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Extension of the generalized n-strong Drazin inverse

Dijana Mosić[a,*], Honglin Zou[b], Long Wang[c]

[a]*Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia*
[b]*College of Basic Science, Zhejiang Shuren University, Hangzhou 310015, China*
[c]*School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, China*

Abstract. The aim of this paper is to present an extension of the generalized n-strong Drazin inverse for Banach algebra elements using a g–Drazin invertible element rather than a quasinilpotent element in the definition of the generalized n-strong Drazin inverse. Thus, we introduce a new class of generalized inverses which is a wider class than the classes of the generalized n-strong Drazin inverse and the extended generalized strong Drazin inverses. We prove a number of characterizations for this new inverse and some of them are based on idempotents and tripotents. Several generalizations of Cline's formula are investigated for the extension of the generalized n-strong Drazin inverse.

1. Introduction

In this paper, $\mathcal{A}$ represents a complex Banach algebra with unit 1. For $a \in \mathcal{A}$, the symbols $\sigma(a)$, $r(a)$ and acc $\sigma(a)$, respectively, will denote the spectrum of a, the spectral radius of a and the set of all accumulation points of $\sigma(a)$. The sets of all invertible, nilpotent and quasinilpotent elements of $\mathcal{A}$, respectively, are denoted by $\mathcal{A}^{-1}$, $\mathcal{A}^{nil}$ and $\mathcal{A}^{qnil}$, respectively. Recall that $a \in \mathcal{A}^{qnil}$ if $\sigma(a) = \{0\}$. We use $\sigma_{\mathcal{B}}(a)$ for the spectrum of $a \in \mathcal{B}$ with respect to $\mathcal{B}$, where $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, and also $a^{-1}_{\mathcal{B}}$ will be the inverse of a in $\mathcal{B}$. It is known that $a \in \mathcal{A}$ is tripotent (or idempotent) if $a^3 = a$ (or $a^2 = a$).

Koliha [9] presented the definition of the g–Drazin inverse for elements of Banach algebras, extending the notion of the Drazin inverse [7]. An element $a \in \mathcal{A}$ is g–Drazin invertible if there exists an element $x \in \mathcal{A}$ which satisfies

$$xax = x, \qquad ax = xa \qquad \text{and} \qquad a - axa \in \mathcal{A}^{qnil}.$$

In this case, x is called the g–Drazin inverse of a (or Koliha–Drazin inverse of a) [9]. The g–Drazin inverse of a is unique, if it exists, and denoted by a^d. Recall that a^d exists if and only if $0 \notin$ acc $\sigma(a)$. The g–Drazin inverse of a doubly commutes with a, that is, a^d commutes with every element of $\mathcal{A}$ that commutes with a (that is, $ab = ba$ implies $a^d b = ba^d$) [9]. We use $\mathcal{A}^d$ to denote the set of all g–Drazin invertible elements of $\mathcal{A}$.

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* Corresponding author: Dijana Mosić
Email addresses: dijana@pmf.ni.ac.rs (Dijana Mosić), honglinzou@163.com (Honglin Zou), lwangmath@yzu.edu.cn (Long Wang)

Since the g–Drazin inverse of a quasinilpotent element is equal to zero, we have that $\mathcal{A}^{qnil} \subseteq \mathcal{A}^d$. For $a \in \mathcal{A}^d$, $a^\pi = 1 - aa^d$ is the spectral idempotent of a corresponding to the set $\{0\}$. More properties of the g–Drazin inverse were given in [4–6].

When $a - axa \in \mathcal{A}^{nil}$ in the definition of the g–Drazin inverse, then $a^d = a^D$ is the Drazin inverse of a. The group inverse of a, denoted by $a^\#$, is a special case of the Drazin inverse for which $a = axa$ is satisfied. The sets of all Drazin invertible and group invertible elements of $\mathcal{A}$ are denoted by $\mathcal{A}^D$ and $\mathcal{A}^\#$, respectively.

One significant property of the Drazin inverse was presented by Cline [2] as: if $ab \in \mathcal{A}^D$, then $ba \in \mathcal{A}^D$ and $(ba)^D = b((ab)^D)^2 a$. This so-called Cline's formula was generalized to many generalized inverses under different assumptions [10, 20].

The concept of a strong Drazin inverse was introduced by Wang [19]. As a generalization of the strong Drazin inverse, a generalized strong Drazin inverse was defined in [12] for Banach algebra elements. For $n \in \mathbb{N}$, the generalized n-strong Drazin inverse was presented in [13] for elements of rings as a new class of generalized inverses which extends the generalized strong Drazin inverse from [12] and the generalized Hirano inverse presented in [18].

Let $n \in \mathbb{N}$. An element $a \in \mathcal{A}$ is called generalized n-strongly Drazin invertible (or *gns*–Drazin invertible) if there exists an element $x \in \mathcal{A}$ such that

$$xax = x, \quad ax = xa \quad \text{and} \quad a^n - ax \in \mathcal{A}^{qnil}.$$

The *gns*–Drazin inverse x of a is unique if it exists [13]. If $a^n - ax \in \mathcal{A}^{nil}$ in the above definition, then x is the n-strong Drazin inverse (or *ns*–Drazin inverse) of a. For $n = 1$, the *gns*–Drazin inverse becomes the generalized strong Drazin inverse [12]. In the case that $n = 2$, the *gns*–Drazin inverse reduces to the generalized Hirano inverse [18]. Some interesting results about (generalized) strong Drazin inverse, (generalized) Hirano inverse and (generalized) n-strongly Drazin inverse can be found in [1, 8, 17, 21, 22].

Using an adequate g–Drazin invertible element rather than a quasinilpotent element in the definition of g–Drazin inverse, the concept of the g–Drazin inverse was extended in [11]. An element $a \in \mathcal{A}$ is called extended g–Drazin invertible (or *eg*–Drazin invertible) if there exists an element $x \in \mathcal{A}$ such that

$$xax = x, \quad xa = ax \quad \text{and} \quad a - axa \in \mathcal{A}^d.$$

In this case, x is an extended g–Drazin inverse (or *eg*–Drazin inverse) of a and it is not uniquely determined. Notice that a is extended g–Drazin invertible if and only if a is g–Drazin invertible [11]. Replacing $a - axa \in \mathcal{A}^d$ with $a - axa \in \mathcal{A}^D$ in the definition of *eg*–Drazin inverse, x is an extended Drazin inverse (or *e*–Drazin inverse) of a. The sets of all *eg*–Drazin invertible and *e*–Drazin invertible elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^{ed}$ and $\mathcal{A}^{eD}$, respectively.

The notion of the generalized strong Drazin inverse was generalized in [16] using the condition $a - ax \in \mathcal{A}^d$ instead of $a - ax \in \mathcal{A}^{qnil}$ in its definition. An element $a \in \mathcal{A}$ is called extended *gs*–Drazin invertible (or *egs*–Drazin invertible) if there exists an element $x \in \mathcal{A}$ such that

$$xax = x, \quad xa = ax \quad \text{and} \quad a - ax \in \mathcal{A}^d.$$

In this case, x is an extended *gs*–Drazin inverse (or *egs*–Drazin inverse) of a. If $a - ax \in \mathcal{A}^D$ in this definition, x is an extended s–Drazin inverse (or *es*–Drazin inverse) of a. The symbols $\mathcal{A}^{esd}$ and $\mathcal{A}^{esD}$, respectively, represent the sets of all *egs*–Drazin invertible and *es*–Drazin invertible elements of $\mathcal{A}$.

Motivated by previous research papers about g–Drazin inverse, generalized strong Drazin inverse and their extensions, our aim is to present a wider class of the *gns*–Drazin inverse and *egs*–Drazin inverse. Precisely, we introduce an extended *gns*–Drazin inverse replacing the condition $a^n - ax \in \mathcal{A}^{qnil}$ in the definition of the *gns*–Drazin inverse with $a^n - ax \in \mathcal{A}^d$. In this way, we define a new class of generalized inverses for elements of Banach algebra. We present different kinds of equivalent conditions for an element to be extended *gns*–Drazin invertible. Some of these characterizations contain idempotent, and some of them involve tripotents. We prove that an element $a \in \mathcal{A}$ is extended *gns*–Drazin invertible if and only if a is *eg*–Drazin invertible if and only if a is g–Drazin invertible. Several extensions of Cline's formula for extended *gns*–Drazin inverse are proposed. Applying these results, we can get new characterizations for *eg*–Drazin invertible and g–Drazin invertible elements. At the end, we define weighted extended *gns*–Drazin invertible and weighted extended *ns*–Drazin invertible Banach algebra elements.

2. Extended *gns*–Drazin inverse

The new class of generalized inverses in a Banach algebra is defined in this section by replacing the condition $a^n - ax \in \mathcal{A}^{qnil}$ in the definition of *gns*–Drazin inverse with $a^n - ax \in \mathcal{A}^d$. In this way, we propose an extension of *gns*–Drazin inverse, i.e. a wider class of generalized inverses.

Definition 2.1. *For $n \in \mathbb{N}$, an element $a \in \mathcal{A}$ is called extended gns–Drazin invertible (or egns–Drazin invertible) if there exists an element $x \in \mathcal{A}$ such that*

$$xax = x, \quad xa = ax \quad \text{and} \quad a^n - ax \in \mathcal{A}^d.$$

In this case, x is an extended gns–Drazin inverse (or egns–Drazin inverse) of a.

Obviously, for $n = 1$, the *egns*–Drazin inverse reduces to the *egs*–Drazin inverse.

In particular, when $a^n - ax \in \mathcal{A}^D$, an extended *gns*–Drazin inverse becomes an extended *ns*–Drazin inverse.

Definition 2.2. *For $n \in \mathbb{N}$, an element $a \in \mathcal{A}$ is called extended ns–Drazin invertible (or ens–Drazin invertible) if there exists an element $x \in \mathcal{A}$ such that*

$$xax = x, \quad xa = ax \quad \text{and} \quad a^n - ax \in \mathcal{A}^D.$$

In this case, x is an extended ns–Drazin inverse (or ens–Drazin inverse) of a.

Denote by $\mathcal{A}^{n,esd}$ (resp. $\mathcal{A}^{n,esD}$) the set of all *egns*–Drazin (resp. *ens*–Drazin) invertible elements of $\mathcal{A}$.

Lemma 2.3. *If $a \in \mathcal{A}^{n,esd}$, then $a \in \mathcal{A}^{ed}$. Furthermore, an egns–Drazin inverse of a is an eg–Drazin inverse of a.*

Proof. Assume that x is an *egns*–Drazin inverse of a. Then $1 - ax$ is an idempotent and so $1 - ax \in \mathcal{A}^\# \subseteq \mathcal{A}^d$. Notice that $a^n - ax \in \mathcal{A}^d$ and, applying [9, Theorem 5.5], $(a - a^2x)^n = a^n(1 - ax) = (a^n - ax)(1 - ax) \in \mathcal{A}^d$. By [10, Corollary 2.2], we deduce that $a - a^2x \in \mathcal{A}^d$ and x is an *eg*–Drazin inverse of a. □

Using Lemma 2.3, we can note that the similar result holds for *ens*–Drazin invertible elements.

Corollary 2.4. *If $a \in \mathcal{A}^{n,esD}$, then $a \in \mathcal{A}^{eD}$. Furthermore, an ens–Drazin inverse of a is an e–Drazin inverse of a.*

According to Lemma 2.3 and [11, Theorem 2.2], we conclude that $\mathcal{A}^{n,esd} \subseteq \mathcal{A}^{ed} = \mathcal{A}^d$. In the following theorem, we show that $\mathcal{A}^{n,esd} = \mathcal{A}^{ed} = \mathcal{A}^d$ and give more characterizations of *egns*–Drazin invertible elements.

Theorem 2.5. *Let $a \in \mathcal{A}$ and $n, m \in \mathbb{N}$. The following statements are equivalent:*

(i) *a is egns–Drazin invertible;*

(ii) *a is eg–Drazin invertible;*

(iii) *a is g–Drazin invertible;*

(iv) *there exists an idempotent $p \in \mathcal{A}$ commuting with a such that $ap \in (p\mathcal{A}p)^{-1}$ and $a^n - p \in \mathcal{A}^d$;*

(v) *there exists an idempotent $p \in \mathcal{A}$ commuting with a such that $ap + 1 - p \in \mathcal{A}^{-1}$ and $a^n - p \in \mathcal{A}^d$;*

(vi) *there exists an idempotent $p \in \mathcal{A}$ commuting with a such that $ap \in (p\mathcal{A}p)^{-1}$ and $a - a^m p \in \mathcal{A}^d$.*

In this case, we have that 0 and $(ap)^{-1}_{p\mathcal{A}p} = (ap + 1 - p)^{-1}p$ are egns–Drazin inverses of a.

Proof. (i) ⇒ (ii): It follows by Lemma 2.3.

(ii) ⇔ (iii): Using [11, Theorem 2], this equivalence is evident.

(iii) ⇒ (i): If $a \in \mathcal{A}^d$, by [10, Corollary 2.2], notice that $a^n \in \mathcal{A}^d$ and so 0 is an *egns*–Drazin inverse of a.

(i) ⇒ (iv) ∧ (v): For an *egns*–Drazin inverse x of a and $p = ax$, we observe that $p^2 = p$, $pa = ap$ and $a^n - p = a^n - ax \in \mathcal{A}^d$. Applying $apx = a^2x^2 = ax = p = xap$, we deduce that ap is invertible in the Banach algebra $p\mathcal{A}p$ and $x = (ap)^{-1}_{p\mathcal{A}p}$. Similarly, we get $(ap + 1 - p)^{-1} = (ap)^{-1}_{p\mathcal{A}p} + 1 - p$.

(iv) ⇒ (i): Let (iv) hold and $x = (ap)^{-1}_{p\mathcal{A}p}$. Then $x = xp = px$ gives $xa = xpa = (ap)^{-1}_{p\mathcal{A}p}ap = p = ap(ap)^{-1}_{p\mathcal{A}p} = ax$, $xax = xp = x$ and $a^n - ax = a^n - p \in \mathcal{A}^d$, i.e. x is an *egns*–Drazin inverse of a.

(v) ⇒ (i): Set $x = (ap + 1 - p)^{-1}p$. The equality $(ap + 1 - p)p = ap$ yields $p = (ap + 1 - p)^{-1}ap = xa = ax$. Now, $xax = px = x$ and $a^n - ax = a^n - p \in \mathcal{A}^d$, that is, x is an *egns*–Drazin inverse of a.

(i) ⇒ (vi): Case 1. $m \geq 2$: By the hypotheses and the proof of (i) ⇒ (iv), there exists an idempotent $p \in \mathcal{A}$ commuting with a such that $ap \in (p\mathcal{A}p)^{-1}$ and $a^{m-1} - p \in \mathcal{A}^d$. Hence, $(a - ap)^{m-1} = (a^{m-1} - p)(1 - p) \in \mathcal{A}^d$, which implies $a - ap \in \mathcal{A}^d$. Note that $ap \in \mathcal{A}^d$. So, $a - a^m p = (a - ap) - (a^{m-1} - p)ap \in \mathcal{A}^d$.

Case 2: $m = 1$. This is clear by the implication of (i) ⇒ (iv).

(vi) ⇒ (ii). Suppose that (vi) holds. Then, $a - ap = (a - a^m p)(1 - p) \in \mathcal{A}^d$. By [11, Theorem 1], we get that (ii) holds. □

By Theorem 2.5, we observe that the *egns*–Drazin inverse is not unique in general. The symbols $a^{n,esd}$ and $a^{n,esD}$ stand for an *egns*–Drazin inverse and *ens*–Drazin inverse of a, respectively. The set of all *egns*–Drazin (or *ens*–Drazin) inverses of a will be denoted by $a\{n, esd\}$ (or $a\{n, esD\}$).

Applying Theorem 2.5, new characterizations for *ens*–Drazin invertible elements can be given.

Corollary 2.6. *Let $a \in \mathcal{A}$ and $n, m \in \mathbb{N}$. The following statements are equivalent:*

(i) *a is ens–Drazin invertible;*

(ii) *a is e–Drazin invertible;*

(iii) *a is Drazin invertible;*

(iv) *there exists an idempotent $p \in \mathcal{A}$ commuting with a such that $ap \in (p\mathcal{A}p)^{-1}$ and $a^n - p \in \mathcal{A}^D$;*

(v) *there exists an idempotent $p \in \mathcal{A}$ commuting with a such that $ap + 1 - p \in \mathcal{A}^{-1}$ and $a^n - p \in \mathcal{A}^D$;*

(iv) *there exists an idempotent $p \in \mathcal{A}$ commuting with a such that $ap \in (p\mathcal{A}p)^{-1}$ and $a - a^m p \in \mathcal{A}^D$.*

In this case, we have that 0 and $(ap)^{-1}_{p\mathcal{A}p} = (ap + 1 - p)^{-1}p$ are ens–Drazin inverses of a.

We now establish some characterizations of *egns*–Drazin invertible elements by means of tripotents.

Theorem 2.7. *Let $a \in \mathcal{A}$ and $n \in \mathbb{N}$. The following statements are equivalent:*

(i) *a is egns–Drazin invertible;*

(ii) *there exists a tripotent $p \in \mathcal{A}$ commuting with a such that $ap \in (p^2\mathcal{A}p^2)^{-1}$ and $a^n - p^2 \in \mathcal{A}^d$;*

(iii) *there exists a tripotent $p \in \mathcal{A}$ commuting with a such that $ap + 1 - p^2 \in \mathcal{A}^{-1}$ and $a^n - p^2 \in \mathcal{A}^d$.*

In this case, we have that $(ap)^{-1}_{p^2\mathcal{A}p^2}p = (ap + 1 - p^2)^{-1}p$ is the egns–Drazin inverse of a.

Proof. (i) ⇒ (ii): According to Theorem 2.5(iv), there exists $p^2 = p \in \mathcal{A}$ commuting with a such that $ap \in (p^2\mathcal{A}p^2)^{-1}$ and $a^n - p^2 \in \mathcal{A}^d$. Hence, $p^3 = p$.

(ii) ⇒ (i): Let $p \in \mathcal{A}$ be a tripotent commuting with a, $ap \in (p^2\mathcal{A}p^2)^{-1}$ and $a^n - p^2 \in \mathcal{A}^d$. For $x = (ap)^{-1}_{p^2\mathcal{A}p^2}p$, we have $xa = (ap)^{-1}_{p^2\mathcal{A}p^2}ap = p^2$ and $ax = ap(ap)^{-1}_{p^2\mathcal{A}p^2} = p^2$. Thus, $ax = xa$, $xax = p^2x = x$ and $a^n - ax = a^n - p^2 \in \mathcal{A}^d$, i.e. x is an *egns*–Drazin inverse of a.

(i) ⇒ (iii): This implication follows similarly as (i) ⇒ (ii) by Theorem 2.5(v).

(iii) ⇒ (ii): Suppose that there exists $p^3 = p \in \mathcal{A}$, $pa = ap$, $ap + 1 - p^2 \in \mathcal{A}^{-1}$ and $a^n - p^2 \in \mathcal{A}^d$. Since $(ap + 1 - p^2)p^2 = ap$, we obtain $p^2 = (ap + 1 - p^2)^{-1}ap = ap(ap + 1 - p^2)^{-1}$, which implies $ap \in (p^2\mathcal{A}p^2)^{-1}$ and $(ap)^{-1}_{p\mathcal{A}p} = (ap + 1 - p^2)^{-1}$. □

According to Theorem 2.7, we characterize *ens*–Drazin invertible elements by tripotents.

Corollary 2.8. *Let $a \in \mathcal{A}$ and $n \in \mathbb{N}$. The following statements are equivalent:*

(i) *a is ens–Drazin invertible;*

(ii) *there exists a tripotent $p \in \mathcal{A}$ commuting with a such that $ap \in (p^2\mathcal{A}p^2)^{-1}$ and $a^n - p^2 \in \mathcal{A}^D$;*

(iii) *there exists a tripotent $p \in \mathcal{A}$ commuting with a such that $ap + 1 - p^2 \in \mathcal{A}^{-1}$ and $a^n - p^2 \in \mathcal{A}^D$.*

In this case, we have that $(ap)^{-1}_{p\mathcal{A}p}p = (ap + 1 - p^2)^{-1}p$ is the ens–Drazin inverse of a.

Applying Theorem 2.5, notice that statements (ii) and (iii) of Theorem 2.7 present new characterizations of *eg*–Drazin and *g*–Drazin invertible elements. Also, for $n = 1$ in Theorem 2.7, we recover [16, Theorem 2.2] for *egs*–Drazin invertible elements.

Basic properties of *egns*–Drazin invertible elements are developed too.

Lemma 2.9. *If $a \in \mathcal{A}^{n,esd}$, then, for arbitrary $a^{n,esd} \in a\{n, esd\}$,*

(i) *$a^{n,esd} \in \mathcal{A}^\#$ and $(a^{n,esd})^\# = a^2a^{n,esd}$;*

(ii) *$a^{n,esd} \in \mathcal{A}^{n,esd}$ and $a^2a^{n,esd} \in a^{n,esd}\{n, esd\}$.*

Proof. (i) It is clear that $a^{n,esd}$ commutes with $a^2a^{n,esd}$. Further, from $(a^2a^{n,esd})a^{n,esd}(a^2a^{n,esd}) = a^2a^{n,esd}$ and $a^{n,esd}(a^2a^{n,esd})a^{n,esd} = a^{n,esd}$, we observe that $a^{n,esd} \in \mathcal{A}^\#$ and $(a^{n,esd})^\# = a^2a^{n,esd}$.

(ii) We know that $a^n - aa^{n,esd} \in \mathcal{A}^d$ and $a^{n,esd}$ commutes with $a^n - aa^{n,esd}$. Since $a^{n,esd} \in \mathcal{A}^\#$ by part (i), then $(a^{n,esd})^n \in \mathcal{A}^\#$. Applying [9, Theorem 5.5], we have that

$$\begin{aligned}(a^{n,esd})^n - a^{n,esd}(a^2a^{n,esd}) &= (a^{n,esd})^n - aa^{n,esd}\\ &= -(a^{n,esd})^n(a^n - aa^{n,esd}) \in \mathcal{A}^d.\end{aligned}$$

□

Lemma 2.9 yields the next properties of a *ens*–Drazin inverse.

Corollary 2.10. *If $a \in \mathcal{A}^{n,esD}$, then, for arbitrary $a^{n,esD} \in a\{n, esD\}$,*

(i) *$a^{n,esD} \in \mathcal{A}^\#$ and $(a^{n,esD})^\# = a^2a^{n,esD}$;*

(ii) *$a^{n,esD} \in \mathcal{A}^{n,esD}$ and $a^2a^{n,esD} \in a^{n,esD}\{n, esD\}$.*

Recall that an arbitrary element $a \in \mathcal{A}$ can be represented by the following matrix form relative to an idempotent $p \in \mathcal{A}$:

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,$$

where $a_{11} = pap$, $a_{12} = pa(1 - p)$, $a_{21} = (1 - p)ap$, $a_{22} = (1 - p)a(1 - p)$. The matrix form of an *egns*–Drazin inverse of $a \in \mathcal{A}^d$ can be developed relative to idempotent aa^d.

Lemma 2.11. *If $a \in \mathcal{A}^d$, then*

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{aa^d} \quad \text{and} \quad a^{n,esd} = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}_{aa^d},$$

where $a_1 \in (aa^d\mathcal{A}aa^d)^{-1}$, $a_2 \in (a^\pi\mathcal{A}a^\pi)^{qnil}$ and $x_i \in a_i\{n, esd\}$ for $i = 1, 2$.

Proof. We have the next representation of $a \in \mathcal{A}^d$:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p,$$

where $p = aa^d$, $a_1 \in (p\mathcal{A}p)^{-1}$ and $a_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}$. Also,

$$a^d = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p.$$

Let $x \in a\{n, esd\}$. Since a^d double commutes with a, then x commutes with p and so

$$x = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}_p.$$

The equalities $ax = xa$ and $xax = x$ imply $a_i x_i = x_i a_i$ and $x_i a_i x_i = x_i$, for $i = 1, 2$. Because

$$a^n - ax = \begin{bmatrix} a_1^n - a_1 x_1 & 0 \\ 0 & a_2^n - a_2 x_2 \end{bmatrix}_p \in \mathcal{A}^d$$

and $\sigma(a^n - ax) = \sigma_{p\mathcal{A}p}(a_1^n - a_1 x_1) \cup \sigma_{(1-p)\mathcal{A}(1-p)}(a_2^n - a_2 x_2)$, we deduce that $a_1^n - a_1 x_1 \in (p\mathcal{A}p)^d$ and $a_2^n - a_2 x_2 \in ((1-p)\mathcal{A}(1-p))^d$. Therefore, $x_i \in a_i\{n, esd\}$, for $i = 1, 2$. □

Lemma 2.11 gives the next matrix form of an *ens*–Drazin inverse of $a \in \mathcal{A}^D$.

Corollary 2.12. *If $a \in \mathcal{A}^D$, then*

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{aa^D} \quad \text{and} \quad a^{n,esD} = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}_{aa^D},$$

where $a_1 \in (aa^D\mathcal{A}aa^D)^{-1}$, $a_2 \in (a^\pi\mathcal{A}a^\pi)^{nil}$ and $x_i \in a_i\{n, esD\}$ for $i = 1, 2$.

New equivalent conditions for an element to be *egns*–Drazin invertible are proposed now.

Theorem 2.13. *Let $a \in \mathcal{A}$ and $n, k \in \mathbb{N}$. The following statements are equivalent:*

(i) *a is egns–Drazin invertible;*

(ii) *there exists an element $y \in \mathcal{A}$ such that $ya^k y = y$, $ya = ay$ and $a^n - a^k y \in \mathcal{A}^d$;*

(iii) *a^k is egns–Drazin invertible;*

In this case, $a^{k-1}y \in a\{n, esd\}$.

Proof. (i) ⇒ (ii): For $x \in a\{n, esd\}$, set $y = x^k$. Then $ya = x^k a = ax^k = ay$, $ya^k y = x^k a^k x^k = (xax)^k = x^k = y$ and $a^n - a^k y = a^n - a^k x^k = a^n - ax \in \mathcal{A}^d$.

(ii) ⇒ (i): Assume that there exists an element $y \in \mathcal{A}$ such that $ya^k y = y$, $ya = ay$ and $a^n - a^k y \in \mathcal{A}^d$. Set $x = a^{k-1}y$. Because $ax = a^k y = a^{k-1}ya = xa$, $xax = a^{k-1}(ya^k y) = a^{k-1}y = x$ and $a^n - ax = a^n - a^k y \in \mathcal{A}^d$, we deduce that $x \in a\{n, esd\}$.

(i) ⇔ (iii): Using Theorem 2.5 and [10, Corollary 2.2], $a \in \mathcal{A}^{n,esd}$ if and only if $a \in \mathcal{A}^d$ if and only if $a^k \in \mathcal{A}^d$ if and only if $a^k \in \mathcal{A}^{n,esd}$. □

Using Theorem 2.13, we obtain the next result.

Corollary 2.14. *Let $a \in \mathcal{A}$ and $n, k \in \mathbb{N}$. The following statements are equivalent:*

(i) *a is ens–Drazin invertible;*

(ii) *there exists an element $y \in \mathcal{A}$ such that $ya^k y = y$, $ya = ay$ and $a^n - a^k y \in \mathcal{A}^D$;*

(iii) *a^k is ens–Drazin invertible;*

In this case, $a^{k-1}y \in a\{n, esD\}$.

3. Cline's formula for the *egns*–Drazin inverse

In this section, a generalization of Cline's formula is considered for the *egns*–Drazin inverse. The next useful result for elements of an associative ring $\mathcal{R}$ with the unit 1, was proposed in [20].

Lemma 3.1. [20, Theorem 2.7] *Let $a, b, c, d \in \mathcal{R}$ satisfy $acd = dbd$ and $dba = aca$. Then $bd \in \mathcal{R}^d \Leftrightarrow ac \in \mathcal{R}^d$. In this case, $(bd)^d = b((ac)^d)^2 d$ and $(ac)^d = d((bd)^d)^3 bac$.*

Under the restrictions $acd = dbd$ and $dba = aca$, an extension of Cline's formula is proved for *egns*–Drazin inverse.

Theorem 3.2. *Let $a, b, c, d \in \mathcal{A}$ satisfy $acd = dbd$ and $dba = aca$. Then*

$$bd \in \mathcal{A}^{n,esd} \quad \Leftrightarrow \quad ac \in \mathcal{A}^{n,esd}.$$

In this case, for arbitrary $(bd)^{n,esd}$ and $(ac)^{n,esd}$, we have $b((ac)^{n,esd})^2 d \in (bd)\{n, esd\}$ and $d((bd)^{n,esd})^3 bac \in (ac)\{n, esd\}$.

Proof. $\Rightarrow$: Let $bd \in \mathcal{A}^{n,esd}$. For arbitrary $(bd)^{n,esd} \in (bd)\{n, esd\}$, $x = d((bd)^{n,esd})^3 bac$ satisfies

$$\begin{aligned} acx &= acd((bd)^{n,esd})^3 bac = dbd((bd)^{n,esd})^3 bac = d((bd)^{n,esd})^3 bdbac \\ &= d((bd)^{n,esd})^3 bacac = xac \end{aligned}$$

and

$$\begin{aligned} xacx &= d((bd)^{n,esd})^2 bacx = d((bd)^{n,esd})^2 bacd((bd)^{n,esd})^3 bac \\ &= d((bd)^{n,esd})^2 bdbd((bd)^{n,esd})^3 bac = d((bd)^{n,esd})^3 bac = x. \end{aligned}$$

In order to check that

$$(ac)^n - acx = (ac)^n - d((bd)^{n,esd})^2 bac = ((db)^{n-1} - d((bd)^{n,esd})^2 b)ac \in \mathcal{A}^d,$$

set $u = ((db)^{n-1} - d((bd)^{n,esd})^2 b)a$ and $v = ((bd)^{n-1} - (bd)^{n,esd})b$. We observe that $vd = (bd)^n - (bd)^{n,esd} bd \in \mathcal{A}^d$,

$$\begin{aligned} ucd &= ((db)^{n-1} - d((bd)^{n,esd})^2 b)acd = ((db)^{n-1} - d((bd)^{n,esd})^2 b)dbd \\ &= d((bd)^{n-1} - (bd)^{n,esd})bd = dvd \end{aligned}$$

and

$$\begin{aligned} dvu &= d((bd)^{n-1} - (bd)^{n,esd})b((db)^{n-1} - d((bd)^{n,esd})^2 b)a \\ &= (d(bd)^{n-1} - d((bd)^{n,esd})^2 bd)(b(db)^{n-1} - bd((bd)^{n,esd})^2 b)a \\ &= ((db)^{n-1} - d((bd)^{n,esd})^2 b)d(b(db)^{n-1} - bd((bd)^{n,esd})^2 b)a \\ &= ((db)^{n-1} - d((bd)^{n,esd})^2 b)(db(db)^{n-1} a - dbd((bd)^{n,esd})^2 ba) \\ &= ((db)^{n-1} - d((bd)^{n,esd})^2 b)ac((db)^{n-1} - d((bd)^{n,esd})^2 b)a \\ &= ucu. \end{aligned}$$

Applying Lemma 3.1, we deduce that $((db)^{n-1} - d((bd)^{n,esd})^2 b)ac = uc \in \mathcal{A}^d$. So, $ac \in \mathcal{A}^{n,esd}$ and $d((bd)^{n,esd})^3 bac \in (ac)\{n, esd\}$.

$\Leftarrow$: Analogously, this implication can be verified. □

Consequently, Theorem 3.2 implies the next extension of Cline's formula for the *ens*–Drazin inverse.

Corollary 3.3. *Let $a, b, c, d \in \mathcal{A}$ satisfy $acd = dbd$ and $dba = aca$. Then*

$$bd \in \mathcal{A}^{n,esD} \quad \Leftrightarrow \quad ac \in \mathcal{A}^{n,esD}.$$

In this case, for arbitrary $(bd)^{n,esD}$ and $(ac)^{n,esD}$, we have $b((ac)^{n,esD})^2 d \in (bd)\{n, esD\}$ and $d((bd)^{n,esD})^3 bac \in (ac)\{n, esD\}$.

In the case that $d = a$ in Theorem 3.2, we obtain a generalization of Cline's formula for the *egns*–Drazin inverse under the assumption $aca = aba$.

Corollary 3.4. *Let $a, b, c \in \mathcal{A}$ satisfy $aca = aba$. Then*

$$ba \in \mathcal{A}^{n,esd} \quad \Leftrightarrow \quad ac \in \mathcal{A}^{n,esd}.$$

In this case, for arbitrary $(ba)^{n,esd}$ and $(ac)^{n,esd}$, $b((ac)^{n,esd})^2 a \in (ba)\{n, esd\}$ and $a((ba)^{n,esd})^2 c \in (ac)\{n, esd\}$.

When $c = b$ in Corollary 3.4, we get Cline's formula for the *egns*–Drazin inverse.

Corollary 3.5. *Let $a, b \in \mathcal{A}$. Then $ba \in \mathcal{A}^{n,esd} \Leftrightarrow ab \in \mathcal{A}^{n,esd}$. In this case, for arbitrary $(ab)^{n,esd}$, $b((ab)^{n,esd})^2 a \in (ba)\{n, esd\}$.*

Applying Corollary 3.4 and Corollary 3.5, we get the following Cline's formula for the *ens*–Drazin inverse as consequences.

Corollary 3.6. *Let $a, b, c \in \mathcal{A}$ satisfy $aca = aba$. Then*

$$ba \in \mathcal{A}^{n,esD} \quad \Leftrightarrow \quad ac \in \mathcal{A}^{n,esD}.$$

In this case, for arbitrary $(ba)^{n,esD}$ and $(ac)^{n,esD}$, $b((ac)^{n,esD})^2 a \in (ba)\{n, esD\}$ and $a((ba)^{n,esD})^2 c \in (ac)\{n, esD\}$.

Corollary 3.7. *Let $a, b \in \mathcal{A}$. Then $ba \in \mathcal{A}^{n,esD} \Leftrightarrow ab \in \mathcal{A}^{n,esD}$. In addition, for arbitrary $(ab)^{n,esD}$, $b((ab)^{esD})^2 a \in (ba)\{n, esD\}$.*

4. Weighted *egns*–Drazin inverse

For $w \in \mathcal{A}\backslash\{0\}$, let $\mathcal{A}_w$ be the complex Banach algebra $\mathcal{A}$ equipped with the w-product $a * b = awb$ and the w-norm $||a||_w = ||a||||w||$, where $a, b \in \mathcal{A}$. Also, we denote by $a^{*n} = a * a * \cdots * a$ (n factors), for $n \in N$ and $a \in \mathcal{A}$.

Lemma 4.1. [3, 14] *Let $\mathcal{A}$ be a complex Banach algebra, and let $w \in \mathcal{A}\backslash\{0\}$. For $a \in \mathcal{A}$, $a \in \mathcal{A}_w^d$ if and only if $aw \in \mathcal{A}^d$ if and only if $wa \in \mathcal{A}^d$.*

We define weighted extended *gns*–Drazin invertible and weighted extended *ns*–Drazin invertible Banach algebra elements.

Definition 4.2. *Let $w \in \mathcal{A}\backslash\{0\}$ and $n \in \mathbb{N}$. An element $a \in \mathcal{A}$ is called:*

(i) *w-weighted extended gns–Drazin invertible (or w-egns–Drazin invertible), if there exists a w-egns-Drazin inverse $a^{n,esd,w} = x \in \mathcal{A}$ such that*

$$x * a * x = x, \quad x * a = a * x \quad \text{and} \quad a^{*n} - a * x \in \mathcal{A}_w^d.$$

(ii) *w-weighted extended ns–Drazin invertible (or w-ens–Drazin invertible), if there exists a w-ens-Drazin inverse $a^{n,esD,w} = x \in \mathcal{A}$ such that*

$$x * a * x = x, \quad x * a = a * x \quad \text{and} \quad a^{*n} - a * x \in \mathcal{A}_w^D.$$

We use $\mathcal{A}^{n,esd,w}$ and $\mathcal{A}^{n,esD,w}$ to denote the sets of all *w-egns*–Drazin invertible and *w-ens*–Drazin invertible elements of $\mathcal{A}$, respectively. Notice that $a \in \mathcal{A}^{n,esd,w}$ if a is generalized n-strongly Drazin invertible in the algebra $\mathcal{A}_w$. When $w = 1$, a *w-egns*-Drazin inverse reduces to *egns*-Drazin inverse.

Some characterizations of *w-egns*-Drazin invertible elements are proved now.

Theorem 4.3. *Let $w \in \mathcal{A}\backslash\{0\}$. Then, for $a \in \mathcal{A}$, the following statements are equivalent:*

(i) $a \in \mathcal{A}^{n,esd,w}$;

(ii) $aw \in \mathcal{A}^{n,esd}$;

(iii) $wa \in \mathcal{A}^{n,esd}$.

In this case, for arbitrary $(aw)^{n,esd}$ *and* $(wa)^{n,esd}$, *we have that* $((aw)^{n,esd})^2a$ *and* $a((wa)^{n,esd})^2$ *are w-egns-Drazin inverses of a.*

Proof. (i) ⇒ (ii): For $x = a^{n,esd,w}$, then $x * a * x = x$, $x * a = a * x$, and $a^{*n} - a * x \in \mathcal{A}^d_w$ which is equivalent to $xwawx = x$, $xwa = awx$ and $(aw)^{n-1}a - awx \in \mathcal{A}^d_w$. Hence, $xw(aw)xw = xw$ and $xw(aw) = (aw)xw$. Applying Lemma 4.1, we have $(aw)^n - (aw)xw \in \mathcal{A}^d$ and so $aw \in \mathcal{A}^{n,esd}$ with $(aw)^{n,esd} = xw$.

(ii) ⇒ (i): Assume that $z = (aw)^{n,esd}$ and $x = z^2a$. Since $z(aw)z = z$ and $(aw)z = z(aw)$, then $a * x = awz^2a = z^2awa = x * a$ and $x * a * x = (z^2aw)(awz^2)a = z^2a = x$. From $((aw)^{n-1}a - za)w = (aw)^n - z(aw) \in \mathcal{A}^d$ and Lemma 4.1, one can see $a^{*n} - a * x = (aw)^{n-1}a - awz^2a = (aw)^{n-1}a - za \in \mathcal{A}^d_w$. Thus, $a \in \mathcal{A}^{n,esd,w}$ and $a^{n,esd,w} = x = z^2a$.

The equivalence (i) ⇔ (iii) can be verified analogously. □

As a consequence of Theorem 4.3, we characterize *w-ens*-Drazin invertible elements.

Corollary 4.4. *Let* $w \in \mathcal{A}\backslash\{0\}$. *Then, for* $a \in \mathcal{A}$, *the following statements are equivalent:*

(i) $a \in \mathcal{A}^{n,esD,w}$;

(ii) $aw \in \mathcal{A}^{n,esD}$;

(iii) $wa \in \mathcal{A}^{n,esD}$.

In this case, for arbitrary $(aw)^{n,esD}$ *and* $(wa)^{n,esD}$, *we have that* $((aw)^{n,esD})^2a$ *and* $a((wa)^{n,esD})^2$ *are w-ens-Drazin inverses of a.*

By Theorem 2.5, Theorem 4.3 and Lemma 4.1, we obtain the following result.

Corollary 4.5. *Let* $w \in \mathcal{A}\backslash\{0\}$. *Then, for* $a \in \mathcal{A}$, *the following statements are equivalent:*

(i) $a \in \mathcal{A}^{n,esd,w}$;

(ii) $aw \in \mathcal{A}^{n,esd}$;

(iii) $wa \in \mathcal{A}^{n,esd}$;

(iv) $aw \in \mathcal{A}^d$;

(v) $wa \in \mathcal{A}^d$;

(v) $a \in \mathcal{A}^d_w$.

More characterizations of *w-egns*-Drazin invertible elements can be found using results proved in [15, 16].

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