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Iterates of $(\alpha, q)-$Bernstein operators

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Abstract. In this paper, the iterates of (α, q)-Bernstein operators are considered. Given fixed $n \in \mathbb{N}$ and $q > 0$, it is shown that for $f \in C[0,1]$ the k-th iterate $T^k_{n,q,\alpha}(f;x)$ converges uniformly on $[0,1]$ to the linear function $L_f(x)$ passing through the points $(0, f(0))$ and $(1, f(1))$. Moreover, it is proved that, when $q \in (0,1)$, the iterates $T^{j_n}_{n,q,\alpha}(f;x)$, in which $\{j_n\} \to \infty$ as $n \to \infty$, also converge to $L_f(x)$. Further, when $q \in (1,\infty)$ and $\{j_n\}$ is a sequence of positive integers such that $j_n/[n]_q \to t$ as $n \to \infty$, where $0 \le t \le \infty$, the convergence of the iterates $T^{j_n}_{n,q,\alpha}(p;x)$ for p being a polynomial is studied.

1. Introduction

Bernstein polynomials and their generalizations based on q-integers are widely used in many branches of mathematics, especially in approximation theory and probability theory. The fundamental property of Bernstein operator is that $B_n(f;.)$ transforms every continuous function f defined on $[0,1]$ into a polynomial of degree n, called Bernstein polynomial. Researchers asked themselves naturally what happens as the Bernstein operator is applied to such f repeatedly and then iterations of Bernstein operators have began to be investigated by researchers. The first study on this subject is done by R.P. Kelinsky and T.J. Rivlin in 1967 [9]. After this date, many researchers get many results related to iterations of Bernstein operators. See, for example, [5–8, 11, 16]. It is known that the iterates of Bernstein operators converge. One of the famous proof is based on the contraction mappings for Banach fixed point theorem used by I.A. Rus at 2004. Moreover, it is shown that Bernstein operators are weakly Picard operators [15]. Another proof is conducted by U.Abel and M. Ivan in 2009, in which the basic properties of Bernstein polynomials and positive linear operators are used [1].

In addition, the iterates of q-Bernstein operators have been considered. In 2002, it is proved that $B^M_{n,q}(f;x)$ converges to the linear interpolating polynomial of f at the endpoints of $[0,1]$ for any fixed $q > 0$ when $M \to \infty$ [12]. The rate of convergence of $B_{n,q}(f,x)$ to an analytic function of f in the norm of $C[0,1]$ has the order q^{-n} was proved by S. Ostrovska in 2003. Moreover, when $\{j_n\}$ is a sequence of positive integers such that $j_n \to \infty$ as $n \to \infty$, the convergence of $B^{j_n}_{n,q}(f,x)$ is investigated as $n \to \infty$. It is shown that for $q \in (0,1)$ the asymptotic behavior of such iterates is quite different from the classical case [13]. There are many research papers related to iterates of generalized Bernstein operators and some of them are shown in the references [2, 14, 17].

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In this study, the iterates of (α, q)-Bernstein operators, introduced by Qing-Bo Cai and Xiao-Wei Xu in [4], are investigated.

2. Preliminaries

The definitions and notations used in this article are adopted from [3, Ch.10]. Let $q > 0$. The q-integer is defined by

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}, \quad [0]_q := 0 \qquad (n = 1, 2, \ldots),$$

the q-factorial of n by

$$[n]_q! := [1]_q[2]_q \ldots [n]_q, \quad [0]_q! := 1 \qquad (n = 1, 2, \ldots).$$

For integers k and n with $0 \leq k \leq n$, the q-binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The q-shifted product is defined by

$$(a; q)_0 := 1, \quad (a; q)_k = \prod_{s=0}^{k-1}(1 - aq^s), \quad (a; q)_\infty = \prod_{s=0}^{\infty}(1 - aq^s).$$

The q-analogue of α-Bernstein operators, called (α, q)-Bernstein operators, are defined in [4] as follows:

Definition 2.1. *[4] Given $q > 0$ and $\alpha \in \mathbb{R}$. For $n \in \mathbb{N}$ and $f : [0, 1] \to \mathbb{R}$, the (α, q)-Bernstein operator is given by $T_{n,q,\alpha} : f \to T_{n,q,\alpha}(f; \cdot)$ such that*

$$T_{n,q,\alpha}(f; x) = \sum_{i=0}^{n} f\left(\frac{[i]_q}{[n]_q}\right) p^{(\alpha)}_{n,q,i}(x), \tag{1}$$

where $p^{(\alpha)}_{n,q,i}(x)$ are the basis (α, q)-Bernstein polynomials of degree n given by $p^{(\alpha)}_{1,q,0}(x) = 1 - x$, $p^{(\alpha)}_{1,q,1}(x) = x$ and for $n \geq 2$,

$$p^{(\alpha)}_{n,q,i}(x) = \left(\begin{bmatrix} n-2 \\ i \end{bmatrix}_q (1-\alpha)x + \begin{bmatrix} n-2 \\ i-2 \end{bmatrix}_q (1-\alpha)q^{n-i}(1 - q^{n-i-1}x) + \begin{bmatrix} n \\ i \end{bmatrix}_q \alpha x(1 - q^{n-i-1}x)\right) x^{i-1}(x; q)_{n-i-1}.$$

It can be shown, as it is done in [13], that for $f \in C[0, 1]$ and $q \in (0, 1)$,

$$\lim_{n\to\infty} T_{n,q,\alpha}(f, x) = B_{\infty,q}(f, x)$$

where $B_{\infty,q}$ is the limit q-Bernstein operator given by

$$B_{\infty,q}(f; x) = \begin{cases} \sum_{i=0}^{\infty} f(1 - q^i) p_{\infty,q,i}(x), & x \in [0, 1) \\ f(1), & x = 1 \end{cases}$$

in which

$$p_{\infty,q,i}(x) = \frac{x^i (x; q)_\infty}{(q; q)_i}, \quad i = 0, 1, 2, \ldots$$

By Euler's identity, see [3, Ch.10, Section 10.2, Corollary 10.2.2],

$$\frac{1}{(x; q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{(q; q)_k}, \quad |x| < 1, \ |q| < 1.$$

Thus, it is evident that

$$\|B_{\infty,q}\| = 1.$$

From the definition it is obvious that $T_{n,q,\alpha}$ satisfy

$$T_{n,q,\alpha}(f;0) = f(0), \quad T_{n,q,\alpha}(f;1) = f(1) \tag{2}$$

for all n which is known as the end-point interpolation. Also, the operators $T_{n,q,\alpha}$ leave the linear functions invariant:

$$T_{n,q,\alpha}(at+b;x) = ax+b. \tag{3}$$

If we take $a = 0$, $b = 1$ we get

$$\sum_{k=0}^{n} p_{n,q,i}^{(\alpha)}(x) = 1 \quad \text{for all} \quad n = 1,2,\ldots$$

Therefore,

$$\|T_{n,q,\alpha}\| = 1. \tag{4}$$

The eigenvalues and corresponding eigenvectors of $T_{n,q,\alpha}$ are studied in [10] where the following results are presented.

Lemma 2.2. *[10] For all $q > 0$ and $\alpha \in [0,1]$, the operator $T_{n,q,\alpha}$ has $n+1$ linearly independent monic eigenvectors $p_{k,q}^{(\alpha,n)}(x)$ of degree $k = 0,1,\ldots,n$ corresponding to the eigenvalues $\lambda_{0,q}^{(\alpha,n)} = \lambda_{1,q}^{(\alpha,n)} = 1$ and*

$$\lambda_{k,q}^{(\alpha,n)} = \frac{q^{\frac{k(k-1)}{2}}[n-2]_q!}{[n-k]_q![n]_q^k}\Big((1-\alpha)[n-k]_q[n-1+k]_q + \alpha[n]_q[n-1]_q\Big)$$

for $k = 2,3,\ldots$.

Lemma 2.3. *[10] The following equality holds:*

$$\lim_{n\to\infty} \lambda_{k,q}^{(\alpha,n)} = \begin{cases} q^{k(k-1)/2} & if \quad q \in (0,1) \\ 1 & if \quad q \in [1,\infty) \end{cases}$$

It has been shown in [10] that

$$T_{n,q,\alpha}(t^m;x) = \lambda_{m,q}^{(\alpha,n)} x^m + P_{m-1}^{(n)} \tag{5}$$

where $\lambda_{k,q}^{(\alpha,n)}$ is given by Lemma 2.2 and $P_{m-1}^{(n)}$ is a polynomial of degree at most $m-1$.

Definition 2.4. *Let $f : [0,1] \to \mathbb{R}$. The k-th iterate of $T_{n,q,\alpha}$ is defined by*

$$T_{n,q,\alpha}^k(f;x) = T_{n,q,\alpha}(T_{n,q,\alpha}^{k-1}(f;x)), \quad k = 2,3,\ldots$$

where $T_{n,q,\alpha}^1(f;x) = T_{n,q,\alpha}(f;x)$.

3. Main Results

In the sequel, π_n stands for the set of polynomials of degree at most n, and $L_f \in \pi_1$ denotes the linear function passing through the $(0, f(0))$ and $(1, f(1))$. That is,

$$L_f(x) = f(0)(1-x) + f(1)x.$$

The next result states the convergence of the iterates.

Theorem 3.1. *Let $q > 0$ and $f \in C[0,1]$. Then, for fixed $n \in \mathbb{N}$,*

$$\lim_{k\to\infty} T^k_{n,q,\alpha}(f;x) = L_f(x)$$

and the convergence is uniform on $[0,1]$.

Proof. As $T_{n,q,\alpha}(f;x) \in \pi_n$ for any function $f : [0,1] \to \mathbb{R}$, it is enough to deal with only the case $f \in \pi_n$. For $n = 1$, using (1), one gets

$$\begin{aligned} T_{1,q,\alpha}(f,x) &= f(0)p^{(\alpha)}_{1,q,0}(x) + f(1)p^{(\alpha)}_{1,q,1}(x) \\ &= f(0)(1-x) + f(1)x \\ &= L_f(x). \end{aligned}$$

For $n \geq 2$, by the help of (3), one can see that the statement is true if $f \in \pi_1$. Now, we will apply induction only for a monomial x^m where $m \leq n$ using again the linearity property of $T_{n,q,\alpha}$. Assume that the statement is true for $f(x) = 1, x, \ldots, x^{m-1}$. From (5), one has

$$T_{n,q,\alpha}(t^m;x) = \eta x^m + P(x) \tag{6}$$

where $\eta \in (0,1)$ and $P \in \pi_{m-1}$. By the induction assumption,

$$\lim_{k\to\infty} T^k_{n,q,\alpha}(P;x) = L_p(x)$$

Write (6) as

$$T_{n,q,\alpha}(t^m;x) = \eta x^m + \rho(x) + L_p(x) \tag{7}$$

where $\rho(x) = P(x) - L_p(x)$. Applying the operator on both sides of (7), as $L_p \in \pi_1$, we get

$$\begin{aligned} T^2_{n,q,\alpha}(t^m;x) &= \eta T_{n,q,\alpha}(t^m;x) + T_{n,q,\alpha}(\rho;x) + L_p(x) \\ &= \eta^2 x^m + \eta\rho(x) + \eta L_p(x) + T_{n,q,\alpha}(\rho;x) + L_p(x) \\ &= \eta^2 x^m + (\eta I + T_{n,q,\alpha})\rho(x) + (1+\eta)L_p(x) \end{aligned}$$

where I denotes the identity operator. If we continue applying the operator successively, we get

$$T^k_{n,q,\alpha}(t^m;x) = \eta^k x^m + \left[\eta^{k-1}I + \eta^{k-2}T_{n,q,\alpha} + \cdots + T^{k-1}_{n,q,\alpha}\right]\rho(x) + (1 + \eta + \cdots + \eta^{k-1})L_P(x).$$

Now, because $\eta \in (0,1)$, for $k \to \infty$, one has $\eta^k x^m \to 0$ and $(1 + \eta + \cdots + \eta^{k-1})L_P(x) \to \frac{L_P(x)}{1-\eta}$ uniformly on $[0,1]$. To show the limit of the term in brackets, choose $\varepsilon > 0$. Clearly, by the linearity of the operator we have

$$\lim_{k\to\infty} T^k_{n,q,\alpha}(\rho;x) = 0.$$

Thus, for $\varepsilon_1 = \frac{\varepsilon(1-\eta)}{2}$, one can find $K \in \mathbb{N}$ such that

$$\|T^j_{n,q,\alpha}(\rho,x)\| < \varepsilon_1, \quad j \geq K.$$

Then,

$$\begin{aligned}\|\eta^{k-1}\rho(x)+\cdots+T^{k-1}_{n,q,\alpha}(\rho;x)\| &\le \|\eta^{k-1}\rho(x)+\cdots+\eta^{k-K}T^{K-1}_{n,q,\alpha}(\rho;x)\| \\ &\quad+\|\eta^{k-K-1}T^{K}_{n,q,\alpha}(\rho;x)+\eta^{k-K-2}T^{K+1}_{n,q,\alpha}(\rho;x)+\cdots+T^{k-1}_{n,q,\alpha}(\rho;x)\| \\ &\le \varepsilon_1+\varepsilon_1\eta+\cdots+\varepsilon_1\eta^{k-K-1}+\eta^{k-K}\|\eta^{K-1}\rho(x)+\cdots+T^{K-1}_{n,q,\alpha}(\rho(x),x)\|.\end{aligned}$$

Set $\|\eta^{K-1}\rho(x)+\eta^{K-2}T_{n,q,\alpha}(\rho;x)+\cdots+T^{K-1}_{n,q,\alpha}(\rho;x)\|=M$ and choose $k_0>K$ such that $M\eta^{k-K}\le\frac{\varepsilon}{2}$ for all $k\ge k_0$. Then, we get

$$\|\eta^{k-1}\rho+\cdots+T^{k-1}_{n,q,\alpha}(\rho;x)\|\le\frac{\varepsilon_1}{1-\eta}+M\eta^{k-K}\le\varepsilon.$$

So,

$$\lim_{k\to\infty}T^{k}_{n,q,\alpha}(t^m;x)=\frac{L_P(x)}{1-\eta},$$

uniformly on $[0,1]$. Using (2), together with the last result, we get

$$\lim_{k\to\infty}T^{k}_{n,q,\alpha}(t^m;x)=x$$

Now, suppose that

$$T_{n,q,\alpha}(f;x)=A_nx^n+A_{n-1}x^{n-1}+\cdots+A_1x+A_0.$$

Then,

$$\begin{aligned}\lim_{k\to\infty}T^{k}_{n,q,\alpha}(f;x)&=\lim_{k\to\infty}\left[A_nT^{k}_{n,q,\alpha}(t^n;x)+\cdots+A_1T^{k}_{n,q,\alpha}(t;x)+A_0\right]\\&=(A_n+\cdots+A_1)x+A_0.\end{aligned}$$

Using (2) once more, one finds $A_0=f(0)$ and $A_0+A_1+\cdots+A_n=f(1)$. Therefore,

$$\lim_{k\to\infty}T^{k}_{n,q,\alpha}(f;x)=[f(1)-f(0)](x)+f(0)=L_f(x)$$

and convergence is uniform on $[0,1]$. □

Example 3.2. *For $f(x)=2x+\sin(9\pi x/2)$ one has $L_f(x)=3x$. The first, third and the fifth iterates of $T_{n,q,\alpha}(f;x)$ when $n=3$, $q=0.9$ and $\alpha=0.5$ are depicted in Figure 1. As it can be observed, $T^{k}_{3,0.9,0.5}(f;x)$ converges to L_f.*

In the previous theorem, n is fixed. Next, we consider the case when n is not fixed. More precisely, when $q\in(0,1)$, we have the following result:

Theorem 3.3. *Let $q\in(0,1)$ and $\{j_n\}$ be a sequence of positive integers such that $j_n\to\infty$ as $n\to\infty$. Then, for $f\in C[0,1]$*

$$\lim_{n\to\infty}T^{j_n}_{n,q,\alpha}(f;x)=L_f(x)$$

uniformly on $[0,1]$.

Proof. Because of end-point interpolation, it suffices to prove that $\lim_{n\to\infty}T^{j_n}_{n,q,\alpha}(f;x)=ax+b$ for some $a,b\in\mathbb{R}$. The theorem is proved in two parts.

1) First, consider the case $f(x)=x^m$ and use induction on m. For $m=0,1$ the statement is obvious due to (3). Assume that $\lim_{n\to\infty}T^{j_n}_{n,q,\alpha}(t^s;x)=a_sx+b_s\in\pi_1$ for $s=0,1,\ldots,m-1$. Then, by (5)

$$T_{n,q,\alpha}(t^m;x)=(\lambda^{(\alpha,n)}_{m,q})^{j_n}x^m+\left[(\lambda^{(\alpha,n)}_{m,q})^{j_n-1}I+(\lambda^{(\alpha,n)}_{m,q})^{j_n-2}T_{n,q,\alpha}+\cdots+T^{j_n-1}_{n,q,\alpha}\right]P^{(n)}_{m-1}(x)$$

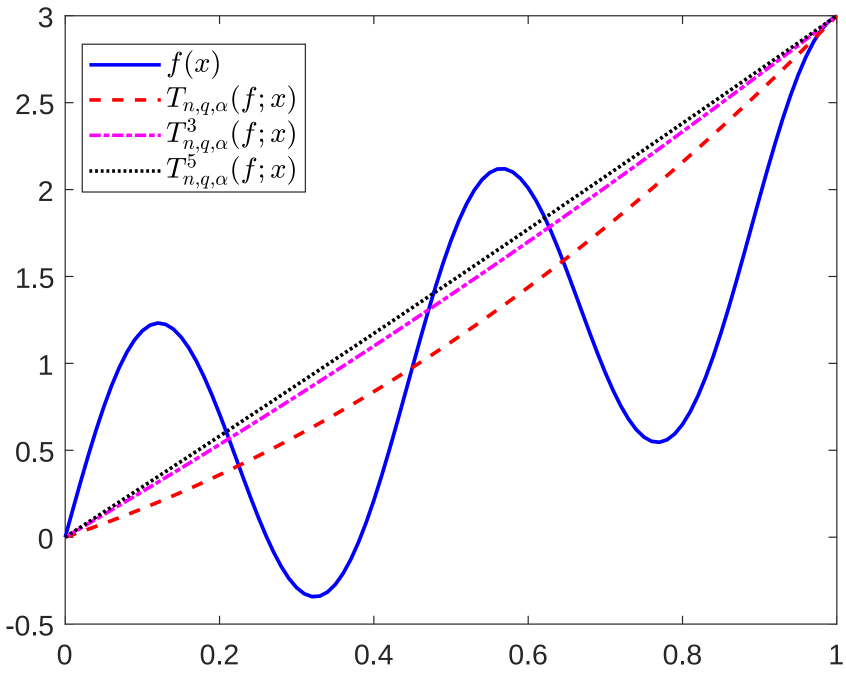

Figure 1: Some iterates of $T_{n,q,\alpha}(f;x)$ for $n = 3$, $q = 0.9$ and $\alpha = 0.5$.

where I denotes the identity operator $P^{(n)}_{m-1} \in \pi_{m-1}$. It follows from Lemma 2.3 that

$$\lim_{n\to\infty} (\lambda^{(\alpha,n)}_{m,q})^{j_n} = 0. \tag{8}$$

The expression in the brackets is a linear operator on the space π_{m-1}.

Consider the sequence of polynomials in π_{m-1},

$$y^{(\alpha,n)}_{m-1} = \left[(\lambda^{(\alpha,n)}_{m,q})^{j_n-1} I + (\lambda^{(\alpha,n)}_{m,q})^{j_n-2} T_{n,q,\alpha} + \cdots + T^{j_n-1}_{n,q,\alpha}\right](P^{(n)}_{m-1}).$$

Then,

$$(\lambda^{(\alpha,n)}_{m,q} I - T_{n,q,\alpha}) y^{(\alpha,n)}_{m-1} = (\lambda^{(\alpha,n)}_{m,q})^{j_n} P^{(n)}_{m-1} - T^{j_n}_{n,q,\alpha} P^{(n)}_{m-1}.$$

It follows from (4) and (5) that $\|P^{(n)}_{m-1}\| \leq 2$. Since (8) holds, we have

$$\lim_{n\to\infty} (\lambda^{(\alpha,n)}_{m,q})^{j_n} P^{(n)}_{m-1} = 0.$$

It can be readily seen from (5) and Lemma 2.3 that

$$\lim_{n\to\infty} P^{(n)}_{m-1}(x) = B_{\infty,q}(t^m;x) - q^{m(m-1)/2} x^m = Q_{m-1}(x) \in \pi_{m-1}$$

In other words,

$$P^{(n)}_{m-1}(x) = Q_{m-1}(x) + \delta_n(x)$$

where $\lim_{n\to\infty} \delta_n(x) = 0$. Thus,

$$T^{j_n}_{n,q,\alpha}(P^{(n)}_{m-1};x) = T^{j_n}_{n,q,\alpha}(Q_{m-1};x) + T^{j_n}_{n,q,\alpha}(\delta_n;x).$$

Due to (4), $\|T^{j_n}_{n,q,\alpha}(\delta_n;x)\| \leq \|\delta_n\|$ which means that $\lim_{n\to\infty} T^{j_n}_{n,q,\alpha}(\delta_n) = 0$. By induction assumption

$$\lim_{n\to\infty} T^{j_n}_{n,q,\alpha}(Q_{m-1};x) = cx + d \in \pi_1.$$

Therefore,

$$\lim_{n\to\infty} (\lambda^{(\alpha,n)}_{m,q} I - T_{n,q,\alpha}) y^{(\alpha,n)}_{m-1} = cx + d$$

or

$$(\lambda^{(\alpha,n)}_{m,q} I - T_{n,q,\alpha}) y^{(\alpha,n)}_{m-1} = cx + d + \omega_n(x)$$

where $\lim_{n\to\infty} \omega_n(x) = 0$ uniformly on $[0,1]$. As stated in Lemma 2.2, the eigenvector of $T_{n,q,\alpha}$ corresponding to the eigenvalue $\lambda^{(\alpha,n)}_{m,q}$ is a polynomial of degree m. Therefore, the operators $\lambda^{(\alpha,n)}_{m,q} I - T_{n,q,\alpha}$ are invertible on π_{m-1} for $n \geq m$ and

$$\lim_{n\to\infty} (\lambda^{(\alpha,n)}_{m,q} I - T_{n,q,\alpha}) = q^{m(m-1)/2} I - B_{\infty,q} =: A_{\infty,q}$$

is also invertible on π_{m-1}. Hence,

$$\lim_{n\to\infty} (\lambda^{(\alpha,n)}_{m,q} I - T_{n,q,\alpha})^{-1} = A^{-1}_{\infty,q}$$

and it follows that

$$\|(\lambda^{(\alpha,n)}_{m,q} I - T_{n,q,\alpha})^{-1}\| \leq M \quad \text{for some } M > 0.$$

Therefore,

$$y^{(\alpha,n)}_{m-1} = (\lambda^{(\alpha,n)}_{m,q} I - T_{n,q,\alpha})^{-1}(cx + d) + (\lambda^{(\alpha,n)}_{m,q} I - T_{n,q,\alpha})^{-1}(\omega_n)$$

Since $\|(\lambda^{(\alpha,n)}_{m,q} I - T_{n,q,\alpha})^{-1}(\omega_n)\| \leq M\|(\omega_n)\| \to 0$ as $n \to \infty$, and $\lim_{n\to\infty} (\lambda^{(\alpha,n)}_{m,q} I - T_{n,q,\alpha})^{-1} = A_{\infty,q}$, we conclude that

$$\lim_{n\to\infty} y^{(\alpha,n)}_{m-1} = A^{-1}_{\infty,q}(cx + d) =: ax + b$$

Thus,

$$\lim_{n\to\infty} T^{j_n}_{n,q,\alpha}(t^m;x) = ax + b.$$

The induction is completed and it follows that for any polynomial p,

$$\lim_{n\to\infty} T^{j_n}_{n,q,\alpha}(p;x) = L_p(x) \quad \text{for} \quad x \in [0,1].$$

2) Let $f \in C[0,1]$, and let $\varepsilon > 0$ be given. Then $f(x) = p(x) + \delta(x)$, where p is a polynomial and $\|\delta(x)\| < \varepsilon$. We have

$$T^{j_n}_{n,q,\alpha}(f;x) = T^{j_n}_{n,q,\alpha}(p;x) + T^{j_n}_{n,q,\alpha}(\delta;x)$$

Since $\lim_{n\to\infty} T^{j_n}_{n,q,\alpha}(p;x) = L_p(x)$, there exists $n_0 \in \mathbb{N}$ such that $\|T^{j_n}_{n,q,\alpha}(p;x) - L_p(x)\| < \varepsilon$ for all $n > n_0$. Obviously, $\|L_\delta(x)\| \leq \|\delta\| < \varepsilon$, and finally, as $L_f = L_p + L_\delta$, we obtain

$$\begin{aligned} \|T^{j_n}_{n,q,\alpha}(f;x) - L_f(x)\| &\leq \|T^{j_n}_{n,q,\alpha}(p;x) - L_p(x)\| + \|T^{j_n}_{n,q,\alpha}(\delta;x) - L_\delta(x)\| \\ &\leq \|T^{j_n}_{n,q,\alpha}(p;x) - L_p(x)\| + \|T^{j_n}_{n,q,\alpha}(\delta;x)\| + \|\delta\| < 3\varepsilon \quad \text{for all} \quad n > n_0. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} T^{j_n}_{n,q,\alpha}(f;x) = L_f(x)$$

uniformly on $[0,1]$. □

In the case $q > 1$, the situation is different. For this purpose, we shall need the following two lemmas.

Lemma 3.4. *Let $\lambda_{k,q}^{(\alpha,n)}$ be as in Lemma 2.2. Then, for $q > 1$,*

$$\lim_{n\to\infty}[n]_q \ln(\lambda_{k,q}^{(\alpha,n)}) = -(1-\alpha)[k]_q(1-q^{1-k}) - \sum_{m=1}^{k-1}[m]_q$$

where an empty sum is taken to be 0.

Proof. As it is shown in [10], the eigenvalues $\lambda_{k,q}^{(\alpha,n)}$ can be expressed as

$$\lambda_{k,q}^{(\alpha,n)} = \left(\alpha + (1-\alpha)\frac{[n-k]_q[n+k-1]_q}{[n]_q[n-1]_q}\right)\prod_{m=1}^{k-1}\left(1-\frac{[m]_q}{[n]_q}\right)$$

Thus,

$$\begin{aligned}\ln(\lambda_{k,q}^{(\alpha,n)}) &= \ln\left(\alpha + (1-\alpha)\frac{[n-k]_q[n+k-1]_q}{[n]_q[n-1]_q}\right) + \sum_{m=1}^{k-1}\ln\left(1-\frac{[m]_q}{[n]_q}\right)\\ &= \ln\left(1-(1-\alpha)\frac{q^{n-k}[k]_q[k-1]_q}{[n]_q[n-1]_q}\right) + \sum_{m=1}^{k-1}\ln\left(1-\frac{[m]_q}{[n]_q}\right).\end{aligned}$$

As $\ln(1+u) = u + O(u^2)$ as $u \to 0$, we get

$$[n]_q \ln(\lambda_{k,q}^{(\alpha,n)}) = -(1-\alpha)\frac{q^{n-k}[k]_q[k-1]_q}{[n-1]_q} - \sum_{m=1}^{k-1}[m]_q + O(1/[n]_q).$$

Taking the limit on both sides as $n \to \infty$, yields

$$\begin{aligned}\lim_{n\to\infty}[n]_q \ln(\lambda_{k,q}^{(\alpha,n)}) &= -(1-\alpha)q^{1-k}(q-1)[k]_q[k-1]_q - \sum_{m=1}^{k-1}[m]_q\\ &= -(1-\alpha)(1-q^{1-k})[k]_q - \sum_{m=1}^{k-1}[m]_q\end{aligned}$$

as desired. □

Lemma 3.5. *Let $q > 1$ and $\{j_n\}$ be a sequence of positive integers such that $\lim_{n\to\infty} j_n/[n]_q = t$. Then,*

$$\lim_{n\to\infty}(\lambda_{k,q}^{(\alpha,n)})^{j_n} = \exp\left\{-\left((1-\alpha)[k]_q(1-q^{1-k}) + \sum_{m=1}^{k-1}[m]_q\right)t\right\}, \quad k = 0,1,\ldots, \tag{9}$$

when $0 \le t < \infty$, and

$$\lim_{n\to\infty}(\lambda_{k,q}^{(\alpha,n)})^{j_n} = 0, \quad k = 2,3,\ldots \tag{10}$$

when $t = \infty$.

Proof. *Let $y = (\lambda_{k,q}^{(\alpha,n)})^{j_n - [n]_q t}$ and $z = (\lambda_{k,q}^{(\alpha,n)})^{[n]_q t}$. Then,*

$$\ln y = (j_n - [n]_q t)\ln(\lambda_{k,q}^{(\alpha,n)}) = \left(\frac{j_n}{[n]_q} - t\right)[n]_q \ln(\lambda_{k,q}^{(\alpha,n)}).$$

As $\lim_{n\to\infty}[n]_q \ln(\lambda_{k,q}^{(\alpha,n)})$ *is finite by Lemma 3.4, we get* $\lim_{n\to\infty}\ln y = 0$*, implying*

$$\lim_{n\to\infty}(\lambda_{k,q}^{(\alpha,n)})^{j_n-[n]_q t} = 1 \tag{11}$$

Also, since $\ln z = t[n]_q \ln(\lambda_{k,q}^{(\alpha,n)})$*, using Lemma 3.4, we get*

$$\lim_{n\to\infty}(\lambda_{k,q}^{(\alpha,n)})^{[n]_q t} = \exp\left\{-\left((1-\alpha)[k]_q(1-q^{1-k}) + \sum_{m=1}^{k-1}[m]_q\right)t\right\}. \tag{12}$$

Using (11) *and* (12)*, we derive* (9)*. To obtain* (10)*, let* $t \to \infty$*.* □

Using Lemmas 3.4 and 3.5, one can state the following theorem, whose proof is omitted here since it is similar to the case $q = 1$ considered in [5].

Theorem 3.6. *Let* $q \in (1,\infty)$ *and* j_n *be a sequence of positive integers such that* $\lim_{n\to\infty} j_n/[n]_q = t$*. Then, for any polynomial* p *and any* $0 \le t \le \infty$*, the sequence* $\{T_{n,q,\alpha}^{j_n}(p,x)\}$ *is uniformly convergent on* $[0,1]$*. Specifically, for* $t = 0$*,*

$$\lim_{n\to\infty} T_{n,q,\alpha}^{j_n}(p;x) = p(x) \qquad on \quad x \in [0,1]$$

and for $t = \infty$*,*

$$\lim_{n\to\infty} T_{n,q,\alpha}^{j_n}(p;x) = L_p(x) \qquad on \quad x \in [0,1].$$

References

[1] U.Abel, M.Ivan, *Over-Iterates of Bernstein's Operators:A.Short and Elementary Proof*, American Mathematical Montly (6)**116**, 535–538.
[2] M.M. Almesbahi, *On properties of q-Bernstein polynomials. Master's thesis*, Ankara, Turkey, Department of Mathematics, Atilim University. 2017.
[3] G.E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.
[4] Qing-Bo Cai, Xiao-Wei Xu. *Shape-preserving properties of a new family of generalized Bernstein operators*, Journal of Inequalities and Applications **241** (2018)
[5] S. Cooper and S. Waldron, *The eigenstructure of the Bernstein operator*, J. Approx. Theory (1) **105** (2000), 133-165.
[6] S. Cooper and S. Waldron, *The diagonalisation of the multivariate Bernstein operator*, J. Approx. Theory (1) **117** (2002), 103-131.
[7] H. H. Gonska and Xin Long Zhou, *Approximation theorems for the iterated Boolean sums of Bernstein operators*, J. Comput. Appl. Math.**53** (1994), 21-31.
[8] S. Karlin and Z. Ziegler, *Iteration of positive approximation operators*, J. Approx.Theory **3** (1970), 310–339.
[9] R.P. Kelisky and T.J. Rivlin, *Iterates of Bernstein polynomials*, Pacific J. Math.**21** (1967), 511-520.
[10] B. Köroğlu, F. T. Yeşildal, *On the eigenstructure of the* (α, q)*-Bernstein operator*, Hacet. J. Math. Stat. (4)**117** (2021), 1111–1122.
[11] C. Micchelli, *The saturation class and iterates of the Bernstein polynomials*, J. Approx. Theory **8** (1973), 1-18.
[12] H. Oruc and N. Tuncer, *On the Convergence and Iterates of q-Bernstein Polynomials*, J. Approx. Theory (2)**117** (2002), 301-313.
[13] S. Ostrovska, *q-Bernstein polynomials and their iterates*, J. Approx. Theory (2)**123** (2003), 232-255.
[14] V. Radu, *Note on the iterates of q- and* (p,q)*-Bernstein operators*, Sci. Stud. Res. Ser. Math. Inform. (2)**26** (2016), 83–93.
[15] I.A. Rus, *Iterates of Bernstein operators,via contraction principle*, J. Math. Anal.Appl. **292** (2004), 259–261.
[16] M. R. da Silva, *Nonnegative order iterates of Bernstein polynomials and their limiting semigroup*, Portugal. Math. **42** (1985) 225-248.
[17] X. Xueyan, He.Qian and W. Yang, *Convergence rate for iterates of q-Bernstein polynomials*, Anal. Theory Appl.(3)**23** (2007) 243–254.