Filomat 37:23 (2023), 7699–7718
https://doi.org/10.2298/FIL2323699B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On characteristic functions of generalized resolvents generated by integral equations with operator measures

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Abstract. We consider a symmetric minimal relation L_0 generated by an integral equation with operators measures. We describe the generalized resolvents of L_0 using the characteristic function $M(\lambda)$ ($\lambda \in \mathbb{C}$), i.e., a function that has the property $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}M(\lambda) \geqslant 0$. We obtain a necessary and sufficient condition for a holomorphic function $M(\lambda)$ to be a characteristic function of a generalized resolvent. We give a detailed example of finding the characteristic function.

1. Introduction

Generalized resolvents of symmetric operators were introduced by M.A. Naimark in 1940 (see, for example, [1]). In the article [24], A.V. Straus described the generalized resolvents of a symmetric operator generated by a formally self-adjoint differential expression of even order in the scalar case. In such a description, a function $M(\lambda)$ ($\lambda \in \mathbb{C}$) plays an essential role. This function has the property $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}M(\lambda) \geqslant 0$. In [24], the function $M(\lambda)$ is called the characteristic function of the generalized resolvent. The characteristic function is used in solving many problems associated with the generalized resolvents (see [24]).

In [5], these results from [24] were extended to the operator case, and in [7], to the case of a differential-operator expression with a non-negative weight operator function. Further, the generalized resolvents of differential operators were studied in many works (a detailed bibliography is available, for example, in [22], [20]).

In this paper, we consider the integral equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}(s)y(s) - iJ\int_a^t d\mathbf{m}(s)f(s), \tag{1}$$

where y is an unknown function, $a \leqslant t \leqslant b$; J is an operator in a separable Hilbert space H, $J = J^*$, $J^2 = E$ (E is the identical operator); $\mathbf{p}$, $\mathbf{m}$ are operator-valued measures defined on Borel sets $\Delta \subset [a,b]$ and taking values in the set of linear bounded operators acting in H; $x_0 \in H$, $f \in L_2(H, d\mathbf{m}; a, b)$. We assume that the measures $\mathbf{p}$, $\mathbf{m}$ have bounded variations and $\mathbf{p}$ is self-adjoint, $\mathbf{m}$ is non-negative.

If the measures $\mathbf{p}$, $\mathbf{m}$ are absolutely continuous (i.e., $\mathbf{p}(\Delta) = \int_\Delta p(t)dt$, $\mathbf{m}(\Delta) = \int_\Delta m(t)dt$ for all Borel sets $\Delta \subset [a,b]$, where $p(t)$, $m(t)$ are bounded operators for fixed t and the functions $\|p(t)\|$, $\|m(t)\|$ belong

2020 *Mathematics Subject Classification*. Primary 47A10; Secondary 47B25, 45N05
Keywords. Hilbert space, integral equation, operator measure, linear relation, symmetric relation, generalized resolvent
Received: 29 November 2022; Accepted: 30 March 2023
Communicated by Dragan S. Djordjević
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to $L_1(a,b)$), then integral equation (1) is transformed to a differential equation with a non-negative weight operator function. Linear relations and operators generated by such differential equations were considered in many works (see [21], [6], [7], further detailed bibliography can be found, for example, in [20], [3]).

The study of integral equation (1) differs essentially from the study of differential equations by the presence of the following features: i) a representation of a solution of equation (1) using an evolutional family of operators is possible if the measures **p**, **m** have not common single-point atoms (see [10]); ii) the Lagrange formula contains summands relating to single-point atoms of the measures **p**, **m** (see [11]).

We consider a symmetric minimal relation L_0 generated by equation (1). We obtain a form of generalized resolvents of L_0. This form contains the characteristic function. The proposed article is a continuation of the works [15], [16]. In [15], continuously invertible relations $T(\lambda)$ with the property $L_0 - \lambda E \subset T(\lambda) \subset L_0^* - \lambda E$ are described. In this description, a function $M(\lambda)$ plays an important role. The function $M(\lambda)$ is determined by the relation $T(\lambda)$ and the choice of a system of functions v_k, where $v_k \in \ker(L_0^* - \lambda E)$ and the linear span of the set $\{v_k\}$ is dense in $\ker(L_0^* - \lambda E)$. In [16], the generalized resolvents of the relation L_0 are described using the function $M(\lambda)$ from [15]. However, this function $M(\lambda)$ is not characteristic (see the example in [16]).

In this article, we construct a system of functions ϑ_k different from the system $\{v_k\}$ in [15], [16], but having the same properties: $\vartheta_k \in \ker(L_0^* - \lambda E)$ and the linear span of the set $\{\vartheta_k\}$ is dense in $\ker(L_0^* - \lambda E)$. We describe generalized resolvents using the system of functions ϑ_k. The corresponding function $M(\lambda)$ is characteristic. We obtain a necessary and sufficient condition for a holomorphic function $M(\lambda)$ to be a characteristic function of a generalized resolvent. We give a detailed example of finding the characteristic function.

We note that this article partially corrects the errors made in the work [9]. Also note that some special cases of the describing of generalized resolvents and related problems for equation (1) are considered in [12], [13], [14].

2. Designations and preliminary assertions

Let **B** be a Hilbert space. A linear relation T is understood as any linear manifold $T \subset \mathbf{B} \times \mathbf{B}$. The terminology on the linear relations can be found, for example, in [18], [22], [2]. In what follows we make use of the following notations: $\{\cdot,\cdot\}$ is an ordered pair; $\mathcal{D}(T)$ is the domain of T; $\mathcal{R}(T)$ is the range of T; $\ker T$ is a set of elements $x \in \mathbf{B}$ such that $\{x, 0\} \in T$; T^{-1} is the relation inverse for T, i.e., the relation formed by the pairs $\{x', x\}$, where $\{x, x'\} \in T$. A relation T is called surjective if $\mathcal{R}(T) = \mathbf{B}$. A relation T is called invertible or injective if $\ker T = \{0\}$ (i.e., the relation T^{-1} is an operator); it is called continuously invertible if it is closed, invertible, and surjective (i.e., T^{-1} is a bounded everywhere defined operator). A relation T^* is called adjoint for T if T^* consists of all pairs $\{y_1, y_2\}$ such that the equality $(x_2, y_1) = (x_1, y_2)$ holds for all pairs $\{x_1, x_2\} \in T$. A relation T is called symmetric if $T \subset T^*$ and self-adjoint if $T = T^*$.

It is known (see, for example, [19, ch.3], [18, ch.1]) that the graph of an operator $T : \mathcal{D}(T) \to \mathbf{B}$ is the set of pairs $\{x, Tx\} \in \mathbf{B} \times \mathbf{B}$, where $x \in \mathcal{D}(T) \subset \mathbf{B}$. Consequently, the linear operators can be treated as linear relations; this is why the notation $\{x_1, x_2\} \in T$ is used also for the operator T. Since all considered relations are linear, we shall often omit the word "linear".

Let T be a closed symmetric relation, $T \subset \mathbf{B} \times \mathbf{B}$, and let $\widetilde{T}$ be a self-adjoint extension of T to $\widetilde{\mathbf{B}}$, where $\widetilde{\mathbf{B}}$ is a Hilbert space, $\widetilde{\mathbf{B}} \supset \mathbf{B}$, and scalar products coincide in $\mathbf{B}$ and $\widetilde{\mathbf{B}}$. By P denote an orthogonal projection of $\widetilde{\mathbf{B}}$ onto $\mathbf{B}$. The function $\lambda \to R_\lambda$ defined by the formula $R_\lambda = P(\widetilde{T} - \lambda E)^{-1}|_{\mathbf{B}}$, $\mathrm{Im}\lambda \neq 0$, is called the generalized resolvent of the relation T (see, for example, [1, ch.9], [17]).

A.V. Straus (see [23]) obtained a formula for all generalized resolvents of a symmetric operator. It is shown in [17] that this formula remains true for symmetric relations also. By $\mathfrak{N}_\lambda$ denote a defect subspace of the closed symmetric relation T, i.e., the orthogonal complement in $\mathbf{B}$ of the range of the relation $T - \lambda E$. We fix some number λ_0 ($\mathrm{Im}\lambda_0 \neq 0$). Let $\lambda \to \mathcal{F}(\lambda)$ be a holomorphic operator function, where $\mathcal{F}(\lambda) : \mathfrak{N}_{\lambda_0} \to \mathfrak{N}_{\overline{\lambda}_0}$ is a bounded operator, $\|\mathcal{F}(\lambda)\| \leqslant 1$, $\mathrm{Im}\lambda \cdot \mathrm{Im}\lambda_0 > 0$. Let $T_{\mathcal{F}(\lambda)}$ be the relation consisting of all pairs of the form $\{y_0 + \mathcal{F}(\lambda)z - z, y_1 + \lambda_0\mathcal{F}(\lambda)z - \overline{\lambda}_0 z\}$, where $\{y_0, y_1\} \in T$, $z \in \mathfrak{N}_{\lambda_0}$. Then (see [23], [17]) the family of operators R_λ is a generalized resolvent of T if and only if R_λ can be represented in the form

$$R_\lambda = (T_{\mathcal{F}(\lambda)} - \lambda E)^{-1}, \quad \mathrm{Im}\lambda \cdot \mathrm{Im}\lambda_0 > 0. \tag{2}$$

In what follows, we use the following notions from measure theory. Let H be a separable Hilbert space with a scalar product $(\cdot,\cdot)$ and a norm $\|\cdot\|$. By $\mathcal{B}$ denote a set of Borel subsets $\Delta\subset[a,b]$. We consider a function $\Delta\to\mathbf{P}(\Delta)$ defined on $\mathcal{B}$ and taking values in the set of linear bounded operators acting in H. The function $\mathbf{P}$ is called an operator measure on $[a,b]$ (see, for example, [4, ch. 5]) if it is zero on the empty set and the equality $\mathbf{P}\left(\bigcup_{n=1}^{\infty}\Delta_n\right)=\sum_{n=1}^{\infty}\mathbf{P}(\Delta_n)$ holds for disjoint Borel sets Δ_n, where the series converges weakly. Further, we extend any measure $\mathbf{P}$ on $[a,b]$ to a segment $[a,b_0]$ $(b_0>b)$ letting $\mathbf{P}(\Delta)=0$ for each Borel sets $\Delta\subset(b,b_0]$.

By $\mathbf{V}_\Delta(\mathbf{P})$ we denote $\mathbf{V}_\Delta(\mathbf{P})=\rho_{\mathbf{P}}(\Delta)=\sup\sum_n\|\mathbf{P}(\Delta_n)\|$, where the supremum is taken over finite sums of disjoint Borel sets $\Delta_n\subset\Delta$. The number $\mathbf{V}_\Delta(\mathbf{P})$ is called the variation of the measure $\mathbf{P}$ on the Borel set Δ. Suppose that the measure $\mathbf{P}$ has the bounded variation on $[a,b]$. Then for $\rho_{\mathbf{P}}$-almost all $\xi\in[a,b]$ there exists an operator function $\xi\to\Psi_{\mathbf{P}}(\xi)$ such that $\Psi_{\mathbf{P}}$ possesses the values in the set of linear bounded operators acting in H, $\|\Psi_{\mathbf{P}}(\xi)\|=1$, and the equality

$$\mathbf{P}(\Delta)=\int_\Delta\Psi_{\mathbf{P}}(s)d\rho_{\mathbf{P}} \tag{3}$$

holds for each set $\Delta\in\mathcal{B}$. The function $\Psi_{\mathbf{P}}$ is uniquely determined up to values on a set of zero $\rho_{\mathbf{P}}$-measure. Integral (3) converges in the sense of usual operator norm ([4, ch. 5]).

Further, $\int_{t_0}^t$ stands for $\int_{[t_0t)}$ if $t_0<t$, for $-\int_{[t,t_0)}$ if $t_0>t$, and for 0 if $t_0=t$. This implies that $y(a)=x_0$ in equation (1). A function h is integrable with respect to the measure $\mathbf{P}$ on a set $\Delta\in\mathcal{B}$ if there exists the Bochner integral $\int_\Delta\Psi_{\mathbf{P}}(t)h(t)d\rho_{\mathbf{P}}=\int_\Delta(d\mathbf{P})h(t)$. Then the function $y(t)=\int_{t_0}^t(d\mathbf{P})h(s)$ is continuous from the left.

By $\mathcal{S}_{\mathbf{P}}$ denote a set of single-point atoms of the measure $\mathbf{P}$ (i.e., a set $t\in[a,b]$ such that $\mathbf{P}(\{t\})\neq0$). The set $\mathcal{S}_{\mathbf{P}}$ is at most countable. The measure $\mathbf{P}$ is continuous if $\mathcal{S}_{\mathbf{P}}=\varnothing$, it is self-adjoint if $(\mathbf{P}(\Delta))^*=\mathbf{P}(\Delta)$ for each Borel set $\Delta\in\mathcal{B}$, it is non-negative if $(\mathbf{P}(\Delta)x,x)\geqslant0$ for all Borel sets $\Delta\in\mathcal{B}$ and for all elements $x\in H$.

In addition, we use the following notation. We construct a continuous measure $\mathbf{P}_0$ from the measure $\mathbf{P}$ in the following way. We set $\mathbf{P}_0(\{t_k\})=0$ for $t_k\in\mathcal{S}_{\mathbf{P}}$ and we set $\mathbf{P}_0(\Delta)=\mathbf{P}(\Delta)$ for all Borel sets such that $\Delta\cap\mathcal{S}_{\mathbf{P}}=\varnothing$. We denote $\widehat{\mathbf{P}}=\mathbf{P}-\mathbf{P}_0$. Then $\widehat{\mathbf{P}}(\{t_k\})=\mathbf{P}(\{t_k\})$ for all $t_k\in\mathcal{S}_{\mathbf{P}}$ and $\widehat{\mathbf{P}}(\Delta)=0$ for all Borel sets Δ such that $\Delta\cap\mathcal{S}_{\mathbf{P}}=\varnothing$.

In following Lemma 2.1, $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}$ are operator measures having bounded variations and taking values in the set of linear bounded operators acting in H. Suppose that the measure $\mathbf{q}$ is self-adjoint and assume that these measures are extended on the segment $[a,b_0]\supset[a,b_0)\supset[a,b]$ in the manner described above.

Lemma 2.1. [11] *Let f, g be functions integrable on $[a,b_0]$ with respect to the measure $\mathbf{q}$ and $y_0,z_0\in H$. Then any functions*

$$y(t)=y_0-iJ\int_{t_0}^t d\mathbf{p}_1(s)y(s)-iJ\int_{t_0}^t d\mathbf{q}(s)f(s),\quad z(t)=z_0-iJ\int_{t_0}^t d\mathbf{p}_2(s)z(s)-iJ\int_{t_0}^t d\mathbf{q}(s)g(s)\quad(a\leqslant t_0<b_0,\ t_0\leqslant t\leqslant b_0)$$

satisfy the following formula (*analogous to the Lagrange one*):

$$\begin{aligned}\int_{c_1}^{c_2}(d\mathbf{q}(t)f(t),z(t))-\int_{c_1}^{c_2}(y(t),d\mathbf{q}(t)g(t))&=(iJy(c_2),z(c_2))-(iJy(c_1),z(c_1))+\int_{c_1}^{c_2}(y(t),d\mathbf{p}_2(t)z(t))-\\ -\int_{c_1}^{c_2}(d\mathbf{p}_1(t)y(t),z(t))&-\sum_{t\in\mathcal{S}_{\mathbf{p}_1}\cap\mathcal{S}_{\mathbf{p}_2}\cap[c_1,c_2)}(iJ\mathbf{p}_1(\{t\})y(t),\mathbf{p}_2(\{t\})z(t))-\sum_{t\in\mathcal{S}_{\mathbf{q}}\cap\mathcal{S}_{\mathbf{p}_2}\cap[c_1,c_2)}(iJ\mathbf{q}(\{t\})f(t),\mathbf{p}_2(\{t\})z(t))-\\ -\sum_{t\in\mathcal{S}_{\mathbf{p}_1}\cap\mathcal{S}_{\mathbf{q}}\cap[c_1,c_2)}(iJ\mathbf{p}_1(\{t\})y(t),\mathbf{q}(\{t\})g(t))&-\sum_{t\in\mathcal{S}_{\mathbf{q}}\cap[c_1,c_2)}(iJ\mathbf{q}(\{t\})f(t),\mathbf{q}(\{t\})g(t)),\quad t_0\leqslant c_1<c_2\leqslant b_0.\end{aligned} \tag{4}$$

Further we assume that measures $\mathbf{p}$, $\mathbf{m}$ have bounded variations and $\mathbf{p}$ is self-adjoint, $\mathbf{m}$ is non-negative. We consider equation (1), where $x_0\in H$, f is integrable with respect to the measure $\mathbf{m}$ on $[a,b]$, $a\leqslant t\leqslant b_0$. We replace $\mathbf{p}$ by $\mathbf{p}_0$ and $\mathbf{m}$ by $\mathbf{m}_0$ in (1). Then we obtain the equation

$$y(t)=x_0-iJ\int_a^t d\mathbf{p}_0(s)y(s)-iJ\int_a^t d\mathbf{m}_0(s)f(s). \tag{5}$$

Equations (1), (5) have unique solutions (see [10]).

By $W(t,\lambda)$ denote an operator solution of the equation

$$W(t,\lambda)x_0 = x_0 - iJ\int_a^t d\mathbf{p}_0(s)W(s,\lambda)x_0 - iJ\lambda\int_a^t d\mathbf{m}_0(s)W(s,\lambda)x_0, \tag{6}$$

where $x_0 \in H$, $\lambda \in \mathbb{C}$ ($\mathbb{C}$ is the set of complex numbers). It follows from Lemma 2.1 that $W^*(t,\overline{\lambda})JW(t,\lambda) = J$. The functions $t \to W(t,\lambda)$ and $t \to W^{-1}(t,\lambda) = JW^*(t,\overline{\lambda})J$ are continuous with respect to the uniform operator topology. Consequently, there exist constants $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that the inequality

$$\varepsilon_1 \|x\|^2 \leqslant \|W(t,\lambda)x\|^2 \leqslant \varepsilon_2 \|x\|^2 \tag{7}$$

holds for all $x \in H$, $t \in [a,b_0]$, $\lambda \in C \subset \mathbb{C}$ (C is a compact set).

Lemma 2.2. [15]. *Suppose that a function f is integrable with respect to the measure $\mathbf{m}$. A function y is a solution of the equation*

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}_0(s)y(s) - iJ\lambda\int_a^t d\mathbf{m}_0(s)y(s) - iJ\int_a^t d\mathbf{m}(s)f(s), \quad x_0 \in H, \quad a \leqslant t \leqslant b_0,$$

if and only if y has the form

$$y(t) = W(t,\lambda)x_0 - W(t,\lambda)iJ\int_a^t W^*(\xi,\overline{\lambda})d\mathbf{m}(\xi)f(\xi).$$

3. Linear relations generated by the integral equation

This article substantially uses the results of [15], [16]. In this section, we provide definitions and statements from [15], [16] that are used in this article. Moreover, in this section, we construct a system of functions $\{\vartheta_k\}$ whose the linear span is dense in $\ker(L_{10}^* - \lambda E)$ (see below). This system is different from the system $\{v_k\}$ introduced in [15], [16]. In the next section, we use the system $\{\vartheta_k\}$ to find the characteristic function $M(\lambda)$.

Let $\mathbf{m}$ is a non-negative operator measure defined on Borel sets $\Delta \subset [a,b]$ and taking values in the set of linear bounded operators acting in the space H. The measure $\mathbf{m}$ is assumed to have a bounded variation on $[a,b]$. We introduce the quasi-scalar product $(x,y)_{\mathbf{m}} = \int_a^{b_0} ((d\mathbf{m})x(t), y(t))$ on a set of step-like functions with values in H defined on the segment $[a,b_0]$. Identifying with zero functions y obeying $(y,y)_{\mathbf{m}} = 0$ and making the completion, we arrive at the Hilbert space denoted by $L_2(H, d\mathbf{m}; a, b) = \mathfrak{H}$. The elements of $\mathfrak{H}$ are the classes of functions identified with respect to the norm $\|y\|_{\mathbf{m}} = (y,y)_{\mathbf{m}}^{1/2}$. In order not to complicate the terminology, the class of functions with a representative y is indicated by the same symbol and we write $y \in \mathfrak{H}$. The equality of functions in $\mathfrak{H}$ is understood as the equality for associated equivalence classes. We denote the scalar product in the space $\mathfrak{H}$ by the symbols $(\cdot,\cdot)_{\mathfrak{H}}$ or $(\cdot,\cdot)_{\mathbf{m}}$. The scalar product generated by the measure $\mathbf{m}_0$ is denoted by $(\cdot,\cdot)_{\mathbf{m}_0}$.

Let us define a *minimal relation* L_0 in the following way. The relation L_0 consists of all pairs $\{\widetilde{y}, \widetilde{f_0}\} \in \mathfrak{H} \times \mathfrak{H}$ satisfying the condition: for each pair $\{\widetilde{y}, \widetilde{f_0}\}$ there exists a pair $\{y, f_0\}$ such that the pairs $\{\widetilde{y}, \widetilde{f_0}\}$, $\{y, f_0\}$ are identical in $\mathfrak{H} \times \mathfrak{H}$ and $\{y, f_0\}$ satisfies equation (1) and the equalities

$$y(a) = y(b_0) = y(\alpha) = 0, \quad \alpha \in \mathcal{S}_{\mathbf{p}}; \quad \mathbf{m}(\{\beta\})f_0(\beta) = 0, \quad \beta \in \mathcal{S}_{\mathbf{m}}. \tag{8}$$

Further, without loss of generality it can be assumed that if $\{y, f_0\} \in L_0$, then equalities (1), (8) hold for this pair. In general, the relation L_0 is not an operator since a function y can happen to be identified with zero in $\mathfrak{H}$, while f is non-zero. The relation L_0 is symmetric and closed. We note that if $y \in \mathcal{D}(L_0)$, then y is continuous and $y(b) = 0$ (see[14], [15]).

By $\overline{\mathcal{S}}_{\mathbf{p}}$ denote the closure of the set $\mathcal{S}_{\mathbf{p}}$. Let $\mathcal{S}_0$ be the set $t \in [a,b]$ such that $y(t) = 0$ for all $y \in \mathcal{D}(L_0)$. The set $\mathcal{S}_0$ is closed and $\overline{\mathcal{S}}_{\mathbf{p}} \cup \{a\} \cup \{b\} \subset \mathcal{S}_0$ (see[15]).

Lemma 3.1. [15]. *Suppose $\{y, f\} \in L_0$. Then $f(t) = 0$ for $\mathbf{m}$-almost all $t \in \mathcal{S}_0$.*

By $\mathfrak{H}_0$ (by $\mathfrak{H}_1$) denote a subspace of functions that vanish on $[a, b] \setminus \mathcal{S}_0$ (on $\mathcal{S}_0$, respectively) with respect to the norm in $\mathfrak{H}$. The subspaces $\mathfrak{H}_0$, $\mathfrak{H}_1$ are orthogonal and $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$. We note that $\mathfrak{H}_0 = \{0\}$ if and only if $\mathbf{m}(\mathcal{S}_0) = 0$. We denote $L_{10} = L_0 \cap (\mathfrak{H}_1 \times \mathfrak{H}_1)$. It follows from Lemma 3.1 that $\mathcal{D}(L_{10}) \subset \mathfrak{H}_1$, $\mathcal{R}(L_{10}) \subset \mathfrak{H}_1$ and $L_0 \cap \mathfrak{H}_0 \times \mathfrak{H}_0 = \{0, 0\}$. Hence,

$$L_0^* = (\mathfrak{H}_0 \times \mathfrak{H}_0) \oplus L_{10}^*, \tag{9}$$

i.e., the relation L_0^* consists of all pairs $\{y, f\} \in \mathfrak{H}$ of the form $\{y, f\} = \{u, v\} + \{z, g\} = \{u + z, v + g\}$, where $u, v \in \mathfrak{H}_0$, $\{z, g\} \in L_{10}^*$.

Consequently, any generalized resolvent $\widetilde{R}_\lambda$ of the relation L_0 has the form $\widetilde{R}_\lambda = R_{0\lambda} \oplus R_\lambda$, where R_λ is some generalized resolvent of L_{10} and $R_{0\lambda}$ is a generalized resolvent of the relation $\{0, 0\}$, i.e., $R_{0\lambda} = (T_{\mathcal{F}_0(\lambda)} - \lambda E)^{-1}$ (see (2)), $T_{\mathcal{F}_0(\lambda)}$ is the relation consisting of pairs of the form $\{\mathcal{F}_0(\lambda)z - z, \lambda_0 \mathcal{F}_0(\lambda)z - \overline{\lambda}_0 z\}$. Here $\mathcal{F}_0(\lambda) : \mathfrak{H}_0 \to \mathfrak{H}_0$ is a bounded operator, $\|\mathcal{F}_0(\lambda)\| \leqslant 1$, $z \in \mathfrak{H}_0$, the operator function $\lambda \to \mathcal{F}_0(\lambda)$ is holomorphic, $\operatorname{Im}\lambda \cdot \operatorname{Im}\lambda_0 > 0$. Therefore, to find the characteristic function of the generalized resolvent, we consider the relation L_{10}.

The set $\mathcal{T}_{\mathbf{p}} = (a, b) \setminus \mathcal{S}_0$ is open and it is the union of at most a countable number of disjoint open intervals $\mathcal{J}_k$, i.e., $\mathcal{T}_{\mathbf{p}} = \bigcup_{k=1}^{\Bbbk_1} \mathcal{J}_k$ and $\mathcal{J}_k \cap \mathcal{J}_j = \varnothing$ for $k \neq j$, where $\Bbbk_1$ is a natural number (equal to the number of intervals if this number is finite) or the symbol ∞ (if the number of intervals is infinite). By $\mathbb{J}$ denote the set of these intervals $\mathcal{J}_k$.

Remark 3.2. *The boundaries α_k, β_k of any interval $\mathcal{J}_k = (\alpha_k, \beta_k) \in \mathbb{J}$ belong to $\mathcal{S}_0$. This follows from the closedness of $\mathcal{S}_0$ and the continuity of functions $y \in \mathcal{D}(L_0)$ (see [15]).*

Let $\mathfrak{X}_A = \mathfrak{X}_A(t)$ be an operator characteristic function of a set A, i.e., $\mathfrak{X}_A(t) = E$ if $t \in A$ and $\mathfrak{X}_A(t) = 0$ if $t \notin A$. We will often omit the argument t in the notation $\mathfrak{X}_A$. We denote

$$w_k(t, \lambda) = \mathfrak{X}_{[\alpha_k, \beta_k)} W(t, \lambda) W^{-1}(\alpha_k, \lambda), \tag{10}$$

where $(\alpha_k, \beta_k) = \mathcal{J}_k \in \mathbb{J}$. Then (see[15])

$$w_k^*(t, \overline{\lambda}) J w_k(t, \lambda) = J, \quad \alpha_k \leqslant t < \beta_k. \tag{11}$$

By $\mathfrak{H}_{10}$ (by $\mathfrak{H}_{11}$) denote a subspace of functions that belong to $\mathfrak{H}_1$ and vanish on $\mathcal{S}_{\mathbf{m}}$ (on $[a, b] \setminus \mathcal{S}_{\mathbf{m}}$, respectively) with respect to the norm in $\mathfrak{H}$. So, $\mathfrak{H}_{10}$ ($\mathfrak{H}_{11}$) consists of functions of the form $\mathfrak{X}_{[a,b]\setminus(\mathcal{S}_0 \cup \mathcal{S}_{\mathbf{m}})} h$ (of the form $\mathfrak{X}_{\mathcal{S}_{\mathbf{m}} \setminus \mathcal{S}_0} h$, respectively), where $h \in \mathfrak{H}$ is an arbitrary function. Therefore,

$$\mathfrak{H}_1 = \mathfrak{H}_{10} \oplus \mathfrak{H}_{11}, \quad \mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_{10} \oplus \mathfrak{H}_{11}.$$

Obviously, the space $\mathfrak{H}_{11}$ is the closure in $\mathfrak{H}$ of the linear span of functions that have the form $\mathfrak{X}_{\{\tau\}}(\cdot)x$, where $x \in H$, $\tau \in \mathcal{S}_{\mathbf{m}} \setminus \mathcal{S}_0$. By (8), it follows that $\mathfrak{H}_{11} \subset \ker L_{10}^*$.

Let $u_k(\cdot, \lambda, \tau) : H \to \mathfrak{H}_1$ be an operator acting by the formula

$$u_k(t, \lambda, \tau)x = -\mathfrak{X}_{[a,b]\setminus(\mathcal{S}_{\mathbf{m}} \cup \mathcal{S}_0)} w_k(t, \lambda) iJ \int_a^t w_k^*(s, \overline{\lambda}) d\mathbf{m}(s) \lambda \mathfrak{X}_{\{\tau\}}(s)x, \tag{12}$$

where $x \in H$, $\tau \in (\alpha_k, \beta_k) \cap \mathcal{S}_{\mathbf{m}}$, $(\alpha_k, \beta_k) = \mathcal{J}_k \in \mathbb{J}$. Then (see[15]) for any $x \in H$ the function

$$u_k(\cdot, \lambda, \tau)x + \mathfrak{X}_{\{\tau\}}(\cdot)x \in \ker(L_{10}^* - \lambda E).$$

Lemma 3.3. [15]. *The linear span of functions of the form $\mathfrak{X}_{[a,b]\setminus(\mathcal{S}_{\mathbf{m}} \cup \mathcal{S}_0)} w_k(\cdot, \lambda) x_0$ and $u_k(\cdot, \lambda, \tau) B_\tau x_j + \mathfrak{X}_{\{\tau\}}(\cdot) B_\tau x_j$ is dense in $\ker(L_{10}^* - \lambda E)$. Here $x_j, x_0 \in H$; $\tau \in (\alpha_k, \beta_k) \cap \mathcal{S}_{\mathbf{m}}$; $B_\tau : H \to H$ is a bounded continuously invertible operator; $k = 1, ..., \Bbbk_1$ if $\Bbbk_1$ is finite and $k \in \mathbb{N}$ if $\Bbbk_1$ is infinite ($\mathbb{N}$ is the set of natural numbers).*

Let $\mathbb{M}$ be a set consisting of intervals $\mathcal{J} \in \mathbb{J}$ and single-point sets $\{\tau\}$, where $\tau \in \mathcal{S}_{\mathbf{m}} \setminus \mathcal{S}_0$. The set $\mathbb{M}$ is at most countable. Let $\Bbbk$ be the number of elements in $\mathbb{M}$. We arrange the elements of $\mathbb{M}$ in the form of a finite or infinite sequence and denote these elements by $\mathcal{E}_k$, where k is any natural number if the number of elements in $\mathbb{M}$ is infinite, and $1 \leqslant k \leqslant \Bbbk$ if the number of elements in $\mathbb{M}$ is finite.

To each element $\mathcal{E}_k \in \mathbb{M}$ assign the operator function ϑ_k in the following way. If $\mathcal{E}_k$ is the interval, $\mathcal{E}_k = \mathcal{J}_k = (\alpha_k, \beta_k) \in \mathbb{J}$, then

$$\vartheta_k(t,\lambda) = \mathfrak{X}_{[\alpha_k,\beta_k)\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda) = \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda). \tag{13}$$

If $\mathcal{E}_k$ is a single-point set, $\mathcal{E}_k = \{\tau_k\}$, then

$$\vartheta_k(t,\lambda) = \lambda^{-1} u_n(t,\lambda,\tau_k) + \lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(t) \text{ if } \lambda \neq 0 \text{ and } \vartheta_k(t,0) = \mathfrak{X}_{\{\tau_k\}}(t) \text{ if } \lambda = 0, \tag{14}$$

where $\tau_k \in (\mathcal{S}_{\mathbf{m}} \setminus \mathcal{S}_0) \cap \mathcal{J}_n$, $\mathcal{J}_n = (\alpha_n, \beta_n) \in \mathbb{J}$, $n = 1, ..., \Bbbk_1$ if $\Bbbk_1$ is finite and $n \in \mathbb{N}$ if $\Bbbk_1$ is infinite. In case (14), using (12), we get

$$\vartheta_k(t,\lambda)x = -\mathfrak{X}_{[a,b]\setminus(\mathcal{S}_{\mathbf{m}}\cup\mathcal{S}_0)} w_n(t,\lambda) iJ \int_a^t w_n^*(s,\overline{\lambda}) d\mathbf{m}(s) \mathfrak{X}_{\{\tau_k\}}(s)x + \lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(t)x \text{ if } \lambda \neq 0 \quad (x \in H). \tag{15}$$

Lemma 3.4. *The linear span of functions $t \to \vartheta_k(t,\lambda)\xi_j$ is dense in $\ker(L_{10}^* - \lambda E)$. (Here $\xi_j \in H$, $k \in \mathbb{N}$ if $\Bbbk = \infty$, and $1 \leqslant k \leqslant \Bbbk$ if $\Bbbk$ is finite.)*

Proof. The required statement follows from Lemma 3.3 immediately. □

Corollary 3.5. *A function $f \in \mathfrak{H}_1$ belongs to the range $\mathcal{R}(L_{10} - \lambda E)$ if and only if the equality $(f, \vartheta_k(\cdot,\overline{\lambda})\xi_j)_{\mathfrak{H}} = 0$ holds for all k and all $\xi_j \in H$. (Here $k \in \mathbb{N}$ if $\Bbbk = \infty$, and $1 \leqslant k \leqslant \Bbbk$ if $\Bbbk$ is finite.)*

Proof. The proof follows from the equality $\mathcal{R}(L_{10} - \lambda E) \oplus \ker(L_{10}^* - \overline{\lambda}E) = \mathfrak{H}_1$ and Lemma 3.4. □

Remark 3.6. *In* [15], [16], *to each element $\mathcal{E}_k \in \mathbb{M}$ was assigned the operator function v_k in the following way. If $\mathcal{E}_k$ is the interval, $\mathcal{E}_k = \mathcal{J}_k = (\alpha_k, \beta_k) \in \mathbb{J}$, then $v_k(t,\lambda) = \vartheta_k(t,\lambda) = \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}} w_k(t,\lambda)$. If $\mathcal{E}_k$ is a single-point set, $\mathcal{E}_k = \{\tau_k\}$, then*

$$v_k(t,\lambda) = u_n(t,\lambda,\tau_k)\mathfrak{X}_{\{\tau_k\}}(t) w_n(\tau_k,\lambda) + \mathfrak{X}_{\{\tau_k\}}(t) w_n(\tau_k,\lambda), \tag{16}$$

where $\tau_k \in (\mathcal{S}_{\mathbf{m}} \setminus \mathcal{S}_0) \cap \mathcal{J}_n$, $\mathcal{J}_n = (\alpha_n, \beta_n) \in \mathbb{J}$. The linear span of functions $t \to v_k(t,\lambda)\xi_j$ ($\xi_j \in H$) is dense in $\ker(L_{10}^* - \lambda E)$. There exist constants $\gamma_{1k} = \gamma_{1k}(\lambda)$, $\gamma_{2k} = \gamma_{2k}(\lambda)$ such that the inequality $\gamma_{1k}\|v_k(\cdot,0)x\|_{\mathfrak{H}} \leqslant \|v_k(\cdot,\lambda)x\|_{\mathfrak{H}} \leqslant \gamma_{2k}\|v_k(\cdot,0)x\|_{\mathfrak{H}}$ holds for all $x \in H$ (see [15]).

Lemma 3.7. *There exist constants $\gamma_1 = \gamma_{1k}(\lambda_1, \lambda_2), \gamma_2 = \gamma_{2k}(\lambda_1, \lambda_2) > 0$ such that the inequality*

$$\gamma_{1k}\|\vartheta_k(\cdot,\lambda_1)x\|_{\mathfrak{H}} \leqslant \|\vartheta_k(\cdot,\lambda_2)x\|_{\mathfrak{H}} \leqslant \gamma_{2k}\|\vartheta_k(\cdot,\lambda_1)x\|_{\mathfrak{H}} \tag{17}$$

holds for all $x \in H$.

Proof. Using Lemma 2.2 and (6), we obtain

$$W(t,\lambda)x_0 = W(t,0)x_0 - W(t,0)iJ\int_a^t W^*(s,0)d\mathbf{m}_0(s)\lambda W(s,\lambda)x_0, \quad x_0 \in H, \tag{18}$$

$$W(t,0)x_0 = W(t,\lambda)x_0 + W(t,\lambda)iJ\int_a^t W^*(\xi,\overline{\lambda})d\mathbf{m}_0(s)\lambda W(s,0)x_0, \quad x_0 \in H. \tag{19}$$

Suppose that ϑ_k has form (13). Using (10), (18), (19), we get

$$\vartheta_k(t,\lambda)x_0 = \vartheta_k(t,0)x_0 - \vartheta_k(t,0)iJ\int_{\alpha_k}^t \vartheta_k^*(s,0)d\mathbf{m}_0(s)\lambda\vartheta_k(s,\lambda)x_0, \quad x_0 \in H, \tag{20}$$

$$\vartheta_k(t,0)x_0 = \vartheta_k(t,\lambda)x_0 + \vartheta_k(t,\lambda)iJ\int_{\alpha_k}^{t}\vartheta_k^*(\xi,\overline{\lambda})d\mathbf{m}_0(s)\lambda\vartheta_k(s,0)x_0, \quad x_0\in H. \tag{21}$$

It follows from (3), (7), (20), (21) that there exist constants $\gamma_3>0$, $\gamma_4>0$ such that

$$\gamma_3\left\|\Psi_{\mathbf{m}}^{1/2}(t)\vartheta_k(t,0)x\right\| \leqslant \left\|\Psi_{\mathbf{m}}^{1/2}(t)\vartheta_k(t,\lambda)x\right\| \leqslant \gamma_4\left\|\Psi_{\mathbf{m}}^{1/2}(t)\vartheta_k(t,0)x\right\|$$

for all $x\in H$, all $t\in[a,b_0]$, and all $\lambda\in C\subset\mathbb{C}$ (C is a compact set). Equalities (3), (20), (21) imply (17) in the case when ϑ_k has form (13).

Suppose that ϑ_k has form (14). Using (14), (15), (12), we get

$$\|\vartheta_k(\cdot,\lambda)x\|_{\mathfrak{H}}^2 = \left\|u_n(\cdot,\lambda,\tau_k)\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(\cdot)x\right\|_{\mathfrak{H}}^2 + \left\|\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(\cdot)x\right\|_{\mathfrak{H}}^2 \geqslant \left\|\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(\cdot)x\right\|_{\mathfrak{H}}^2 \geqslant \gamma_5\left\|\mathfrak{X}_{\{\tau_k\}}(\cdot)x\right\|_{\mathfrak{H}}^2, \quad \gamma_5=\gamma_5(\lambda)>0. \tag{22}$$

On the other hand, using (12), we obtain

$$\|\vartheta_k(\cdot,\lambda)x\|_{\mathfrak{H}} \leqslant \left\|u_n(\cdot,\lambda,\tau_k)\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(\cdot)x\right\|_{\mathfrak{H}} + \left\|\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(\cdot)x\right\|_{\mathfrak{H}} \leqslant \gamma_6\left\|\mathfrak{X}_{\{\tau_k\}}(\cdot)x\right\|_{\mathfrak{H}}, \quad \gamma_6=\gamma_6(\lambda)>0, \tag{23}$$

for all $x\in H$, all $t\in[a,b_0]$, and all $\lambda\neq 0$. Inequalities (22), (23) imply (17) in the case when ϑ_k has form (14). The lemma is proved. □

Lemma 3.8. *There exist constants $\delta_{1k}=\delta_{1k}(\lambda),\delta_{2k}=\delta_{2k}(\lambda)>0$ such that the inequality*

$$\delta_{1k}\|\vartheta_k(\cdot,\lambda)x\|_{\mathfrak{H}} \leqslant \|v_k(\cdot,\lambda)x\|_{\mathfrak{H}} \leqslant \delta_{2k}\|\vartheta_k(\cdot,\lambda)x\|_{\mathfrak{H}} \tag{24}$$

holds for all $x\in H$.

Proof. According to Remark 3.6, $v_k=\vartheta_k$ if $\mathcal{E}_k$ is the interval, $\mathcal{E}_k=\mathcal{J}_k=(\alpha_k,\beta_k)\in\mathbb{J}$. Obviously, inequality (24) holds in this case. Suppose that v_k, ϑ_k have form (16), (14), respectively. Using (16), (12), (7), (10), we obtain

$$\|v_k(\cdot,\lambda)x\|_{\mathfrak{H}}^2 = \left\|u_n(\cdot,\lambda,\tau_k)\mathfrak{X}_{\{\tau_k\}}(\cdot)w_n(\tau_k,\lambda)x\right\|_{\mathfrak{H}}^2 + \left\|\mathfrak{X}_{\{\tau_k\}}(\cdot)w_n(\tau_k,\lambda)x\right\|_{\mathfrak{H}}^2 \geqslant \delta_{3k}\left\|\mathfrak{X}_{\{\tau_k\}}(\cdot)x\right\|_{\mathfrak{H}}^2, \quad \delta_{3k}=\delta_{3k}(\lambda)>0. \tag{25}$$

On the other hand, using (12), we get

$$\|v_k(\cdot,\lambda)x\|_{\mathfrak{H}} \leqslant \left\|u_n(\cdot,\lambda,\tau_k)\mathfrak{X}_{\{\tau_k\}}(\cdot)w_n(\tau_k,\lambda)x\right\|_{\mathfrak{H}} + \left\|\mathfrak{X}_{\{\tau_k\}}(\cdot)w_n(\tau_k,\lambda)x\right\|_{\mathfrak{H}} \leqslant \delta_{4k}\left\|\mathfrak{X}_{\{\tau_k\}}(\cdot)x\right\|_{\mathfrak{H}}, \quad \delta_{4k}=\delta_{4k}(\lambda)>0, \tag{26}$$

for all $x\in H$. Inequalities (25), (26), and (22), (23) imply (24) in the case when v_k, ϑ_k have form (16), (14), respectively. The lemma is proved. □

We fix some point $\lambda_0\in\mathbb{C}$ and by $Q_{k,0}$ denote a set $x\in H$ such that the functions $t\to\vartheta_k(t,\lambda_0)x$ are identical with zero in $\mathfrak{H}$. We put $Q_k=H\ominus Q_{k,0}$. On the linear space Q_k we introduce a norm $\|\cdot\|_-$ by the equality

$$\|\xi_k\|_- = \|\vartheta_k(\cdot,\lambda_0)\xi_k\|_{\mathfrak{H}}, \quad \xi_k\in Q_k. \tag{27}$$

By Q_k^- denote the completion of Q_k with respect to norm (27). Using (7), (13), (14), we get the inequality $\|\xi_k\|_- \leqslant \gamma\|\xi_k\|$, where $\gamma>0$. Hence the space Q_k^- can be treated as a space with a negative norm with respect to Q_k ([4, ch. 1], [18, ch.2]). By Q_k^+ denote the associated space with a positive norm. The definition of spaces with positive and negative norms implies that $Q_k^+\subset Q_k\subset Q_k^-$. By $(\cdot,\cdot)_+$ and $\|\cdot\|_+$ we denote the scalar product and the norm in Q_k^+, respectively.

Remark 3.9. *It follows from Remark 3.6, Lemmas 3.7, 3.8 that the set $Q_{k,0}$ will not change if the function $\vartheta_k(\cdot,\lambda_0)$ is replaced by $\vartheta_k(\cdot,\lambda)$ or by $v_k(\cdot,\lambda)$ in the definition of $Q_{k,0}$. Moreover, with such replacement, the space Q_k^- will not change in the following sense: the set Q_k^- will not change, and the norm in it will be replaced by the equivalent one. The similar statement holds for the space Q_k^+.*

Suppose that a sequence $\{x_{kn}\}$, $x_{kn} \in Q_k$, converges in the space Q_k^- to $x_0 \in Q_k^-$ as $n \to \infty$. Then the sequence $\{\vartheta_k(\cdot,\lambda)x_{kn}\}$ is fundamental in $\mathfrak{H}_1$. Therefore this sequence converges to some element in $\mathfrak{H}_1$. By $\vartheta_k(\cdot,\lambda)x_0$ we denote this element.

Let $\widetilde{Q}_N^- = Q_1^- \times \ldots \times Q_N^-$ ($\widetilde{Q}_N^+ = Q_1^+ \times \ldots \times Q_N^+$) be the Cartesian product of the first N sets Q_k^- (Q_k^+, respectively) and let $\mathrm{V}_N(t,\lambda) = (\vartheta_1(t,\lambda), \ldots, \vartheta_N(t,\lambda))$ be the operator one-row matrix. It is convenient to treat elements from $\widetilde{Q}_N^-$ as one-column matrices, and to assume that $\mathrm{V}_N(t,\lambda)\widetilde{\xi}_N = \sum_{k=1}^N \vartheta_k(t,\lambda)\xi_k$, where $\widetilde{\xi}_N = \mathrm{col}(\xi_1, \ldots, \xi_N) \in \widetilde{Q}_N^-$, $\xi_k \in Q_k^-$. Let $\ker_k(\lambda)$ be a linear space of functions $t \to \vartheta_k(t,\lambda)\xi_k$, $\xi_k \in Q_k^-$. The space $\ker_k(\lambda)$ is closed in $\mathfrak{H}$. We denote $\mathcal{K}_N(\lambda) = \ker_1(\lambda) \dot{+} \ldots \dot{+} \ker_N(\lambda)$. Obviously, $\mathcal{K}_{N_1}(\lambda) \subset \mathcal{K}_{N_2}(\lambda)$ for $N_1 < N_2$. By $\mathbf{V}_N(\lambda)$ denote the operator $\widetilde{\xi}_N \to \mathrm{V}_N(\cdot,\lambda)\widetilde{\xi}_N$, where $\widetilde{\xi}_N \in \widetilde{Q}_N^-$. The operator $\mathbf{V}_N(\lambda)$ maps continuously and one-to-one $\widetilde{Q}_N^-$ onto $\mathcal{K}_N(\lambda) \subset \mathfrak{H}_1 \subset \mathfrak{H}$.

Let $\boldsymbol{Q}_-, \boldsymbol{Q}_+, \boldsymbol{Q}$ be linear spaces of sequences, respectively, $\widetilde{\eta} = \{\eta_k\}$, $\widetilde{\varphi} = \{\varphi_k\}$, $\widetilde{\xi} = \{\xi_k\}$, where $\eta_k \in Q_k^-$, $\varphi_k \in Q_k^+$, $\xi_k \in Q_k$; $k \in \mathbb{N}$ if $\Bbbk = \infty$, and $1 \leqslant k \leqslant \Bbbk$ if $\Bbbk$ is finite; $\Bbbk$ is the number of elements in $\mathbb{M}$. We assume that the series $\sum_{k=1}^\infty \|\eta_k\|_-^2$, $\sum_{k=1}^\infty \|\varphi_k\|_+^2$, $\sum_{k=1}^\infty \|\xi_k\|^2$ converge if $\Bbbk = \infty$. These spaces become Hilbert spaces if we introduce the scalar products by the formulas

$$(\widetilde{\eta},\widetilde{\zeta})_- = \sum_{k=1}^{\Bbbk}(\eta_k,\zeta_k)_-, \quad \widetilde{\eta},\widetilde{\zeta} \in \boldsymbol{Q}_-; \quad (\widetilde{\varphi},\widetilde{\psi})_+ = \sum_{k=1}^{\Bbbk}(\varphi_k,\psi_k)_+, \quad \widetilde{\varphi},\widetilde{\psi} \in \boldsymbol{Q}_+; \quad (\widetilde{\xi},\widetilde{\sigma}) = \sum_{k=1}^{\Bbbk}(\xi_k,\sigma_k), \quad \widetilde{\xi},\widetilde{\sigma} \in \boldsymbol{Q}.$$

The spaces $\boldsymbol{Q}_+, \boldsymbol{Q}_-$ can be treated as spaces with positive and negative norms with respect to $\boldsymbol{Q}$ ([4, ch. 1], [18, ch.2]). So, $\boldsymbol{Q}_+ \subset \boldsymbol{Q} \subset \boldsymbol{Q}_-$ and $\gamma_1 \|\widetilde{\varphi}\|_- \leqslant \|\widetilde{\varphi}\| \leqslant \gamma_2 \|\widetilde{\varphi}\|_+$, where $\widetilde{\varphi} \in \boldsymbol{Q}_+$, $\gamma_1, \gamma_2 > 0$. The "scalar product" $(\widetilde{\eta},\widetilde{\varphi})$ is defined for all $\widetilde{\varphi} \in \boldsymbol{Q}_+$, $\widetilde{\eta} \in \boldsymbol{Q}_-$. If $\widetilde{\eta} \in \boldsymbol{Q}$, then $(\widetilde{\eta},\widetilde{\varphi})$ coincides with the scalar product in $\boldsymbol{Q}$.

Let $\mathcal{M} \subset \boldsymbol{Q}_-$ be a set of sequences vanishing starting from a certain number (its own for each sequence). The set $\mathcal{M}$ is dense in the space $\boldsymbol{Q}_-$. The operator $\mathbf{V}_N(\lambda)$ is the restriction of $\mathbf{V}_{N+1}(\lambda)$ to $\widetilde{Q}_N^-$. By $\mathbf{V}'(\lambda)$ denote an operator in $\mathcal{M}$ such that $\mathbf{V}'(\lambda)\widetilde{\eta} = \mathbf{V}_N(\lambda)\widetilde{\eta}_N$ for all $N \in \mathbb{N}$, where $\widetilde{\eta} = (\widetilde{\eta}_N, 0, \ldots)$, $\widetilde{\eta}_N \in \widetilde{Q}_N^-$. The operator $\mathbf{V}'(\lambda)$ admits an extension by continuity to the space $\boldsymbol{Q}_-$. By $\mathbf{V}(\lambda)$ denote the extended operator. This operator maps continuously and one-to-one $\boldsymbol{Q}_-$ onto $\ker(L_{10}^* - \lambda E) \subset \mathfrak{H}_1 \subset \mathfrak{H}$. Moreover, we denote $\widetilde{\mathrm{V}}(t,\lambda)\widetilde{\eta} = (\mathbf{V}(\lambda)\widetilde{\eta})(t)$, where $\widetilde{\eta} = \{\eta_k\} \in \boldsymbol{Q}_-$.

Remark 3.10. *The operator $\mathbf{V}(\lambda)$ is constructed by the functions ϑ_k (13), (14) in the same way that an operator $\mathcal{V}(\lambda)$ is constructed by the functions v_k in [15] (see Remark 3.6). The operators $\mathcal{V}(\lambda)$, $\mathbf{V}(\lambda)$ have the same properties formulated in Lemma 3.11.*

Lemma 3.11. *The operator $\mathbf{V}(\lambda)$ maps $\boldsymbol{Q}_-$ onto $\ker(L_{10}^* - \lambda E)$ continuously and one to one. A function z belongs to $\ker(L_{10}^* - \lambda E)$ if and only if there exists an element $\widetilde{\eta} = \{\eta_k\} \in \boldsymbol{Q}_-$ such that $z(t) = (\mathbf{V}(\lambda)\widetilde{\eta})(t) = \widetilde{\mathrm{V}}(t,\lambda)\widetilde{\eta}$. The adjoint operator $\mathbf{V}^*(\lambda)$ maps $\mathfrak{H}$ onto $\boldsymbol{Q}_+$ continuously, and acts by the formula*

$$\mathbf{V}^*(\lambda)f = \int_a^{b_0} \widetilde{\mathrm{V}}^*(t,\lambda)d\mathbf{m}(t)f(t), \tag{28}$$

and $\ker \mathbf{V}^*(\lambda) = \mathfrak{H}_0 \oplus \mathcal{R}(L_{10} - \overline{\lambda}E)$. Moreover, $\mathbf{V}^*(\lambda)$ maps $\ker(L_{10}^* - \lambda E)$ onto $\boldsymbol{Q}_+$ one to one.

Proof. The proof repeats verbatim the corresponding proof from [15] with the change of $\mathcal{V}(\lambda)$ to $\mathbf{V}(\lambda)$. □

4. The description of generalized resolvents

Following Theorems 4.1 and 4.2 are proved in [15] for functions v_k (see Remark 3.6). For functions ϑ_k (13), (14), the proof is the same. We have changed some designations from [15] to shorten the record.

Theorem 4.1. *A pair $\{\widetilde{y},\widetilde{f}\}\in\mathfrak{H}\times\mathfrak{H}$ belongs to $L_0^*-\lambda E$ if and only if there exist a pair $\{\widehat{y},\widehat{f}\}\in\mathfrak{H}\times\mathfrak{H}$, functions $y_0, y_0'\in\mathfrak{H}_0$, $y,f\in\mathfrak{H}_1$ and an element $\widetilde{\eta}\in Q_-$ such that the pairs $\{\widetilde{y},\widetilde{f}\}$, $\{\widehat{y},\widehat{f}\}$ are identical in $\mathfrak{H}\times\mathfrak{H}$ and the equalities*

$$\widehat{y}=y_0+y,\ \widehat{f}=y_0'+f,$$

$$y(t)=\widetilde{\mathrm{V}}(t,\lambda)\widetilde{\eta}-\sum_{k=1}^{\Bbbk_1}\mathfrak{X}_{[\alpha_k,\beta_k)\setminus\mathcal{S}_{\mathbf{m}}}w_k(t,\lambda)iJ\int_{\alpha_k}^{t}w_k^*(s,\overline{\lambda})d\mathbf{m}(s)f(s) \tag{29}$$

hold, where the series in (29) *converges in $\mathfrak{H}_1$, $\Bbbk_1$ is the number of intervals $\mathcal{J}_k\in\mathbb{J}$.*

Theorem 4.2. *Let $T(\lambda)$ be a linear relation such that $L_{10}-\lambda E\subset T(\lambda)\subset L_{10}^*-\lambda E$ and $\lambda\neq 0$. The relation $T(\lambda)$ is continuously invertible in the space $\mathfrak{H}_1$ if and only if there exists a bounded operator $M(\lambda):Q_+\to Q_-$ such that the following equality holds for any pair $\{y,f\}\in T(\lambda)$ and for any λ*

$$y(t)=y(t,\lambda)=\int_a^b\widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{\mathrm{V}}^*(s,\overline{\lambda})d\mathbf{m}(s)f(s)+$$

$$+2^{-1}\sum_{k=1}^{\Bbbk_1}\int_a^b\mathfrak{X}_{[\alpha_k,\beta_k)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_k(t,\lambda)\mathrm{sgn}(s-t)iJw_k^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{[\alpha_k,\beta_k)\setminus\mathcal{S}_{\mathbf{m}}}(s)f(s)-\lambda^{-1}\sum_{k=1}^{\Bbbk_1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_k,\beta_k)}(t)f(t). \tag{30}$$

Lemma 4.3. *In Theorem 4.2, the function $\lambda\to T^{-1}(\lambda)f$ is holomorphic for any $f\in\mathfrak{H}_1$ at a point λ_1 ($\lambda_1\neq 0$) if and only if the function $\lambda\to M(\lambda)\widetilde{x}$ is holomorphic for any element $\widetilde{x}\in Q_+$ at the same point λ_1.*

Proof. First we assume that the function $\lambda\to M(\lambda)\widetilde{x}$ is holomorphic. Then all functions on the right side of equality (30) are holomorphic. Consequently, the function $y=T^{-1}(\lambda)f$ is holomorphic for any $f\in\mathfrak{H}_1$.

Now suppose that $y=T^{-1}(\lambda)f$ is a holomorphic function for any $f\in\mathfrak{H}_1$. Let us prove that the function $\lambda\to M(\lambda)\widetilde{x}$ is holomorphic for any $\widetilde{x}\in Q_+$. We use the notation from Theorem 4.2. Moreover, we denote $S(\lambda)=M(\lambda)\mathbf{V}^*(\overline{\lambda})$. According to Lemma 3.11, $\mathbf{V}^*(\overline{\lambda})f$ is defined by equality (28). It follows from the holomorphicity of the function $\lambda\to T^{-1}(\lambda)$ that the function $\lambda\to\mathbf{V}(\lambda)S(\lambda)f$ is holomorphic. Using Lemma 3.11, we obtain that the function $\lambda\to S(\lambda)f$ is holomorphic. Now the holomorphicity of the function $\lambda\to M(\lambda)$ follows from Lemma 4.4 that is formulated after the proof of this lemma. In Lemma 4.4 it should be taken that $\mathcal{B}_1=\mathfrak{H}_1$, $\mathcal{B}_2=Q_+$, $\mathcal{B}_3=Q_-$, $T_1(\lambda)=\mathbf{V}^*(\overline{\lambda})$, $T_2(\lambda)=M(\lambda)$, $T_3(\lambda)=S(\lambda)$. The lemma is proved. □

Lemma 4.4. [8]. *Let $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ be Banach spaces. Suppose bounded operators $T_3(\lambda):\mathcal{B}_1\to\mathcal{B}_3$, $T_1(\lambda):\mathcal{B}_1\to\mathcal{B}_2$, $T_2(\lambda):\mathcal{B}_2\to\mathcal{B}_3$ satisfy the equality $T_3(\lambda)=T_2(\lambda)T_1(\lambda)$ for every fixed λ belonging to some neighborhood of a point λ_1 and suppose the range of operator $T_1(\lambda_1)$ coincides with $\mathcal{B}_2$. If the functions $T_1(\lambda)$, $T_3(\lambda)$ are strongly differentiable at the point λ_1, then the function $T_2(\lambda)$ is are strongly differentiable at λ_1.*

It follows from Lemma 3.11 that for any element $\widetilde{x}\in Q_+$ there exists a function $f\in\mathfrak{H}_1$ such that $\mathbf{V}^*(\overline{\lambda})f=\widetilde{x}$. We denote

$$z(t)=z(t,f,\lambda)=\widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{x}-\sum_{n=1}^{\Bbbk_1}2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{\beta_n}w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s), \tag{31}$$

where $\widetilde{x}=\mathbf{V}^*(\overline{\lambda})f$. It follows from Lemmas 3.4, 3.11 that $z\in\ker(L_{10}^*-\lambda E)$.

Theorem 4.5. *Suppose that the relation $T(\lambda)$ satisfy the conditions of Theorem 4.2, and $T(\lambda)$ is continuously invertible, and $R(\lambda)=T^{-1}(\lambda)$, and $\mathrm{Im}\lambda\neq 0$. Then the equality*

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(M(\lambda)\widetilde{x},\widetilde{x})-(z,z)_{\mathbf{m}}=(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(R(\lambda)f,f)_{\mathbf{m}}-(R(\lambda)f,R(\lambda)f)_{\mathbf{m}} \tag{32}$$

holds for all $f\in\mathfrak{H}_1$ and $\widetilde{x}=\mathbf{V}^*(\overline{\lambda})f\in Q_+$.

Proof. Using (30), we get

$$y(t) = (R(\lambda)f)(t) = \widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\int_a^b \widetilde{\mathrm{V}}^*(s,\overline{\lambda})d\mathbf{m}(s)f(s)+$$

$$+\sum_{n=1}^{\Bbbk_1}\left(-2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^t w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)+2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_t^{\beta_n} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)\right)-$$

$$-\lambda^{-1}\sum_{n=1}^{\Bbbk_1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_n,\beta_n)}(t)f(t). \quad (33)$$

By standard transformations, equality (33) is reduced to the form

$$y(t)=\widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\int_a^b \widetilde{\mathrm{V}}^*(s,\overline{\lambda})d\mathbf{m}(s)f(s)-\sum_{n=1}^{\Bbbk_1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^t w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)+$$

$$+\sum_{n=1}^{\Bbbk_1}2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{\beta_n} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f(s)-\lambda^{-1}\sum_{n=1}^{\Bbbk_1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_n,\beta_n)}(t)f(t). \quad (34)$$

We fix an interval $(\alpha_n,\beta_n)\in\mathbb{J}$ and denote $f_n=\mathfrak{X}_{[\alpha_n,\beta_n)}f$. Let $\mathbb{M}_n\subset\mathbb{M}$ be a set consisting of the interval $(\alpha_n,\beta_n)\in\mathbb{J}$ and single-point sets $\{\tau\}$, where $\tau\in\mathcal{S}_{\mathbf{m}}\cap(\alpha_n,\beta_n)$. Suppose $\widetilde{\Bbbk}_n$ is a natural number equal to the number of elements in $\mathbb{M}_n$ if this number is finite or $\widetilde{\Bbbk}_n$ is the symbol ∞ if the number of elements in $\mathbb{M}_n$ is infinite.

We arrange the elements of $\mathbb{M}_n$ in the form of a finite or infinite sequence and denote these elements by $\mathcal{E}_{nk}$, where k is any natural number if the number of elements in $\mathbb{M}_n$ is infinite, and $1\leqslant k\leqslant\widetilde{\Bbbk}_n$ if the number of elements in $\mathbb{M}_n$ is finite; $\mathcal{E}_{n1}$ is the interval (α_n,β_n), $\mathcal{E}_{nk}$ $(k\geqslant 2)$ is the single-point set $\{\tau_k\}$, $\tau_k\in\mathcal{S}_{\mathbf{m}}\cap(\alpha_n,\beta_n)$.

To each element $\mathcal{E}_{nk}\in\mathbb{M}_n$ assign the operator function ϑ_{nk} in the following way (see also (10), (13), (14)). If $k=1$, then

$$\vartheta_{n1}(t,\lambda)=\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}w_n(t,\lambda)=\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}w_n(t,\lambda),\quad w_n(t,\lambda)=\mathfrak{X}_{[\alpha_n,\beta_n)}W(t,\lambda)W^{-1}(\alpha_n,\lambda); \quad (35)$$

if $k\geqslant 2$, then

$$\vartheta_{nk}(t,\lambda)=\lambda^{-1}u_n(t,\lambda,\tau_k)+\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(t)=-\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}w_n(t,\lambda)iJ\int_{\alpha_n}^t w_n^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{\{\tau_k\}}(s)+\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(t)=$$

$$=\begin{cases}0 \text{ for } t<\tau_k,\\ \lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(t) \text{ for } t=\tau_k,\\ -\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}w_n(t,\lambda)iJw_n^*(\tau_k,\overline{\lambda})\mathbf{m}(\{\tau_k\}) \text{ for } t>\tau_k.\end{cases} \quad (36)$$

Using (35), (36), we get

$$\vartheta_{n1}^*(t,\overline{\lambda})=\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}\mathbf{m}}w_n^*(t,\overline{\lambda});\quad \vartheta_{nk}^*(t,\overline{\lambda})=\begin{cases}0 \text{ for } t<\tau_k,\\ \lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(t) \text{ for } t=\tau_k,\\ \mathbf{m}(\{\tau_k\})w_n(\tau_k,\lambda)iJ\mathfrak{X}_{(\tau_k,b]}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}w_n^*(t,\overline{\lambda}) \text{ for } t>\tau_k.\end{cases}$$

We denote $\mathbf{V}^*(\overline{\lambda})f_n=\widetilde{x}_n$ and

$$x_{n1}=\int_{\alpha_n}^{\beta_n}\vartheta_{n1}^*(s,\overline{\lambda})d\mathbf{m}(s)f_n(s)=\int_{\alpha_n}^{\beta_n}w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s), \quad (37)$$

$$x_{nk}=\int_{\alpha_n}^{\beta_n}\vartheta_{nk}^*(s,\overline{\lambda})d\mathbf{m}(s)f_n(s)=$$

$$=\lambda^{-1}\int_{\{\tau_k\}}d\mathbf{m}(s)\mathfrak{X}_{\{\tau_k\}}(s)f(\tau_k)+\mathbf{m}(\{\tau_k\})w_n(\tau_k,\lambda)iJ\int_{\tau_k}^b w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s),\ k\geqslant 2. \quad (38)$$

It follows from (28) and Lemma 3.11 that $\widetilde{x}_n \in Q_+ \subset Q \subset Q_-$ is a sequence $\widetilde{x}_n = \{x_j\}$ with elements x_{n1}, x_{nk} and zeros. Suppose $p_j : Q_- \to Q_j^-$ is an operator defined by the formula $p_j\widetilde{\eta} = \eta_j$, where $\widetilde{\eta} = \{\eta_j\} \in Q_-$. Let $x_{j_{n1}} = x_{n1}$, $x_{j_{nk}} = x_{nk}$. Then by $p_{j_{n1}}$ (by $p_{j_{nk}}$ for $k \geqslant 2$) we denote the operator such that

$$p_{j_{n1}}\widetilde{\eta} = \eta_{n1} \quad (p_{j_{nk}}\widetilde{\eta} = \eta_{nk} \text{ for } k \geqslant 2, \text{ respectively}). \tag{39}$$

Hence $p_{j_{n1}}\widetilde{x}_n = x_{n1}$ and $p_{j_{nk}}\widetilde{x}_n = x_{nk}$ for $k \geqslant 2$. Let $\mu_{n1} = p_{j_{n1}}M(\lambda)\widetilde{x}_n$ and $\mu_{nk} = p_{j_{nk}}M(\lambda)\widetilde{x}_n$ for $k \geqslant 2$. So, $(M(\lambda)\widetilde{x}_n, \widetilde{x}_n) = \sum_{k=1}^{\widetilde{\mathbb{k}}_n}(\mu_{nk}, x_{nk})$.

Let $y_n = R(\lambda)f_n$. It follows from (34) that the equality

$$y_n(t) = \widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{x}_n - \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s) +$$
$$+ 2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{\beta_n} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s) - \lambda^{-1}\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}\cap(\alpha_n,\beta_n)}(t)f_n(t) \tag{40}$$

holds. Using (35), (36), (40), and the equality

$$\widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{x}_n = \mathfrak{X}_{[a,b]\setminus(\mathcal{S}_0\cup[\alpha_n,\beta_n))}(t)\widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{x}_n + \mathfrak{X}_{[\alpha_n,\beta_n)}(t)\widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{x}_n, \tag{41}$$

we get

$$y_n(t) = \mathfrak{X}_{[a,b]\setminus(\mathcal{S}_0\cup[\alpha_n,\beta_n))}(t)\widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{x}_n + \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)\mu_{n1} +$$
$$+ \sum_{k=2}^{\widetilde{\mathbb{k}}_n}\left(-\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{\{\tau_k\}}(s)\mu_{nk} + \lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(t)\mu_{nk}\right) -$$
$$- \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s) + 2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJx_{n1} - \sum_{k=2}^{\widetilde{\mathbb{k}}_n}\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(t)f_n(t), \tag{42}$$

where $\tau_k \in \mathcal{S}_{\mathbf{m}} \cap (\alpha_n, \beta_n)$. We denote

$$z_n(t) = \widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{x}_n - 2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{\beta_n} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s).$$

By (31), so that $z(t) = \sum_{n=1}^{\mathbb{k}_1} z_n(t)$. It follows from (37), (41) that

$$z_n(t) = \mathfrak{X}_{[a,b]\setminus(\mathcal{S}_0\cup[\alpha_n,\beta_n))}(t)\widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{x}_n + \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}w_n(t,\lambda)\mu_{n1} +$$
$$+ \sum_{k=2}^{\widetilde{\mathbb{k}}_n}\left(-\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{\{\tau_k\}}(s)\mu_{nk} + \lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(t)\mu_{nk}\right) - 2^{-1}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJx_{n1}, \tag{43}$$

where $\tau_k \in \mathcal{S}_{\mathbf{m}} \cap (\alpha_n, \beta_n)$. We note that $z_n \in \ker(L_{10}^* - \lambda E)$.

Using (42), (43), we obtain

$$y_n(t) - z_n(t) = -\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s) + \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJx_{n1} - \sum_{k=2}^{\widetilde{\mathbb{k}}_n}\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}(t)f_n(t);$$

$$y_n(t) + z_n(t) = 2\mathfrak{X}_{[a,b]\setminus(\mathcal{S}_0\cup[\alpha_n,\beta_n))}(t)\widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{x}_n + 2\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}w_n(t,\lambda)\mu_{n1} +$$
$$+ 2\sum_{k=2}^{\widetilde{\mathbb{k}}_n}\left(-\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}(s)\mu_{nk} + \lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}(t)\mu_{nk}\right) -$$
$$- \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s) - \sum_{k=2}^{\widetilde{\mathbb{k}}_n}\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}(t)f_n(t).$$

We decompose the functions $y_n - z_n$, $y_n + z_n$ into terms to which Lagrange formula (4) is applicable. Let us introduce the following designations:

$$\varphi_1(t) = -\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s) + \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJx_{n1};$$

$$\varphi_2(t) = -\sum_{k=2}^{\widetilde{\mathbb{k}}_n} \lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}(t)f_n(t); \tag{44}$$

$$\psi_1(t) = 2\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)\mu_{n1} - \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}(t)w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s);$$

$$\psi_2(t) = 2\lambda^{-1}\sum_{k=2}^{\widetilde{\mathbb{k}}_n}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}(t)\mu_{nk} - \lambda^{-1}\sum_{k=2}^{\widetilde{\mathbb{k}}_n}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}(t)f_n(\tau_k); \tag{45}$$

$$\psi_3(t) = -2\sum_{k=2}^{\widetilde{\mathbb{k}}_n}\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}(s)\mu_{nk} = \sum_{k=2}^{\widetilde{\mathbb{k}}_n}\psi_{3k}(t),$$

where

$$\psi_{3k}(t) = -2\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}w_n(t,\lambda)iJ\int_{\alpha_n}^{t} w_n^*(s,\overline{\lambda})d\mathbf{m}(s)\mathfrak{X}_{\{\tau_k\}}(s)\mu_{nk} = \begin{cases} 0 \ \text{ for } \ t \leqslant \tau_k, \\ -2\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}w_n(t,\lambda)iJw_n^*(\tau_k,\overline{\lambda})\mathbf{m}(\{\tau_k\})\mu_{nk} \text{ for } t > \tau_k. \end{cases}$$

Then

$$y_n - z_n = \varphi_1 + \varphi_2, \quad y_n + z_n = \psi_1 + \psi_2 + \psi_3 + 2\mathfrak{X}_{[a,b]\setminus(\mathcal{S}_0\cup[\alpha_n,\beta_n))}(t)\widetilde{\mathrm{V}}(t,\lambda)M(\lambda)\widetilde{x}_n. \tag{46}$$

Since $z_n \in \ker(L_{10}^* - \lambda E)$, $\{y_n, f_n\} \in L_{10}^* - \lambda E$, it follows that $\{y_n - z_n, f_n\} \in L_{10}^* - \lambda E$, $\{y_n + z_n, f_n\} \in L_{10}^* - \lambda E$. Theorem 4.1 implies that $\{\varphi_1, \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}f_n\} \in L_{10}^* - \lambda E$, $\{\psi_1, \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}f_n\} \in L_{10}^* - \lambda E$. By (8), so that $\varphi_2 \in \ker L_{10}^*$, $\psi_2 \in \ker L_{10}^*$. Therefore,

$$\{y_n - z_n, \lambda(y_n - z_n) + f_n\} \in L_{10}^*, \ \ \{y_n + z_n, \lambda(y_n + z_n) + f_n\} \in L_{10}^*,$$

$$\{\varphi_1, \lambda\varphi_1 + \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}f_n\} \in L_{10}^*, \ \ \{\psi_1, \lambda\psi_1 + \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}f_n\} \in L_{10}^*.$$

We denote

$$u_- = \lambda(y_n - z_n) + f_n, \ \ u_+ = \lambda(y_n + z_n) + f_n. \tag{47}$$

Then $\{y_n - z_n, u_-\} \in L_{10}^*$, $\{y_n + z_n, u_+\} \in L_{10}^*$. Moreover, $(\lambda\varphi_1 + \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}f_n, \psi_2)_{\mathbf{m}} = 0$, $(\varphi_2, \lambda\psi_1 + \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}}f_n)_{\mathbf{m}} = 0$.

We note that for all functions $g, h \in \mathfrak{H}$ the equalities

$$(\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}g, \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}h)_{\mathbf{m}} = 0; \ \ (\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}g, h)_{\mathbf{m}} = (\mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}g, \mathfrak{X}_{[a,b]\setminus\mathcal{S}_{\mathbf{m}}}h)_{\mathbf{m}} = (g,h)_{\mathbf{m}_0}; \ \ (\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}g, h)_{\mathbf{m}} = (\mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}g, \mathfrak{X}_{\mathcal{S}_{\mathbf{m}}}h)_{\mathbf{m}}$$

hold. Moreover,

$$(\mathfrak{X}_{[a,b]\setminus(\mathcal{S}_0\cup[\alpha_n,\beta_n))}\widetilde{\mathrm{V}}(\cdot,\lambda)M(\lambda)\widetilde{x}_n, y_n - z_n)_{\mathbf{m}} = 0, \ \ (\mathfrak{X}_{[a,b]\setminus(\mathcal{S}_0\cup[\alpha_n,\beta_n))}\widetilde{\mathrm{V}}(\cdot,\lambda)M(\lambda)\widetilde{x}_n, f_n)_{\mathbf{m}} = 0.$$

Therefore by (46), (47), we get

$$\begin{aligned}(u_-, y_n + z_n)_{\mathbf{m}} - (y_n - z_n, u_+)_{\mathbf{m}} &= (\lambda(\varphi_1 + \varphi_2) + f_n, \psi_1 + \psi_2 + \psi_3)_{\mathbf{m}} - (\varphi_1 + \varphi_2, \lambda(\psi_1 + \psi_2 + \psi_3) + f_n)_{\mathbf{m}} = \\ &= (\lambda\varphi_1, \psi_1)_{\mathbf{m}_0} + (f_n, \psi_1)_{\mathbf{m}_0} + (\lambda\varphi_2, \psi_2)_{\mathbf{m}} + (f_n, \psi_2)_{\mathbf{m}} + (\lambda\varphi_1, \psi_3)_{\mathbf{m}_0} + (f_n, \psi_3)_{\mathbf{m}_0} - \\ &\quad - (\varphi_1, \lambda\psi_1)_{\mathbf{m}_0} - (\varphi_1, \lambda\psi_3)_{\mathbf{m}_0} - (\varphi_1, f_n)_{\mathbf{m}_0} - (\varphi_2, \lambda\psi_2)_{\mathbf{m}} - (\varphi_2, f_n)_{\mathbf{m}}.\end{aligned} \tag{48}$$

It follows from (44), (45) that

$$(\lambda\varphi_2,\psi_2)_{\mathbf{m}}=\sum_{k=2}^{\widetilde{\mathbb{k}}_n}(-\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}f_n(\tau_k),\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}(2\mu_{nk}-f_n(\tau_k)))_{\mathbf{m}}; \tag{49}$$

$$(f_n,\psi_2)_{\mathbf{m}}=\sum_{k=2}^{\widetilde{\mathbb{k}}_n}(\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}f_n(\tau_k),\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}(2\mu_{nk}-f_n(\tau_k)))_{\mathbf{m}}; \tag{50}$$

$$-(\varphi_2,\lambda\psi_2)_{\mathbf{m}}=\sum_{k=2}^{\widetilde{\mathbb{k}}_n}(\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}f_n(\tau_k),\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}(2\mu_{nk}-f_n(\tau_k)))_{\mathbf{m}}; \tag{51}$$

$$-(\varphi_2,f_n)_{\mathbf{m}}=\sum_{k=2}^{\widetilde{\mathbb{k}}_n}(\lambda^{-1}\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}f_n(\tau_k),\mathfrak{X}_{\{\tau_k\}\cap(\alpha_n,\beta_n)}f_n(\tau_k)))_{\mathbf{m}}. \tag{52}$$

Summing (49), (50), (51), (52), we get

$$((\lambda\varphi_2,\psi_2)_{\mathbf{m}}+(f_n,\psi_2)_{\mathbf{m}}-(\varphi_2,\lambda\psi_2)_{\mathbf{m}}-(\varphi_2,f_n)_{\mathbf{m}}=$$
$$=2\sum_{k=2}^{\widetilde{\mathbb{k}}_n}(\lambda^{-1}\mathfrak{X}_{\{\tau_k\}}f_n(\tau_k),\mathfrak{X}_{\{\tau_k\}}\mu_{nk})_{\mathbf{m}}=2\sum_{k=2}^{\widetilde{\mathbb{k}}_n}(\lambda^{-1}f_n(\tau_k),\mathbf{m}(\{\tau_k\})\mu_{nk}),\quad \tau_k\in(\alpha_n,\beta_n). \tag{53}$$

Taking into account (53), we continue equality (48)

$$(u_-,y+z)_{\mathbf{m}}-(y-z,u_+)_{\mathbf{m}}=[(\lambda\varphi_1+f_n,\psi_1)_{\mathbf{m}_0}-(\varphi_1,\lambda\psi_1+f_n)_{\mathbf{m}_0}]+[(\lambda\varphi_1+f_n,\psi_3)_{\mathbf{m}_0}-(\varphi_1,\lambda\psi_3)_{\mathbf{m}_0}]+$$
$$+2\sum_{k=2}^{\widetilde{\mathbb{k}}_2}(\lambda^{-1}f_n(\tau_k),\mathbf{m}(\{\tau_k\})\mu_{nk}),\quad \tau_k\in(\alpha_n,\beta_n). \tag{54}$$

We denote

$$\widetilde{\varphi}_1(t)=w_n(t,\lambda)iJx_{n1}-w_n(t,\lambda)iJ\int_{\alpha_n}^t w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s); \tag{55}$$

$$\widetilde{\psi}_1(t)=2w_n(t,\lambda)\mu_{n1}-w_n(t,\lambda)iJ\int_{\alpha_n}^t w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s); \tag{56}$$

$$\widetilde{\psi}_{3k}(t)=\begin{cases}0 \text{ for } t<\tau_k,\\ -2w_n(t,\lambda)iJw_n^*(\tau_k,\overline{\lambda})\mathbf{m}(\{\tau_k\})\mu_{nk} \text{ for } t\geqslant\tau_k\end{cases};\quad \widetilde{\psi}_3(t)=\sum_{k=2}^{\widetilde{\mathbb{k}}_n}\widetilde{\psi}_{3k}(t),$$

where $\tau_k\in\mathcal{S}_{\mathbf{m}}\cap(\alpha_n,\beta_n)$. Then

$$\widetilde{\varphi}_1(t)-\varphi_1(t)=\mathfrak{X}_{[\alpha_n,\beta_n)\cap\mathcal{S}_{\mathbf{m}}}(t)\widetilde{\varphi}_1(t),\ \widetilde{\psi}_1(t)-\psi_1(t)=\mathfrak{X}_{[\alpha_n,\beta_n)\cap\mathcal{S}_{\mathbf{m}}}(t)\widetilde{\psi}_1(t),\ \widetilde{\psi}_{3k}(t)-\psi_{3k}(t)=\mathfrak{X}_{\{\tau_k\}}(t)\widetilde{\psi}_{3k}(t).$$

This implies that

$$(\varphi_1,\psi_1)_{\mathbf{m}_0}=(\widetilde{\varphi}_1,\widetilde{\psi}_1)_{\mathbf{m}_0},\ (\varphi_1,f_n)_{\mathbf{m}_0}=(\widetilde{\varphi}_1,f_n)_{\mathbf{m}_0},\ (f_n,\psi_1)_{\mathbf{m}_0}=(f_n,\widetilde{\psi}_1)_{\mathbf{m}_0},\ (\varphi_1,\psi_{3k})_{\mathbf{m}_0}=(\widetilde{\varphi}_1,\widetilde{\psi}_{3k})_{\mathbf{m}_0}. \tag{57}$$

By (37), it follows that $\lim_{t\to\beta_n-0}\widetilde{\varphi}_1(t)=\widetilde{\varphi}_1(\beta_n)=0$. Using (57), we continue equality (54)

$$(u_-,y_n+z_n)_{\mathbf{m}}-(y_n-z_n,u_+)_{\mathbf{m}}=[(\lambda\widetilde{\varphi}_1+f_n,\widetilde{\psi}_1)_{\mathbf{m}_0}-(\widetilde{\varphi}_1,\lambda\widetilde{\psi}_1+f_n)_{\mathbf{m}_0}]+$$
$$+\sum_{k=2}^{\widetilde{\mathbb{k}}_n}[(\lambda\widetilde{\varphi}_1+f_n,\widetilde{\psi}_{3k})_{\mathbf{m}_0}-(\widetilde{\varphi}_1,\lambda\widetilde{\psi}_{3k})_{\mathbf{m}_0}]+2\lambda^{-1}\sum_{k=2}^{\widetilde{\mathbb{k}}_n}(f_n(\tau_k),\mathbf{m}(\{\tau_k\}\mu_{nk}),\quad \tau_k\in\mathcal{S}_{\mathbf{m}\cap(\alpha_n,\beta_n)}. \tag{58}$$

Let us transform the formula $(\lambda\widetilde{\varphi}_1+f_n,\widetilde{\psi}_1)_{\mathbf{m}_0}-(\widetilde{\varphi}_1,\lambda\widetilde{\psi}_1+f_n)_{\mathbf{m}_0}$. It follows from Lemma 2.2 and equalities (55), (56) that

$$\widetilde{\varphi}_1(t)=iJx_{n1}-iJ\int_{\alpha_n}^{t}d\mathbf{p}_0(s)\widetilde{\varphi}_1(s)-iJ\lambda\int_{\alpha_n}^{t}d\mathbf{m}_0(s)\widetilde{\varphi}_1(s)-iJ\int_{\alpha_n}^{t}d\mathbf{m}_0(s)f_n(s), \tag{59}$$

$$\widetilde{\psi}_1(t)=2\mu_{n1}-iJ\int_{\alpha_n}^{t}d\mathbf{p}_0(s)\widetilde{\psi}_1(s)-iJ\lambda\int_{\alpha_n}^{t}d\mathbf{m}_0(s)\widetilde{\psi}_1(s)-iJ\int_{\alpha_n}^{t}d\mathbf{m}_0(s)f_n(s). \tag{60}$$

We denote $\widetilde{\Phi}_1=\lambda\widetilde{\varphi}_1+f_n$, $\widetilde{\Psi}_1=\lambda\widetilde{\psi}_1+f_n$. Then

$$(\lambda\widetilde{\varphi}_1+f_n,\widetilde{\psi}_1)_{\mathbf{m}_0}-(\widetilde{\varphi}_1,\lambda\widetilde{\psi}_1+f_n)_{\mathbf{m}_0}=(\widetilde{\Phi}_1,\widetilde{\psi}_1)_{\mathbf{m}_0}-(\widetilde{\varphi}_1,\widetilde{\Psi}_1)_{\mathbf{m}_0}.$$

Using (59), (60), we obtain that the pairs $\{\widetilde{\varphi}_1,\widetilde{\Phi}_1\}$, $\{\widetilde{\psi}_1,\widetilde{\Psi}_1\}$ satisfy equations (61), (62), respectively (see below)

$$\widetilde{\varphi}_1(t)=iJx_{n1}-iJ\int_{\alpha_n}^{t}d\mathbf{p}_0(s)\widetilde{\varphi}_1(s)-iJ\int_{\alpha_n}^{t}d\mathbf{m}_0(s)\widetilde{\Phi}_1(s), \tag{61}$$

$$\widetilde{\psi}_1(t)=2\mu_{n1}-iJ\int_{\alpha_n}^{t}d\mathbf{p}_0(s)\widetilde{\psi}_1(s)-iJ\int_{\alpha_n}^{t}d\mathbf{m}_0(s)\widetilde{\Psi}_1(s). \tag{62}$$

Therefore we can apply Lagrange formula (4) to the functions $\widetilde{\varphi}_1$, $\widetilde{\Phi}_1$, $\widetilde{\psi}_1$, $\widetilde{\Psi}_1$ for $c_1=\alpha_n$, $c_2=\beta_n$, $\mathbf{q}=\mathbf{m}_0$, $\mathbf{p}_1=\mathbf{p}_2=\mathbf{p}_0$. Using (4) and the equality $\lim_{t\to\beta_n-0}\widetilde{\varphi}_1(t)=\widetilde{\varphi}_1(\beta_n)=0$, we get

$$(\lambda\widetilde{\varphi}_1+f_n,\widetilde{\psi}_1)_{\mathbf{m}_0}-(\widetilde{\varphi}_1,\lambda\widetilde{\psi}_1+f_n)_{\mathbf{m}_0}=\int_{\alpha_n}^{\beta_n}(d\mathbf{m}_0(t)\widetilde{\Phi}_1(t),\widetilde{\psi}_1(t))-\int_{\alpha_n}^{\beta_n}(d\mathbf{m}_0(t)\widetilde{\varphi}_1(t),\widetilde{\Psi}_1(t))=$$
$$=(iJ\widetilde{\varphi}_1(\beta_n),\widetilde{\psi}_1(\beta_n))-(iJ\widetilde{\varphi}_1(\alpha_n),\widetilde{\psi}_1(\alpha_n))=-(iJiJx_{n1},2\mu_{n1})=2(x_{n1},\mu_{n1}). \tag{63}$$

Now let us transform the formula $(\lambda\widetilde{\varphi}_1+f_n,\widetilde{\psi}_{3k})_{\mathbf{m}_0}-(\widetilde{\varphi}_1,\lambda\widetilde{\psi}_{3k})_{\mathbf{m}_0}$. For this we denote

$$\widetilde{\psi}_{4k}(t)=-2^{-1}\widetilde{\psi}_{3k}(t)=\begin{cases}0 \ \text{ for } \ t<\tau_k,\\ w_n(t,\lambda)iJw_n^*(\tau_k,\overline{\lambda})\mathbf{m}(\{\tau_k\})\mu_{nk} \ \text{ for } \ t\geqslant\tau_k,\end{cases} \tag{64}$$

where $\tau_k\in\mathcal{S}_m\cap(\alpha_n,\beta_n)$. It follows from Lemma 2.2, equalities (55), (64), and (10), (11) that the equality

$$\widetilde{\psi}_{4k}(t)=iJ\mathbf{m}(\{\tau_k\})\mu_{nk}-iJ\int_{\tau_k}^{t}d\mathbf{p}_0(s)\widetilde{\psi}_{4k}(s)-iJ\lambda\int_{\tau_k}^{t}d\mathbf{m}_0(s)\widetilde{\psi}_{4k}(s)$$

holds. We denote $\widetilde{\Psi}_{4k}=\lambda\widetilde{\psi}_{4k}$. Then the pair $\{\widetilde{\psi}_{4k},\widetilde{\Psi}_{4k}\}$ satisfies the equation

$$\widetilde{\psi}_{4k}(t)=iJ\mathbf{m}(\{\tau_k\})\mu_{nk}-iJ\int_{\tau_k}^{t}d\mathbf{p}_0(s)\widetilde{\psi}_{4k}(s)-iJ\int_{\tau_k}^{t}d\mathbf{m}_0(s)\widetilde{\Psi}_{4k}(s).$$

We can apply Lagrange formula (4) to the functions $\widetilde{\varphi}_1$, $\widetilde{\Phi}_1$, $\widetilde{\psi}_{4k}$, $\widetilde{\Psi}_{4k}$ for $c_1=\tau_k$, $c_2=\beta_n$, $\mathbf{q}=\mathbf{m}_0$, $\mathbf{p}_1=\mathbf{p}_2=\mathbf{p}_0$. We first note that equalities (55), (37) imply

$$\widetilde{\varphi}_1(\tau_k)=-w_n(\tau_k,\lambda)iJ\int_{\alpha_n}^{\tau_k}w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s)+$$
$$+w_n(\tau_k,\lambda)iJ\int_{\alpha_n}^{\beta_n}w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s)=w_n(\tau_k,\lambda)iJ\int_{\tau_k}^{\beta_n}w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s).$$

Now using (4) and the equality $\lim_{t\to\beta_n-0}\widetilde{\varphi}_1(t)=\widetilde{\varphi}_1(\beta_n)=0$, we get

$$(\lambda\widetilde{\varphi}_1+f_n,\widetilde{\psi}_{3k})_{\mathbf{m}_0}-(\widetilde{\varphi}_1,\lambda\widetilde{\psi}_{3k})_{\mathbf{m}_0}=-2[(\widetilde{\Phi}_1,\widetilde{\psi}_{4k})_{\mathbf{m}_0}-(\widetilde{\varphi}_1,\widetilde{\Psi}_{4k})_{\mathbf{m}_0}]=$$
$$=-2\left[\int_{\tau_k}^{\beta_n}(d\mathbf{m}_0(t)\widetilde{\Phi}_1(t),\widetilde{\psi}_{4k}(t))-\int_{\tau_k}^{\beta_n}(d\mathbf{m}_0(t)\widetilde{\varphi}_1(t),\widetilde{\Psi}_{4k}(t))\right]=-2[(iJ\widetilde{\varphi}_1(\beta_n),\widetilde{\psi}_{4k}(\beta_n))-(iJ\widetilde{\varphi}_1(\tau_k),\widetilde{\psi}_{4k}(\tau_k))]=$$
$$=-2[-iJw_n(\tau_k,\lambda)iJ\int_{\tau_k}^{\beta_n}w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s),iJ\mathbf{m}(\{\tau_k\})\mu_{nk})]=$$
$$=2(w_n(\tau_k,\lambda)iJ\int_{\tau_k}^{\beta_n}w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s),\mathbf{m}(\{\tau_k\})\mu_{nk}). \quad (65)$$

Using (37), (38), (63), (65), and the equality $\mathbf{m}(\{\tau_k\})=\mathbf{m}^*(\{\tau_k\})$, we continue equality (58):

$$(u_-,y_n+z_n)_{\mathbf{m}}-(y_n-z_n,u_+)_{\mathbf{m}}=2(x_{n1},\mu_{n1})+$$
$$+2\sum_{k=2}^{\widetilde{\mathbb{k}}_n}\left((\lambda^{-1}f_n(\tau_k),\mathbf{m}(\{\tau_k\}\mu_{nk})+(w_n(\tau_k,\lambda)iJ\int_{\tau_k}^{\beta_n}w_n^*(s,\overline{\lambda})d\mathbf{m}_0(s)f_n(s),\mathbf{m}(\{\tau_k\}\mu_{nk})\right)=2(\widetilde{x}_n,M(\lambda)\widetilde{x}_n). \quad (66)$$

On the other hand, using (47), we get

$$(f_n,y_n+z_n)_{\mathfrak{H}}-(y_n-z_n,f_n)_{\mathfrak{H}}=(u_--\lambda(y_n-z_n),y_n+z_n)_{\mathfrak{H}}-(y_n-z_n,u_+-\lambda(y_n+z_n))_{\mathfrak{H}}=$$
$$=(u_-,y_n+z_n)_{\mathfrak{H}}-(y_n-z_n,u_+)_{\mathfrak{H}}-(\lambda y_n,y_n)_{\mathfrak{H}}+(\lambda z_n,y_n)_{\mathfrak{H}}-(\lambda y_n,z_n)_{\mathfrak{H}}+$$
$$+(\lambda z_n,z_n)_{\mathfrak{H}}+(y_n,\lambda y_n)_{\mathfrak{H}}-(z_n,\lambda y_n)_{\mathfrak{H}}+(y_n,\lambda z_n)_{\mathfrak{H}}-(z_n,\lambda z_n)_{\mathfrak{H}}. \quad (67)$$

Combining (66) and (67), we obtain

$$(f_n,y_n+z_n)_{\mathfrak{H}}-(y_n-z_n,f_n)_{\mathfrak{H}}=2(\widetilde{x}_n,M(\lambda)\widetilde{x}_n)-(\lambda y_n,y_n)_{\mathfrak{H}}+(\lambda z_n,y_n)_{\mathfrak{H}}-$$
$$-(\lambda y_n,z_n)_{\mathfrak{H}}+(\lambda z_n,z_n)_{\mathfrak{H}}+(y_n,\lambda y_n)_{\mathfrak{H}}-(z_n,\lambda y_n)_{\mathfrak{H}}+(y_n,\lambda z_n)_{\mathfrak{H}}-(z_n,\lambda z_n)_{\mathfrak{H}}.$$

This implies

$$\mathrm{Im}(f_n,y_n)_{\mathfrak{H}}=\mathrm{Im}(\widetilde{x}_n,M(\lambda)\widetilde{x}_n)-(\mathrm{Im}\lambda)[(y_n,y_n)_{\mathfrak{H}}-(z_n,z_n)_{\mathfrak{H}}]. \quad (68)$$

Using (68), we get

$$\mathrm{Im}(f,y)_{\mathfrak{H}}=\mathrm{Im}(\widetilde{x},M(\lambda)\widetilde{x})-(\mathrm{Im}\lambda)[(y,y)_{\mathfrak{H}}-(z,z)_{\mathfrak{H}}].$$

Since $y=R(\lambda)f$, we obtain

$$\mathrm{Im}(M(\lambda)\widetilde{x},\widetilde{x})-(\mathrm{Im}\lambda)(z,z)_{\mathbf{m}}=(R(\lambda)f,f)_{\mathbf{m}}-(\mathrm{Im}\lambda)(R(\lambda)f,R(\lambda)f)_{\mathbf{m}}. \quad (69)$$

Equality (69) implies (32). The theorem is proved. □

Corollary 4.6. *If*

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(R(\lambda)f,f)_{\mathbf{m}}-(R(\lambda)f,R(\lambda)f)_{\mathbf{m}}\geqslant 0 \quad (70)$$

for all $f\in\mathfrak{H}$, then

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(M(\lambda)\widetilde{x},\widetilde{x})\geqslant 0 \quad (71)$$

for all $\widetilde{x}\in Q_+$.

Theorem 4.7. *Let $R(\lambda)$ $(\mathrm{Im}\lambda \neq 0)$ be a generalized resolvent of the relation L_{10} and $y = R(\lambda)f$. Then y has the form* (30), *where $M(\lambda): \mathcal{Q}_+ \to \mathcal{Q}_-$ is the bounded operator such that $M(\overline{\lambda}) = M^*(\lambda)$, $\mathrm{Im}\lambda \neq 0$. The function $\lambda \to M(\lambda)\widetilde{x}$ is holomorphic for every $\widetilde{x} \in \mathcal{Q}_+$ on the half-planes $\mathrm{Im}\lambda \neq 0$ and inequality* (71) *holds for every λ $(\mathrm{Im}\lambda \neq 0)$ and for every $\widetilde{x} \in \mathcal{Q}_+$.*

Proof. Equality (30) follows from (2) and Theorem 4.2. The equality $R^*(\lambda) = R(\overline{\lambda})$ implies $M(\overline{\lambda}) = M^*(\lambda)$. The generalized resolvent is holomorphic. Then it follows from Lemma 4.3 that the function $M(\lambda)$ is holomorphic. Inequality (70) holds for the generalized resolvent $R(\lambda)$ (see, for example, [17]). Using Corollary 4.6, we obtain that inequality (71) holds for the function $M(\lambda)$. The theorem is proved. □

Theorem 4.8. *Suppose that the relation $T(\lambda)$ satisfies the conditions of Theorem 4.2, and $T(\lambda)$ is continuously invertible, and $R(\lambda) = T^{-1}(\lambda)$, and $M(\lambda)$ is the operator from Theorem 4.2. Assume that the function $\lambda \to M(\lambda)$ is holomorphic on the half-planes $\mathrm{Im}\lambda \neq 0$ and $M(\lambda) = M^*(\overline{\lambda})$. Then the function $\lambda \to R(\lambda)$ is a generalized resolvent of the relation L_{10} if and only if the inequality*

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}M(\lambda)-$$
$$-\left(M^*(\lambda)\mathbf{V}^*(\lambda) + 2^{-1}i\sum_{n=1}^{\Bbbk_1} p^*_{j_{n1}} J\mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}} w^*_n(\cdot,\lambda)\right)\left(\mathbf{V}(\lambda)M(\lambda) - 2^{-1}i\sum_{n=1}^{\Bbbk_1} \mathfrak{X}_{[\alpha_n,\beta_n)\setminus\mathcal{S}_{\mathbf{m}}} w_n(\cdot,\lambda)Jp_{j_{n1}}\right) \geqslant 0 \quad (72)$$

holds, i.e., the operator on the left side of inequality (72) *is non-negative.*

Proof. By $\mathcal{A}$ denote the operator on the left side (72). Then $(\mathcal{A}\widetilde{x},\widetilde{x}) = (\mathrm{Im}\lambda)^{-1}\mathrm{Im}(M(\lambda)\widetilde{x},\widetilde{x}) - (z,z)_{\mathbf{m}}$, where $\widetilde{x} = \mathbf{V}^*(\overline{\lambda})f \in \mathcal{Q}_+ \subset \mathcal{Q}_-$, $f \in \mathfrak{H}_1$, z is defined by equality (31), $p_{j_{n1}}$ is the operator such that $p_{j_{n1}}\widetilde{x} = x_{n1}$ (see (39)), and x_{n1} is defined by equality (37). Using (32), we get

$$(\mathcal{A}\widetilde{x},\widetilde{x}) = (\mathrm{Im}\lambda)^{-1}\mathrm{Im}(R(\lambda)f,f)_{\mathbf{m}} - (R(\lambda)f,R(\lambda)f)_{\mathbf{m}}.$$

Thus, inequality (72) is equivalent to the following

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(R(\lambda)f,f)_{\mathbf{m}} - (R(\lambda)f,R(\lambda)f)_{\mathbf{m}} \geqslant 0. \quad (73)$$

It follows from Lemma 4.3 that the function $\lambda \to R(\lambda)$ is holomorphic on the half-planes $\mathrm{Im}\lambda \neq 0$. The equality $M(\lambda) = M^*(\overline{\lambda})$ is equivalent to the equality $R(\lambda) = R^*(\overline{\lambda})$. Now the desired assertion follows from [17], where it was established that a function $\lambda \to R(\lambda)$ is a generalized resolvent if and only if this function is holomorphic on the half-planes $\mathrm{Im}\lambda \neq 0$, $R(\lambda) = R^*(\overline{\lambda})$, and inequality (73) holds. The theorem is proved. □

5. The example

We consider equation (1) on a segment $[0,b]$ and assume that $H = \mathbb{C}$, $J = E = 1$, $\mathbf{p} = 0$, $\mathbf{m} = \mathbf{m}_0 + \widehat{\mathbf{m}}$, where $\mathbf{m}_0$ is the usual Lebesque measure (i.e., $\mathbf{m}([\alpha,\beta)) = \beta - \alpha$ for $0 \leqslant \alpha < \beta \leqslant b$; we write ds instead of $d\mathbf{m}_0(s)$), $0 < \tau < b$, $\widehat{\mathbf{m}}(\{\tau\}) = 1$ and $\widehat{\mathbf{m}}(\Delta) = 0$ for all Borel sets such that $\tau \notin \Delta$. So, $\mathcal{S}_{\mathbf{m}} = \{\tau\}$. Thus equation (1) has the form

$$y(t) = x_0 - i\int_0^t d\mathbf{m}(s)f(s). \quad (74)$$

In this example, we consider the minimal operator L_0 generated by equation (74). We find a function $M(\lambda)$ corresponding to the resolvent of a self-adjoint extension of the operator L_0 and use the system of functions $\{\vartheta_k\}$ (13), (14) to find $M(\lambda)$. This function $M(\lambda)$ has the property $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}M(\lambda) \geqslant 0$. In the example from [16], a function $M(\lambda)$ was found using the system of functions $\{v_k\}$ (see (16) and Remark 3.6). This function from [16] does not have the property $(\mathrm{Im}\lambda)^{-1}\mathrm{Im}M(\lambda) \geqslant 0$ (see [16]).

It follows from the definition of L_0 and (8), (74) that L_0 is an operator and if $y = L_0 f$, then

$$y(t) = -i\int_0^t f(s)ds, \quad y(b) = 0, \quad f(\tau) = 0 \quad \Leftrightarrow \quad y'(t) = -if(t), \quad y(0) = y(b) = 0, \quad f(\tau) = 0.$$

Since $\mathcal{S}_0 = \{0, b\}$ and $\mathbf{m}(\mathcal{S}_0) = 0$, we have $\mathfrak{H}_0 = \{0\}$ and $L^*_{10} = L^*_0$ in equality (9).

Equation (6) (for $x_0 = 1$) takes the form

$$W(t,\lambda) = 1 - i\lambda\int_0^t W(s,\lambda)ds, \quad \lambda \in \mathbb{C}.$$

Therefore, $W(t,\lambda) = e^{-i\lambda t}$. Obviously, if $\lambda = 0$, then $W(t, 0) = 1$. The number of intervals $\mathcal{J}_k \in \mathbb{J}$ is $\mathbb{k}_1 = 1$. We write $w(t,\lambda)$ instead of $w_1(t,\lambda)$. Then $w(t,\lambda) = \mathfrak{X}_{[0,b)}W(t,\lambda)$. Without loss of generality it can be assumed that $w(t,\lambda) = W(t,\lambda) = e^{-i\lambda t}$.

The set $\mathbb{M}$ consists of the interval $(0, b)$ and the single-point set $\{\tau\}$. Hence the number of elements of $\mathbb{M}$ is $\mathbb{k} = 2$. By (27), it follows that $Q_{10} = Q_{20} = \{0\}$ and $Q_1 = Q_1^- = Q_1^+ = Q_2 = Q_2^- = Q_2^+ = H = \mathbb{C}$. Therefore, $\boldsymbol{Q} = \boldsymbol{Q}_- = \boldsymbol{Q}_+ = \mathbb{C}^2$.

The domain $\mathcal{D}(L_0)$ of L_0 is dense in $\mathfrak{H} = L_2(\mathbb{C}, d\mathbf{m}; 0, b)$. This yields that L^*_0 is an operator. A function y belongs to $\mathcal{D}(L^*_0)$ if and only if there exist numbers $\eta_1, \eta_2 \in \mathbb{C}$ such that

$$y(t) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}(t)\eta_1 + \mathfrak{X}_{\{\tau\}}(t)\eta_2 - \mathfrak{X}_{[0,b]\setminus\{\tau\}}(t)i\int_0^t d\mathbf{m}(s)u(s), \tag{75}$$

where $u = L^*_0 y$ (see [16]).

With each pair $\{y, u\} \in L^*_0$ we associate a pair of boundary values $\{Y, Y'\} \in \mathbb{C} \times \mathbb{C}$ by the formulas

$$Y = \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} - 2^{-1}i\begin{pmatrix}\int_0^b d\mathbf{m}(s)u(s)\\ \int_0^b d\mathbf{m}(s)u(s)\end{pmatrix} + 2^{-1}i\begin{pmatrix}0\\ u(\tau)\end{pmatrix} + i\begin{pmatrix}0\\ \int_0^\tau d\mathbf{m}(s)u(s)\end{pmatrix}, \quad Y' = \begin{pmatrix}\int_0^b u(s)ds\\ u(\tau)\end{pmatrix}, \tag{76}$$

where y has form (75), $u = L^*_0 y$. Let $\mathcal{L}$ be a restriction of L^*_0 to a set of functions satisfying the condition $Y = 0$. Then $\mathcal{L}$ is the self-adjoint operator and $L_0 \subset \mathcal{L} \subset L^*_0$ (see [16]). Let us find the function

$$M(\lambda) = \begin{pmatrix}M_{11}(\lambda) & M_{12}(\lambda)\\ M_{21}(\lambda) & M_{22}(\lambda)\end{pmatrix} : \mathbb{C}^2 \to \mathbb{C}^2 \tag{77}$$

that corresponds to the resolvent $R_\lambda = (\mathcal{L} - \lambda E)^{-1}$.

Using (13), (14), and the equality $\mathbf{m}(\{\tau\}) = 1$, we get

$$\vartheta_1(t,\lambda) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}w(t,\lambda) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{-i\lambda t} = \begin{cases} e^{-i\lambda t} \text{ for } t \neq \tau,\\ 0 \text{ for } t = \tau \end{cases}; \tag{78}$$

$$\begin{aligned}\vartheta_2(t,\lambda) &= u_1(t,\lambda,\tau)\lambda^{-1} + \mathfrak{X}_{\{\tau\}}(t)\lambda^{-1} = \\ &= -\mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{-i\lambda t}i\int_0^t e^{i\lambda s}d\mathbf{m}(s)\lambda\mathfrak{X}_{\{\tau\}}(s)\lambda^{-1}ds + \mathfrak{X}_{\{\tau\}}(t)\lambda^{-1} = \begin{cases} 0 \text{ for } t < \tau,\\ \lambda^{-1} \text{ for } t = \tau,\\ -ie^{-i\lambda t}e^{i\lambda\tau} \text{ for } t > \tau; \ \lambda \neq 0.\end{cases}\end{aligned} \tag{79}$$

Therefore,

$$\vartheta_1^*(t,\overline{\lambda}) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{i\lambda t} = \begin{cases} e^{i\lambda t} \text{ for } t \neq \tau,\\ 0 \text{ for } t = \tau \end{cases}; \quad \vartheta_2^*(t,\overline{\lambda}) = \begin{cases} 0 \text{ for } t < \tau,\\ \lambda^{-1} \text{ for } t = \tau,\\ ie^{-i\lambda\tau}\mathfrak{X}_{(\tau,b]}e^{i\lambda t} \text{ for } t > \tau; \ \lambda \neq 0.\end{cases}$$

We denote

$$x_1 = x_1(f,\lambda) = \int_0^b \vartheta_1^*(s,\overline{\lambda})d\mathbf{m}(s)f(s) = \int_0^b e^{i\lambda s}f(s)ds, \tag{80}$$

$$x_2 = x_2(f,\lambda) = \int_0^b \vartheta_2^*(s,\overline{\lambda})d\mathbf{m}(s)f(s) = \lambda^{-1}f(\tau) + \int_\tau^b ie^{i\lambda s}e^{-i\lambda\tau}f(s)ds. \tag{81}$$

Using (34), we get

$$y(t) = \vartheta_1(t,\lambda)(M_{11}(\lambda)x_1 + M_{12}(\lambda)x_2) + \vartheta_2(t,\lambda)(M_{21}(\lambda)x_1 + M_{22}(\lambda)x_2) - \\ - \mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{-i\lambda t}i\int_0^t e^{i\lambda s}f(s)ds + 2^{-1}\mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{-i\lambda t}i\int_0^b e^{i\lambda s}f(s)ds - \lambda^{-1}\mathfrak{X}_{\{\tau\}}(t)f(\tau). \tag{82}$$

By (78), (79), it follows that

$$\vartheta_1(t,\lambda)(M_{11}(\lambda)x_1 + M_{12}(\lambda)x_2) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{-i\lambda t}(M_{11}(\lambda)x_1 + M_{12}(\lambda)x_2), \tag{83}$$

$$\vartheta_2(t,\lambda)(M_{21}(\lambda)x_1 + M_{22}(\lambda)x_2) = \begin{cases} 0 \text{ for } t < \tau, \\ \lambda^{-1}(M_{21}(\lambda)x_1 + M_{22}(\lambda)x_2) \text{ for } t = \tau, \\ -ie^{-i\lambda t}e^{i\lambda\tau}(M_{21}(\lambda)x_1 + M_{22}(\lambda)x_2) \text{ for } t > \tau; \ \lambda \neq 0. \end{cases} \tag{84}$$

To find $M_{12}(\lambda)$, $M_{22}(\lambda)$, we take the function $f_1(t) = \mathfrak{X}_{\{\tau\}}(t)$, i.e., $f_1(t) = 1$ if $t = \tau$ and $f_1(t) = 0$ if $t \neq \tau$ (by $\{Y_1, Y_1'\}$ denote the corresponding pair of boundary values). It follows from (80), (81) that $x_1 = 0$, $x_2 = \lambda^{-1}$. We denote $y_1 = R_\lambda f_1$. Using (82), (83), (84), we obtain

$$y_1(t) = \vartheta_1(t,\lambda)M_{12}(\lambda)\lambda^{-1} + \vartheta_2(t,\lambda)M_{22}(\lambda)\lambda^{-1} - \lambda^{-1}\mathfrak{X}_{\{\tau\}}(t). \tag{85}$$

We denote $u_1 = L_0^* y_1 = \lambda y_1 + f_1$. Then using (85), (78), (79), we get

$$u_1(t) = \vartheta_1(t,\lambda)M_{12}(\lambda) + \vartheta_2(t,\lambda)M_{22}(\lambda) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}(t)e^{-i\lambda t}M_{12}(\lambda) + \mathfrak{X}_{\{\tau\}}(t)\lambda^{-1}M_{22}(\lambda) - \mathfrak{X}_{(\tau,b]}(t)ie^{-i\lambda t}e^{i\lambda\tau}M_{22}(\lambda). \tag{86}$$

Taking into account (75), (85), we obtain

$$y_1(t) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}(t)M_{12}(\lambda)\lambda^{-1} + \mathfrak{X}_{\{\tau\}}(t)\lambda^{-1}(M_{22}(\lambda)\lambda^{-1} - 1) - \mathfrak{X}_{[0,b]\setminus\{\tau\}}(t)i\int_0^t d\mathbf{m}(s)u_1(s). \tag{87}$$

Using (86), by direct calculations, we get

$$\int_0^b d\mathbf{m}(s)u_1(s) = \lambda^{-1}(i(e^{-i\lambda b} - 1)M_{12}(\lambda) + e^{-i\lambda b}e^{i\lambda\tau}M_{22}(\lambda)); \quad \int_0^\tau d\mathbf{m}(s)u_1(s) = i\lambda^{-1}(e^{-i\lambda\tau} - 1)M_{12}(\lambda). \tag{88}$$

By (76), (87), (88), so that

$$Y_1 = \begin{pmatrix} M_{12}(\lambda)\lambda^{-1} \\ \lambda^{-1}(M_{22}(\lambda)\lambda^{-1} - 1) \end{pmatrix} - 2^{-1}i \begin{pmatrix} \lambda^{-1}(i(e^{-i\lambda b} - 1)M_{12}(\lambda) + e^{-i\lambda b}e^{i\lambda\tau}M_{22}(\lambda)) \\ \lambda^{-1}(i(e^{-i\lambda b} - 1)M_{12}(\lambda) + e^{-i\lambda b}e^{i\lambda\tau}M_{22}(\lambda)) \end{pmatrix} + \\ + 2^{-1}i \begin{pmatrix} 0 \\ \lambda^{-1}M_{22}(\lambda) \end{pmatrix} + i \begin{pmatrix} 0 \\ i\lambda^{-1}(e^{-i\lambda\tau} - 1)M_{12}(\lambda) \end{pmatrix}.$$

The equality $Y_1 = 0$ is equivalent to two equalities

$$\begin{cases} M_{12}(\lambda)\lambda^{-1} + 2^{-1}\lambda^{-1}((e^{-i\lambda b} - 1)M_{12}(\lambda) - ie^{-i\lambda b}e^{i\lambda\tau}M_{22}(\lambda)) = 0, \\ \lambda^{-1}(M_{22}(\lambda)\lambda^{-1} - 1) + 2^{-1}\lambda^{-1}((e^{-i\lambda b} - 1)M_{12}(\lambda) - ie^{-i\lambda b}e^{i\lambda\tau}M_{22}(\lambda)) + 2^{-1}\lambda^{-1}iM_{22}(\lambda) - \lambda^{-1}(e^{i\lambda\tau} - 1)M_{12}(\lambda) = 0. \end{cases}$$

Solving this system of equations, we get

$$M_{12}(\lambda) = i\frac{2\lambda e^{i\lambda\tau}e^{-i\lambda b}}{e^{-i\lambda b}(2-i\lambda)+(2+i\lambda)}\ ;\quad M_{22}(\lambda) = \frac{2\lambda(e^{-i\lambda b}+1)}{e^{-i\lambda b}(2-i\lambda)+(2+i\lambda)}\ . \tag{89}$$

To find $M_{11}(\lambda)$, $M_{21}(\lambda)$, we take the function $f_2(t) = \mathfrak{X}_{[0,\tau)}(t)$, i.e., $f_2(t) = 1$ if $t < \tau$ and $f_2(t) = 0$ if $t \geqslant \tau$ (by $\{Y_2, Y_2'\}$ denote the corresponding pair of boundary values). It follows from (80), (81) that $x_1 = i\lambda^{-1}(1-e^{i\lambda\tau})$, $x_2 = 0$. We denote $y_2 = R_\lambda f_2$. Using (82), (83), (84), we obtain

$$y_2(t) = \vartheta_1(t,\lambda)M_{11}(\lambda)x_1 + \vartheta_2(t,\lambda)M_{21}(\lambda)x_1 - \mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{-i\lambda t}i\int_0^t e^{i\lambda s}f_2(s)ds + 2^{-1}\mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{-i\lambda t}i\int_0^b e^{i\lambda s}f_2(s)ds. \tag{90}$$

The equality $f_2(t) = \mathfrak{X}_{[0,\tau)}(t)$ implies

$$-\mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{-i\lambda t}i\int_0^t e^{i\lambda s}f_2(s)ds = \begin{cases}\lambda^{-1}(e^{-i\lambda t}-1) \text{ for } t<\tau,\\ 0 \text{ for } t=\tau,\\ \lambda^{-1}e^{-i\lambda t}(1-e^{i\lambda\tau}) \text{ for } t>\tau;\end{cases} \tag{91}$$

$$2^{-1}\mathfrak{X}_{[0,b]\setminus\{\tau\}}e^{-i\lambda t}i\int_0^b e^{i\lambda s}f_2(s)ds = \begin{cases}2^{-1}ie^{-i\lambda t}x_1 \text{ for } t\neq\tau,\\ 0 \text{ for } t=\tau.\end{cases} \tag{92}$$

By (75), (90), (91), (92), it follows that

$$y_2(t) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}(M_{11}(\lambda)x_1 + 2^{-1}ix_1) + \mathfrak{X}_{\{\tau\}}(t)\lambda^{-1}M_{21}(\lambda)x_1 - \mathfrak{X}_{[0,b]\setminus\{\tau\}}(t)i\int_0^t d\mathbf{m}(s)u_2(s), \tag{93}$$

where $u_2 = L_0^*y_2 = \lambda y_2 + f_2$. Equalities (78), (79), (90), (91), (92) imply that $u_2(t) = u_{21}(t)+u_{22}(t)+u_{23}(t)+u_{24}(t)$, where

$$u_{21}(t) = \mathfrak{X}_{[0,b]\setminus\{\tau\}}\lambda e^{-i\lambda t}M_{11}(\lambda)x_1;\quad u_{22}(t) = \begin{cases}0 \text{ for } t<\tau,\\ M_{21}(\lambda)x_1 \text{ for } t=\tau,\\ -\lambda ie^{i\lambda\tau}e^{-i\lambda t}M_{21}(\lambda)x_1 \text{ for } t>\tau;\end{cases} \tag{94}$$

$$u_{23}(t) = \begin{cases}e^{-i\lambda t} \text{ for } t<\tau,\\ 0 \text{ for } t=\tau,\\ e^{-i\lambda t}(1-e^{i\lambda\tau}) \text{ for } t>\tau;\end{cases}\qquad u_{24}(t) = \begin{cases}2^{-1}\lambda ie^{-i\lambda t}x_1 \text{ for } t\neq\tau,\\ 0 \text{ for } t=\tau.\end{cases}\ . \tag{95}$$

Using (94), (95), and the equality $x_1 = i\lambda^{-1}(1-e^{i\lambda\tau})$, by direct calculations, we obtain

$$\int_0^b d\mathbf{m}(s)u_2(s) = i(e^{-i\lambda b}-1)M_{11}(\lambda)x_1 + e^{i\lambda\tau}e^{-i\lambda b}M_{21}(\lambda)x_1 + 2^{-1}(e^{-i\lambda b}+1)x_1, \tag{96}$$

$$\int_0^\tau d\mathbf{m}(s)u_2(s) = i(e^{-i\lambda\tau}-1)M_{11}x_1 + 2^{-1}(e^{-i\lambda\tau}+1)x_1. \tag{97}$$

By (76), (93), so that

$$Y_2 = \begin{pmatrix}M_{11}(\lambda)x_1 + 2^{-1}ix_1\\ \lambda^{-1}M_{21}(\lambda)x_1\end{pmatrix} - 2^{-1}i\begin{pmatrix}\int_0^b d\mathbf{m}(s)u_2(s)\\ \int_0^b d\mathbf{m}(s)u_2(s)\end{pmatrix} + 2^{-1}i\begin{pmatrix}0\\ M_{21}(\lambda)x_1\end{pmatrix} + i\begin{pmatrix}0\\ \int_0^\tau d\mathbf{m}(s)u_2(s)\end{pmatrix},$$

where the integrals $\int_0^b d\mathbf{m}(s)u_2(s)$, $\int_0^\tau d\mathbf{m}(s)u_2(s)$ are calculated by formulas (96), (97), respectively.

The equality $Y_2 = 0$ is equivalent to two equalities

$$\begin{cases} M_{11}(\lambda)x_1 + 2^{-1}ix_1 - 2^{-1}i\int_0^b d\mathbf{m}(s)u_2(s) = 0, \\ \lambda^{-1}M_{21}(\lambda)x_1 - 2^{-1}i\int_0^b d\mathbf{m}(s)u_2(s) + 2^{-1}iM_{21}(\lambda)x_1 + i\int_0^\tau d\mathbf{m}(s)u_2(s) = 0. \end{cases}$$

Solving this system of equations, we obtain

$$M_{11}(\lambda) = i\frac{(2-i\lambda)e^{-i\lambda b} - (2+i\lambda)}{2((2-i\lambda)e^{-i\lambda b} + (2+i\lambda))}\ ; \quad M_{21}(\lambda) = i\frac{-2\lambda e^{-i\lambda\tau}}{(2-i\lambda)e^{-i\lambda b} + (2+i\lambda)}\ . \tag{98}$$

Thus the matrix $M(\lambda)$ (77) is calculated by equalities (98), (89).

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