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Judgement of two Weyl type theorems for bounded linear operators

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Abstract. Let H be an infinite dimensional separable complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. $T \in B(H)$ is said to satisfy property (UW_{Π}) if $\sigma_a(T)\backslash\sigma_{ea}(T) = \Pi(T)$, where $\sigma_a(T)$ and $\sigma_{ea}(T)$ denote the approximate point spectrum and the essential approximate point spectrum of T respectively, $\Pi(T)$ denotes the set of all poles of T. $T \in B(H)$ satisfies a-Weyl's theorem if $\sigma_a(T)\backslash\sigma_{ea}(T) = \pi_{00}^a(T)$, where $\pi_{00}^a(T) = \{\lambda \in \mathrm{iso}\sigma_a(T) : 0 < n(T-\lambda I) < \infty\}$. In this paper, we give necessary and sufficient conditions for a bounded linear operator and its function calculus to satisfy both property (UW_{Π}) and a-Weyl's theorem by topological uniform descent. In addition, the property (UW_{Π}) and a-Weyl's theorem under perturbations are also discussed.

1. Introduction and preliminaries

Throughout this paper, $\mathbb{C}$ and $\mathbb{N}$ denote the set of complex numbers and the set of nonnegative integers. The unit closed disk and unit circle on the complex plane $\mathbb{C}$ are denoted by $\mathbb{D}$ and Γ, respectively. Let H be a complex separable infinite dimensional Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Let $T \in B(H)$. We denote by $n(T)$ the dimension of the kernel $N(T)$ and by $d(T)$ the codimension of the range $R(T)$. If $R(T)$ is closed and $n(T) < \infty$, then T is called an upper semi-Fredholm operator. T is said to be a lower semi-Fredholm operator if $d(T) < \infty$. An operator T is said to be Fredholm operator if it is both lower and upper semi-Fredholm. Especially, if T is an upper semi-Fredholm operator and $n(T) = 0$, then T is called a bounded below operator. The index of T is defined by $\mathrm{ind}(T) = n(T) - d(T)$. An operator T is said to be an upper semi-Weyl operator if it is an upper semi-Fredholm operator with $\mathrm{ind}(T) \leq 0$. If T is an upper semi-Fredholm operator and $\mathrm{ind}(T) = 0$, then T is called Weyl operator. The spectrum of T, the approximate point spectrum $\sigma_a(T)$, the essential approximate point spectrum $\sigma_{ea}(T)$, the upper semi-Fredholm spectrum $\sigma_{SF_+}(T)$ are defined by

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\},$$

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a bounded below operator}\},$$

$$\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not an upper semi-Weyl operator}\},$$

$$\sigma_{SF_+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not an upper semi-Fredholm operator}\}.$$

The ascent and descent of T are defined by $\mathrm{asc}(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$ and $\mathrm{des}(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$. If the infimum does not exist, then we write $\mathrm{asc}(T) = \infty$(resp.$\mathrm{des}(T) = \infty$). If

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$\mathrm{asc}(T) = \mathrm{des}(T) < \infty$, then T is Drazin invertible. T is called a Browder operator if T is both Fredholm operator and Drazin invertible. The Drazin spectrum $\sigma_D(T)$, the left Browder spectrum $\sigma_{ab}(T)$ and the Browder spectrum $\sigma_b(T)$ are defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\},$$

$$\sigma_{ab}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Fredholm or asc(T} - \lambda\text{I}) = \infty\},$$

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Browder operator}\}.$$

Let $\rho(T) = \mathbb{C}\backslash\sigma(T)$, $\rho_a(T) = \mathbb{C}\backslash\sigma_a(T)$, $\rho_b(T) = \mathbb{C}\backslash\sigma_b(T)$. We denote by $\sigma_0(T)$ the set of all normal eigenvalues of T, thus $\sigma_0(T) = \sigma(T)\backslash\sigma_b(T)$. For a set $E \subseteq \mathbb{C}$, we write $\mathrm{iso}E$, $\mathrm{acc}E$ and ∂E as the set of isolated points, accumulation points and boundary points of E.

For a Cauchy domain Ω, if all the curves of $\partial\Omega$ are regular analytic Jordan curves, we say that Ω is an analytic Cauchy domain. For $T \in B(H)$, if σ is a clopen subset of $\sigma(T)$, then there exists an analytic Cauchy domain Ω such that $\sigma \subseteq \Omega$ and $[\sigma(T) \setminus \sigma] \cap \overline{\Omega} = \emptyset$, where $\overline{\Omega}$ is the closure of Ω. We denote by $E(\sigma; T)$ the Riesz idempotent of T corresponding to σ, i.e.,

$$E(\sigma; T) = \frac{1}{2\pi i}\int_{\Gamma}(\lambda I - T)^{-1}d\lambda,$$

where $\Gamma = \partial\Omega$ is positively oriented with respect to Ω in the sense of complex variable theory. In this case, we have $H(\sigma; T) = R(E(\sigma; T))$. Clearly, if $\lambda \in \mathrm{iso}\sigma(T)$, then $\{\lambda\}$ is a clopen subset of $\sigma(T)$. We write $H(\lambda; T)$ instead of $H(\{\lambda\}; T)$; if in addition, $\dim H(\lambda; T) < \infty$, then $\lambda \in \sigma_0(T)$.

Spectral theory of operators is an important part of operator theory. Weyl's theorem, as an important conclusion in spectral theory, is discovered by H.Weyl in 1909 ([16]) when he studied the spectral set of self-adjoint operators on Hilbert spaces. As one of the research focuses of spectral theory in recent years, scholars have made various modifications to it.

The variation of Weyl's theorem, namely, a-Weyl's theorem ([13, 14]) were given by Rakočevič. We say that the a-Weyl's theorem holds for T if

$$\sigma_a(T)\backslash\sigma_{ea}(T) = \pi_{00}^a(T),$$

where $\pi_{00}^a(T) = \{\lambda \in \mathrm{iso}\sigma_a(T) : 0 < n(T - \lambda I) < \infty\}$.

Property (UW_Π), as well as a-Weyl's theorem, is also a variant of Weyl's theorem. In [6], Berkani and Kachad introduced the definition of property (UW_Π). $T \in B(H)$ satisfies property (UW_Π) and denoted by $T \in (UW_\Pi)$, if

$$\sigma_a(T)\backslash\sigma_{ea}(T) = \Pi(T),$$

where $\Pi(T) = \sigma(T)\backslash\sigma_D(T)$. If $\lambda \in \Pi(T)$, then λ is a pole of T.

The concept of topological uniform descent was first proposed by Sandy Grabiner ([9]). The introduction of this concept provides a new tool for the study of operator theory, and many scholars have achieved corresponding research results by using topological uniform descent ([8, 11, 15]). If $T \in B(H)$, then for each nonnegative integer n, T induces a linear transformation

$$\Gamma_n : R(T^n)/R(T^{n+1}) \longrightarrow R(T^{n+1})/R(T^{n+2}),$$

we will let $k_n(T)$ be the dimension of the null space of the induced map and let $k(T) = \sum_{n=0}^{\infty} k_n(T)$. The operator range topology on $R(T^n)$ is defined by the norm $\| y \|_n = \inf\{\| x \|, x \in H, y = T^n x\}$. If there is a nonnegative integer d for which $k_n(T) = 0$ for $n \geq d$ and $R(T^n)$ is closed in the operator range topology of $R(T^d)$ for $n \geq d$, then we say that T has topological uniform descent.

It can be shown that if T is semi-Fredholm, then T has topological uniform descent. If $T - \lambda I$ has topological uniform descent and $\lambda \in \partial\sigma(T)$, then $\lambda \in \Pi(T)$([9, Corollary 4.9]). The topological uniform descent spectrum of T is defined by

$$\sigma_\tau(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ has not topological uniform descent}\},$$

and $\rho_\tau(T) = \mathbb{C}\backslash\sigma_\tau(T)$.

Example 1.1. *(i) It is easy to see that if $T \in (UW_{\Pi})$, then $\sigma_a(T)\backslash\sigma_{ea}(T) \subseteq \pi^a_{00}(T)$. But property (UW_{Π}) does not imply a-Weyl's theorem. Let $A, B \in B(\ell^2)$ be defined by*

$$A(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots), B(x_1, x_2, \cdots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots).$$

Put $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Then we have $\sigma(T) = \mathbb{D}$, $\Pi(T) = \emptyset$, $\sigma_a(T) = \sigma_{ea}(T) = \{0\} \cup \Gamma$, $\pi^a_{00}(T) = \{0\}$. It follows that $T \in (UW_{\Pi})$, but the a-Weyl's theorem does not hold for T.

(ii) A-Weyl's theorem does not imply property (UW_{Π}). Let $A, B \in B(\ell^2)$ be defined by

$$A(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots), B(x_1, x_2, \cdots) = (0, x_2, x_3, \cdots).$$

Put $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Then we have $\sigma(T) = \mathbb{D}$, $\Pi(T) = \emptyset$, $\sigma_a(T) = \{0\} \cup \Gamma$, $\sigma_{ea}(T) = \Gamma$ $\pi^a_{00}(T) = \{0\}$. It follows that T satisfies a-Weyl's theorem, but $T \notin (UW_{\Pi})$.

(iii) There exists $T \in B(H)$ such that neither property (UW_{Π}) nor a-Weyl's theorem holds for T. Let $A, B \in B(\ell^2)$ be defined by

$$A(x_1, x_2, \cdots) = (0, x_2, 0, x_4, \cdots),\ B(x_1, x_2, \cdots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots).$$

Put $T = \begin{pmatrix} A & 0 \\ 0 & B - I \end{pmatrix}$. Then we have $\sigma(T) = \{-1, 0, 1\}$, $\Pi(T) = \{0, 1\}$, $\sigma_a(T) = \sigma_{ea}(T) = \{-1, 0, 1\}$, $\pi^a_{00}(T) = \{-1\}$. Thus, neither property (UW_{Π}) nor a-Weyl's theorem holds for T.

We have seen in Example 1.1 that there is no relationship between $T \in (UW_{\Pi})$ and T satisfies a-Weyl's theorem although the forms of property (UW_{Π}) and a-Weyl's theorem are similar.

In this paper, we will give necessary and sufficient conditions for bounded linear operators to satisfy both property (UW_{Π}) and a-Weyl's theorem by topological uniform descent in section 2. What's more, we also discuss both property (UW_{Π}) and a-Weyl's theorem under quasi-nilpotent perturbation for bounded linear operators. In section 3, we will talk about operator functions to satisfy both property (UW_{Π}) and a-Weyl's theorem in terms of topological uniform descent. In addition, we also discuss the case that Drazin invertible operators satisfy both property (UW_{Π}) and a-Weyl's theorem.

2. Property (UW_{Π}) and a-Weyl's theorem of bounded linear operators

In this section, we will describe both property (UW_{Π}) and a-Weyl's theorem hold for T by means of the property of topological uniform descent.

Theorem 2.1. *Let $T \in B(H)$. The following statements are equivalent:*

(1) T satisfies both the property (UW_{Π}) and a-Weyl's theorem;

(2) $\sigma_b(T) = [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I)\}] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [acc\sigma_a(T) \cap \sigma_{ea}(T)] \cup \{\{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cap acc[\rho_a(T) \cap \sigma(T)]\}$.

Proof. (1) ⇒ (2). The inclusion "⊇" is obvious. For the opposite inclusion, take arbitrarily λ_0 that does not belong to the right side of (2). Without loss of generality, suppose that $\lambda_0 \in \sigma(T)$. Then we have $n(T - \lambda_0 I) > 0$.

Case 1 Suppose that $\lambda_0 \notin \sigma_{ea}(T)$. Then $\lambda_0 \in \sigma_a(T)\backslash\sigma_{ea}(T)$. Since $T \in (UW_{\Pi})$, we have $\lambda_0 \notin \sigma_b(T)$.

Case 2 Suppose that $\lambda_0 \notin \mathrm{acc}\sigma_a(T) \cup \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$. Then $\lambda_0 \in \pi^a_{00}(T)$. Since T satisfies both property (UW_{Π}) and a-Weyl's theorem, we can get that $\lambda_0 \notin \sigma_b(T)$.

Case 3 Suppose that $\lambda_0 \notin \sigma_\tau(T) \cup \mathrm{acc}\sigma_a(T) \cup \mathrm{acc}[\rho_a(T) \cap \sigma(T)]$. Then $\lambda_0 \in \rho_\tau(T) \cap \partial\sigma(T)$, we can get $\lambda_0 \in \Pi(T)$([9, Corollary 4.9]). From $T \in (UW_{\Pi})$ we get that $\lambda_0 \notin \sigma_b(T)$.

Case 4 Suppose that $\lambda_0 \notin \{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I)\} \cup \mathrm{acc}\sigma_a(T) \cup \mathrm{acc}[\rho_a(T) \cap \sigma(T)]$. We have $0 < n(T - \lambda_0 I) < \infty$, thus $\lambda_0 \in \pi^a_{00}(T)$. Since T satisfies both property (UW_{Π}) and a-Weyl's theorem, we get that $\lambda_0 \notin \sigma_b(T)$.

(2) ⇒ (1). It is clear that $\{[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \Pi(T) \cup \pi_{00}^a(T)\} \cap [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = d(T-\lambda I)\}] = \emptyset$, $\{[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \Pi(T) \cup \pi_{00}^a(T)\} \cap \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} = \emptyset$, $\{[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \Pi(T) \cup \pi_{00}^a(T)\} \cap [\mathrm{acc}\sigma_a(T) \cap \sigma_{ea}(T)] = \emptyset$, $\{[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \Pi(T) \cup \pi_{00}^a(T)\} \cap \{\{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cap \mathrm{acc}[\rho_a(T) \cap \sigma(T)] = \emptyset$. Hence $[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \Pi(T) \cup \pi_{00}^a(T) = \sigma_0(T)$. It follows that T satisfies both property (UW_Π) and a-Weyl's theorem. □

Remark 2.2. *(i) In Theorem 2.1, suppose $T \in B(H)$ satisfies both property (UW_Π) and a-Weyl's theorem, then each part of the decomposition of $\sigma_b(T)$ can not be deleted.*

(a) Let $A, B \in B(\ell^2)$ be defined by

$$A(x_1, x_2, \cdots) = (0, x_1, 0, \frac{x_3}{3}, 0, \frac{x_5}{5}, \cdots), B(x_1, x_2, \cdots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots).$$

Put $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Then we have $\sigma(T) = \{0\}$, $\Pi(T) = \emptyset$, $\sigma_a(T) = \sigma_{ea}(T) = \{0\}$, $\pi_{00}^a(T) = \emptyset$. Hence T satisfies both property (UW_Π) and a-Weyl's theorem. But $\{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} = [acc\sigma_a(T) \cap \sigma_{ea}(T)] = \{\{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cap acc[\rho_a(T) \cap \sigma(T)]\} = \emptyset$. Thus $\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = d(T-\lambda I)\}$ cannot be deleted.

(b) Let $T \in B(\ell^2)$ be defined by

$$T(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots).$$

Then we have $\sigma(T) = \mathbb{D}$, $\Pi(T) = \emptyset$, $\sigma_a(T) = \sigma_{ea}(T) = \Gamma$, $\pi_{00}^a(T) = \emptyset$, and so T satisfies both property (UW_Π) and a-Weyl's theorem. But $\sigma_b(T) \neq [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = d(T-\lambda I)\}] \cup [acc\sigma_a(T) \cap \sigma_{ea}(T)] \cup \{\{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cap acc[\rho_a(T) \cap \sigma(T)]\}$. It follows that $\{\lambda \in \sigma(T) : n(T-\lambda I) = 0\}$ cannot be deleted.

(c) Let $T \in B(\ell^2)$ be defined by

$$T(x_1, x_2, \cdots) = (0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots).$$

Then we have $\sigma(T) = \sigma_a(T) = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$, $\sigma_{ea}(T) = \{0\}$, $\Pi(T) = \pi_{00}^a(T) = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$. So property (UW_Π) and a-Weyl's theorem hold for T. But $\sigma_b(T) \neq [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = d(T-\lambda I)\}] \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup \{\{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cap acc[\rho_a(T) \cap \sigma(T)]\}$. Thus $acc\sigma_a(T) \cap \sigma_{ea}(T)$ cannot be deleted.

(d) Let $A, B \in B(\ell^2)$ be defined by

$$A(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots), B(x_1, x_2, \cdots) = (0, x_2, 0, x_4, \cdots).$$

Put $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Then we have $\sigma(T) = \mathbb{D}$, $\Pi(T) = \emptyset$, $\sigma_a(T) = \sigma_{ea}(T) = \{0\} \cup \Gamma$, $\pi_{00}^a(T) = \emptyset$. It follows that T satisfies both property (UW_Π) and a-Weyl's theorem. However, $\sigma_b(T) \neq [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = d(T-\lambda I)\}] \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup [acc\sigma_a(T) \cap \sigma_{ea}(T)]$, which means that $\{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cap acc[\rho_a(T) \cap \sigma(T)]$ cannot be deleted.

(ii) It is clear that $\sigma_{ea}(T) = \sigma_{SF_+}(T) \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) > d(T-\lambda I)\}$. From Theorem 2.1, we can get that T satisfies both property (UW_Π) and a-Weyl's theorem if and only if $\sigma_b(T) = [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = d(T-\lambda I)\}] \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup [acc\sigma_a(T) \cap \sigma_{SF_+}(T)] \cup \{\{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cap acc[\rho_a(T) \cap \sigma(T)]\} \cup \{\lambda \in \mathbb{C} : n(T-\lambda I) > d(T-\lambda I)\}$.

(iii) If $\sigma_\tau(T) = \emptyset$, we claim that $int\sigma(T) = \emptyset$. If not, there exists a continuous curve segment $L \subseteq \partial\sigma(T)$. Take $\lambda_0 \in L$, from $\sigma_\tau(T) = \emptyset$ we can get that $\lambda_0 \in \Pi(T)$. Then $\lambda_0 \in iso\sigma(T)$, a contradiction. Thus, $\sigma(T) = \partial\sigma(T)$. Take arbitrarily $\lambda \in \sigma(T) = \partial\sigma(T)$. It follows from $\lambda \notin \sigma_\tau(T)$ that $\lambda \in \Pi(T)$ and $\lambda \in iso\sigma(T)$. Since $\sigma(T)$ is a bounded set, we can get that $\sigma(T)$ consists of finite points. Therefore, if $\sigma_\tau(T) = \emptyset$, then $\sigma(T) = \Pi(T)$.

From Theorem 2.1, we can obtain this result: If $\sigma_\tau(T) = \emptyset$ and T satisfies both property (UW_Π) and a-Weyl's theorem(or only property (UW_Π) is required), then $\sigma_b(T) = \emptyset$, a contradiction with the fact that $\sigma_b(T)$ is nonempty. Therefore, if T satisfies both property (UW_Π) and a-Weyl's theorem(or only property (UW_Π)), $\sigma_\tau(T) \neq \emptyset$.

Corollary 2.3. *Let $T \in B(H)$. The following statements are equivalent:*

(1) T satisfies both property (UW_Π) and a-Weyl's theorem;

(2) $\sigma_b(T) = [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}] \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup [acc\sigma_a(T) \cap \sigma_{ea}(T)] \cup \{\{\lambda \in \mathbb{C} : n(T-\lambda I) = d(T-\lambda I)\} \cap acc[\rho_a(T) \cap \sigma(T)]\}$.

Proof. (1) ⇒ (2). The inclusion "⊇" is obvious. For the opposite inclusion, we know that $\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I) < \infty\} = \emptyset$. Hence $[\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I)\}] = [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I) = \infty\}] \subseteq \sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\}$. From $\{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty, d(T - \lambda I) < \infty\} \cap \text{acc}[\rho_a(T) \cap \sigma(T)] = \emptyset$ we can get that $\{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cap \text{acc}[\rho_a(T) \cap \sigma(T)] = \{\{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I) = \infty\} \cap \text{acc}[\rho_a(T) \cap \sigma(T)]\} \subseteq \{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I)\} \cap \text{acc}[\rho_a(T) \cap \sigma(T)]$. According to Theorem 2.1, the inclusion "⊆" is obvious.

(2) ⇒ (1). Similar to the proof of Theorem 2.1, this result is trivial. □

It is easy to get that $[\rho_a(T) \cap \sigma(T)] \subseteq \{\lambda \in \mathbb{C} : n(T - \lambda I) < d(T - \lambda I)\}$ and $\{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\} \subseteq \text{acc}\{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\}$. From (ii) in Remark 2.2 we obtain the following corollary.

Corollary 2.4. *Let $T \in B(H)$. The following statements are equivalent:*

(1) T satisfies both property (UW_Π) and a-Weyl's theorem;

(2) $\sigma_b(T) = [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I)\}] \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [acc\sigma_a(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cap acc\{\lambda \in \mathbb{C} : n(T - \lambda I) < d(T - \lambda I)\}] \cup acc\{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\}$.

Corollary 2.5. *Let $T \in B(H)$. The following statements are equivalent:*

(1) T satisfies both property (UW_Π) and a-Weyl's theorem;

(2) $\pi_{00}^a(T) \subseteq \rho_\tau(T) \subseteq \rho_b(T) \cup \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [acc\sigma_a(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cap acc\{\lambda \in \mathbb{C} : n(T - \lambda I) < d(T - \lambda I)\}] \cup acc\{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\}$.

Proof. (1) ⇒ (2). It is obvious that $\pi_{00}^a(T) \subseteq \rho_\tau(T)$. Suppose that $\lambda_0 \in \rho_\tau(T)$. Then $\lambda_0 \notin [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T - \lambda I) = d(T - \lambda I)\}]$. If $\lambda_0 \notin \{\lambda \in \sigma(T) : n(T - \lambda I) = 0\} \cup [\text{acc}\sigma_a(T) \cap \sigma_{SF_+}(T)] \cup [\{\lambda \in \mathbb{C} : n(T - \lambda I) = \infty\} \cap \text{acc}\{\lambda \in \mathbb{C} : n(T - \lambda I) < d(T - \lambda I)\}] \cup \text{acc}\{\lambda \in \mathbb{C} : n(T - \lambda I) > d(T - \lambda I)\}$, from Corollary 2.4 we can get that $\lambda_0 \in \rho_b(T)$.

(2) ⇒ (1). It is clear that $\{[\sigma_a(T) \backslash \sigma_{ea}(T)] \cup \Pi(T)\} \subseteq \rho_\tau(T)$. From Corollary 2.4 and the proof of Theorem 2.1, we have $\{[\sigma_a(T) \backslash \sigma_{ea}(T)] \cup \Pi(T)\} \subseteq \sigma_0(T)$ and $\pi_{00}^a(T) \subseteq \sigma_0(T)$. Thus T satisfies both property (UW_Π) and a-Weyl's theorem. □

Weyl type Theorem and its perturbation problems have attracted extensive attention in recent years([7, 10, 17]). In the following, we will discuss quasi-nilpotent perturbation of both property (UW_Π) and a-Weyl's theorem.

We call $R \in B(H)$ is Riesz operator if $R - \lambda I$ is Fredholm operator for every nonzero λ. In [3, Theorem 4.7], we have that

$$\sigma_*(T) = \sigma_*(T + R)$$

for every Riesz operator R commuting with $T \in B(H)$, where $* \in \{ea, ab, b\}$. It is clear that quasi-nilpotent operators are Riesz operators. $T \in B(H)$ is said to be a-isoloid operator if $\text{iso}\sigma_a(T) \subseteq \sigma_p(T)$, where $\sigma_p(T) = \{\lambda \in \mathbb{C} : n(T - \lambda I) > 0\}$. If $\text{iso}\sigma_a(T) \subseteq \Pi(T)$, then T is called a-polaroid operator.

Example 2.6. *(1) Let $T, Q \in B(\ell^2)$ be defined by*

$$T(x_1, x_2, x_3 \cdots) = (0, x_1, x_2, x_3, \cdots), Q(x_1, x_2, x_3, \cdots) = (0, -x_1, 0, 0, \cdots).$$

We have that T satisfies both property (UW_Π) and a-Weyl's theorem. However, $\sigma(T+Q) = \sigma_a(T+Q) = \mathbb{D}$, $\sigma_{ea}(T) = \Gamma$, $\Pi(T) = \pi_{00}^a(T) = \emptyset$. It follows that both property (UW_Π) and a-Weyl's theorem don't hold for $T + Q$.

(2)Let $A, B \in B(\ell^2)$ be defined by

$$A(x_1, x_2, \cdots) = (0, x_1, 0, \frac{x_3}{3}, 0, \frac{x_5}{5}, \cdots), B(x_1, x_2, \cdots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots).$$

Put $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 \\ 0 & -B \end{pmatrix}$. *Then we have $QT = TQ$, T is a-isoloid operator, $\sigma(T) = \sigma_a(T) = \sigma_{ea}(T) = \{0\}$, $\pi_{00}^a(T) = \Pi(T) = \emptyset$. It follows that T satisfies both property (UW_Π) and a-Weyl's theorem, but we can see that $T + Q \notin (UW_\Pi)$.*

From Example 2.6 we know that the commutativity of T is indispensable, and we can't also induce that $T+Q$ satisfies both property (UW_{Π}) and a-Weyl's theorem if T is a-isoloid operator. Now, let Q be a quasi-nilpotent operator with $QT = TQ$. For $T \in B(H)$, the quasi-nilpotent part of T is defined by

$$H_0(T) = \{x \in H : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

It is known that T is a quasi-nilpotent operator if and only if $H_0(T) = H$. Thus we have the following lemma.

Lemma 2.7. *[3, Theorem 4.9] Let $T \in B(H)$ and Q a quasi-nilpotent operator with $QT = TQ$, then $\sigma(T) = \sigma(T+Q)$ and $\sigma_a(T) = \sigma_a(T+Q)$.*

Proof. Since $-Q$ is quasi-nilpotent operator, we only need $T+Q$ is bounded below if T is bounded below. We claim that $H_0(T) = \{0\}$ if T is bounded below. In fact, since T is bounded below, there exists $k > 0$ such that $\|Tx\| \geq k\|x\|$, $\forall x \in H$. Suppose that $x_0 \in H_0(T)$, then $\lim_{n\to\infty} \|T^n x_0\|^{\frac{1}{n}} = 0$ and $\|T^n x_0\| \geq k^n \|x_0\|$. It follows that $\|T^n x_0\|^{\frac{1}{n}} \geq k\|x_0\|^{\frac{1}{n}}$. Thus, $x_0 = 0$. Since $T+Q$ is upper semi-Weyl operator, we only need $N(T+Q) = \{0\}$. For all $x \in N(T+Q)$ we have that $Qx = -Tx$. Thus $Q^n x_0 = (-1)^n T^n x_0$. From Q is quasi-nilpotent operator we can get that $\lim_{n\to\infty} \|Q^n x\|^{\frac{1}{n}} = 0$, so $\lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0$. It follows that $x \in H_0(T)$. Since T is bounded below, $H_0(T) = \{0\}$. So, $x = 0$. Hence, $T+Q$ is bounded below.

If T is invertible, we know that $T+Q$ is Weyl operator. From $T+Q$ is bounded below we can get that $T+Q$ is invertible. □

Theorem 2.8. *Let $T \in B(H)$ and Q a quasi-nilpotent operator with $QT = TQ$. Then the following statements are equivalent:*

(1) T satisfies both property (UW_{Π}) and a-Weyl's theorem, and T is a-polaroid operator;

(2) $T+Q$ satisfies both property (UW_{Π}) and a-Weyl's theorem, and $T+Q$ is a-polaroid operator.

Proof. Since $-Q$ is quasi-nilpotent operator, we only need to show (1) $\Rightarrow$ (2). Let $\lambda \in \sigma_a(T+Q)\backslash\sigma_{ea}(T+Q)$, from Lemma 2.7 we can get that $\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T)$. It follows from $T \in (UW_{\Pi})$ that $\lambda \in \sigma_0(T)$ and so $\lambda \in \sigma_0(T+Q)$. Let $\lambda_0 \in \Pi(T+Q)$, then $\lambda_0 \in \text{iso}\sigma(T+Q)$. From Lemma 2.7 and T is a-polaroid operator we have $\lambda_0 \in \Pi(T)$. By $T \in (UW_{\Pi})$ we can get $\lambda_0 \in \sigma_0(T)$. Then $\lambda_0 \in \sigma_0(T+Q)$. Let $\mu_0 \in \pi_{00}^a(T+Q)$, from Lemma 2.7 and T is a-polaroid operator, we get that $\mu_0 \in \Pi(T)$. By $T \in (UW_{\Pi})$ we can get $\mu_0 \in \sigma_0(T)$ and $\mu_0 \in \sigma_0(T+Q)$. Let $\mu \in \text{iso}\sigma_a(T+Q)$. Similar to the above proof, it is clear that $\mu \in \Pi(T+Q)$. Thus, $T+Q \in (UW_{\Pi})$ and satisfies a-Weyl's theorem, and $T+Q$ is a-polaroid operator. □

In the following, we will discuss the quasi-nilpotent perturbation of both property (UW_{Π}) and a-Weyl's theorem according to topological uniform descent.

Theorem 2.9. *Let $T \in B(H)$. Then the following statements are equivalent:*

(1) T satisfies both property (UW_{Π}) and a-Weyl's theorem, and T is a-polaroid operator;

(2) $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup [\rho_a(T) \cap \sigma(T)]$.

Proof. (1) $\Rightarrow$ (2). The inclusion "$\supseteq$" is obvious. For the opposite inclusion, take arbitrarily λ_0 that does not belong to the right side of (2). Without loss of generality, suppose that $\lambda_0 \in \sigma(T)$. Then we have $\lambda_0 \in \sigma_a(T)$.

Case 1 Suppose $\lambda_0 \notin \sigma_\tau(T) \cup \text{acc}\sigma_{ea}(T)$. Then there exists $\epsilon > 0$ such that $\lambda \in \rho_a(T) \cup [\sigma_a(T)\backslash\sigma_{ea}(T)]$ when $0 < \|\lambda - \lambda_0\| < \epsilon$. From T is a-polaroid operator and $T \in (UW_{\Pi})$, we can get that $\lambda_0 \in \Pi(T)$. Thus $\lambda_0 \notin \sigma_b(T)$.

Case 2 Suppose $\lambda_0 \notin \text{acc}\sigma_a(T) \cup \text{acc}\sigma_{ea}(T)$. Then $\lambda_0 \in \text{iso}\sigma_a(T)$. From T is a-polaroid operator and $T \in (UW_{\Pi})$, we get that $\lambda_0 \notin \sigma_b(T)$.

(2) $\Rightarrow$ (1). It is clear that $\{[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \Pi(T) \cup \pi_{00}^a(T)\} \cap [\sigma_\tau(T) \cap \text{acc}\sigma_a(T)] = \emptyset$, $\{[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \Pi(T) \cup \pi_{00}^a(T)\} \cap \text{acc}\sigma_{ea}(T) = \emptyset$, $\{[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \Pi(T) \cup \pi_{00}^a(T)\} \cap [\rho_a(T) \cap \sigma(T)] = \emptyset$. Thus, $[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \Pi(T) \cup \pi_{00}^a(T) = \sigma_0(T)$. And $\text{iso}\sigma_a(T) \cap \{[\sigma_\tau(T) \cap \text{acc}\sigma_a(T)] \cup \text{acc}\sigma_{ea}(T) \cup [\rho_a(T) \cap \sigma(T)]\} = \emptyset$. It follows that both property (UW_{Π}) and a-Weyl's theorem hold for T, and T is a-polaroid operator. □

From Theorem 2.8 and Theorem 2.9 we finally get the following result.

Corollary 2.10. *Let $T \in B(H)$ and Q a quasi-nilpotent operator with $QT = TQ$. Then the following statements are equivalent:*

(1) $T + Q$ satisfies both property (UW_Π) and a-Weyl's theorem, and $T + Q$ is a-polaroid operator;

(2) $\sigma_b(T) = [\sigma_\tau(T) \cap acc\sigma_a(T)] \cup acc\sigma_{ea}(T) \cup [\rho_a(T) \cap \sigma(T)]$.

3. Property (UW_Π) and a-Weyl's theorem of operator functions

For $T \in B(H)$, we use $Hol(\sigma(T))$ to denote the class of all complex-valued functions analytic on a neighborhood of $\sigma(T)$ and not constant on any components of $\sigma(T)$.

Remark 3.1. *(i) T satisfies both property (UW_Π) and a-Weyl's theorem does not imply $f(T)$ satisfies both property (UW_Π) and a-Weyl's theorem, where $f \in Hol(\sigma(T))$.*

Let $A, B \in B(\ell^2)$ be defined by

$$A(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots), B(x_1, x_2, \cdots) = (x_2, x_3, x_4, \cdots).$$

Put $T = \begin{pmatrix} A+I & 0 \\ 0 & B-I \end{pmatrix}$. Then T satisfies both property (UW_Π) and a-Weyl's theorem. Let $f(z) = (z-1)(z+1), z \in \mathbb{C}$, we can get $0 \in \sigma_a(f(T)) \backslash \sigma_{ea}(f(T))$. But $0 \notin \Pi(T)$, $0 \notin \pi^a_{00}(T)$. We know that both property (UW_Π) and a-Weyl's theorem don't hold for $f(T)$.

(ii) $f(T)$ satisfies both property (UW_Π) and a-Weyl's theorem for some $f \in Hol(\sigma(T))$ does not imply T satisfies both property (UW_Π) and a-Weyl's theorem. Let $A, B, C \in B(\ell^2)$ be defined by

$$A(x_1, x_2, \cdots) = (0, x_2, 0, x_4, \cdots), B(x_1, x_2, \cdots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \cdots), C(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots).$$

Put $T = \begin{pmatrix} A+I & 0 & 0 \\ 0 & B-I & 0 \\ 0 & 0 & C+I \end{pmatrix}$. We know that $\sigma_a(T^2) = \sigma_{ea}(T^2) = \{re^{i\theta} : r = 2(1 + \cos\theta)\} \cup \{1, \frac{1}{9}\}$, $\Pi(T^2) = \emptyset$, $\pi^a_{00}(T^2) = \emptyset$. So $T^2 \in (UW_\Pi)$ and satisfies a-Weyl's theorem. But $\Pi(T) = \{\frac{1}{3}\}$, $\pi^a_{00}(T) = \{-1\}$, $\sigma_a(T) = \sigma_{ea}(T) = \{-1, -\frac{1}{3}\} \cup \{\lambda \in \mathbb{C} : \|\lambda - 1\| = 1\}$. Thus both property (UW_Π) and a-Weyl's theorem don't hold for T.

From the above Remark, T and $f(T)$ satisfy both property (UW_Π) and a-Weyl's theorem are not directly connected. In the following, we will discuss the property (UW_Π) and a-Weyl's theorem for operator functions through the relation between $\sigma_b(T)$ and $\sigma_\tau(T)$.

First we have this fact: For any $f \in Hol(\sigma(T))$, $f(\sigma_{ea}(T)) = \sigma_{ea}(f(T))$ if and only if for any $\lambda, \mu \in \rho_{SF_+}(T)$, $\text{ind}(T - \lambda I) \cdot \text{ind}(T - \mu I) \geq 0$. Next, we will use topological uniform descent to describe the properties of Fredholm index.

Lemma 3.2. *Let $T \in B(H)$ and $f \in Hol(\sigma(T))$. If $f(T) \in (UW_\Pi)$, then for any $\lambda, \mu \in \rho_{SF_+}(T)$, $ind(T - \lambda I) \cdot ind(T - \mu I) \geq 0$.*

Proof. If not, then there exist $\lambda_0, \mu_0 \in \rho_{SF_+}(T)$ such that $\text{ind}(T - \lambda_0 I) = m > 0$, $\text{ind}(T - \mu_0 I) = -n < 0$. Suppose that $f(z) = (z - \lambda_0)^n (z - \mu_0)^m$ when $n < \infty$ and $f(z) = (z - \lambda_0)(z - \mu_0)$ when $n = \infty$. In both instances, we can get $0 \in \sigma_a(f(T)) \backslash \sigma_{ea}(f(T))$. From $f(T) \in (UW_\Pi)$, we know that $f(T)$ is Browder operator. Thus $\lambda_0 \notin \sigma_b(T)$, a contradiction. □

Lemma 3.3. *Let $T \in B(H)$ and $T \in (UW_\Pi)$. Then the following statements hold:*

(1) $\rho_\tau(T) \subseteq \rho_b(T) \cup acc\sigma_{ea}(T)$ if and only if for any $\lambda \in \rho_{SF_+}(T)$, $ind(T - \lambda I) \geq 0$;

(2) $\rho_\tau(T) \subseteq \rho_b(T) \cup acc\sigma_{SF_+}(T) \cup acc\{\lambda \in \mathbb{C} : n(T - \lambda I) = 0\}$ if and only if for any $\lambda \in \rho_{SF_+}(T)$, $ind(T - \lambda I) \leq 0$.

Proof. (1). "$\Rightarrow$". If not, there exist $\lambda_0 \in \rho_{SF_+}(T)$ such that $\text{ind}(T - \lambda_0 I) < 0$. We can get that $\lambda_0 \in \rho_\tau(T)$ and $\lambda_0 \notin \text{acc}\sigma_{ea}(T)$, then $\lambda_0 \in \rho_b(T)$, a contradiction.

"$\Leftarrow$". Suppose that for any $\lambda \in \rho_{SF_+}(T)$, $\text{ind}(T-\lambda I) \geq 0$. Take $\lambda_0 \in \rho_\tau(T)$ but $\lambda_0 \notin \text{acc}\sigma_{ea}(T)$, then there exists $\epsilon > 0$ such that $T-\lambda I$ is an upper semi-Weyl operator when $0 < \|\lambda - \lambda_0\| < \epsilon$. By $\text{ind}(T-\lambda I) \geq 0$ and $T \in (UW_\Pi)$ we get that $T-\lambda I$ is a Browder operator. It follows that $\lambda_0 \in \partial\sigma(T) \cap \rho_\tau(T)$ and $\lambda_0 \in \Pi(T)$. Therefore, $\lambda_0 \in \rho_b(T)$.

(2). Similar to the proof of (1), this result is obvious. □

Theorem 3.4. *Let $T \in B(H)$. Then for any $f \in Hol(\sigma(T))$, $f(T)$ satisfies both property (UW_Π) and a-Weyl's theorem if and only if:*

(1) T satisfies both property (UW_Π) and a-Weyl's theorem;

(2) $\rho_\tau(T) \subseteq \rho_b(T) \cup acc\sigma_{ea}(T)$ or $\rho_\tau(T) \subseteq \rho_b(T) \cup acc\sigma_{SF_+}(T) \cup acc\{\lambda \in \mathbb{C} : n(T-\lambda I) = 0\}$;

(3) If $\sigma_0(T) \neq \emptyset$, then $\sigma_b(T) = [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}] \cup [acc\sigma_a(T) \cap \sigma_{ea}(T)]$.

Proof. "$\Rightarrow$". From Lemma 3.2 and Lemma 3.3, we only need to prove (3) holds. The inclusion "$\supseteq$" is clear. For the converse, we first claim that $\{\lambda \in \text{iso}\sigma_a(T) : n(T-\lambda I) < \infty\} = \sigma_0(T)$. In fact, take $\lambda_1 \in \sigma_0(T)$, $\lambda_2 \in \{\lambda \in \text{iso}\sigma_a(T) : n(T-\lambda I) < \infty\}$. Set $\sigma_1 = \{\lambda_1\}$, $\sigma_2 = \{\lambda_2\}$ and $\sigma_3 = \sigma(T)\backslash[\sigma_1 \cup \sigma_2]$. Then by [12, Theorem 2.10] T can be represented as $T = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix}$, where $\sigma(T_i) = \sigma_i$, $i = 1, 2, 3$. Put $f_0(z) = (z-\lambda_1)(z-\lambda_2)$. Then $f_0(T) = \begin{pmatrix} f_0(T_1) & 0 & 0 \\ 0 & f_0(T_2) & 0 \\ 0 & 0 & f_0(T_3) \end{pmatrix}$. Therefore $0 \in \text{iso}\sigma_a(f_0(T))$ and $0 < n(f_0(T)) < \infty$. It follows that $0 \in \pi_{00}^a(f_0(T))$. From $f_0(T)$ satisfies both property (UW_Π) and a-Weyl's theorem, we obtain that $f_0(T)$ is a Browder operator, and so is $T - \lambda_2 I$. The inclusion "$\supseteq$" is clear. So $\{\lambda \in \text{iso}\sigma_a(T) : n(T-\lambda I) < \infty\} = \sigma_0(T)$.

Then we prove $\sigma(T) = \sigma_a(T)$. If not, put $\lambda_1 \in \sigma(T)\backslash\sigma_a(T)$. Let $\lambda_2 \in \sigma_0(T)$ and $f_1(T) = (T-\lambda_1 I)(T-\lambda_2 I)$, then $0 \in \sigma_a(f_1(T))\backslash\sigma_{ea}(f_1(T))$. Since $f_1(T) \in (UW_\Pi)$, we can get that $f_1(T)$ is a Browder operator. It implies that $\lambda_1 \in \rho(T)$, a contradiction.

Take arbitrarily λ_0 that does not belong to the right side of (3). Without loss of generality, suppose that $\lambda_0 \in \sigma(T)$.

Case 1 Suppose that $\lambda_0 \notin \sigma_\tau(T) \cup \text{acc}\sigma_a(T)$. From $\sigma(T) = \sigma_a(T)$ we can get that $\lambda_0 \in \rho_\tau(T) \cap \text{iso}\sigma(T)$, then $\lambda_0 \in \Pi(T)$. It follows that $\lambda_0 \notin \sigma_b(T)$.

Case 2 Suppose that $\lambda_0 \notin \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup \text{acc}\sigma_a(T)$. It follows that $\lambda_0 \in \{\lambda \in \text{iso}\sigma_a(T) : n(T-\lambda I) < \infty\}$. Thus $\lambda_0 \in \sigma_0(T)$.

Case 3 Suppose that $\lambda_0 \notin \sigma_{ea}(T)$. Then $\lambda_0 \in \rho_a(T) \cup [\sigma_a(T)\backslash\sigma_{ea}(T)]$. Since $\sigma(T) = \sigma_a(T)$ and $T \in (UW_\Pi)$, we can get $\lambda_0 \notin \sigma_b(T)$.

"$\Leftarrow$". **Case 1** Suppose that $\sigma_0(T) = \emptyset$. Since T satisfies both property (UW_Π) and a-Weyl's theorem, we know that $\sigma_a(T) = \sigma_{ea}(T)$, $\Pi(T) = \pi_{00}^a(T) = \emptyset$. From the condition (2) and Lemma 3.3 we can get that $\sigma_a(f(T)) = f(\sigma_a(T)) = f(\sigma_{ea}(T)) = \sigma_{ea}(f(T))$. Thus $\sigma_a(f(T))\backslash\sigma_{ea}(f(T)) = \emptyset$. Meanwhile, $\Pi(f(T)) \subseteq f(\Pi(T)) = \emptyset$, $\pi_{00}^a(f(T)) \subseteq f(\pi_{00}^a(T)) = \emptyset$. So $f(T)$ satisfies both property (UW_Π) and a-Weyl's theorem.

Case 2 Suppose that $\sigma_0(T) \neq \emptyset$. The fact $\sigma_b(T) = [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}] \cup [\text{acc}\sigma_a(T) \cap \sigma_{ea}(T)]$ implies that $\sigma_{ea}(T) = \sigma_b(T)$. Take $\mu_0 \in \sigma_a(f(T)) \setminus \sigma_{ea}(f(T))$ and suppose that

$$f(T) - \mu_0 I = a(T-\lambda_1 I)^{n_1}(T-\lambda_2 I)^{n_2} \cdots (T-\lambda_t I)^{n_t} g(T),$$

where $\lambda_i \neq \lambda_j$ if $i \neq j$ and $g(T)$ is invertible. From the condition (2) and Lemma 3.3 we have $\lambda_i \in \rho_a(T) \cup [\sigma_a(T)\backslash\sigma_{ea}(T)]$ for $1 \leq i \leq t$. Since $\sigma_{ea}(T) = \sigma_b(T)$ and $T \in (UW_\Pi)$, we know that $\lambda_i \in \rho_b(T)$ for $1 \leq i \leq t$. It follows that $\mu_0 \in \sigma_0(f(T))$. Take arbitrarily $\mu_0 \in \Pi(f(T))$ and suppose that $f(T) - \mu_0 I$ has the same decomposition as above. Then $T - \lambda_i I$ is Drazin invertible for $1 \leq i \leq t$. Since $T \in (UW_\Pi)$, we can get that $\mu_0 \in \sigma_0(f(T))$. Take arbitrarily $\mu_0 \in \pi_{00}^a(f(T))$ and suppose that $f(T) - \mu_0 I$ has the same decomposition as above. Then $\lambda_i \in \rho_a(T) \cup \text{iso}\sigma_a(T)$ and $n(T-\lambda_i I) < \infty$. From the condition (3) we have $\mu_0 \in \sigma_0(f(T))$. Hence for any $f \in Hol(\sigma(T))$, $f(T)$ satisfies both property (UW_Π) and a-Weyl's theorem. □

From (3) in Theorem 3.4 we can get that T satisfies both property (UW_Π) and a-Weyl's theorem, and for any $\lambda \in \rho_{SF_+}(T)$, $\text{ind}(T-\lambda I) \geq 0$. Hence we have the following fact:

Corollary 3.5. *Let $T \in B(H)$ and $\sigma_0(T) \neq \emptyset$. Then for any $f \in Hol(\sigma(T))$, $f(T)$ satisfies both property (UW_Π) and a-Weyl's theorem if and only if $\sigma_b(T) = [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}] \cup [acc\sigma_a(T) \cap \sigma_{ea}(T)]$.*

Corollary 3.6. *Let $T \in B(H)$. Then $\sigma_0(T) = \emptyset$ and for any $f \in Hol(\sigma(T))$, $f(T)$ satisfies both property (UW_Π) and a-Weyl's theorem if and only if one of the following conditions holds:*

(1) $\sigma(T) = [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}] \cup \{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\} \cup [acc\sigma_a(T) \cap \sigma_{ea}(T)]$;

(2) $\sigma(T) = [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = d(T-\lambda I)\}] \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup [acc\sigma_a(T) \cap \sigma_{SF_+}(T)] \cup \{\{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cap acc[\rho_a(T) \cap \sigma(T)]\}$.

Proof. "$\Rightarrow$". The inclusion "$\supseteq$" is clear. For the converse, by Lemma 3.2 we know that for any $\lambda, \mu \in \rho_{SF_+}(T)$, $\text{ind}(T-\lambda I) \cdot \text{ind}(T-\mu I) \geq 0$.

Case 1 Suppose that $\lambda \in \rho_{SF_+}(T)$, $\text{ind}(T-\lambda I) \geq 0$. Take arbitrarily λ_0 that does not belong to the right side of (1). We claim that $\lambda_0 \notin \sigma_a(T)$. In fact, if $\lambda_0 \in \sigma_a(T)$, then $n(T-\lambda_0 I) > 0$. If $\lambda_0 \notin \sigma_\tau(T) \cup \text{acc}\sigma_a(T)$, from the proof of Theorem 3.4 we can get $\sigma(T) = \sigma_a(T)$. Then $\lambda_0 \in \text{iso}\sigma(T)$ and $\lambda_0 \in \Pi(T)$. It follows from $T \in (UW_\Pi)$ that $\lambda_0 \in \sigma_0(T)$. If $\lambda_0 \notin \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cup \text{acc}\sigma_a(T)$, then $\lambda_0 \in \pi_{00}^a(T)$ and hence $\lambda_0 \in \sigma_0(T)$. If $\lambda_0 \notin \sigma_{ea}(T)$, then $\lambda_0 \in \sigma_a(T)\backslash\sigma_{ea}(T)$. It follows that $\lambda_0 \in \sigma_0(T)$. But $\sigma_0(T) = \emptyset$. This contradiction shows that $\lambda_0 \notin \sigma_a(T)$. From $\text{ind}(T-\lambda_0 I) \geq 0$ we get that $\lambda_0 \notin \sigma(T)$.

Case 2 Suppose that $\lambda \in \rho_{SF_+}(T)$, $\text{ind}(T-\lambda I) \leq 0$. Take arbitrarily λ_0 that does not belong to the right side of (2). We claim that $\lambda_0 \notin \sigma(T)$. Similar to the proof of case 1, this claim is clear.

"$\Leftarrow$". **Case 1** If condition (1) holds, we obtain that for any $\lambda \in \rho_{SF_+}(T)$, $\text{ind}(T-\lambda I) \geq 0$. If not, there exist $\lambda_0 \in \rho_{SF_+}(T)$, $\text{ind}(T-\lambda_0 I) < 0$. It follows that $\lambda_0 \notin [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}] \cup \{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\} \cup [\text{acc}\sigma_a(T) \cap \sigma_{ea}(T)]$, then $\lambda_0 \notin \sigma(T)$, a contradiction. If there exist $\mu \in \sigma_0(T)$, then $\mu \notin [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}] \cup \{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\} \cup [\text{acc}\sigma_a(T) \cap \sigma_{ea}(T)]$. By condition (1) we can get $\mu \notin \sigma(T)$, a contradiction. Hence $\sigma_0(T) = \emptyset$. It is clear that $\{[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \Pi(T) \cup \pi_{00}^a(T)\} \cap [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}] = \emptyset$, $\{[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \Pi(T) \cup \pi_{00}^a(T)\} \cap \{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\} = \emptyset$, $\{[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \Pi(T) \cup \pi_{00}^a(T)\} \cap [\text{acc}\sigma_a(T) \cap \sigma_{ea}(T)] = \emptyset$. Thus $\{[\sigma_a(T)\backslash\sigma_{ea}(T)] \cup \Pi(T) \cup \pi_{00}^a(T)\} \subseteq \rho(T)$, a contradiction. So we can get $\sigma_a(T)\backslash\sigma_{ea}(T) = \Pi(T) = \pi_{00}^a(T) = \emptyset$, then T satisfies both property (UW_Π) and a-Weyl's theorem. From Theorem 3.4 we know that for any $f \in Hol(\sigma(T))$, $f(T)$ satisfies both property (UW_Π) and a-Weyl's theorem.

Case 2 If condition (2) holds, we get that for any $\lambda \in \rho_{SF_+}(T)$, $\text{ind}(T-\lambda I) \leq 0$. Similar to the proof of case 1, the result is trivial. □

From Corollary 3.5 and Corollary 3.6 we can describe the property (UW_Π) and a-Weyl's theorem for operator functions through the relation between $\sigma_b(T)$ and $\sigma_\tau(T)$.

Theorem 3.7. *Let $T \in B(H)$. Then for any $f \in Hol(\sigma(T))$, $f(T)$ satisfies both property (UW_Π) and a-Weyl's theorem if and only if one of the following statements holds:*

(1) $\sigma(T) = [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}] \cup \{\lambda \in \sigma_a(T) : n(T-\lambda I) = 0\} \cup [acc\sigma_a(T) \cap \sigma_{ea}(T)]$;

(2) $\sigma(T) = [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = d(T-\lambda I)\}] \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup [acc\sigma_a(T) \cap \sigma_{SF_+}(T)] \cup \{\{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cap acc[\rho_a(T) \cap \sigma(T)]\}$;

(3) $\sigma_b(T) = [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}] \cup [acc\sigma_a(T) \cap \sigma_{ea}(T)]$.

We can get that property (UW_Π) is transmitted from Drazin invertible operator to its Drazin inverse in [2]. Now, we will discuss S satisfies both property (UW_Π) and a-Weyl's theorem by topological uniform descent.

If $T \in B(H)$ is Drazin invertible with inverse S, then $\text{asc}(T) = \text{des}(T) = p$ for any $p \in \mathbb{N}$. We know that $R(T^p)$ is closed and $H = N(T^p) \oplus R(T^p)$. Under this space decomposition, $T = T_1 \oplus T_2$, where T_1 is nilpotent operator and T_2 is invertible. Thus $S = 0 \oplus T_2^{-1}$. In [1, 4, 5], we get that

$$\sigma(S)\backslash\{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma(T) \setminus \{0\}\}, \sigma_*(S)\backslash\{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma_*(T) \setminus \{0\}\}, * \in \{b, ea, a, \tau, D\}.$$

Besides, one can verify that for any $\lambda \neq 0$, $n(S-\lambda I) = n(T - \frac{1}{\lambda}I)$, $d(S-\lambda I) = d(T - \frac{1}{\lambda}I)$ and $\text{acc}\sigma_a(S) \setminus \{0\} = \{\frac{1}{\lambda} : \lambda \in \text{acc}\sigma_a(T) \setminus \{0\}\}$.

Theorem 3.8. *Let $T \in B(H)$ be Drazin invertible with inverse S. Then*

(1) T satisfies both property (UW_{Π}) and a-Weyl's theorem if and only if S satisfies both property (UW_{Π}) and a-Weyl's theorem;

(2) For any $f \in Hol(\sigma(T)) \cap Hol(\sigma(S))$, $f(T)$ satisfies both property (UW_{Π}) and a-Weyl's theorem if and only if $f(S)$ satisfies both property (UW_{Π}) and a-Weyl's theorem.

Proof. (1). "⇒". From Theorem 2.1, we only need $\sigma_b(S) = [\sigma_\tau(S) \cap \{\lambda \in \mathbb{C} : n(S-\lambda I) = d(S-\lambda I)\}] \cup \{\lambda \in \sigma(S) : n(S-\lambda I) = 0\} \cup [\mathrm{acc}\sigma_a(S) \cap \sigma_{ea}(S)] \cup \{\{\lambda \in \mathbb{C} : n(S-\lambda I) = \infty\} \cap \mathrm{acc}[\rho_a(S) \cap \sigma(S)]\}$. If T is invertible, then $S = T^{-1}$. The conclusion is clear. In the following, we assume T is not invertible but Drazin invertible.

Let λ does not belong to the right side. If $\lambda = 0$. Since $0 \in \Pi(T)$ and $T \in (UW_{\Pi})$, we get that $0 \notin \sigma_b(T)$. Thus $0 \notin \sigma_b(S)$([2, Lemma 4.9]). If $\lambda \neq 0$. Now, $S - \lambda I = \begin{pmatrix} -\lambda I & 0 \\ 0 & \lambda T_2^{-1}(\frac{1}{\lambda}I - T_2) \end{pmatrix}$, then $\frac{1}{\lambda} \notin [\sigma_\tau(T_2) \cap \{\lambda \in \mathbb{C} : n(T_2-\lambda I) = d(T_2-\lambda I)\}] \cup \{\lambda \in \sigma(T_2) : n(T_2-\lambda I) = 0\} \cup [\mathrm{acc}\sigma_a(T_2) \cap \sigma_{ea}(T_2)] \cup \{\{\lambda \in \mathbb{C} : n(T_2-\lambda I) = \infty\} \cap \mathrm{acc}[\rho_a(T_2) \cap \sigma(T_2)]\}$. Under above space decomposition, we know that $T - \frac{1}{\lambda}I = \begin{pmatrix} T_1 - \frac{1}{\lambda}I & 0 \\ 0 & T_2 - \frac{1}{\lambda}I \end{pmatrix}$, where $T_1 - \lambda I$ is invertible. So $\frac{1}{\lambda} \notin [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = d(T-\lambda I)\}] \cup \{\lambda \in \sigma(T) : n(T-\lambda I) = 0\} \cup [\mathrm{acc}\sigma_a(T) \cap \sigma_{ea}(T)] \cup \{\{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\} \cap \mathrm{acc}[\rho_a(T) \cap \sigma(T)]\}$. By Theorem 2.1 we can get $\frac{1}{\lambda} \notin \sigma_b(T)$ and so $\frac{1}{\lambda} \notin \sigma_b(S)$.

"⇐". Suppose that S satisfies both property (UW_{Π}) and a-Weyl's theorem. The Drazin inverse of S is $U := T^2S = TST$ and Drazin inverse of U is T ([1, Chapter 1]). Thus, T satisfies both property (UW_{Π}) and a-Weyl's theorem.

(2). "⇒". It is obvious that T satisfies both property (UW_{Π}) and a-Weyl's theorem. Then from (1), we get that S satisfies both property (UW_{Π}) and a-Weyl's theorem. Suppose that $\sigma_0(S) \neq \emptyset$, we claim that $\sigma_0(T) \neq \emptyset$. In fact, let $\lambda \in \sigma_0(S)$. If $\lambda = 0$, then $0 < \dim N(T^p) < \infty$. It follows that $n(T_1) > 0$ and T is not invertible. By $T \in (UW_{\Pi})$ we have that T_1 is Browder operator. So T is Browder operator and $0 \in \sigma_0(T)$. If $\lambda \neq 0$. Now, $S - \lambda I = \begin{pmatrix} -\lambda I & 0 \\ 0 & \lambda T_2^{-1}(\frac{1}{\lambda}I - T_2) \end{pmatrix}$. It follows that $\frac{1}{\lambda}I - T_2$ is Browder operator but not invertible. We know that $T - \frac{1}{\lambda}I = \begin{pmatrix} T_1 - \frac{1}{\lambda}I & 0 \\ 0 & T_2 - \frac{1}{\lambda}I \end{pmatrix}$, then $\frac{1}{\lambda} \in \sigma_0(T)$. From Corollary 3.5 we get $\sigma_b(T) = [\sigma_\tau(T) \cap \{\lambda \in \mathbb{C} : n(T-\lambda I) = \infty\}] \cup [\mathrm{acc}\sigma_a(T) \cap \sigma_{ea}(T)]$. By using the similar way of (1) we get that $\sigma_b(S) = [\sigma_\tau(S) \cap \{\lambda \in \mathbb{C} : n(S-\lambda I) = \infty\}] \cup [\mathrm{acc}\sigma_a(S) \cap \sigma_{ea}(S)]$. Moreover, $\sigma_{ea}(f(S)) = f(\sigma_{ea}(S))$ for any $f \in Hol(\sigma(T)) \cap Hol(\sigma(S))$. From Lemma 3.3 and Theorem 3.4 $f(S)$ satisfies both property (UW_{Π}) and a-Weyl's theorem.

"⇐". The same as the above proof. □

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