+\varepsilon}\big|\zeta(\sigma+iT)L(\mathrm{sym}^2 f,\sigma+iT)\big|x^{b+c+1+\sigma}T^{-1}d\sigma \\ &\ll x^{b+c+1}\int_{\frac12+\varepsilon}^{1+\varepsilon}\big|\zeta(\sigma+iT)L(\mathrm{sym}^2 f,\sigma+iT)\big|x^{\sigma}T^{-1}d\sigma. \\ &\ll x^{b+c+1}\max_{\frac12+\varepsilon\leq\sigma\leq1+\varepsilon}x^{\sigma}T^{(\frac{13}{42}+\frac{6}{5})(1-\sigma)+\varepsilon}T^{-1} \\ &\ll \frac{x^{b+c+2+\varepsilon}}{T}+x^{b+c+\frac32+\varepsilon}T^{-\frac{103}{420}+\varepsilon}. \end{aligned} \tag{21}$$

For I_1^*, by Lemma 2.7, we have

$$\begin{aligned} I_1^* &\ll x^{b+c+\frac{3}{2}+\varepsilon}\int_1^T \left|\zeta\Big(\frac{1}{2}+it\Big)L\Big(\mathrm{sym}^2 f,\frac{1}{2}+it\Big)\right|t^{-1}dt + x^{b+c+\frac{3}{2}+\varepsilon} \\ &\ll x^{b+c+\frac{3}{2}+\varepsilon}\log T \max_{1\le T_1\le T}\left\{\frac{1}{T_1}\max_{T_1/2\le t\le T_1} T_1\left|\zeta\Big(\frac{1}{2}+it\Big)L\Big(\mathrm{sym}^2 f,\frac{1}{2}+it\Big)\right|\right\} + x^{b+c+\frac{3}{2}+\varepsilon} \\ &\ll x^{b+c+\frac{3}{2}+\varepsilon}T^{\frac{13}{42}\times\frac{1}{2}+\frac{6}{5}\times\frac{1}{2}+\varepsilon} \\ &\ll x^{b+c+\frac{3}{2}+\varepsilon}T^{\frac{317}{420}+\varepsilon}. \end{aligned} \tag{22}$$

Therefore, from (19), (21) and (22), we have

$$\sum_{n\le x}\lambda_f^2(n)\sigma^b(n)\varphi^c(n)r_4(n) = \widetilde{c}_f x^{b+c+2} + O\Big(\frac{x^{b+c+2+\varepsilon}}{T}\Big) + O\Big(x^{b+c+\frac{3}{2}+\varepsilon}T^{\frac{317}{420}+\varepsilon}\Big). \tag{23}$$

On taking $\frac{x^{b+c+2}}{T} = x^{b+c+\frac{3}{2}}T^{\frac{317}{420}}$, i.e., $T = x^{\frac{210}{737}}$, we get

$$\sum_{n\le x}\lambda_f^2(n)\sigma^b(n)\varphi^c(n)r_4(n) = \widetilde{c}_f x^{b+c+2} + O\Big(x^{b+c+\frac{1264}{737}+\varepsilon}\Big).$$

This proves the case $j = 2$ in Theorem 1.1.

Now we consider the case for $j \ge 3$ by applying Lemma 2.1. For $j = 2m$, by (13) in Lemma 2.5, we see that

$$G_{j,b,c}^*(s) := \sum_{n=1}^{\infty}\frac{b(n)}{n^s}$$

is an L-function of degree 2^{j+1} which can be analytically extended to the whole complex plane except for poles at $s = b + c + 1$ and $s = b + c + 2$ of order A_m.

By combining a modification of Lemma 2.1 and the proof in [40, Sect. 3], i.e., by shifting the line of integration from $\Re(s) = b + c + 2 + \varepsilon$ to the parallel line with $\Re(s) = b + c + \frac{3}{2} + \varepsilon$, we get

$$\sum_{n\le x} b(n) = x^{b+c+2}P''_{A_m-1}(\log x) + O\Big(x^{b+c+2-2^{-j+1}+\varepsilon}\Big),$$

where the main term $x^{b+c+2}P''_{A_m-1}(\log x)$ is given by

$$x^{b+c+2}P''_{A_m-1}(\log x) = \mathrm{Res}_{s=b+c+2}\Big\{G_{2m,b,c}^*(s)\frac{x^s}{s}\Big\}.$$

Here $P''_{\omega}(t)$ denotes a polynomial in t of degree ω, and A_m is defined as (15).

By Lemma 2.5 we know that

$$\lambda_f^j(n)\sigma^b(n)\varphi^c(n)r(n) = \sum_{n=uv} c(v)b(u)$$

satisfying the relations

$$\sum_{v\ge1}|c(v)|v^{-\sigma} \ll_\sigma 1 \quad \text{for any} \quad \sigma > b + c + \frac{3}{2}. \tag{24}$$

Hence, we can obtain

$$\begin{aligned}&\sum_{n\leq x}\lambda_f^j(n)\sigma^b(n)\varphi^c(n)r_4(n)\\ =\ & 8\sum_{n\leq x}\lambda_f^j(n)\sigma^b(n)\varphi^c(n)r(n)\\ =\ & 8\sum_{v\leq x}c(v)\sum_{u\leq x/v}b(u)\\ =\ & 8\sum_{v\leq x}c(v)\Big(\Big(\frac{x}{v}\Big)^{b+c+2}P''_{A_m-1}\Big(\log\big(x/v\big)\Big)+O\Big((x/v)^{b+c+2-2^{-j+1}+\varepsilon}\Big)\Big)\\ =\ & x^{b+c+2}P_{A_m-1}(\log x)+O\Big(x^{b+c+2-2^{-j+1}+\varepsilon}\Big)\end{aligned}$$

by noting the relation (24). Here $P_\omega(t)$ is another polynomial in t of degree ω.

Now we compute the explicit form of the coefficients of the polynomial $P_{A_m-1}(\log x)$. From (13), we have

$$\begin{aligned}G^*_{2m,b,c}(s) \;=\;& \zeta(s-b-c)^{A_m}L(\mathrm{sym}^{2m}f,s-b-c)\\ &\times\Big(\Big(1-\frac{3}{2^{s-b-c-1}}\Big)^{-1}\Big(1-\frac{1}{2^{s-b-c-1}}\Big)^2\zeta(s-b-c-1)\Big)^{A_m}\\ &\times L(\mathrm{sym}^{2m}f\otimes\tilde{\chi}_0,s-b-c-1)\\ &\times\prod_{1\leq r\leq m-1}L(\mathrm{sym}^{2r}f,s-b-c)^{C_m(r)}L(\mathrm{sym}^{2r}f\otimes\tilde{\chi}_0,s-b-c-1)^{C_m(r)}.\end{aligned}$$

From [19, (1.11)], we learn that $\zeta(s)$ has the Laurent expansion at the simple pole $s=1$:

$$\zeta(s)=\frac{1}{s-1}+\gamma_0+\sum_{n=1}^{\infty}\gamma_j(s-1)^j,$$

where γ_j, $j=0,1,\ldots$ are suitable constants. In particular, $\gamma:=\gamma_0$ is Euler's constant.

By the Leibniz's rule and the method for the computation of residue at the pole $s=b+c+2$ for integrand function, we have

$$\begin{aligned}&x^{b+c+2}P_{A_m-1}(\log x)=8\mathrm{Res}_{s=b+c+2}\Big\{L^*_{j,b,c}(s)\frac{x^s}{s}\Big\}\\ =\ &\Big(\frac{8}{a+b+2}\Big)\frac{(-1/2)^{A_m}}{(A_m-1)!}\zeta(2)^{A_m}L(\mathrm{sym}^{2m}f,2)L(\mathrm{sym}^{2m}f\otimes\tilde{\chi}_0,1)\\ &\times\prod_{1\leq r\leq m-1}L(\mathrm{sym}^{2r}f,2)^{C_m(r)}L(\mathrm{sym}^{2r}f\otimes\tilde{\chi}_0,1)^{C_m(r)}U_{j,b,c}(b+c+2)x^{b+c+22}(\log x)^{A_m-1}\\ &+\ldots+c_f^*x^{b+c+2},\end{aligned}$$

where c_f^* is some suitable constant depending on f and various associated L-functions.

For $j=2m+1$, by (14) in Lemma 2.5, we know that the L-function $G^*_{2m+1,b,c}(s)$ can be extended to the whole complex plane as an entire function and satisfies certain Riemann type functional equation. By Lemma 2.1 and arguing as above, we can derive the desired conclusion.

We complete the proof of Theorem 1.1.

4. Proof of Theorem 1.2

We can argue similarly as that of Theorem 1.1 with some modifications. Let $j\geq 2$ be any fixed integer. By applying Lemma 2.2, we obtain

$$\sum_{n\leq x}\lambda^2_{\mathrm{sym}^j}(n)\sigma^b(n)\varphi^c(n)r_4(n)=\frac{8}{2\pi i}\int_{b+c+2+\varepsilon-iT}^{b+c+2+\varepsilon+iT}L_{j,b,c}(s)\frac{x^s}{s}ds+O\Big(\frac{x^{b+c+2+\varepsilon}}{T}\Big),$$

where $s = \sigma + it$ and $1 \le T \le x$ is some parameter to be specified later.

By shifting the line of integration to the parallel segment with $\Re(s) = b+c+\frac{3}{2}+\varepsilon$ and invoking Cauchy's residue theorem, by Lemma 2.3 we have

$$\begin{aligned}
&\sum_{n\le x} \lambda^2_{\mathrm{sym}^j}(n)\sigma^b(n)\varphi^c(n)r_4(n)\\
= \quad & 8\mathrm{Res}_{s=b+c+2}\Big\{L_{j,b,c}(s)\frac{x^s}{s}\Big\}\\
&+\frac{8}{2\pi i}\Big\{\int_{b+c+\frac{3}{2}+\varepsilon-iT}^{b+c+\frac{3}{2}+\varepsilon+iT}+\int_{b+c+2+\varepsilon-iT}^{b+c+\frac{3}{2}+\varepsilon-iT}+\int_{b+c+\frac{3}{2}+\varepsilon+iT}^{b+c+2+\varepsilon+iT}\Big\}L^*_{j,b,c}(s)\frac{x^s}{s}ds+O\Big(\frac{x^{b+c+2+\varepsilon}}{T}\Big)\\
:= \quad & c_{f,j}x^{b+c+2}+J_1+J_2+J_3+O\Big(\frac{x^{b+c+2+\varepsilon}}{T}\Big),
\end{aligned} \tag{25}$$

here $c_{f,j}$ is some suitable constant depending on f and various associated L-functions. In fact,

$$\begin{aligned}
c_{f,j} &= 8\lim_{s\to(b+c+2)}\Big\{(s-(b+c+2))\frac{L_{j,b,c}(s)}{s}\Big\}\\
&= \Big(\frac{-4}{b+c+2}\Big)\zeta(2)\prod_{n=1}^{j}L(\mathrm{sym}^{2n}f,2)L(\mathrm{sym}^{2n}f\otimes\tilde{\chi}_0,1)H_j(b+c+2).
\end{aligned}$$

Next, we evaluate these three integrals J_1, J_2 and J_3. Let

$$\widetilde{G}_j(s)=\zeta(s)L(\mathrm{sym}^2f,s)L_{3,j}(s),$$

where

$$L_{3,j}(s):=\prod_{n=2}^{j}L(\mathrm{sym}^{2n}f,s)$$

be an L-function of degree $(j+1)^2-4$.

For J_1, by Lemmas 2.6-2.7, (17) and (20), along with Hölder's inequality, we have

$$\begin{aligned}
J_1 \ll\ & x^{b+c+\frac{3}{2}+\varepsilon}\log T\max_{1\le T_1\le T}\Big\{T_1^{-1}\int_{T_1/2}^{T_1}\Big|\widetilde{G}_j\Big(\frac{1}{2}+it\Big)\Big|dt\Big\}+x^{b+c+\frac{3}{2}+\varepsilon}\\
\ll\ & x^{b+c+\frac{3}{2}+\varepsilon}\log T\max_{1\le T_1\le T}\Big\{\frac{1}{T_1}\Big(\int_{T_1/2}^{T_1}\Big|\zeta\Big(\frac{1}{2}+it\Big)\Big|^{12}dt\Big)^{\frac{1}{12}}\\
&\times\Big(\int_{T_1/2}^{T_1}\Big|L\Big(\mathrm{sym}^2f,\frac{1}{2}+it\Big)\Big|^2dt\Big)^{\frac{1}{2}}\Big(\int_{T_1/2}^{T_1}\Big|L_{3,j}\Big(\frac{1}{2}+it\Big)\Big|^{\frac{12}{5}}dt\Big)^{\frac{5}{12}}\Big\}+x^{b+c+\frac{3}{2}+\varepsilon}\\
\ll\ & x^{b+c+\frac{3}{2}+\varepsilon}\log T\max_{1\le T_1\le T}\Big\{\frac{1}{T_1}\Big(\int_{T_1/2}^{T_1}\Big|\zeta\Big(\frac{1}{2}+it\Big)\Big|^{12}dt\Big)^{\frac{1}{12}}\Big(\int_{T_1/2}^{T_1}\Big|L_{3,j}\Big(\frac{1}{2}+it\Big)\Big|^2dt\Big)^{\frac{1}{2}}\\
&\times\Big(\max_{T_1/2\le t\le T_1}\Big|L\Big(\mathrm{sym}^2f,\frac{1}{2}+it\Big)\Big|^{2/5}\cdot\int_{T_1/2}^{T_1}\Big|L\Big(\mathrm{sym}^2f,\frac{1}{2}+it\Big)\Big|^2dt\Big)^{\frac{5}{12}}\Big\}+x^{b+c+\frac{3}{2}+\varepsilon}\\
\ll\ & x^{b+c+\frac{3}{2}+\varepsilon}T^{-1+2\times\frac{1}{12}+((j+1)^2-4)\times\frac{1}{2}\times\frac{1}{2}+(\frac{2}{5}\times\frac{6}{5}\times\frac{1}{2}+3\times\frac{1}{2})\times\frac{5}{12}+\varepsilon}\\
\ll\ & x^{b+c+\frac{3}{2}+\varepsilon}T^{\frac{1}{4}(j+1)^2-\frac{133}{120}+\varepsilon}.
\end{aligned} \tag{26}$$

For the integrals over the horizontal segments J_2 and J_3, by Lemma 2.7 and (18), along with (20), we have

$$\begin{aligned} J_2 + J_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^{\sigma+b+c+1} \left| \zeta(\sigma+it) \prod_{n=1}^{j} L(\mathrm{sym}^{2n} f, \sigma+it) \right| T^{-1} d\sigma \\ &\ll \max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} x^{\sigma+b+c+1} T^{(\frac{13}{42}+\frac{6}{5}+\frac{1}{2}((j+1)^2-4))(1-\sigma)+\varepsilon} T^{-1} \\ &\ll \frac{x^{b+c+2+\varepsilon}}{T} + x^{b+c+\frac{3}{2}+\varepsilon} T^{\frac{1}{4}(j+1)^2-\frac{523}{420}+\varepsilon}. \end{aligned} \tag{27}$$

Combining (25)-(27), we obtain

$$\sum_{n\le x} \lambda^2_{\mathrm{sym}^j}(n)\sigma^b(n)\varphi^c(n)r_4(n) = c_{f,j}x^{b+c+2} + O\Big(\frac{x^{b+c+2+\varepsilon}}{T}\Big) + O\Big(x^{b+c+\frac{3}{2}+\varepsilon} T^{\frac{1}{4}(j+1)^2-\frac{133}{120}+\varepsilon}\Big).$$

On taking $\frac{x^{b+c+2}}{T} = x^{b+c+\frac{3}{2}} T^{\frac{1}{4}(j+1)^2-\frac{133}{120}}$, i.e., $T = x^{\frac{60}{30(j+1)^2-13}}$, we get

$$\sum_{n\le x} \lambda^2_{\mathrm{sym}^j}(n)\sigma^b(n)\varphi^c(n)r_4(n) = c_{f,j}x^{b+c+2} + O\Big(x^{b+c+2-\frac{60}{30(j+1)^2-13}+\varepsilon}\Big).$$

This completes the proof of Theorem 1.2.

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