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An exact computation for mixed multifractal dimensions of sets and measures

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Abstract. In the present work, we are concerned with the estimation of some mixed variants of multifractal dimensions for a special class of measures characterized by a weak Ahlfors assumption applying mixed multifractal generalizations of Hausdorff and packing measures. Exact computation of such dimensions is shown to be valid for a class of Moran-type measures in some special cases.

1. Introduction

In recent years, fractals have been widely mentioned in many fields of science. Their application solves a lot of problems. The most commonly used concepts to characterize fractals are their dimensions, such as Hausdorff, packing, box-counting, and Rényi dimensions ([15–17, 21]). Different variants, generalizations and simplified forms have been derived according to the need and the use. One of the most known practical cases are the so-called Moran sets and measures ([8, 20, 25]).

In [9], Dai applied these concepts to a class of Moran sets satisfying the strong separation condition, and showed that the Hausdorff and packing measures are equivalent, although their corresponding dimensions are different. In [1, 2], some density results have been generalized based on the multifractal formalism introduced by Cole [7]. Inspired from [26], Selmi and collaborators developed different approaches for the computation of the exact multifractal dimension, and the equivalence between the Hausdorff and packing measures on a class of Moran sets satisfying the strong separation condition. The authors in [22, 23] extended the mixed multifractal analysis to a large class of non necessary Gibbs cases. The authors in [3, 4, 18, 19] conducted an extension to the case of mixed multifractal generalization of Hausdorff and packing measures, and introduced a simultaneous density characteristic of finitely many measures, where one measure at least satisfies a quasi-Ahlfors property.

In [12] a decomposition theorem of Besicovitch's type is established for the regularities of general Hausdorff and packing functions, which is applied by the next to describe a Tricot's density theorem.

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In [13], projection related results are established for relative multifractal box-dimensions, density and multifractal dimensions. Selmi in [27] improve many known density results by connecting various relative multifractal measures and density theorems. The equivalence of the relative multifractal Hausdorff and packing measures on cookie-cutter-like sets is discussed in the case of a relative strong separation condition.

In [28], the authors are concerned with some multifractal dimensions estimations of vector-valued measures where the measures are no longer Gibbs.

In [29], relative multifractal formalism is established in the case of generalized Hausdorff and packing measures in a special case where such measures differ.

In [5], some non-necessary Gibbs vector-valued measures are studied in the framework of the mixed multifractal analysis based on gauge functions and suitable multifractal densities.

In [14], some non-regular homogeneous Moran measures are investigated. Sufficient conditions are shown to yield an explicit computation of the relative multifractal dimensions of the level sets. Mutual singularities of the relative multifractal measures are investigated for the homogeneous Moran case with different multifractal dimensions.

We recall finally that the results established in the present paper are discussed in some special cases in [30, 31, 33, 34] improving results in [1, 2].

In the present paper, we analyse the correlation between the mixed generalized Hausdorff and packing measures relatively to a suitable vector-valued measure $\xi = (\mu, \nu) = (\mu_1, \mu_2, \dots, \mu_k, \nu)$ composed of Borel probability measures on $\mathbb{R}^n$, $n, k \in \mathbb{N}$ fixed. We estimate a corresponding entropy-type dimension in the case where the vector-valued measure satisfies some weak Ahlfors assumption on the regularity. We also analyse the equivalence between the mixed generalized Hausdorff and packing measures on a class of Moran sets in some special cases. Our results join and improved many cases already developed such as [8–11]

Finally, before developing our main results, it is worthy to mention that by adopting both the proofs in [31] with necessary modifications, our results may be extended effectively to non-atomic measures. However, we stress on the fact that in our main results we already assumed that the gauge measure ν is non-atomic. It is also easily noticeable that in [30, 31], the case of a single measure is considered. The results may be extended to our mixed case in a natural way. The quantity $\mu(B(x,r))^q$, $\mu \in \mathcal{P}(\mathbb{R}^n)$ and $q \in \mathbb{R}$ will be replaced by $\mu_1(B(x,r))^{q_1} \dots \mu_k(B(x,r))^{q_k}$. In general when, $q \geq 0$ or the $q_i \geq 0$, $\forall i$, the results is an adoption of the same proofs. For $q < 0$ (the single case), or some of the $q_i < 0$ (the mixed case), we may use the doubling property or the property $\mathcal{P}^{q,t}_{\mu,\nu} < \infty$.

The present paper is organized as follows. Section 2 is devoted to the development of some general concepts to be used later. Section 3 is concerned with the presentation of our main results. In particular, new and direct proofs of main results are provided. Section 4 is devoted to the development of a practical case due to our theoretical results provided previously. An exact computation of the fractal dimension of a Moran type construction is provided.

2. General settings

In this section, we aim to introduce the general tools that will be applied next. We will review in brief the notion of mixed multifractal generalizations of Hausdorff and packing measures and the associated dimensions already introduced in [3, 18], and next recall the mixed multifractal generalisations of densities associated to vector valued measures developed in [4, 19].

For $n \geq 1$ an integer, consider on $\mathbb{R}^n$, the set $\mathcal{P}(\mathbb{R}^n)$ of Borel probability measures, and the set of Borel quasi-Ahlfors probability measures defined by

$$\mathcal{QAH}(\mathbb{R}^n) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^n);\ \exists \alpha > 0,\ \text{such that,}\ \limsup_{|U| \longrightarrow 0} \frac{\mu(U)}{|U|^\alpha} < +\infty \right\}.$$

U being a subset of $\mathbb{R}^n$, and $|U|$ its diameter, $|U| = \sup_{x,y \in U} \|x - y\|_2$, where $\|x - y\|_2$ is the usual Euclidean norm on $\mathbb{R}^n$. For $\mu \in \mathcal{QAH}(\mathbb{R}^n)$, the corresponding exponent α is called the quasi-Ahlfors exponent, or quasi-Ahlfors regularity, or also the quasi-Ahlfors index of μ, and μ is said to be α-quasi-Ahlfors regular.

Let $k \in \mathbb{N}$ be fixed. For a vector-valued measure $\mu = (\mu_1, \mu_2, ..., \mu_k) \in \mathcal{P}(\mathbb{R}^n)^k$, $q = (q_1, q_2, ..., q_k) \in \mathbb{R}^k$. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and $r > 0$, denote

$$\mu(B(x,r)) = (\mu_1(B(x,r)), ..., \mu_k(B(x,r))),$$

and

$$(\mu(B(x,r))^q = \prod_{i=1}^{k} (\mu_i(B(x,r))^{q_i},$$

where $B(x,r)$ is the ball of center x and radius r in $\mathbb{R}^n$. Next, for a Borel vector-valued probability measure $\xi = (\mu, \nu) = (\mu_1, \mu_2, ..., \mu_k, \nu) \in \mathcal{P}(\mathbb{R}^n)^k \times \mathcal{QAH}(\mathbb{R}^n)$ and $(q,t) = (q_1, q_2, ..., q_k, t) \in \mathbb{R}^{k+1}$, we write

$$\Phi_{\xi}^{q,t}(B(x,r)) = \mu(B(x,r)))^q (\nu(B(x,r)))^t.$$

For $E \subset \mathbb{R}^n$, and $\varepsilon > 0$, let

$$\overline{\mathcal{H}}_{\xi}^{q,t}(E) = \lim_{\varepsilon \downarrow 0} \overline{\mathcal{H}}_{\xi,\varepsilon}^{q,t}(E) = \lim_{\varepsilon \downarrow 0} \left(\inf \sum_i \Phi_{\xi}^{q,t}(B(x_i, r_i)) \right),$$

and

$$\mathcal{H}_{\xi}^{q,t}(E) = \sup_{F \subseteq E} \overline{\mathcal{H}}_{\xi}^{q,t}(F).$$

The lower bound above is taken over all centred ε-coverings of E. We mean here by an ε-covering any collection of balls $(B(x_i, r_i))_i$, with $x_i \in E$, $r_i \leq \varepsilon$, and $E \subset \underset{i}{\cup} B(x_i, r_i)$. The quantity $\mathcal{H}_{\xi}^{q,t}$ is known as the generalized mixed Hausdorff measure relatively to ξ. Similarly, let

$$\overline{\mathcal{P}}_{\xi}^{q,t}(E) = \lim_{\varepsilon \downarrow 0} \overline{\mathcal{P}}_{\xi,\varepsilon}^{q,t}(E) = \lim_{\varepsilon \downarrow 0} \left(\sup \sum_i \Phi_{\xi}^{q,t}(B(x_i, r_i)) \right),$$

and

$$\mathcal{P}_{\xi}^{q,t}(E) = \inf_{E \subseteq \cup_i E_i} \sum_i \overline{\mathcal{P}}_{\xi}^{q,t}(E_i).$$

The upper bound above is taken over all centred ε-packings of E. We mean by centred ε-packings any collection of balls $(B(x_i, r_i))_i$, with $x_i \in E$, $r_i \leq \varepsilon$, and $B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$, for all i, j; $i \neq j$. The quantity $\mathcal{P}_{\xi}^{q,t}$ is known as the generalized mixed packing measure relatively to ξ. See [3, 18] for backgrounds on such measures. The authors showed there that the quasi-Ahlfors assumption allows to associate to these measures some mixed generalisations of Hausdorff and packing dimensions, and a mixed generalizations of the Rényi entropy dimension as follows,

$$dim_{\xi}^{q}(E) = \inf\left\{t \in \mathbb{R};\ \mathcal{H}_{\xi}^{q,t}(E) = 0\right\} = \sup\left\{t \in \mathbb{R};\ \mathcal{H}_{\xi}^{q,t}(E) = \infty\right\},$$

$$Dim_{\xi}^{q}(E) = \inf\left\{t \in \mathbb{R};\ \mathcal{P}_{\xi}^{q,t}(E) = 0\right\} = \sup\left\{t \in \mathbb{R};\ \mathcal{P}_{\xi}^{q,t}(E) = \infty\right\},$$

and

$$\Delta_{\xi}^{q}(E) = \inf\left\{t \in \mathbb{R};\ \overline{\mathcal{P}}_{\xi}^{q,t}(E) = 0\right\} = \sup\left\{t \in \mathbb{R};\ \overline{\mathcal{P}}_{\xi}^{q,t}(E) = \infty\right\}.$$

We now recall the generalized mixed multifractal density of measures introduced in [4, 19]. Original definitions are introduced in [6, 7]. Let $\theta \in \mathcal{P}(\mathbb{R}^n)$, and $x \in Support(\theta)$, The upper and lower generalized mixed multifractal (q,t)-densities of θ at x, with respect to ξ, are defined respectively by

$$\overline{d}_{\xi}^{q,t}(x, \theta) = \limsup_{r \to 0} \frac{\theta(B(x,r))}{\Phi_{\xi}^{q,t}(B(x,r))},$$

and

$$\underline{d}_{\xi}^{q,t}(x,\theta)=\liminf_{r\to 0}\frac{\theta(B(x,r))}{\Phi_{\xi}^{q,t}(B(x,r))}.$$

Whenever $\overline{d}_{\xi}^{q,t}(x,\theta)=\underline{d}_{\xi}^{q,t}(x,\theta)$, we denote the common value by $d_{\xi}^{q,t}(x,\theta)$, and call it the (q,t)-density of θ at x with respect to ξ.

The following result is proved in [4, 19] as a general case of [1, 2, 26], and it provides lower and upper bounds for the generalized mixed multifractal (q,t)-density.

Theorem 2.1. *[4, 19] Let E be a Borel subset of Support(ξ). The following assertions hold.*

1. *If $\mathcal{H}_{\xi}^{q,t}(E)<\infty$, there exists constants K_1 and K_2; $0<K_1\leq K_2<\infty$ such that*

$$K_1\mathcal{H}_{\xi}^{q,t}(E)\inf_{x\in E}\overline{d}_{\xi}^{q,t}(x,\theta)\leq\theta(E)\leq K_2\mathcal{H}_{\xi}^{q,t}(E)\sup_{x\in E}\overline{d}_{\xi}^{q,t}(x,\theta).\tag{2.1}$$

2. *If $\mathcal{P}_{\xi}^{q,t}(E)<\infty$, then*

$$\mathcal{P}_{\xi}^{q,t}(E)\inf_{x\in E}\underline{d}_{\xi}^{q,t}(x,\theta)\leq\theta(E)\leq\mathcal{P}_{\xi}^{q,t}(E)\sup_{x\in E}\underline{d}_{\xi}^{q,t}(x,\theta).\tag{2.2}$$

Next, we need to introduce the following quantities which will be useful later. For a single measure $\mu\in\mathcal{P}(\mathbb{R}^n)$, and $a>1$, we write

$$P_a(\mu)=\limsup_{r\downarrow 0}\left(\sup_{x\in S_\mu}\frac{\mu(B(x,ar))}{\mu(B(x,r))}\right),$$

and for a vector-valued measure $\mu=(\mu_1,\mu_2,...,\mu_k)\in(\mathcal{P}(\mathbb{R}^n))^k$, we write

$$P_a(\mu)=\bigcap_{i=1}^{k}P_a(\mu_i).$$

Finally, we denote the set of the so-called doubling vector-valued measures on $\mathbb{R}^n$ by

$$P_D(\mathbb{R}^n)=\bigcup_{a>1}\{\mu\in\mathcal{P}(\mathbb{R}^n);\ P_a(\mu)<\infty\}.$$

We denote also

$$\mathcal{QAH}_D(\mathbb{R}^n)=\mathcal{QAH}(\mathbb{R}^n)\cap P_D(\mathbb{R}^n).$$

3. Main results

In this section, we propose to provide a generalization of a result proved in [2]. Let μ and ν be Borel probability measures on $\mathbb{R}^n$. We say that μ is absolutely continuous with respect to ν if for any set $A\subset\mathbb{R}^n$, $\nu(A)=0$ implies $\mu(A)=0$. We write $\nu\ll\mu$. We also say that μ and ν are mutually singular if there exists $A\subset\mathbb{R}^n$, such that $\mu(A)=\nu(\mathbb{R}^n\setminus A)=0$, and we write $\mu\perp\nu$. Finally we say that μ is non-atomic if $\mu(\{x\})=0$, $\forall x\in\mathbb{R}^n$.

Theorem 3.1. *[2] Let $\alpha,q\in\mathbb{R}$, $\mu,\nu\in\mathcal{P}_D(\mathbb{R}^n)$ and θ a probability measure on $\mathbb{R}^n$. If ν is non-atomic, then, the following assertions are equivalent.*

1. *θ has α as a multifractal exact dimension.*
2. *We have*
   (a) *There exist a set $S\subset\mathbb{R}^n$ with $dim_{\mu,\nu}^q(S)=\alpha$, such that $\theta(S)=1$.*
   (b) *If $E\subset\mathbb{R}^n$ satisfies $dim_{\mu,\nu}^q(E)<\alpha$, then, $\theta(E)=0$.*

3. *We have*
   (a) *θ is absolutely continuous with respect to $\mathcal{H}^{q,\gamma}_{\mu,\nu}$ for all $\gamma < \alpha$.*
   (b) *θ and $\mathcal{H}^{q,\beta}_{\mu,\nu}$ are mutually singular for all $\beta > \alpha$.*

The first extension of [1, 2] to the mixed case starts with the following definition.

Definition 3.2. *Let $\xi = (\mu, \nu) = (\mu_1, \mu_2, ..., \mu_k, \nu)$ be a vector-valued measure composed of Borel probability measures on $\mathbb{R}^n$, $q = (q_1, q_2, \dots, q_k) \in \mathbb{R}^k$, and $\alpha \in \mathbb{R}$. A Borel probability measure θ on $\mathbb{R}^n$ is said to posses α as mixed q-multifractal exact dimension if*

$$\underline{\lim}_{r\to 0} \frac{\log\theta(B(x,r)) - \log\mu(B(x,r))^q}{\log\nu(B(x,r))} = \alpha, \text{ for } \theta.\ a.\ a\ x \in \mathbb{R}^n.$$

Next, as in the single multifractal case, we speak here also about the mixed multifractal extensions of Hausdorff and packing dimensions of Borel probability measures, and relate them to the mixed multifractal exact dimension introduced in Definition 3.2 above.

For a vector-valued Borel probability measure $\xi = (\mu, \nu) = (\mu_1, \mu_2, \dots, \mu_k, \nu)$, and θ on $\mathbb{R}^n$, we write

$$\mathcal{H}^{q,t}_{\xi}(\theta) = \inf_E \left\{\mathcal{H}^{q,t}_{\xi}(E),\ \theta(E) = 1\right\}, \tag{3.1}$$

and the dual measure

$$\mathcal{P}^{q,t}_{\xi}(\theta) = \inf_E \left\{\mathcal{P}^{q,t}_{\xi}(E),\ \theta(E) = 1\right\}. \tag{3.2}$$

Write also

$$dim^q_{\xi}\theta = \inf_E \left\{dim^q_{\xi}E,\ \theta(E) = 1\right\}, \tag{3.3}$$

and

$$Dim^q_{\xi}\theta = \inf_E \left\{Dim^q_{\xi}E,\ \theta(E) = 1\right\}. \tag{3.4}$$

As in the single multifractal case, we may also show here that

$$dim^q_{\xi}\theta = \sup\left\{t \geq 0,\ \mathcal{H}^{q,t}_{\xi}(\theta) = \infty\right\} = \inf\left\{t \geq 0,\ \mathcal{H}^{q,t}_{\xi}(\theta) = 0\right\}, \tag{3.5}$$

and

$$Dim^q_{\xi}\theta = \sup\left\{t \geq 0,\ \mathcal{P}^{q,t}_{\xi}(\theta) = \infty\right\} = \inf\left\{t \geq 0,\ \mathcal{P}^{q,t}_{\xi}(\theta) = 0\right\}. \tag{3.6}$$

We get also the inequalities

$$dim^q_{\xi}(support(\mu) \cap support(\nu)) \geq dim^q_{\xi}\theta,$$

and

$$Dim^q_{\xi}(support(\mu) \cap support(\nu)) \geq Dim^q_{\xi}\theta.$$

As [1, 2], Proposition 1, we get here an analogue for the mixed case.

Proposition 3.3. *There exists a minimum-dimension support S_θ satisfying*

$$dim^q_{\xi}S_\theta = dim^q_{\xi}\theta.$$

The proof is a simple compilation of the one due to [2], Proposition 1, and thus omitted here, and left to the reader.

Now, a first main extension to the case of mixed multifractal densities introduced in [4, 19] consists as in the single case in relating these densities to the concept of mixed multifractal dimensions. We get the following result.

Proposition 3.4. *Let $x \in support(\xi)$, $q = (q_1, q_2, ..., q_k) \in \mathbb{R}^k$, and $\alpha \in \mathbb{R}$. Assume further that ν is non-atomic, and that there exists a real number α_0 satisfying*

$$\overline{d}_{\xi}^{q,\alpha}(x,\theta) = \begin{cases} \infty \text{ if } \alpha > \alpha_0, \\ 0 \text{ if } \alpha < \alpha_0. \end{cases}$$

Then, θ has α_0 as a mixed q-multifractal exact dimension.

Proof. Denote $\alpha_\theta = \underline{\lim}_{r\to 0} \dfrac{\log\theta(B(x,r)) - \log\mu(B(x,r))^q}{\log\nu(B(x,r))}$, and let $\alpha > \alpha_0$. So, $\overline{d}_{\xi}^{q,\alpha}(x,\theta) = \infty$. There exists $\delta > 0$, small enough, for which

$$\frac{\theta(B(x,r))}{\Phi_{\xi}^{q,\alpha}(B(x,r))} \geq 1, \ \forall\, r,\ 0 < r < \delta.$$

As a result,

$$\theta(B(x,r)) \geq \Phi_{\xi}^{q,\alpha}(B(x,r)), \ \forall\, r,\ 0 < r < \delta.$$

Applying next the logarithm in both sides, we get

$$\frac{\log\theta(B(x,r)) - \log(\mu(B(x,r))^q}{\log\nu(B(x,r))} < \alpha, \ \forall\, r,\ 0 < r < \delta.$$

which gives easily $\alpha_\theta \leq \alpha$, $\forall\alpha > \alpha_0$, which in turns yields that $\alpha_\theta \leq \alpha_0$.
The converse is similar. So, we get $\alpha_\theta = \alpha_0$.

Now, our second main result in the present paper constitutes a mixed multifractal generalization of Theorem 3.1, and is subject of Theorem 3.5 below. The second improvement of [2] consists in the development of a different and direct proof for our extension in Theorem 3.5.

Theorem 3.5. *With the same notations as in Definition 3.2, assume that θ is non-atomic, and that $\xi \in (P_D(\mathbb{R}^n))^k \times \mathcal{QAH}_D(\mathbb{R}^n)$. The following assertions are equivalent.*

1. *α is the the mixed q-multifractal exact dimension of θ.*
2. *There exists a set $S \subset \mathbb{R}^n$ with $dim_{\xi}^{q} S = \alpha$, and $\theta(S) = 1$. Moreover, for any $E \subset \mathbb{R}^n$ with $dim_{\xi}^{q} E < \alpha$, we have $\theta(E) = 0$.*
3. *θ is absolutely continuous with respect to $\mathcal{H}_{\xi}^{q,\gamma}$, $\forall\,\gamma < \alpha$, and θ and $\mathcal{H}_{\xi}^{q,\beta}$ are mutually singular, $\forall\,\beta > \alpha$.*

Proof.
(1) $\Longrightarrow$ (2) follows immediately from Proposition 3.3, and equations (3.1), (3.3) and (3.5).
(2) $\Longrightarrow$ (3) Let $A \subset \mathbb{R}^n$ be such that $\mathcal{H}_{\xi}^{q,\gamma}(A) = 0$, with $\gamma < \alpha$. We immediately observe that $dim_{\xi}^{q} A \leq \gamma < \alpha$. Therefore, Assertion 2 implies that $\theta(A) = 0$, which shows that $\theta \ll \mathcal{H}_{\xi}^{q,\gamma}$.
Now, take $A = S$ in Assertion 2. We get in one hand $\theta(A) = 1$. On the other hand, again from Assertion 2, we have $\beta > \alpha = dim_{\xi}^{q} A$, which yields that $\mathcal{H}_{\xi}^{q,\beta}(A) = 0$. Therefore, $\theta \perp \mathcal{H}_{\xi}^{q,\beta}$.
(3) $\Longrightarrow$ (1) Observe firstly that

$$\overline{\lim_{r\to 0}} \frac{\theta(B(x,r))}{\Phi_{\xi}^{q,t}(B(x,r))} < \infty.$$

As a consequence,

$$\overline{d}_{\xi}^{q,\gamma-\varepsilon}(x,\theta) = 0, \ \forall\varepsilon > 0.$$

Hence, $dim_{\xi}^{q}(\theta) \leq \gamma - \varepsilon$, $\forall\varepsilon > 0$, and $\forall\gamma < \alpha$. Therefore, $dim_{\xi}^{q}(\theta) \leq \alpha$.
On the other hand, by using similarly the fact that $\theta \perp \mathcal{H}_{\xi}^{q,\beta}$, $\forall\beta > \alpha$, we deduce that $dim_{\xi}^{q}(\theta) \geq \alpha$. So as the result.

4. A Moran case for the mixed q-multifractal exact dimension

In this section, we propose to construct a special case of Moran sets to apply our theoretical findings already exposed in the previous section. Denote $n_k = 2^k$, $k \in \mathbb{N}$, and consider a vector of real numbers $(c_{i_1}, c_{i_2}, ..., c_{i_{n_k}}) \subset (0,1)^{n_k}$, with $\sum_{j=1}^{n_k} c_{i_j} \leq 1$,. Write next $D_k = \left\{(i_1, i_2, ...i_k) : 1 \leq i_j \leq n_j,\ 1 \leq j \leq k\right\}$. Denote finally, $D_0 = \emptyset$, and $D = \underset{k\geq 0}{\cup} D_k$. For $\sigma = (i_1, i_2, ...i_k) \in D_k$, and $\tau = (j_1, j_2, ...j_m) \in D_l$, we write $\sigma * \tau = (i_1, i_2, ...i_k, j_1, j_2, ...j_m)$. Consider next the unit interval $I = [0,1]$, and the collection $\mathcal{F} = \{I_\sigma, \sigma \in D\}$ of sub-intervals of I for which $I_\emptyset = I$, and for all $k \geq 1$, $\sigma, \tau \in D_k$, $\sigma \neq \tau$, we have $int(I_\sigma) \cap int(I_\tau) = \emptyset$, where $int(.)$ denotes the topological interior. Assume further that for all $k \geq 1$, for all $1 \leq j \leq n_k$, and $i \in D_k$, we have $|I_{i*j}| = c_j |I_i|$, where $|.|$ denotes the diameter of I. The system $(I, \mathcal{F})$ is known as a Moran structure. The set $E = \underset{k\geq 1}{\cap} \underset{\sigma\in D_k}{\cup} I_\sigma$ is called Moran set associated to $\mathcal{F}$. For $k \in \mathbb{N}$, the collection $\mathcal{F}_k$=$\{I_\sigma, \sigma \in D_k\}$ is the k-order fundamental sets of E. The interval I is called the original set of E. Whenever $\lim_{k\to\infty} \max_{\sigma\in D_k} |I_\sigma| = 0$, then, for all $i \in D$, the intersection set $\underset{n\geq 1}{\cap} I_{i_1 i_2 .....i_k}$ is reduced to a single point denoted by $\varphi(i)$. Next, for $k \in \mathbb{N}$, and $\sigma = (i_1, i_2, ...i_k, ...) \in D$, we write $\sigma|_k$ to designate the truncation $\sigma|_k = (i_1, i_2, ...i_k)$, and we denote accordingly, $I_k(\sigma) = I_{\sigma|_k} = I_{i_1 i_2 .....i_k}$. Finally, we assume that the Moran sets E satisfies a strong separation condition in the sense that for all $\sigma \in D$,

$$dist(I_{\sigma*i}, I_{\sigma*j}) \geq a_k \times |I_\sigma|,\ \forall\, i \neq j, \tag{4.1}$$

where $(a_k)_k$ is a sequence of positive real numbers, bounded away from 0.

Let next $(\xi, \theta) = (\mu, \nu, \theta) = (\mu_1, \mu_2, \ldots, \mu_k, \nu, \theta) \in \mathcal{P}(\mathbb{R}^n)^{k+2}$ be such that $support(\theta) \subset E$. We will show that the Moran construction permits the computation of the mixed q-multifractal exact dimension of sets in the sense of the following definition.

Definition 4.1. *We say that the collection $\mathcal{F}$ permits the computation of the mixed q-multifractal exact dimension of sets if*

$$\overline{d}^{q,t}_{\xi,\mathcal{F}}(x,\theta) = \underline{\lim}_{n\to+\infty} \frac{\theta(I_n(i))}{\mu^q(I_n(i))\nu^t(I_n(i))} = \begin{cases} 0 \text{ if } t < \alpha \\ \infty \text{ if } t > \alpha \end{cases} \text{ for any } i \in D, \tag{4.2}$$

where α is the mixed q-multifractal exact dimension of θ.

Definition 4.2. *We say that two Borel measures μ and ν are equivalent and we write $\mu \sim \nu$ if for any Borel set A, we have*

$$\mu(A) = 0 \ \Leftrightarrow \nu(A) = 0.$$

The following theorem confirms effectively that the collection $\mathcal{F}$ of the Moran construction above permits the computation of the mixed q-multifractal exact dimension of the measure θ.

Theorem 4.3. *Let $E \subset I$ be the Moran set described above for which the assumption (4.1) holds, and $\alpha \in \mathbb{R}$ satisfying (4.2) with $\mathcal{P}^{q,\alpha}_\xi(E) < \infty$. The following assertions hold.*

1. *The collection $\mathcal{F}$ permits the computation of the mixed q-multifractal exact dimension of θ. Furthermore, $dim^q_\xi(E) = dim^q_\xi(\theta) = \alpha$.*
2. *Assume that $0 < \overline{d}^{q,t}_{\xi,\mathcal{F}}(x,\theta) < \infty$, $\forall\, i \in D$, then the restrictions of θ and $\mathcal{H}^{q,\alpha}_\xi$ on E are equivalent.*

Proof.
1) Using (4.1), we deduce that $\overline{d}^{q,\alpha}_{\xi}(x,\theta) = \overline{d}^{q,\alpha}_{\xi,\mathcal{F}}(x,\theta)$, $\forall\, x = \varphi(i) \in E$.
Next, using Theorem 2.1, Assertion 1, we get $dim^q_\xi(E) = dim^q_\xi(\theta)$. Using 3.4, we have $\alpha = dim^q_\xi \theta$. Assertion 1 is thus proved.

2) Let $B \subset E$ be a Borel set with $\theta(B) = 0$. We shall show that $\mathcal{H}_{\xi}^{q,\alpha}(B) = 0$. Consider a sequence of open sets $(B_k)_k$ such that

$$B \subset B_k \text{ and } \theta(B_k) \leq 2^{-k}, \ \forall\, k \geq 0.$$

Denote next for $k \in \mathbb{N}$, $\theta_k(.) = \theta(. \cap B_k)$, and for $n \in \mathbb{N}$,

$$\Omega_n = \left\{ i \in D;\ \overline{d}_{\xi}^{q,t}(\varphi(i), \theta) < n \right\}.$$

From Theorem 2.1, Assertion 1, We obtain

$$\begin{aligned} K_1 \mathcal{H}_{\xi}^{q,\alpha}(B \cap \varphi(\Omega_n)) &\leq \theta_k(B) \times \sup_{x \in B \cap \varphi(\Omega_n)} \frac{1}{d_{\xi}^{q,\alpha}(x, \theta_k)} \\ &\leq \theta(B) \times \sup_{x \in B \cap \varphi(\Omega_n)} \frac{1}{d_{\xi}^{q,\alpha}(x, \theta)} \\ &\leq n 2^{-k}, \ \forall\, n, k. \end{aligned}$$

As a results,

$$\mathcal{H}_{\xi}^{q,\alpha}(B \cap \varphi(\Omega_n)) = 0, \forall n \in \mathbb{N},$$

which leads to $\mathcal{H}_{\xi}^{q,\alpha}(B) = 0$.
The opposite sense ($\mathcal{H}_{\xi}^{q,\alpha}(B) = 0 \implies \theta(b) = 0$) may be proved by similar techniques.

By similar arguments and techniques as in Theorem 4.3, we can prove the following result.

Theorem 4.4. *Let $E \subset I$ be the Moran set described above for which the assumption (4.1) holds, and $\alpha \in \mathbb{R}$ satisfying (4.2) with $\mathcal{P}_{\xi}^{q,\alpha}(E) < \infty$. The following assertions hold.*

1. *$Dim_{\xi}^{q} E = Dim_{\xi}^{q} \theta = \alpha$.*
2. *Assume that $0 < \overline{d}_{\xi,\mathcal{F}}^{q,t}(x, \theta) < \infty$, $\forall\, i \in D$, then the restrictions of θ and $\mathcal{P}_{\xi}^{q,\alpha}$ on E are equivalent.*

Finally, as a consequence of Theorem 4.3, and Theorem 4.4, we deduce the following results which gives a case of equivalence for the mixed Hausdorff measure $\mathcal{H}_{\xi}^{q,t}$ and the mixed packing measure $\mathcal{P}_{\xi}^{q,t}$.

Corollary 4.5. *Let $E \subset I$ be a Moran set satisfying (4.1). Let also $(\xi, \theta) = (\mu, \nu, \theta) \in \mathcal{P}(\mathbb{R}^n)^{k+1} \times \mathcal{AHP}(\mathbb{R}^n)$ be a vector-valued probability measure on $\mathbb{R}^n$, such that support$(\theta) \subset E$, and α satisfying (4.2). Then the restrictions of the measures θ, $\mathcal{H}_{\xi}^{q,\alpha}$ and $\mathcal{P}_{\xi}^{q,\alpha}$ are equivalent.*

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