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# Bounded approximate version of module character contractibility of Banach algebras

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**Abstract.** The (bounded) approximate version of module character contractibility of Banach algebras is introduced and studied. This new concept is characterized by several different concepts such as bounded approximate module character diagonals. Moreover, this new concept is investigated for second dual, unitization, tensor product and $l^p$-direct sums of Banach algebras.

## 1. Introduction and preliminaries

Througout this paper, $A$ is a Banach algebra. For a Banach $A$-bimodule $X$, a *derivation* is a bounded linear map $D : A \to X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in A).$$

For each $x \in X$, the derivation $D_x : A \to X$ given by $D_x(a) = a \cdot x - x \cdot a$ is called an *inner derivation*. A derivation $D : A \to X$ is called *approximately inner*, if there exists a net $(x_i) \subset X$ such that

$$D(a) = \lim_i D_{x_i}(a) \qquad (a \in A),$$

if also there is $L > 0$ such that

$$\sup \left\| D_{x_i}(a) \right\| \leq L\|a\| \qquad (a \in A),$$

then $D$ is called *boundedly approximately inner*.

Let $\phi \in \sigma(A)$ be a character on $A$, and let $\mathcal{M}_\phi^A$ [resp. ${}_\phi\mathcal{M}^A$] denotes the class of Banach $A$-bimodules $X$ such that $x \cdot a = \phi(a)x$ [resp. $a \cdot x = \phi(a)x$] for all $a \in A$ and $x \in X$, [10].

**Definition 1.1.** *Let $A$ be a Banach algebra and $\phi \in \sigma(A)$. Then*

*(i) $A$ is called (approximately) (boundedly approximately) contractible if for each $A$-bimodule $X$, every derivation $D : A \to X$ is (approximately) (boundedly approximately) inner.*

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*(ii)* *$A$ is called left [right] (approximately) (boundedly approximately) $\phi$-contractible if for each $X \in_\phi \mathcal{M}^A$ [resp. $\mathcal{M}_\phi^A$], every derivation $D : A \to X$ is (approximately) (boundedly approximately) inner.*

*(iii)* *$A$ is called left [right] (approximately) (boundedly approximately) character contractible if it is left [right] (approximately) (boundedly approximately) $\phi$-contractible for each $\phi \in \sigma(A)$.*

*(iv)* *$A$ is called (approximately) (boundedly approximately) character contractible if it is both left and right (approximately) (boundedly approximately) character contractible.*

Throughout this paper, $\mathfrak{A}$ is a Banach algebra such that $A$ is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b \quad , \quad (ab) \cdot \alpha = a(b \cdot \alpha) \qquad (a, b \in A \ , \ \alpha \in \mathfrak{A}).$$

Let $X$ be a Banach $A$-bimodule and Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x \quad , \quad a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x \quad , \quad (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a) \qquad (a \in A \ , \ \alpha \in \mathfrak{A} \ , \ x \in X),$$

and similarly for the right and two-sided actions, in this case we say that $X$ is a Banach *$A$-$\mathfrak{A}$-module*. If moreover, $\alpha \cdot x = x \cdot \alpha$ for all $\alpha \in \mathfrak{A}$ and $x \in X$, then $X$ is called a *commutative $A$-$\mathfrak{A}$-module*.

A bounded map $D : A \to X$ is called an *$\mathfrak{A}$-module derivation* if it is $\mathfrak{A}$-bimodule homomorphism and

$$D(a \pm b) = D(a) \pm D(b) \quad , \quad D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in A).$$

The boundedness of $D$ means that there is $L > 0$ such that $\|D(a)\| \le L\|a\|$, for all $a \in A$.

When $X$ is a commutative $A$-$\mathfrak{A}$-module, then for each $x \in X$ the map $D_x : A \to X$ given by $D_x(a) = a \cdot x - x \cdot a$ is called *inner $\mathfrak{A}$-module derivation* [1].

**Definition 1.2.** *The Banach algebra $A$ is called (approximately) $\mathfrak{A}$-module contractible if for any commutative Banach $A$-$\mathfrak{A}$-module $X$, each $\mathfrak{A}$-module derivation $D : A \to X$ is (approximately) inner [2, 18].*

The concepts of *contractible* and *amenable* Banach algebras was introduced by Johnson in [12]. Then, Bodaghi et al. investigated the concepts of *module contractibility*, *module amenability* and (*n-weak*) *module amenability* in [2, 4–6, 8, 11, 17]. The concepts of *character contractibilily* and *character amenability* for Banach algebras was introduced by Kaniut, Lau and Pym in [13] and by Monfared and Isfahani in [10, 14, 16]. The *approximate* versions of these notions were introduced and studied by several authors, see [18, 19]. Furthermore, the authors in [3, 7, 9], introduced and investigated the concepts of *module character contractibility* and *module $(\varphi, \psi)$-amenability*. They showed that such Banach algebras posses module character diagonals. Finally, the authors in [20] studied the bounded version of *approximate character contractibilily*.

In this paper, we define and study the concept of approximate module character contractibility and its bounded version. In addition, we have some results for second dual, unitization, tensor products and $l^p$-direct sums of Banach algebras. One of the consequences of this paper will be the bounded version of approximate module contractibility.

## 2. Bounded approximate-module-character-contractibility

Throughout this section $A$ and $\mathfrak{A}$ are Banach algebras and $A$ is Banach $\mathfrak{A}$-bimodule with compatible actions. At first, we will define the concepts: approximate-module-character-contractibility and also its bounded version.

Let $\varphi \in \sigma(\mathfrak{A})$ be a character on $\mathfrak{A}$ and consider the multiplicative linear map $\phi : A \to \mathfrak{A}$ such that

$$\phi(a \cdot \alpha) = \phi(\alpha \cdot a) = \varphi(\alpha)\phi(a) \qquad (a \in A \ , \ \alpha \in \mathfrak{A}),$$

we denote the set of all such maps by $\Omega_A$.

**Definition 2.1.** *Let $\varphi \in \sigma(\mathfrak{A})$ and $\phi \in \Omega_A$. We say that the Banach space $X$ is a $\Big((\phi,\varphi), A\text{-}\mathfrak{A}\Big)$-module or $X \in_{(\phi,\varphi)} \mathcal{M}^{A,\mathfrak{A}}$, if left module action of $A$ on $X$ is given by*

$$a \cdot x = \phi(a) \cdot x \quad (a \in A \ , \ x \in X),$$

*and the actions of $\mathfrak{A}$ on $X$ is given by*

$$\alpha \cdot x = x \cdot \alpha = \varphi(\alpha) x \quad (\alpha \in \mathfrak{A} \ , \ x \in X).$$

*Note that in this case we can write $a \cdot x = \phi(a) \cdot x = \varphi \circ \phi(a) x$, for all $a \in A$ and $x \in X$. Similarly, we say that $X$ is $\Big(A\text{-}\mathfrak{A}, (\phi,\varphi)\Big)$-module or $X \in \mathcal{M}^{A,\mathfrak{A}}_{(\phi,\varphi)}$, if right module action of $A$ on $X$ is given by*

$$x \cdot a = \phi(a) \cdot x \quad (a \in A \ , \ x \in X),$$

*and the actions of $\mathfrak{A}$ on $X$ is given by*

$$\alpha \cdot x = x \cdot \alpha = \varphi(\alpha) x \quad (\alpha \in \mathfrak{A} \ , \ x \in X).$$

The authors in [9], introduced the concept of module-character-contractibility. In the following, we will introduce the concept of approximate-module-character contractibility and also its bounded version.

**Definition 2.2.** *Let $A$ be a Banach $\mathfrak{A}$-bimodule, $\phi \in \Omega_A$ and $\varphi \in \sigma(\mathfrak{A})$. Then*

*(i) $A$ is called left (boundedly) approximately-module-$(\phi,\varphi)$-contractible, if every $\mathfrak{A}$-module derivation $D : A \to X$ is (boundedly) approximately inner, for all $X \in_{(\phi,\varphi)} \mathcal{M}^{A,\mathfrak{A}}$. There is a similar definition for right (boundedly) approximately-module-$(\phi,\varphi)$-contractible Banach $\mathfrak{A}$-bimodule.*

*(ii) $A$ is called (boundedly) approximately-module-$(\phi,\varphi)$-contractible, if it is left and right (boundedly) approximately-module-$(\phi,\varphi)$-contractible.*

*(iii) $A$ is called (boundedly) approximately-module-character-contractible, if it is (boundedly) approximately-module-$(\phi,\varphi)$-contractible for all $\phi \in \Omega_A$ and all $\varphi \in \sigma(\mathfrak{A})$.*

**Notation.** We will use the abbreviated symbol ($b \cdot app \cdot m \cdot (\phi,\varphi)$-cont.) for bounded approximate-module-$(\phi,\varphi)$-contractibility.

We remind that, if $\mathfrak{A} = \mathbb{C}$ and $\varphi$ is the identity map, then all of the above definitions coincide with their classical case.

**Proposition 2.3.** *Let $A$ be a Banach $\mathfrak{A}$-bimodule, $\phi \in \Omega_A$ and $\varphi \in \sigma(\mathfrak{A})$. Then the following statements are equivalent:*

*(i) $A$ is right $b \cdot app \cdot m \cdot (\phi,\varphi)$-cont.*

*(ii) There exist a net $(m_i) \subset A$ and $L, L' > 0$ such that: $\varphi \circ \phi(m_i) = 1$, $am_i - \phi(a) \cdot m_i \to 0$, $\alpha \cdot m_i - \varphi(\alpha) m_i \to 0$, $\|am_i - \phi(a) \cdot m_i\| \le L\|a\|$ and $\|\alpha \cdot m_i - \varphi(\alpha) m_i)\| \le L'\|\alpha\|$, for all $a \in A$ and $\alpha \in \mathfrak{A}$.*

*(iii) There exist a net $(m_i) \subset A$ and $L, L' > 0$ such that: $\varphi \circ \phi(m_i) \to 1$, $am_i - \phi(a) \cdot m_i \to 0$, $\alpha \cdot m_i - \varphi(\alpha) m_i \to 0$, $\|am_i - \phi(a) \cdot m_i\| \le L\|a\|$ and $\|\alpha \cdot m_i - \varphi(\alpha) m_i)\| \le L'\|\alpha\|$, for all $a \in A$ and $\alpha \in \mathfrak{A}$.*

*There is a similar statements for the "left" version.*

*Proof.* $(i) \Rightarrow (ii)$: We define the right $A$-module action on $X =: A$ by $x \cdot a = \phi(a) \cdot x$ and the left $A$-module action is naturally, and we define $\mathfrak{A}$-module actions on $X = A$ by $\alpha \cdot x = x \cdot \alpha = \varphi(\alpha) x$, for $\alpha \in \mathfrak{A}$ and $a, x \in A$. Take $b \in A$ such that $\varphi \circ \phi(b) = 1$ and define a module derivation $D : A \to X$ by $D(a) = ab - \phi(a) \cdot b$. Obviousely, $D(A) \subseteq \ker \varphi \circ \phi$. By $(i)$, $D$ is boundedly approximately inner. It follows that, there exist a net $(n_i) \subset \ker \varphi \circ \phi \subset A$ and $L'' > 0$ such that $D(a) = \lim_i D_{n_i}(a)$ and $\|D_{n_i}(a)\| \le L''\|a\|$, for all $a \in A$. Set $m_i = b - n_i$, then $\varphi \circ \phi(m_i) = 1$ and for all $a \in A$ and $\alpha \in \mathfrak{A}$ we have

$$\begin{aligned}
\|am_i - \phi(a) \cdot m_i\| &= \|ab - an_i - \phi(a) \cdot b + \phi(a) \cdot n_i\| \\
&= \|D(a) - D_{n_i}(a)\| \to 0, \\
\|am_i - \phi(a) \cdot m_i\| &\le \|D\|\|a\| + L''\|a\| = (\|D\| + L'')\|a\|,
\end{aligned}$$

and also

$$\|\alpha \cdot m_i - \varphi(\alpha)m_i\| = 0,$$

$(ii) \Rightarrow (iii)$: is obvious.
$(iii) \Rightarrow (i)$: Let $(m_i) \subset A$ be a net satisfies in $(iii)$. Without loss of generality we can assume that $\|\varphi \circ \phi(m_i)\| < 1$ for each $i$. Let $X$ be a $\big(A\text{-}\mathfrak{A}, (\phi, \varphi)\big)$-bimodule and $D : A \to X$ be an $\mathfrak{A}$-module derivation. Set $x_i =: D(m_i)$. Since $D$ is an $\mathfrak{A}$-module derivation, then for all $a \in A$ we have $D\big(\phi(a) \cdot m_i\big) = \phi(a) \cdot D(m_i)$ and

$$\begin{aligned} \|a \cdot x_i - x_i \cdot a + D(a)\| &= \|a \cdot x_i - \phi(a) \cdot x_i + D(a)\| \\ &= \|a \cdot D(m_i) - \phi(a) \cdot D(m_i) + D(a)\| \\ &= \|D(am_i) - D(a) \cdot m_i - \phi(a) \cdot D(m_i) + D(a)\| \\ &= \|D(am_i - \varphi \circ \phi(m_i)D(a) - D\big(\phi(a) \cdot m_i\big) + D(a)\| \\ &= \|D\big(am_i - \phi(a) \cdot m_i\big) - \varphi \circ \phi(m_i)D(a) + D(a)\| \to 0, \end{aligned}$$

also

$$\begin{aligned} \|a \cdot x_i - x_i \cdot a\| &\leq \|D\|\|am_i - \phi(a) \cdot m_i\| + \|\varphi \circ \phi(m_i)\|\|D\|\|a\| \\ &\leq \|D\|L\|a\| + \|D\|\|a\| \\ &= \big(\|D\|L + \|D\|\big)\|a\|. \end{aligned}$$

This shows that $D$ is boundedly approximately inner. Since $D$ was arbitrary, it follows that $A$ is right $b \cdot app \cdot m \cdot (\phi, \varphi)$-cont. □

Consider the module projective tensor product $A\hat{\otimes}_{\mathfrak{A}}A \cong A\hat{\otimes}A/I_A$, where $I_A$ is the closed ideal of $A\hat{\otimes}A$ generated by

$$\{a \cdot \alpha \otimes b - a \otimes \alpha \cdot b \ : \ a, b \in A \ , \ \alpha \in \mathfrak{A}\},$$

and consider the closed ideal $J_A$ of $A$ generated by

$$\{(a \cdot \alpha)b - a(\alpha \cdot b) \ : \ a, b \in A \ , \ \alpha \in \mathfrak{A}\}.$$

Then $I_A$ and $J_A$ are $A$-submodules and $\mathfrak{A}$-submodules of $A\hat{\otimes}A$ and $A$, respectively. Then the quotients $A/J_A$ and $A\hat{\otimes}A/I_A \cong A\hat{\otimes}_{\mathfrak{A}}A$ will be $A$-bimodules and $\mathfrak{A}$-bimodules [9].

Let $\phi \in \Omega_A$ and $\varphi \in \sigma(\mathfrak{A})$. It is obvious that $\phi = 0$ on $J_A$. So $\tilde{\phi} : A/J_A \to \mathfrak{A}\big(\tilde{\phi}(a + J_A) =: \phi(a)\big)$ is well defined and $\tilde{\phi} \in \Omega_{A/J_A}$.

Consider the map $\omega : A\hat{\otimes}A \to A\big(\omega(a \otimes b) = ab\big)$ and

$$\tilde{\omega} : A\hat{\otimes}_{\mathfrak{A}}A \cong A\hat{\otimes}A/I_A \to A/J_A$$

defined by $\tilde{\omega}(a \otimes b + I_A) =: ab + J_A$, which is $A$-module and $\mathfrak{A}$-module homomorphism [9]. The authors in [9] defined the concept *module-$(\phi, \varphi)$-diagonal* for $A$, and now we extend this definition.

**Definition 2.4.** *Let $A$ be a Banach $\mathfrak{A}$-bimodule, $\phi \in \Omega_A$ and $\varphi \in \sigma(\mathfrak{A})$. A net $(\tilde{m}_i) \subset A\hat{\otimes}_{\mathfrak{A}}A$ is called a left multiplier bounded approximate-module-$(\phi, \varphi)$-diagonal ($mul \cdot b \cdot app \cdot m \cdot (\phi, \varphi)$-diag.) for $A$ if*

*(i)* $\langle \varphi \circ \tilde{\phi}, \tilde{\omega}(\tilde{m}_i)\rangle = 1,$

*(ii)* $\tilde{m}_i \cdot a - \phi(a) \cdot \tilde{m}_i \to 0,$

*(iii)* $\alpha \cdot \tilde{m}_i - \varphi(\alpha)\tilde{m}_i \to 0,$

*(iv)* $\exists\, L' > 0 \ : \ \|\tilde{m}_i \cdot a - \phi(a) \cdot \tilde{m}_i\| \leq L'\|a\|,$

*(v)* $\exists\, L'' > 0 \ : \ \|\alpha \cdot \tilde{m}_i - \varphi(\alpha)\tilde{m}_i\| \leq L''\|\alpha\|,$

*for each* $a \in A$ *and* $\alpha \in \mathfrak{A}$.

*By using above conditions we can write*

$$\tilde{m}_i \cdot a - \varphi \circ \phi(a)\tilde{m}_i \to 0 \quad , \quad \|\tilde{m}_i \cdot a - \varphi \circ \phi(a)\tilde{m}_i\| \leq L'\|a\|.$$

*There is a similar definition for the "right" case.*

**Proposition 2.5.** *Let* $A$ *be a Banach left [right] essential* $\mathfrak{A}$*-bimodule,* $\phi \in \Omega_A$ *and* $\varphi \in \sigma(\mathfrak{A})$*. Then* $A$ *is left [right]* $b \cdot app \cdot m \cdot (\phi, \varphi)$*-cont. if and only if* $A$ *has left [right]* $mul \cdot b \cdot app \cdot m \cdot (\phi, \varphi)$*-diag.*

*Proof.* Suppose that $A$ is left $b \cdot app \cdot m \cdot (\phi, \varphi)$-*cont*. We consider $X = A\hat{\otimes}_{\mathfrak{A}}A$ with left $A$-module action $a \cdot x = \phi(a) \cdot x$, for $a \in A$ and $x \in X$, and the right $A$-module action naturally. We define $\mathfrak{A}$-module actions on $X$ by $\alpha \cdot x = x \cdot \alpha = \varphi(\alpha)x$ for all $x \in X$ and $\alpha \in \mathfrak{A}$. Let $\tilde{m}_0 \in A\hat{\otimes}_{\mathfrak{A}}A$ such that $\langle \varphi \circ \tilde{\phi}, \tilde{\omega}(\tilde{m}_0)\rangle = 1$. Since $A$ is a left essential $\mathfrak{A}$-module, then the map $\varphi \circ \phi$ is $\mathbb{C}$-linear by the proof of Theorem 3.14 in [6] and we conclude that $\varphi \circ \phi(a)\tilde{m}_0 - \tilde{m}_0 \cdot a \in \ker(\varphi \circ \tilde{\phi} \circ \tilde{\omega})$. Now, we can define the $\mathfrak{A}$-module derivation $D_{\tilde{m}_0} : A \to \ker(\varphi \circ \tilde{\phi} \circ \tilde{\omega}) \subset A\hat{\otimes}_{\mathfrak{A}}A$ by

$$D_{\tilde{m}_0}(a) =: \phi(a) \cdot \tilde{m}_0 - \tilde{m}_0 \cdot a \Big( = \varphi \circ \phi(a)\tilde{m}_0 - \tilde{m}_0 \cdot a\Big).$$

Thus, by the hypothesis there exist a net $(\tilde{m}_i) \subset \ker(\varphi \circ \tilde{\phi} \circ \tilde{\omega})$ and $L > 0$ such that for all $a \in A$ we have

$$\begin{aligned} D_{\tilde{m}_0}(a) &= \lim_i \Big(\phi(a) \cdot \tilde{m}_i - \tilde{m}_i \cdot a\Big) \\ &= \lim_i \Big(\varphi \circ \phi(a)\tilde{m}_i - \tilde{m}_i \cdot a\Big), \end{aligned}$$

and

$$\|\phi(a) \cdot \tilde{m}_i - \tilde{m}_i \cdot a\| \leq L\|a\|.$$

Put $\tilde{M}_i = \tilde{m}_0 - \tilde{m}_i$. It is easy to check that $\langle \varphi \circ \tilde{\phi}, \tilde{\omega}(\tilde{M}_i)\rangle = 1$, and for all $a \in A$ we have

$$\begin{aligned} \|\phi(a) \cdot \tilde{M}_i - \tilde{M}_i \cdot a\| &= \|\phi(a) \cdot \tilde{m}_0 - \phi(a) \cdot \tilde{m}_i - \tilde{m}_0 \cdot a + \tilde{m}_i \cdot a\| \\ &= \|D_{\tilde{m}_0}(a) + \tilde{m}_i \cdot a - \phi(a) \cdot \tilde{m}_i\| \to 0, \end{aligned}$$

and we conclude that

$$\|\phi(a) \cdot \tilde{M}_i - \tilde{M}_i \cdot a\| \leq \Big[\|\tilde{m}_0\|(\|\varphi \circ \phi\| + 1) + L\Big]\|a\|.$$

We also have $\alpha \cdot \tilde{M}_i = \tilde{M}_i \cdot \alpha = \varphi(\alpha)\tilde{M}_i$ for all $\alpha \in \mathfrak{A}$. Finally, this shows that $(\tilde{M}_i)$ is a left mul· b· app· m ·$(\phi, \varphi)$-diag for $A$.

Conversely, let there exist a net $(\tilde{m}_i) \subset A\hat{\otimes}_{\mathfrak{A}}A$ and $L', L'' > 0$ such that $\langle \varphi \circ \tilde{\phi}, \tilde{\omega}(\tilde{m}_i)\rangle = 1$, and for all $a \in A$ and $\alpha \in \mathfrak{A}$

$$\begin{aligned} &\phi(a) \cdot \tilde{m}_i - \tilde{m}_i \cdot a \to 0 \quad , \quad \|\phi(a) \cdot \tilde{m}_i - \tilde{m}_i \cdot a\| \leq L'\|a\|, \\ &\varphi(\alpha)\tilde{m}_i - \tilde{m}_i \cdot \alpha \to 0 \quad , \quad \|\varphi(\alpha)\tilde{m}_i - \tilde{m}_i \cdot \alpha\| \leq L''\|\alpha\|. \end{aligned}$$

Suppose that $X$ is a Banach $A$-bimodule and $\mathfrak{A}$-bimodule with module actions $a \cdot x =: \phi(a) \cdot x$ and $\alpha \cdot x = x \cdot \alpha = \varphi(\alpha)x$ for $x \in X$, $a \in A$ and $\alpha \in \mathfrak{A}$, and let $D : A \to X$ be an $\mathfrak{A}$-module derivation. We consider $X$ as an $A/J_A$-bimodule by defining

$$x \cdot \tilde{a} =: x \cdot a \ , \ \tilde{a} \cdot x =: a \cdot x\Big( = \phi(a) \cdot x = \tilde{\phi}(\tilde{a}) \cdot x = \varphi \circ \tilde{\phi}(\tilde{a})x\Big) \quad (\tilde{a} = a + J_A \in A/J_A \ , \ x \in X).$$

We also define the map $\tilde{D} : A/J_A \to X \left(\tilde{D}(\tilde{a}) =: D(a)\right)$, which is an $\mathfrak{A}$-module derivation. Put $x_i =: \tilde{D}\left(\tilde{w}(\tilde{m}_i)\right) \in X$, then for all $\tilde{a} \in A/J_A$ we have

$$\tilde{D}\left(\tilde{\phi}(\tilde{a}) \cdot \tilde{w}(\tilde{m}_i)\right) = \tilde{\phi}(\tilde{a}) \cdot \tilde{D}\left(\tilde{w}(\tilde{m}_i)\right),$$

and

$$\begin{aligned}
& \|a \cdot x_i - x_i \cdot a - D(a)\| = \|\tilde{a} \cdot x_i - x_i \cdot \tilde{a} - \tilde{D}(\tilde{a})\| \\
= \;& \|\tilde{\phi}(\tilde{a}) \cdot \tilde{D}\left(\tilde{w}(\tilde{m}_i)\right) - \tilde{D}\left(\tilde{w}(\tilde{m}_i)\right) \cdot \tilde{a} - \tilde{D}(\tilde{a})\| \\
= \;& \|\tilde{D}\left(\tilde{\phi}(\tilde{a}) \cdot \tilde{w}(\tilde{m}_i)\right) - \tilde{D}\left(\tilde{w}(\tilde{m}_i) \cdot \tilde{a}\right) + \tilde{w}(\tilde{m}_i) \cdot \tilde{D}(\tilde{a}) - \tilde{D}(\tilde{a})\| \\
= \;& \|\tilde{D}\left[\tilde{\phi}(\tilde{a}) \cdot \tilde{w}(\tilde{m}_i) - \tilde{w}(\tilde{m}_i) \cdot \tilde{a}\right] + \varphi \circ \tilde{\phi}\left(\tilde{w}(\tilde{m}_i)\right)\tilde{D}(\tilde{a}) - \tilde{D}(\tilde{a})\| \\
= \;& \|\tilde{D}\left[\varphi \circ \tilde{\phi}(\tilde{a}) \cdot \tilde{w}(\tilde{m}_i) - \tilde{w}(\tilde{m}_i) \cdot \tilde{a}\right]\| \\
= \;& \|\tilde{D}\left[\tilde{w}\left(\varphi \circ \tilde{\phi}(\tilde{a})\tilde{m}_i - \tilde{m}_i \cdot \tilde{a}\right)\right]\| \\
= \;& \|\tilde{D}\left[\tilde{w}\left(\varphi \circ \phi(a)\tilde{m}_i - \tilde{m}_i \cdot a\right)\right]\| \to 0,
\end{aligned}$$

and

$$\begin{aligned}
\|a \cdot x_i - x_i \cdot a\| &= \|\tilde{a} \cdot x_i - x_i \cdot \tilde{a}\| \\
&= \|\tilde{D}\left[\tilde{w}\left(\varphi \circ \phi(a)\tilde{m}_i - \tilde{m}_i \cdot a\right)\right] + D(a)\| \\
&\leq (\|\tilde{D}\|\|\tilde{w}\|L' + \|D\|)\|a\|.
\end{aligned}$$

This proves that $A$ is left $b \cdot app \cdot m \cdot (\phi, \varphi)$-cont. □

In the next proposition we consider the spacial case $\phi \equiv 0$.

**Proposition 2.6.** *Let $A$ be a Banach $\mathfrak{A}$-bimodule and $\varphi \in \sigma(\mathfrak{A})$, then $A$ is left [right] $b \cdot app \cdot m \cdot (0, \varphi)$-cont. if and only if it has multiplier bounded left [right] approximate identity.*

*Proof.* At first, suppose that $A$ is left $b \cdot app \cdot m \cdot (0, \varphi)$-cont. Let $X =: A \oplus_1 A$ with the following $A$-module and $\mathfrak{A}$-module actions:

$$\begin{aligned}
& a \cdot (b, c) = (0, 0) \quad , \quad (b, c) \cdot a = (ba, ca), \\
& \alpha \cdot (b, c) = (b, c) \cdot \alpha =: \left(\varphi(\alpha)b, \varphi(\alpha)c\right),
\end{aligned}$$

for $a, b, c \in A$ and $\alpha \in \mathfrak{A}$. Then, $X$ is a Banach $A$-$\mathfrak{A}$-module with the compactible actions. We consider the bounded $\mathfrak{A}$-module derivation $D : A \to A \oplus_1 A$ by $D(a) =: (a, a)$. It follows from the assumption that there is a net $\left((a_i, b_i)\right) \subset A \oplus_1 A$ and $L > 0$ such that for all $a \in A$ we have

$$(a, a) = D(a) = \lim_i D_{(a_i, b_i)}(a) = \lim_i \left(a \cdot (a_i, b_i) - (a_i, b_i) \cdot a\right) = \lim_i (-a_i a, -b_i a),$$

and $\|D_{(a_i, b_i)}(a)\| \leq L\|a\|$. Therefore,

$$a = \lim_i (-a_i a) = \lim_i (-b_i a).$$

This shows that $\{-a_i\}$ and $\{-b_i\}$ are left approximate identities for $A$. We have for $L > 0$

$$\begin{aligned}
\|D_{(a_i, b_i)}(a)\| &= \|a \cdot (a_i, b_i) - (a_i, b_i) \cdot a\| \\
&= \| - a_i a\| + \| - b_i a\| \leq L\|a\|,
\end{aligned}$$

and we conclude that $A$ has multiplier bounded left approximate identity.

Conversely, let $(a_i) \subset A$ is a multiplier bounded left approximate identity for $A$. Consider the bounded $\mathfrak{A}$-module derivation $D : A \to X$, where $X$ is an $A$-$\mathfrak{A}$-module with the following actions

$$a \cdot x =: \phi(a) \cdot x = 0 \quad , \quad \alpha \cdot x = x \cdot \alpha =: \varphi(\alpha)x,$$

for $a \in A$ , $x \in X$ and $\alpha \in \mathfrak{A}$. Now, set $x_i =: -D(a_i) \in X$, then we have

$$\begin{aligned} D(a) &= D\Big(\lim_i a_i a\Big) = \lim_i \Big[D(a_i a)\Big] \\ &= \lim_i \Big[a_i \cdot D(a) + D(a_i) \cdot a\Big] \\ &= \lim_i [0 + (-x_i) \cdot a] = \lim_i (-x_i \cdot a) \\ &= \lim_i \Big(a \cdot x_i - x_i \cdot a\Big) = \lim_i D_{x_i}(a), \end{aligned}$$

and also there is $L > 0$ such that

$$\begin{aligned} &\|D_{x_i}(a)\| = \|a \cdot x_i - x_i \cdot a\| = \|D(a_i \cdot a)\| \\ &\leq \|D\|\|a_i a\| \leq \|D\|L\|a\|. \end{aligned}$$

Thus $D$ is boundedly approximately inner. □

**Corollary 2.7.** *A Banach algebra A is left [right] boundedly approximately-module-character-contratible if and only if it has a multiplier-bounded left [right] approximate identity.*

**Proposition 2.8.** *Let $A$ and $B$ be $\mathfrak{A}$-bimodules, and $\theta : A \to B$ be [norm-preserving] continuous $\mathfrak{A}$-module epimorphism. Then left [right] [bounded] approximate-module-$(\phi \circ \theta, \varphi)$-contractibility of $A$ implies left [right] [bounded] approximate-module-$(\phi, \varphi)$-contractibility of $B$.*

*Proof.* Let $X \in_{(\phi,\varphi)} \mathcal{M}^{B,\mathfrak{A}}$ and $D : B \to X$ be an $\mathfrak{A}$-module derivation.

Thus $X \in_{(\phi\circ\theta,\varphi)} \mathcal{M}^{A,\mathfrak{A}}$ by defining the following $A$-bimodule actions on $X$

$$a \cdot x =: \theta(a) \cdot x = \phi\Big(\theta(a)\Big) \cdot x \quad , \quad x \cdot a =: x \cdot \theta(a) \quad (a \in A \ , \ x \in X),$$

and also $D \circ \theta : A \to X$ is an $\mathfrak{A}$-module derivation. By hypothesis, there is a net $(x_i) \subset X$ such that $D \circ \theta(a) = \lim_i (a \cdot x_i - x_i \cdot a) = \lim_i \Big(\theta(a) \cdot x_i - x_i \cdot \theta(a)\Big)$, for all $a \in A$. Since $\theta$ is surjective, for each $b \in B$ there is $a \in A$ such that $\theta(a) = b$. Now, we can write

$$D(b) = (D \circ \theta)(a) = \lim_i (b \cdot x_i - x_i \cdot b),$$

this shows that $D$ is approximately inner. For proving the bounded part, by hypothesis we can find $L > 0$ such that

$$\|a \cdot x_i - x_i \cdot a\| \leq L\|a\| \quad (a \in A).$$

Since $\theta$ is norm-preserving, we have $\|b\| = \|\theta(a)\| = \|a\|$ and

$$\begin{aligned} \|b \cdot x_i - x_i \cdot b\| &= \|\theta(a) \cdot x_i - x_i \cdot \theta(a)\| \\ &= \|a \cdot x_i - x_i \cdot a\| \\ &\leq L\|a\| = L\|b\|. \end{aligned}$$

□

**Corollary 2.9.** *Let $I$ be a closed ideal and $\mathfrak{A}$-submodule of a Banach $\mathfrak{A}$-bimodule $A$, and $\pi : A \to A/I$ be the canonical projection. If $A$ is approximately-module-$(\phi\circ\pi, \varphi)$-contractible then $A/I$ is approximately-module-$(\phi, \varphi)$-contractible. The boundedness holds only if $I = \{0\}$.*

**Proposition 2.10.** *Let $I$ be a closed left ideal and $\mathfrak{A}$-submodule of a Banach $\mathfrak{A}$-bimodule $A$. If $\phi \in \Omega_A$ and $\varphi \in \sigma(\mathfrak{A})$ such that $I \nsubseteq \ker(\varphi \circ \phi)$, then the following statements are equivalant:*

*(i) $A$ is right [left] $b \cdot app \cdot m \cdot (\phi, \varphi)$-cont.*
*(ii) $I$ is right [left] $b \cdot app \cdot m \cdot (\phi |_I, \varphi)$-cont.*

*Proof.* Suppose that $A$ is right $b \cdot app \cdot m \cdot (\phi, \varphi)$-cont., then by Proposition 2.3, there exist a net $(m_j) \subset A$ and $L, L^{'} > 0$ such that $\varphi \circ \phi(m_j) = 1$ and for all $a \in A$ and $\alpha \in \mathfrak{A}$

$$\begin{array}{ccc} \|am_j - \phi(a) \cdot m_j\| \to 0 & , & \|am_j - \phi(a) \cdot m_j\| \leq L\|a\|; \\ \|\alpha \cdot m_j - \varphi(\alpha)m_j\| \to 0 & , & \|\alpha \cdot m_j - \varphi(\alpha)m_j\| \leq L^{'}\|\alpha\|. \end{array}$$

Choose $b \in I$ such that $\varphi \circ \phi(b) = 1$, and set $n_j =: m_j b$. Now for the net $(n_j) \subset I$, we have $\Big(\varphi \circ \phi |_I \Big)(n_j) = 1$ and for all $i \in I$ we can write

$$\begin{aligned} \|in_j - \phi |_I (i) \cdot n_j\| &= \|i(m_j b) - \phi(i)(m_j b)\| \\ &\leq \|im_j - \phi(i)m_j\|\|b\| \to 0, \end{aligned}$$

and

$$\|in_j - \phi |_I (i) \cdot n_j\| \leq L\|b\|\|i\|,$$

also for all $\alpha \in \mathfrak{A}$ we have

$$\begin{aligned} \|\alpha \cdot n_j - \varphi(\alpha)n_j\| &= \|\alpha \cdot (m_j b) - \varphi(\alpha)(m_j b)\| \\ &\leq \|\alpha \cdot m_j - \varphi(\alpha)m_j\|\|b\| \to 0, \end{aligned}$$

and

$$\|\alpha \cdot n_j - \varphi(\alpha)n_j\| \leq L^{'}\|b\|\|\alpha\|,$$

and we conclude that $(ii)$ is true by Proposition 2.3. Conversely, suppose that $I$ is right $b \cdot app \cdot m \cdot \Big(\phi |_I, \varphi\Big)$-cont. then by Proposition 2.3, there exist a net $(m_j) \subset I$ and $L, L^{'} > 0$ such that $\varphi \circ \phi |_I (m_j) = 1$ and for all $i \in I$ and $\alpha \in \mathfrak{A}$

$$\begin{array}{ccc} \|im_j - \phi |_I (i) \cdot m_j\| \to 0 & , & \|im_j - \phi |_I (i) \cdot m_j\| \leq L\|i\|; \\ \|\alpha \cdot m_j - \varphi(\alpha)m_j\| \to 0 & , & \|\alpha \cdot m_j - \varphi(\alpha)m_j\| \leq L^{'}\|\alpha\|. \end{array}$$

Choose $s \in I$ such that $\varphi \circ \phi(s) = 1$ and set $n_j =: sm_j$. Now for the net $(n_j) \subset A$, $\varphi \circ \phi(n_j) = 1$ and for all $a \in A$ we have

$$\begin{aligned} \|an_j - \phi(a) \cdot n_j\| &= \|a(sm_j) - \phi(a) \cdot (sm_j)\| \\ &\leq \|(as)m_j - \phi(as) \cdot m_j\| + \|\phi(as) \cdot m_j - \phi(a) \cdot sm_j\| \\ &\leq \|(as)m_j - \phi(as) \cdot m_j\| + \|\phi(s) \cdot m_j - sm_j\|\|\phi(a)\| \to 0 \end{aligned}$$

and

$$\begin{aligned} \|an_j - \phi(a) \cdot n_j\| &\leq L\|as\| + L\|s\|\|\phi(a)\| \\ &\leq \Big(L\|s\|(1 + \|\phi\|)\Big)\|a\|. \end{aligned}$$

Moreover, for all $\alpha \in \mathfrak{A}$ we have

$$\begin{aligned}
&\|\alpha \cdot n_j - \varphi(\alpha)n_j\| = \|\alpha \cdot (sm_j) - \varphi(\alpha)sm_j\| \\
&\|(\alpha \cdot s)m_j - \phi(\alpha s) \cdot m_j\| + \|\phi(\alpha s) \cdot m_j - \varphi(\alpha)sm_j\| \\
&\|(\alpha \cdot s)m_j - \phi(\alpha s) \cdot m_j\| + \|\phi(\alpha)\phi(s) \cdot m_j - \varphi(\alpha)sm_j\| \\
&\|(\alpha \cdot s)m_j - \phi(\alpha s) \cdot m_j\| + |\varphi(\alpha)|\|\phi(s) \cdot m_j - sm_j\| \to 0
\end{aligned}$$

and

$$\begin{aligned}
\|\alpha \cdot n_j - \varphi(\alpha)n_j\| &\leq L\|\alpha \cdot s\| + \|\varphi\|\|\alpha\|L\|s\| \\
&\leq \Big[L\|s\|(1 + \|\varphi\|)\Big]\|\alpha\|.
\end{aligned}$$

Thus $(i)$ is true by Proposition 2.3. □

**Proposition 2.11.** *Let $A$ be a Banach $\mathfrak{A}$-bimodule. If $A/J_A$ is left [right] $b \cdot app \cdot m \cdot (\tilde{\phi}, \varphi)$-cont. then $A$ is left [right] $b \cdot app \cdot m \cdot (\phi, \varphi)$-cont.*

*Proof.* Let $X \in {}_{(\phi,\varphi)}\mathcal{M}^{A,\mathfrak{A}}$ and $D : A \to X$ be an $\mathfrak{A}$-module derivation. We can assume that $X \in {}_{(\tilde{\phi},\varphi)}\mathcal{M}^{A/J_A,\mathfrak{A}}$ by the following $A/J_A$-bimodule actions on $X$

$$\begin{aligned}
&(a + J_A) \cdot x =: a \cdot x = \phi(a) \cdot x = \tilde{\phi}(a + J_A) \cdot x, \\
&x \cdot (a + J_A) =: x \cdot a \quad (a \in A \ , \ x \in X),
\end{aligned}$$

note that the above actions are well-defined because $XJ_A = J_AX = 0$. On the other hand, we can extend $D$ to $\mathfrak{A}$-module derivation $\tilde{D} : A/J_A \to X \left(\tilde{D}(a + J_A) =: D(a)\right)$, and $\tilde{D}$ is well-defined because $D \mid_{J_A} \equiv 0$.
Now, by hypothesis, there is a net $(x_i) \subset X$ and $L > 0$ such that for all $a \in A$

$$\begin{aligned}
&\tilde{D}(a + J_A) = \lim_i \Big[(a + J_A) \cdot x_i - x_i \cdot (a + J_A)\Big], \\
&\|D_{x_i}(a + J_A)\| \leq L\|a + J_A\|,
\end{aligned}$$

so we have

$$\begin{aligned}
&D(a) = \lim_i (a \cdot x_i - x_i \cdot a), \\
&\|D_{x_i}(a)\| \leq L\|a + J_A\| \leq L\|a\|.
\end{aligned}$$

This shows that $D$ is boundedly approximately inner. □

**Corollary 2.12.** *For a Banach $\mathfrak{A}$-bimodule $A$, $A/J_A$ is left [right] approximately-module-$(\tilde{\phi}, \varphi)$-contractible if and only if $A$ is left [right] approximately-module-$(\phi, \varphi)$-contractible.*

*Proof.* This is a consequence of Proposition 2.11 and Corollarly 2.9. □

**Proposition 2.13.** *Let $A$ be a Banach $\mathfrak{A}$-bimodule, $\varphi \in \sigma(\mathfrak{A})$, $\phi \in \Omega_A$ such that $\ker\phi = \ker\varphi \circ \phi$ and $\phi$ is surjective [norm-preserving]. If $\ker\varphi \circ \phi$ has a multiplier bounded right [left] approximate identity, then $A$ is right [left] [boundedly] approximately-module-$(\phi, \phi)$-contractible.*

*Proof.* Suppose that $(b_i) \in \ker\varphi \circ \phi$ be a multiplier bounded right approximate identity ($i \cdot e \cdot$ there is $k > 0$ such that for all $b \in \ker\varphi \circ \phi : \|b - bb_i\| \to 0$ and $\|bb_i\| \leq k\|b\|$). Choose $u_0 \in A$ such that $\varphi \circ \phi(u_0) = 1$, then

$A = \mathbb{C}u_0 \oplus \ker\varphi\circ\phi$. We set $a_0 =: u_0^2 - \phi(u_0)\cdot u_0$ and $m_i =: u_0 - u_0 b_i$. Hence, $a_0 \in \ker\varphi\circ\phi = \ker\phi$ and for each $a = \lambda u_0 + b \in A$ ($\lambda \in \mathbb{C}$ , $b \in \ker\varphi\circ\phi$) we have $\varphi\circ\phi(a) = \lambda$, $\phi(a) = \lambda\phi(u_0)$ and $\varphi\circ\phi(m_i) = 1$. Furthermore

$$\begin{aligned}
\|am_i - \phi(a)\cdot m_i\| &= \|(\lambda u_0 + b)m_i - \lambda\phi(u_0)\cdot m_i\| \\
&\leq |\lambda|\|u_0 m_i - \phi(u_0)\cdot m_i\| + \|bm_i\| \\
&= |\lambda|\|u_0(u_0 - u_0 b_i) - \phi(u_0)\cdot(u_0 - u_0 b_i)\| + \|b(u_0 - u_0 b_i)\| \\
&= |\lambda|\|u_0^2 - u_0^2 b_i - \phi(u_0)\cdot u_0 + \phi(u_0)\cdot u_0 b_i\| + \|bu_0 - bu_0 b_i\| \\
&= |\lambda|\|(u_0^2 - \phi(u_0)\cdot u_0) - (u_0^2 - \phi(u_0)\cdot u_0)b_i\| + \|bu_0 - bu_0 b_i\| \\
&= |\lambda|\|a_0 - a_0 b_i\| + \|bu_0 - bu_0 b_i\| \to 0
\end{aligned}$$

and

$$\begin{aligned}
\|am_i - \phi(a)\cdot m_i\| &\leq |\lambda|\|a_0 - a_0 b_i\| + \|bu_0 - bu_0 b_i\| \\
&\leq |\lambda|(\|a_0\| + \|a_0 b_i\|) + \|bu_0\| + \|bu_0 b_i\| \\
&\leq |\lambda|(\|a_0\| + k\|a_0) + \|b\|\|u_0\| + k\|b\|\|u_0\| \\
&= |\lambda|\|a_0\|(1+k) + \|b\|\|u_0\|(1+k) \\
&= |\lambda|\|u_0\|\frac{1}{\|u_0\|}\|a_0\|(1+k) + \|b\|\|u_0\|(1+k) \\
&\leq L(|\lambda|\|u_0\| + \|b\|) = L\|a\|,
\end{aligned}$$

in which $L =: \operatorname{Max}\Big\{\frac{1}{\|u_0\|}\|a_0\|(1+k) \quad , \quad \|u_0\|(1+k)\Big\}$. On the other hand, for each $\alpha \in \mathfrak{A} = \phi(A) = \phi(\mathbb{C}u_0 \oplus \ker\varphi\circ\phi) = \mathbb{C}\varphi(u_0)$, there is $\lambda \in \mathbb{C}$ such that $\alpha = \lambda\phi(u_0)$. Then $\varphi(\alpha) = \lambda$ and since $u_0^2 - u_0 \in \ker\varphi\circ\phi$, we can write

$$\begin{aligned}
\|\alpha\cdot m_i - \varphi(\alpha)m_i\| &= \|\lambda\phi(u_0)\cdot m_i - \lambda m_i\| \\
&\leq |\lambda|\|\phi(u_0)\cdot m_i - m_i\| \\
&= |\lambda|\Big(\|\phi(u_0)\cdot m_i - u_0 m_i\| + \|u_0 m_i - m_i\|\Big) \\
&= |\lambda|\Big(\|\phi(u_0)\cdot m_i - u_0 m_i\| + \|u_0(u_0 - u_0 b_i) - (u_0 - u_0 b_i)\|\Big) \\
&= |\lambda|\Big(\|\phi(u_0)\cdot m_i - u_0 m_i\| + \|u_0^2 - u_0 - (u_0^2 - u_0)b_i\|\Big) \to 0.
\end{aligned}$$

If $\phi$ be norm-preserving then $\|\alpha\| = |\lambda|\|u_0\|$ and we have

$$\begin{aligned}
\|\alpha\cdot m_i - \varphi(\alpha)m_i\| &\leq |\lambda|\Big(\|\phi(u_0)\cdot m_i - u_0 m_i\| + \|u_0^2 - u_0 - (u_0^2 - u_0)b_i\|\Big) \\
&\leq |\lambda|\Big(L\|u_0\| + \|u_0\|\|1 - u_0\| + k\|u_0^2 - u_0\|\Big) \\
&= \Big(L + \|1 - u_0\|(1+k)\Big)|\lambda|\|u_0\| \\
&= \Big(L + \|1 - u_0\|(1+k)\Big)\|\alpha\|.
\end{aligned}$$

The proof is completed by using Proposition 2.3. □

**Proposition 2.14.** *Let $A$ be right [left] $b\cdot app\cdot m\cdot(\phi,\varphi)$-cont. Banach $\mathfrak{A}$-bimodule for some $\varphi \in \sigma(\mathfrak{A})$ and $\phi \in \Omega_A$ and let $A$ has a multiplier bounded right [left] approximate identity. Then $\ker\varphi\circ\phi$ has a multiplier bounded right [left] approximate identity.*

*Proof.* Choose $u_0 \in A$ such that $\varphi\circ\phi(u_0) = 1$, then $A = \mathbb{C}u_0 \oplus \ker\varphi\circ\phi$. Let $(n_\beta = \lambda_\beta u_0 + b_\beta) \subset A$ be a multiplier bounded right approximate identity for $A$ with multiplier bound $k > 0$, where $(b_\beta) \subset \ker\varphi\circ\phi$

and $\lambda_\beta = \varphi \circ \phi(n_\beta) \to 1$. By using Proposition 2.3, we can find a net $(m_i) = (\lambda_i u_0 + b_i) \subset A$ such that $\varphi \circ \phi(m_i) = \lambda_i = 1$ and there exist $L, L' > 0$ such that for all $a \in A$ and $\alpha \in \mathfrak{A}$

$$\|am_i - \phi(a)\cdot m_i\| \to 0 \quad , \quad \|am_i - \phi(a)\cdot m_i\| \leq L\|a\|;$$
$$\|\alpha \cdot m_i - \varphi(\alpha)m_i\| \to 0 \quad , \quad \|\alpha \cdot m_i - \varphi(\alpha)m_i\| \leq L'\|\alpha\|.$$

So

$$\|am_i - \varphi \circ \phi(a)m_i\| \to 0 \quad , \quad \|am_i - \varphi \circ \phi(a)m_i\| \leq L\|a\|.$$

Set $e_{i,\beta} =: b_\beta - b_i$, where $b_\beta = n_\beta - \lambda_\beta u_0$ and $b_i = m_i - \lambda_i u_0$. Then, for all $b \in \ker \varphi \circ \phi$ we have

$$\begin{aligned} \|bm_i - \phi(b)\cdot m_i\| &= \|bm_i - \varphi \circ \phi(b)m_i\| \\ &= \|bm_i\| \to 0. \end{aligned}$$

and

$$\|bm_i - \phi(b)\cdot m_i\| = \|bm_i\| \leq L\|b\|.$$

Therefore

$$\begin{aligned} \|be_{i,\beta} - b\| &= \|bb_\beta - bb_i - b\| \\ &= \|b(n_\beta - \lambda_\beta u_0) - b(m_i - \lambda_i u_0) - b\| \\ &\leq \|bn_\beta - b\| + \|bu_0\|\|\lambda_i - \lambda_\beta\| + \|bm_i\| \to 0. \end{aligned}$$

Since $(\lambda_i - \lambda_\beta) \to 1$, it is a bounded net with bound $k'$ and

$$\begin{aligned} \|be_{i,\beta}\| &= \|bb_\beta - bb_i\| \\ &= \|b(n_\beta - \lambda_\beta u_0) - b(m_i - \lambda_i u_0)\| \\ &\leq \|bu_0\|\|\lambda_i - \lambda_\beta\| + \|bn_\beta\| + \|bm_i\| \\ &\leq \|b\|\|u_0\|k' + k\|b\| + L\|b\| \\ &= (\|u_0\|k' + k + L)\|b\|. \end{aligned}$$

This shows that $(e_{i,\beta})$ is a multiplier bounded right approximate identity for $\ker \varphi \circ \phi$. □

## 3. Unitization and second dual of Banach algebras

In this section, $A^\# = A \oplus \mathbb{C}$ and $\mathfrak{A}^\# = \mathfrak{A} \oplus \mathbb{C}$ are unitizations of $A$ and $\mathfrak{A}$, respectively. Similar to notations in [7], let $B = A \oplus \mathfrak{A}^\#$ with following multiplication

$$(a, u)(b, v) =: (ab + a\cdot v + u\cdot b, uv) \qquad (a, b \in A \ , \ u, v \in \mathfrak{A}^\#),$$

in which $\mathfrak{A}^\#$-module actions on $A$ defined by

$$a\cdot(\alpha, \lambda) =: a\cdot\alpha + \lambda a \ , \ (\alpha, \lambda)\cdot a =: \alpha\cdot a + \lambda a \qquad (a \in A \ , \ (\alpha, \lambda) \in \mathfrak{A}^\#).$$

Moreover, we can define $\mathfrak{A}^\#$-module actions on $B$ by

$$u\cdot(a, v) =: (u\cdot a, uv) \ , \ (a, v)\cdot u =: (a\cdot u, vu) \qquad (a \in A \ ; \ u, v \in \mathfrak{A}^\#).$$

Then, $B$ is a unital Banach algebra and Banach $\mathfrak{A}^\#$-bimodule with compatible actions and with identity $e_B = (0, e_{\mathfrak{A}^\#})$ , where $e_{\mathfrak{A}^\#} = (0, 1)$ is the identity of $\mathfrak{A}^\#$. Now, suppose that $\phi \in \Omega_A$ and $\varphi \in \sigma(\mathfrak{A})$. We can define the extensions of $\phi$ and $\varphi$ by

$$\tilde{\varphi} : \mathfrak{A}^\# \to \mathbb{C} \quad , \quad \tilde{\varphi}(\alpha, \lambda) =: \varphi(\alpha) + \lambda.$$
$$\phi_e : A \to \mathfrak{A}^\# \quad , \quad \phi_e(a) =: \big(\phi(a), 0\big).$$
$$\tilde{\tilde{\phi}} : B = A \oplus \mathfrak{A}^\# \to \mathfrak{A}^\# \quad , \quad \tilde{\tilde{\phi}}(a, u) = \big(\phi(a), \tilde{\varphi}(u)\big).$$

It is easy to check that $\phi_e \in \Omega_A$, $\tilde{\varphi} \in \sigma(\mathfrak{A}^\#)$ and $\tilde{\tilde{\phi}} \in \Omega_B$.

Now, suppose that $X$ be a Banach $A$-$\mathfrak{A}$-module. We define $B$-module and $\mathfrak{A}^\#$-module actions on $X$ by

$$(a, u) \cdot x =: a \cdot x + u \cdot x \quad , \quad x \cdot (a, u) = x \cdot a + x \cdot u \qquad (x \in X \;,\; a \in A \;,\; u \in \mathfrak{A}^\#).$$
$$(\alpha, \lambda) \cdot x =: \alpha \cdot x + \lambda x \quad , \quad x \cdot (\alpha, \lambda) =: x \cdot \alpha + \lambda x \qquad (x \in X \;,\; (\alpha, \lambda) \in \mathfrak{A}^\#).$$

Therefore

$$(a, 0) \cdot x = a \cdot x \quad , \quad x \cdot (a, 0) = x \cdot a \qquad (x \in X \;,\; a \in A),$$
$$(0, u) \cdot x = u \cdot x \quad , \quad x \cdot (0, u) = x \cdot u \qquad (x \in X \;,\; u \in \mathfrak{A}^\#).$$

On the other hand, if $D : B = A \oplus \mathfrak{A}^\# \to X$ is a $\mathfrak{A}^\#$-module derivation, then for each $u, v \in \mathfrak{A}^\#$

$$\begin{aligned} D(0, uv) = D\big[(0, u)(0, v)\big] &= D(0, u) \cdot (0, v) + (0, u) \cdot D(0, v) \\ &= D(0, u) \cdot v + u \cdot D(0, v), \end{aligned}$$

also

$$\begin{aligned} D(0, uv) &= D\big[u \cdot (0, v)\big] = u \cdot D(0, v), \\ D(0, uv) &= D\big[(0, u) \cdot v\big] = D(0, u) \cdot v. \end{aligned}$$

We conclude that $u \cdot D(0, v) = D(0, u) \cdot v = 0$, hence

$$\begin{aligned} D(0, u) &= D\big((0, u)e_B\big) \\ &= D\big((0, u)(0, e_{\mathfrak{A}^\#})\big) \\ &= D(0, ue_{\mathfrak{A}^\#}) \\ &= u \cdot D(0, e_{\mathfrak{A}^\#}) = 0, \end{aligned}$$

so $D\mid_{\mathfrak{A}^\#} \equiv 0$.

**Proposition 3.1.** *The Banach algebra and $\mathfrak{A}$-bimodule $A$ is left [right] $b \cdot app \cdot m \cdot (\phi, \varphi)$-cont. if and only if the Banach algebra $B =: A \oplus \mathfrak{A}^\#$ as an $\mathfrak{A}^\#$-bimodule is left [right] $b \cdot app \cdot m \cdot (\tilde{\tilde{\phi}}, \tilde{\varphi})$-cont.*

*Proof.* Let $A$ be *left $b \cdot app \cdot m \cdot (\phi, \varphi)$-cont*. Suppose that $X \in {}_{(\tilde{\tilde{\phi}}, \tilde{\varphi})}\mathcal{M}^{B, \mathfrak{A}^\#}$ and $D : B \to X$ be an $\mathfrak{A}^\#$-module derivation. We can define $A$-module and $\mathfrak{A}$-module actions on $X$ by

$$a \cdot x =: (a, 0) \cdot x = \tilde{\tilde{\phi}}(a, 0) \cdot x = \big(\phi(a), \tilde{\varphi}(0)\big) \cdot x = \phi(a) \cdot x,$$
$$x \cdot a =: x \cdot (a, 0) \qquad (x \in X \;,\; a \in A),$$

and

$$\alpha \cdot x = x \cdot \alpha =: \big(0, (\alpha, 0)\big) \cdot x = \tilde{\varphi}(\alpha, 0) \cdot x = \varphi(\alpha)x \qquad (x \in X \;,\; \alpha \in \mathfrak{A}).$$

We consider $\tilde{D} = D\mid_A\colon A \to X$ by $\tilde{D}(a) =: D(a, O_{\mathfrak{A}^\#})$. It is easy to check that $X \in {}_{(\phi,\varphi)}\mathcal{M}^{A,\mathfrak{A}}$ and $\tilde{D}$ is an $\mathfrak{A}$-module derivation. By hypothesis, there exist a net $(x_i) \subset X$ and $L > 0$ such that for all $a \in A$

$$\tilde{D}(a) = \lim_i (a \cdot x_i - x_i \cdot a),$$
$$\|a \cdot x_i - x_i \cdot a\| \leq L\|a\|.$$

Since $D\mid_{\mathfrak{A}^\#} \equiv 0$, for each $(a, u) \in B$ we have

$$\begin{aligned} D(a,u) &= D\Big[(a,0) + (0,u)\Big] = D(a,0) \\ &= \lim_i \Big[(a,0)\cdot x_i - x_i \cdot (a,0)\Big] \\ &= \lim_i (a\cdot x_i - x_i \cdot a) \\ &= \lim_i (a \cdot x_i + u\cdot x_i - x_i \cdot u - x_i \cdot a) \\ &= \lim_i \Big[(a,u)\cdot x_i - x_i \cdot (a,u)\Big], \end{aligned}$$

and

$$\begin{aligned} \|(a,u)\cdot x_i - x_i\cdot(a,u)\| &= \|a\cdot x_i - x_i \cdot a\| \\ &\leq L\|a\| \leq L\Big(\|a\| + \|u\|\Big) \\ &= L\|(a,u)\|. \end{aligned}$$

This shows that $D$ is boundedly approximately inner.

For the converse, suppose that $X \in {}_{(\phi,\varphi)}\mathcal{M}^{A,\mathfrak{A}}$ and $D : A \to X$ is an $\mathfrak{A}$-module derivation. We define $B$-module and $\mathfrak{A}^\#$-module actions on $X$ by

$$\begin{aligned} (a,u)\cdot x =: a\cdot x + u \cdot x &= \phi(a)\cdot x + \tilde{\varphi}(u)\cdot x \\ &= \Big[\phi(a) + \tilde{\varphi}(u)\Big]\cdot x \\ &= \tilde{\tilde{\phi}}(a,u)\cdot x, \\ x\cdot(a,u) =: x\cdot a + x\cdot u & \qquad (x \in X \ , \ a \in A \ , \ u \in \mathfrak{A}^\#), \end{aligned}$$

and

$$\begin{aligned} u\cdot x = x \cdot u =: \alpha\cdot x + \lambda x &= \varphi(\alpha)x + \lambda x \\ &= \Big[\varphi(\alpha) + \lambda\Big]x \\ &= \tilde{\varphi}(u)x \qquad (x \in X \ , \ u = (\alpha,\lambda) \in \mathfrak{A}^\#). \end{aligned}$$

We consider $\tilde{D} : B = A \oplus \mathfrak{A}^\# \to X$ by $\tilde{D}(a,u) =: D(a)$. It is easy to check that $X \in {}_{(\tilde{\tilde{\phi}},\tilde{\varphi})}\mathcal{M}^{B,\mathfrak{A}^\#}$ and $\tilde{D}$ is an $\mathfrak{A}^\#$-module derivation. By hypothesis, there exist a net $(x_i) \subset X$ and $L > 0$ such that for all $(a, u) \in B$

$$\tilde{D}(a,u) = \lim_i \Big[(a,u)\cdot x_i - x_i\cdot(a,u)\Big],$$
$$\|(a,u)\cdot x_i - x_i \cdot (a,u)\| \leq L\|(a,u)\| = L\Big(\|a\| + \|u\|\Big).$$

Hence

$$\begin{aligned} D(a) = \tilde{D}(a,0) &= \lim_i \Big[(a,0)\cdot x_i - x_i\cdot(a,0)\Big] \\ &= \lim_i (a\cdot x_i - x_i \cdot a), \end{aligned}$$

and

$$\begin{aligned} \|a\cdot x_i - x_i\cdot a\| &= \|(a,0)\cdot x_i - x_i\cdot(a,0)\| \\ &\leq L\Big(\|a\| + \|0\|\Big) = L\|a\|. \end{aligned}$$

This shows that $D$ is boundedly approximately inner. □

**Proposition 3.2.** *The Banach algebra $A$ is right [left] $b \cdot app \cdot m \cdot (\phi, \varphi)$-cont. as an $\mathfrak{A}$-bimodule if and only if it is right [left] $b \cdot app \cdot m \cdot (\phi_e, \tilde{\varphi})$-cont. as an $\mathfrak{A}^\#$-bimodule.*

*Proof.* It is easy to check that for each $a, m \in A$

$$\tilde{\varphi} \circ \phi_e(m) = \varphi \circ \phi(m) \quad , \quad \phi_e(a) \cdot m = \phi(a) \cdot m.$$

Suppose that $A$ is *right $b \cdot app \cdot m \cdot (\phi, \varphi)$-cont*. Then by Proposition 2.3, there exist a net $(m_i) \subset A$ and $L, L' > 0$ such that $\varphi \circ \phi(m_i) = 1$ and for all $a \in A$ and $\alpha \in \mathfrak{A}$

$$\begin{array}{lll} \|am_i - \phi(a) \cdot m_i\| \to 0 & , & \|am_i - \phi(a) \cdot m_i\| \leq L\|a\|; \\ \|\alpha \cdot m_i - \varphi(\alpha) \cdot m_i\| \to 0 & , & \|\alpha \cdot m_i - \varphi(\alpha) \cdot m_i\| \leq L'\|\alpha\|. \end{array}$$

Therefore, $\tilde{\varphi} \circ \phi_e(m_i) = \varphi \circ \phi(m_i) = 1$ and for all $a \in A$ and $u = (\alpha, \lambda) \in \mathfrak{A}^\#$

$$\begin{aligned} \|am_i - \phi_e(a) \cdot m_i\| &= \|am_i - \phi(a) \cdot m\| \to 0, \\ \|am_i - \phi_e(a) \cdot m_i\| &= \|am_i - \phi(a) \cdot m\| \leq L\|a\|, \\ \\ \|u \cdot m_i - \tilde{\varphi}(u)m_i\| &= \|\alpha \cdot m_i + \lambda m_i - \varphi(\alpha)m_i - \lambda m_i\| \\ &= \|\alpha \cdot m_i - \varphi(\alpha)m_i\| \to 0, \\ \\ \|u \cdot m_i - \tilde{\varphi}(u)m_i\| &= \|\alpha \cdot m_i - \varphi(\alpha)m_i\| \\ &\leq L'\|\alpha\| \leq L'\Big(\|\alpha\| + \|\lambda\|\Big) = L'\|u\|. \end{aligned}$$

This shows that $A$ is right $b \cdot app \cdot m \cdot (\phi_e, \tilde{\varphi})$-cont. The proof for the converse is similar and it is omitted. □

**Proposition 3.3.** *Let $A$ be a banach algebra and $\mathfrak{A}^\#$-bimodule. Then $A$ is left [right] $b \cdot app \cdot m \cdot (\phi_e, \tilde{\varphi})$-cont. if and only if $B = A \oplus \mathfrak{A}^\#$ is left [right] $b \cdot app \cdot m \cdot (\tilde{\tilde{\phi}}, \tilde{\varphi})$-cont.*

*Proof.* We can suppose that $A$ is an right [and left] ideal and $\mathfrak{A}^\#$-submodule of $B = A \oplus \mathfrak{A}^\#$ because

$$a \cdot (b, u) =: (a, 0)(b, u) = (ab + a \cdot u, 0) \qquad (a \in A \;\; , \;\; (b, u) \in B).$$

Furthermore

$$\tilde{\tilde{\phi}}\mid_A (a, 0) = \Big(\phi(a), \tilde{\varphi}(0)\Big) = \Big(\phi(a), 0\Big) = \phi_e(a) \qquad (a \in A).$$

So, this proposition is a consequence of Proposition 2.10. □

In the next proposition, we assume that $A^{**}$, the second dual of $A$ is equipped with the first Arens product, and we denote it by $\Box$. The canonical image of $a \in A$ in $A^{**}$ is denoted by $\hat{a}$ , and $\hat{A} = \{\hat{a} \;:\; a \in A\}$. Let $F = w^* - \lim_i \hat{a}_i$ and $G = w^* - \lim_j \hat{b}_j$ are members of $A^{**}$ and $\Lambda = w^* - \lim_k \hat{\alpha}_k \in \mathfrak{A}^{**}$, where $(a_i)$ and $(b_j)$ are nets in $A$ and $(\alpha_k)$ is a net in $\mathfrak{A}$. We consider the module $\mathfrak{A}^{**}$ actions on $A^{**}$ by

$$\Lambda \cdot F = w^* - \lim_k w^* - \lim_i (\alpha_k \cdot a_i)\hat{} \quad , \quad F \cdot \Lambda = w^* - \lim_i w^* - \lim_k (a_i \cdot \alpha_k)\hat{},$$

and also for the first Arens product $\Box$ on $A^{**}$ we have

$$F \Box G = w^* - \lim_i w^* - \lim_j (a_i b_j)\hat{}.$$

Let $\varphi \in \sigma(\mathfrak{A})$ and $\phi \in \Omega_A$. It is easy to check that $\varphi^{**} \in \sigma(\mathfrak{A}^{**})$ and $\phi^{**} \in \Omega_{A^{**}}$.

**Proposition 3.4.** *Let $A^{**}$ be right [left] $b \cdot app \cdot m \cdot (\phi^{**}, \varphi^{**})$-cont. and $A$ is a right [left] ideal of $A^{**}$, then $A$ is right [left] $b \cdot app \cdot m \cdot (\phi, \varphi)$-cont.*

*Proof.* By hypothesis, there is a net $(M_i) \subset A^{**}$ and $L, L' > 0$ such that: $\varphi^{**} \circ \phi^{**}(M_i) = 1$, and for all $F \in A^{**}$ and $\Lambda \in \mathfrak{A}^{**}$

$$\|F \Box M_i - \phi^{**}(F) \cdot M_i\| \to 0 \quad , \quad \|F \Box M_i - \phi^{**}(F) \cdot M_i\| \leq L\|F\|;$$
$$\|\Lambda \cdot M_i - \varphi^{**}(\Lambda) M_i\| \to 0 \quad , \quad \|\Lambda \cdot M_i - \varphi^{**}(\Lambda) M_i\| \leq L'\|\Lambda\|.$$

Now, choose $b \in A$ such that $\varphi \circ \phi(b) = 1$. Since $A$ is right ideal in $A^{**}$, we can choose the net $(n_i) \subset A$ such that $\hat{n}_i = bM_i\big(= \hat{b} \Box M_i\big)$. Hence

$$(\varphi \circ \phi)(n_i) = (\varphi \circ \phi)^{**}(\hat{n}_i) = (\varphi \circ \phi)^{**}(\hat{b})(\varphi \circ \phi)^{**}(M_i) = 1,$$

and also for all $a \in A$ and $\alpha \in \mathfrak{A}$

$$\begin{aligned} &\|an_i - \phi(a) \cdot n_i\| = \|abM_i - \phi(a) \cdot bM_i\| \\ &\leq \|abM_i - \phi(ab) \cdot M_i\| + \|\phi(a)\phi(b) \cdot M_i - \phi(a) \cdot bM_i\| \\ &\leq \|abM_i - \phi(ab) \cdot M_i\| + \|\phi(b) \cdot M_i - bM_i\|\|\phi(a)\| \to 0, \end{aligned}$$

$$\begin{aligned} \|an_i - \phi(a) \cdot n_i\| &\leq L\|ab\| + L\|b\|\|\phi\|\|a\| \\ &\leq \Big[L\|b\|(1 + \|\phi\|)\Big]\|a\|, \end{aligned}$$

and

$$\begin{aligned} &\|\alpha \cdot n_i - \varphi(\alpha) n_i\| = \|\alpha \cdot bM_i - \varphi(\alpha) bM_i\| \\ &\leq \|\alpha \cdot bM_i - \phi(\alpha b) M_i\| + \|\phi(\alpha b) M_i - \varphi(\alpha) bM_i\| \\ &\leq \|\alpha \cdot bM_i - \phi(\alpha b) M_i\| + \|\phi(b) M_i - bM_i\|\|\varphi(\alpha)\| \to 0, \end{aligned}$$

$$\begin{aligned} \|\alpha \cdot n_i - \varphi(\alpha) n_i\| &\leq L\|\alpha b\| + L\|b\|\|\varphi\|\|\alpha\| \\ &\leq \Big[L\|b\|(1 + \|\varphi\|\Big]\|\alpha\|. \end{aligned}$$

This proves that $A$ is right $b \cdot app \cdot m \cdot (\phi, \varphi)$-cont. by Proposition 2.3. □

## 4. Projective tensor product and $l^p$-direct sum of Banach algebras

In this section, $A$ and $B$ are Banach $\mathfrak{A}$-bimodules. The projective tensor product $A \hat{\otimes} B$ of $A$ and $B$ is a Banach $\mathfrak{A} \hat{\otimes} \mathfrak{A}$-bimodule with following actions

$$\begin{aligned} &(\alpha \otimes \beta) \cdot (a \otimes b) =: (\alpha \cdot a) \otimes (\beta \cdot b), \\ &(a \otimes b) \cdot (\alpha \otimes \beta) =: (a \cdot \alpha) \otimes (b \cdot \beta) \quad (a \in A \ , \ b \in B \ ; \ \alpha, \beta \in \mathfrak{A}). \end{aligned}$$

For $\phi_1 \in \Omega_A$, $\phi_2 \in \Omega_B$ and $\varphi_1, \varphi_2 \in \sigma(\mathfrak{A})$, consider

$$\phi_1 \otimes \phi_2 : A \hat{\otimes} B \to \mathfrak{A} \hat{\otimes} \mathfrak{A} \left(\phi_1 \otimes \phi_2(a \otimes b) =: \phi_1(a) \otimes \phi_2(b)\right),$$

and

$$\varphi_1 \otimes \varphi_2 : \mathfrak{A} \hat{\otimes} \mathfrak{A} \to \mathbb{C} \left(\varphi_1 \otimes \varphi_2(\alpha \otimes \beta) =: \varphi_1(\alpha)\varphi_2(\beta)\right).$$

Clearly, $\phi_1 \otimes \phi_2 \in \Omega_{A \hat{\otimes} B}$ and $\varphi_1 \otimes \varphi_2 \in \sigma(\mathfrak{A} \hat{\otimes} \mathfrak{A})$.

**Proposition 4.1.** *If $A \hat{\otimes} B$ is right [left] $b \cdot app \cdot m \cdot (\phi_1 \otimes \phi_2, \varphi_1 \otimes \varphi_2)$-cont. then $A$ is right [left] $b \cdot app \cdot m \cdot (\phi_1, \varphi_1)$-cont. and $B$ is right [left] $b \cdot app \cdot m \cdot (\phi_2, \varphi_2)$-cont.*

*Proof.* By Proposition 2.3, there exist a net $(m_i) \subset A\hat{\otimes}B$ and $L, L' > 0$ such that $\big[(\varphi_1 \otimes \varphi_2) \circ (\phi_1 \otimes \phi_2)\big](m_i) = 1$, and for all $w \in A\hat{\otimes}B$ and $\omega \in \mathfrak{A}\hat{\otimes}\mathfrak{A}$

$$\begin{array}{lll} \|wm_i - (\phi_1 \otimes \phi_2)(w) \cdot m_i\| \to 0 & , & \|wm_i - (\phi_1 \otimes \phi_2)(w) \cdot m_i\| \le L\|w\|, \\ \|\omega \cdot m_i - (\varphi_1 \otimes \varphi_2)(\omega)m_i\| \to 0 & , & \|\omega \cdot m_i - (\varphi_1 \otimes \varphi_2)(\omega)m_i\| \le L'\|\omega\|. \end{array}$$

Now, consider the linear map $p_A : A\hat{\otimes}B \to A$ $\big(p_A(a \otimes b) =: \varphi_2 \circ \phi_2(b)a\big)$. Then, for $a \otimes b \in A\hat{\otimes}B$

$$\begin{aligned} (\varphi_1 \circ \phi_1)\big(p_A(a \otimes b)\big) &= (\varphi_1 \circ \phi_1)\big(\varphi_2 \circ \phi_2(b)a\big) \\ &= (\varphi_2 \circ \phi_2)(b)(\varphi_1 \circ \phi_1)(a) \\ &= \big((\varphi_1 \circ \phi_1) \otimes (\varphi_2 \circ \phi_2)\big)(a \otimes b) \\ &= \big((\varphi_1 \otimes \varphi_2) \circ (\phi_1 \otimes \phi_2)\big)(a \otimes b), \end{aligned}$$

so for each $m_i \in A\hat{\otimes}B$ we have

$$(\varphi_1 \circ \phi_1)\big(p_A(m_i)\big) = \big((\varphi_1 \otimes \varphi_2) \circ (\phi_1 \otimes \phi_2)\big)(m_i) = 1.$$

Now, choose $\alpha_0, \beta_0 \in \mathfrak{A}$ such that $\varphi_1(\alpha_0) = \varphi_2(\beta_0) = 1$. Then

$$\begin{aligned} \|(\alpha_0 \otimes \beta_0) \cdot m_i - m_i\| &= \|(\alpha_0 \otimes \beta_0) \cdot m_i - (\varphi_1 \otimes \varphi_2)(\alpha_0 \otimes \beta_0)m_i\| \to 0, \\ \|(\alpha_0 \otimes \beta_0) \cdot m_i - m_i\| &\le L'\|\alpha_0 \otimes \beta_0\|, \end{aligned}$$

and for all $\alpha \in \mathfrak{A}$ we have

$$\begin{aligned} \|(\alpha\alpha_0 \otimes \beta_0) \cdot m_i - \varphi_1(\alpha)m_i\| &= \|(\alpha\alpha_0 \otimes \beta_0) \cdot m_i - (\varphi_1 \otimes \varphi_2)(\alpha\alpha_0 \otimes \beta_0)m_i\| \to 0, \\ \|(\alpha\alpha_0 \otimes \beta_0) \cdot m_i - \varphi_1(\alpha)m_i\| &\le L'\|\alpha\alpha_0 \otimes \beta_0\|. \end{aligned}$$

Since $\alpha \cdot p_A(m_i) = p_A(\alpha \cdot m_i)$ and $p_A$ is linear, then we have

$$\begin{aligned} &\|\alpha \cdot p_A(m_i) - \varphi_1(\alpha)p_A(m_i)\| \le \|p_A\|\|\alpha \cdot m_i - \varphi_1(\alpha)m_i\| \\ &\le \|p_A\|\Big[\|\alpha \cdot m_i - (\alpha\alpha_0 \otimes \beta_0)m_i\| + \|(\alpha\alpha_0 \otimes \beta_0)m_i - \varphi_1(\alpha)m_i\|\Big] \\ &\le \|p_A\|\Big[\|\alpha\|\|m_i - (\alpha_0 \otimes \beta_0)m_i\| + \|(\alpha\alpha_0 \otimes \beta_0)m_i - \varphi_1(\alpha)m_i\|\Big] \to 0, \end{aligned}$$

and

$$\begin{aligned} \|\alpha \cdot p_A(m_i) - \varphi_1(\alpha)p_A(m_i)\| &\le \|p_A\|\Big[\|\alpha\|L'\|\alpha_0\|\|\beta_0\| + L'\|\alpha\|\|\alpha_0\|\|\beta_0\|\Big] \\ &= \big(2\|p_A\|L'\|\alpha_0\|\|\beta_0\|\big)\|\alpha\|. \end{aligned}$$

For the rest of proof, choose $a_0 \otimes b_0 \in A\hat{\otimes}B$ such that

$$\varphi_1 \circ \phi_1(a_0) = \varphi_2 \circ \phi_2(b_0) = 1.$$

Then

$$\begin{aligned} &\|(a_0 \otimes b_0)m_i - m_i\| = \|(a_0 \otimes b_0)m_i - (\varphi_1 \circ \phi_1)(a_0)(\varphi_2 \circ \phi_2)(b_0)m_i\| \\ &\|(a_0 \otimes b_0)m_i - \big((\varphi_1 \otimes \varphi_2) \circ (\phi_1 \otimes \phi_2)\big)(a_0 \otimes b_0)\| \to 0, \\ &\|(a_0 \otimes b_0)m_i - m_i\| \le L\|a_0 \otimes b_0\|, \end{aligned}$$

and for all $a \in A$

$$\begin{aligned} &\|(aa_0 \otimes b_0)m_i - \varphi_1 \circ \phi_1(a)m_i\| = \|(aa_0 \otimes b_0)m_i - \varphi_1 \circ \phi_1(aa_0)\varphi_2 \circ \phi_2(b_0)m_i\| \\ &= \|(aa_0 \otimes b_0)m_i - \big((\varphi_1 \otimes \varphi_2) \circ (\phi_1 \otimes \phi_2)\big)(aa_0 \otimes b_0)\| \to 0, \\ &\|(aa_0 \otimes b_0)m_i - \varphi_1 \circ \phi_1(a)m_i\| \le L\|aa_0 \otimes b_0\|. \end{aligned}$$

Since $ap_A(m_i) = p_A(am_i)$, we conclude that

$$\begin{aligned}&\|ap_A(m_i) - \varphi_1 \circ \phi_1(a)p_A(m_i)\| \leq \|p_A\|\|am_i - \varphi_1 \circ \phi_1(a)m_i\|\\&\leq \|p_A\|\Big[\|am_i - (aa_0 \otimes b_0)m_i\| + \|(aa_0 \otimes b_0)m_i - \varphi_1 \circ \phi_1(a)m_i\|\Big]\\&\leq \|p_A\|\Big[\|a\|\|m_i - (a_0 \otimes b_0)m_i\| + (aa_0 \otimes b_0)m_i - \varphi_1 \circ \phi_1(a)m_i\|\Big] \to 0,\end{aligned}$$

and also

$$\begin{aligned}&\|ap_A(m_i) - \varphi_1 \circ \phi_1(a)p_A(m_i)\| \leq \|p_A\|\Big[\|a\|L\|a_0\|\|b_0\| + L\|a\|\|a_0\|\|b_0\|\Big]\\&\Big(2\|p_A\|L\|a_0\|\|b_0\|\Big)\|a\|.\end{aligned}$$

Finally, this shows that $A$ is right $b \cdot app \cdot m \cdot (\phi_1, \varphi_1)$-cont. by Proposition 2.3. There is a similar proof for $B$. □

Now let $\phi \in \Omega_A$ , $\psi \in \Omega_B$, $\varphi \in \sigma(\mathfrak{A})$ and $1 \leq p \leq +\infty$. The *$l^p$-direct sums* $A \oplus_\infty B$ and $A \oplus_p B$ are Banach algebras with respect to multiplication defined by

$$(a, b)(c, d) =: (ac, bd) \qquad (a, c \in A \;,\; b, d \in B),$$

and norms

$$\|(a, b)\|_\infty =: \max\{\|a\|, \|b\|\} \quad , \quad \|(a, b)\|_p = \Big(\|a\|^p + \|b\|^p\Big)^{1/p} \qquad (a \in A \;,\; b \in B).$$

Furthermore, $A \oplus_\infty B$ and $A \oplus_p B$ are Banach $\mathfrak{A}$-bimodules under the following $\mathfrak{A}$-module actions

$$\alpha \cdot (a, b) =: (\alpha \cdot a, \alpha \cdot b) \quad , \quad (a, b) \cdot \alpha =: (a \cdot \alpha, b \cdot \alpha) \qquad (a \in A \;,\; b \in B \;,\; \alpha \in \mathfrak{A}).$$

We define

$$\begin{aligned}&(\phi, 0) : A \oplus_p B \to \mathfrak{A} \qquad , \qquad (\phi, 0)(a, b) =: \phi(a),\\&(0, \psi) : A \oplus_p B \to \mathfrak{A} \qquad , \qquad (0, \psi)(a, b) =: \psi(b),\end{aligned}$$

for $(a, b) \in A \oplus_p B$ and $1 \leq p \leq +\infty$. Then $(0, \psi), (\phi, 0) \in \Omega_{A\oplus_p B}$ for $1 \leq p \leq +\infty$, and $(\phi, 0) \mid_A = \phi$ , $(0, \psi) \mid_B = \psi$.

**Proposition 4.2.** *Let $A$ and $B$ be Banach algebras and $\mathfrak{A}$-bimodules, $\phi \in \Omega_A$, $\psi \in \Omega_B$, $\varphi \in \sigma(\mathfrak{A})$ and $1 \leq p \leq +\infty$. Then*

*(i) $A \oplus_p B$ is right [left] $b \cdot app \cdot m \cdot \Big((\phi, 0), \varphi\Big)$-cont. if and only if $A$ is right [left] $b \cdot app \cdot m \cdot (\phi, \varphi)$-cont.*

*(ii) $A \oplus_p B$ is right [left] $b \cdot app \cdot m \cdot \Big((0, \psi), \varphi\Big)$-cont. if and only if $B$ is right [left] $b \cdot app \cdot m \cdot (\psi, \varphi)$-cont.*

*Proof.* This is a consequence of Proposition 2.10. □

## 5. Examples

We start this section with following definitions.

**Definition 5.1.** *[1] A discrete semigroup $S$ is called an inverse semigroup if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of all idempotents of $S$ is denoted by $E$.*

It is easy to see that $E$ is a commutative subsemigroup of $S$ and $l^1(E)$ is a subalgebra of $l^1(S)$. Suppose that $l^1(S)$ is a $l^1(E)$-bimodule by following actions, that is multiplication from right and trivially from left

$$\delta_e \cdot \delta_s =: \delta_s \quad , \quad \delta_s \cdot \delta_e =: \delta_{se}\Big( = \delta_s * \delta_e\Big) \qquad (s \in S \;,\; e \in E).$$

We denote $J_{l^1(s)}$ by $J$ that is the closed ideal of $l^1(s)$ generated by $\Big\{\delta_{set} - \delta_{st} \;\; : \;\; s, t \in S \;,\; e \in E\Big\}$.

Next, we consider the congruence relation $\backsim$ on $S$ by

$$s \backsim t \;\Leftrightarrow\; \exists\, e \in E \;:\; se = te \qquad (s, t \in S).$$

The quotient semigroup $G_S := S/\backsim$ is a group by Theorem 1 in [15]. Furthermore, $l^1(G_S)$ is a quotient of $l^1(S)$ by Lemma 3.2 in [1]. Indeed $l^1(G_S) \cong l^1(S)/J$, and by lifting the $l^1(E)$-module actions on $l^1(S)$ to $l^1(G_S)$ it becomes a Banach $l^1(E)$-bimodule. But, the right and left $l^1(E)$-module actions on $l^1(G_S)$ are trivial, so we have

$$l^1(G_S)\hat{\otimes}_{l^1(E)} l^1(G_S) \cong l^1(G_S)\hat{\otimes} l^1(G_S),$$

see Lemma 3.3 in [1].

Now we are ready to show the main results of this section.

**Proposition 5.2.** *Let $S$ be an inverse semigroup with idempotents $E$. Consider $l^1(S)$ as a Banach $l^1(E)$-bimodule with multiplication right action and the trivial left action. Suppose that $\phi \in \Omega_{l^1(S)}$ and $\varphi \in \sigma\big(l^1(E)\big)$, such that $J \subset \ker\phi$. The following statements are equivalent:*

*(i) $l^1(S)$ is left [right] approximately-module-$(\phi, \varphi)$-contractible.*
*(ii) $l^1(G_S)$ is left [right] approximately-module-$(\tilde{\phi}, \varphi)$-contractible.*
*(iii) $l^1(G_S)$ is left [right] approximately-$\varphi \circ \tilde{\phi}$-contractible.*

*Proof.* The equivalence of (*i*) and (*ii*) follows from Colloary 2.12. Since $L^1(G_S)$ is a commutative Banach $l^1(G_S) - l^1(E)$-module and

$$l^1(G_S)\hat{\otimes}_{l^1(E)} l^1(G_S) \cong l^1(G_S)\hat{\otimes} l^1(G_S),$$

then every approximate-module-$(\phi, \varphi)$-diagonal for $l^1(G_S)$ is an approximate-$\varphi \circ \tilde{\phi}$-diagonal and vice versa. So (*ii*) and (*iii*) are equivalent by Proposition 2.5 and by Theorem 2.7. in [20]. □

**Corollary 5.3.** *With the setting of above proposition, the following statements are equivalent:*

*(i) $l^1(G_S)$ is left [right] $b \cdot app \cdot m \cdot (\tilde{\phi}, \varphi)$-cont.*
*(ii) $l^1(G_S)$ is left [right] $b \cdot app \cdot (\varphi \circ \tilde{\phi})$-cont.*

*Proof.* This a consequence of Proposition 2.5 and Proposition 5.2. □

**Corollary 5.4.** *With the setting of above proposition, if $l^1(G_S)$ is left [right] $b \cdot app \cdot m \cdot (\tilde{\phi}, \varphi)$-cont then $l^1(S)$ is left [right] $b \cdot app \cdot m \cdot (\phi, \varphi)$-cont.*

*Proof.* This is a consequence of Proposition 2.11. □

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