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Upper and lower estimations of Popoviciu's difference via weighted Hadamard inequality with applications

S. I. Butt[a,*], T. Rasheed[a], Đ. Pečarić[b], J. Pečarić[c]

[a]*COMSATS University Islamabad, Lahore Campus Pakistan*
[b]*Department of Media and Communication, University North, 48000 Koprivnica, Croatia*
[c]*Croatian Academy of Sciences and Arts, 10000 Zagreb, Croatia*

Abstract. We consider differences coming from Popoviciu's inequality and give upper and lower bounds by employing weighted Hermite-Hadamard inequality along with the approximations of Montgomery two point formula. We also give bounds for Popoviciu's inequality by employing weighted Hermite-Hadamard inequality along with the approximations of Montgomery one point formula. We testify this scenario by utilizing the theory of n-times differentiable convex functions. Our results hold for all $n \geq 2$ and we provide explicit examples to show the correctness of the bounds obtained for special cases. Last but not least, we provide applications in information theory by providing new uniform estimations of the generalized Csiszár divergence, Rényi-divergence, Shannon-entropy, Kullback-Leibler divergence, Zipf and Hybrid Zipf-Mandelbrot entropies.

1. Introduction

Convex functions (see [1, 2]) show a vital role in several areas of mathematics. There is a nice connection between mathematical inequalities and the theory of convex functions. Like Jensen's inequality, Popoviciu's inequality also characterizes convex functions. Nowadays, Popoviciu's inequality is also a topic of interest for many mathematicians since last fifty years.
In (1965), Tiberiu Popoviciu (see [3]) gave the following characterization of convex function.

Theorem 1.1. *Let* $\mathfrak{f} : [\varrho_1, \varrho_2] \to \mathbb{R}$ *is convex. Then, for each* $z_1, z_2, z_3 \in [\varrho_1, \varrho_2]$ *and all* $q_1, q_2, q_3 > 0$, *the following result holds*

$$\begin{aligned}&(q_1 + q_2 + q_3)\mathfrak{f}\left(\frac{q_1 z_1 + q_2 z_2 + q_3 z_3}{q_1 + q_2 + q_3}\right)\\ &- (q_2 + q_3)\mathfrak{f}\left(\frac{q_2 z_2 + q_3 z_3}{q_2 + q_3}\right) - (q_3 + q_1)\mathfrak{f}\left(\frac{q_3 z_3 + q_1 z_1}{q_3 + q_1}\right) - (q_1 + q_2)\mathfrak{f}\left(\frac{q_1 z_1 + q_2 z_2}{q_1 + q_2}\right)\\ &\qquad + q_1\mathfrak{f}(z_1) + q_2\mathfrak{f}(z_2) + q_3\mathfrak{f}(z_3) \geq 0.\end{aligned}$$

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* Corresponding author: S. I. Butt
Email addresses: saadihsanbutt@gmail.com (S. I. Butt), tahirtishna24@gmail.com (T. Rasheed), djilda.pecaric@unin.hr (Đ. Pečarić), pecaric@element.hr (J. Pečarić)

The ability to maximise and/or minimise a particular function (or specific real numbers) without requiring derivatives is one of Popoviciu inequality's most useful applications, making this kind of inequality a crucial tool in approximations and optimizations. Another important application is the generalization of several well-known classical inequalities. For instance, Popoviciu's inequality is a beautiful generalization of Hlawka's inequality [4] that makes use of the geometry concept of convexity.

Without the aid of Hlawka's inequality, Popoviciu's inequality could not have been extended to many variables since the authors were motivated to create a higher dimensional [5] counterpart of Popoviciu's inequality based on his characterization. Popoviciu's inequality has several intriguing generalizations and analogues with some ramifications, which may be found in [6].

Nowadays, a convex function axiom that T. Popoviciu proved in [3] is the subject of extensive research (see [1] and references with in). Recent developments in Popoviciu's inequality are introduced by M. V. Mihai in 2016 (see [7]). Popoviciu's inequality for functions of several variables was provided by M. Bencze et al. in [8] in 2010. The integral form of Popoviciu's inequality was presented by C. P. Niculescu in 2009 (see [9]). Popoviciu's inequality in [10] was further developed by C. P. Niculescu in 2006. Recently, Butt et al. provide a few recent updates and enhancements to Popoviciu's inequality in [11–14].

The following is Vasić and Stanković shape of Popoviciu's inequality given in [15] (see also [1] page 173).

Theorem 1.2. *Consider* $\mathfrak{m}, \mathfrak{r} \in \mathbb{N}$ *where* $\mathfrak{m} \geq 3$ *and* $2 \leq \mathfrak{r} \leq \mathfrak{m}-1$ *also* $[\varrho_1, \varrho_2] \subset \mathbb{R}$, $\mathfrak{z} = (\mathfrak{z}_1, ..., \mathfrak{z}_\mathfrak{m}) \in [\varrho_1, \varrho_2]^\mathfrak{m}$, $\mathfrak{q} = (\mathfrak{q}_1, ..., \mathfrak{q}_\mathfrak{m})$ *be positive* $\mathfrak{m}$*-tuple in such a way that* $\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i = 1$. *If* $\mathfrak{f} : [\varrho_1, \varrho_2] \to \mathbb{R}$ *is convex function, then*

$$\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}} \sum_{1 \leq i_1 < ... < i_\mathfrak{r} \leq \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j} \right) \mathfrak{f} \left(\frac{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j} \mathfrak{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}} \right) \leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} \sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \mathfrak{f}(\mathfrak{z}_i) + \frac{\mathfrak{r}-1}{\mathfrak{m}-1} \mathfrak{f} \left(\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \mathfrak{z}_i \right),$$

or

$$P_\mathfrak{r}^\mathfrak{m}(\mathfrak{z}, \mathfrak{q}; \mathfrak{f}) \leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} P_1^\mathfrak{m}(\mathfrak{z}, \mathfrak{q}; \mathfrak{f}) + \frac{\mathfrak{r}-1}{\mathfrak{m}-1} P_\mathfrak{m}^\mathfrak{m}(\mathfrak{z}, \mathfrak{q}; \mathfrak{f}), \tag{1}$$

where

$$P_\mathfrak{r}^\mathfrak{m}(\mathfrak{z}, \mathfrak{q}; \mathfrak{f}) = P_\mathfrak{r}^\mathfrak{m}(\mathfrak{z}, \mathfrak{q}; \mathfrak{f}(\mathfrak{z})) := \frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}} \sum_{1 \leq i_1 < ... < i_\mathfrak{r} \leq \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j} \right) \mathfrak{f} \left(\frac{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j} \mathfrak{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}} \right),$$

is the linear w.r.t. $\mathfrak{f}$.

From inequality (1) we can write

$$\mathbf{F}(\mathfrak{z}, \mathfrak{q}; \mathfrak{f}) := \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} P_1^\mathfrak{m}(\mathfrak{z}, \mathfrak{q}; \mathfrak{f}) + \frac{\mathfrak{r}-1}{\mathfrak{m}-1} P_\mathfrak{m}^\mathfrak{m}(\mathfrak{z}, \mathfrak{q}; \mathfrak{f}) - P_\mathfrak{r}^\mathfrak{m}(\mathfrak{z}, \mathfrak{q}; \mathfrak{f}). \tag{2}$$

The following is the generalized Montgomery identity:

Theorem 1.3. *Let* $n \in \mathbb{N}$, $\mathfrak{f} : I \to \mathbb{R}$ *be such that* $\mathfrak{f}^{(n-1)}$ *is absolutely continuous,* $I \subset \mathbb{R}$ *an open interval,* $\varrho_1, \varrho_2 \in I$, $\varrho_1 < \varrho_2$. *Then, the following identity holds*

$$\mathfrak{f}(\mathfrak{z}) = \frac{1}{\varrho_2 - \varrho_1} \int_{\varrho_1}^{\varrho_2} \mathfrak{f}(t)\, dt + \sum_{w=0}^{n-2} \frac{\mathfrak{f}^{(w+1)}(\varrho_1)}{w!\,(w+2)} \frac{(\mathfrak{z}-\varrho_1)^{w+2}}{\varrho_2 - \varrho_1} - \sum_{w=0}^{n-2} \frac{\mathfrak{f}^{(w+1)}(\varrho_2)}{w!\,(w+2)} \frac{(\mathfrak{z}-\varrho_2)^{w+2}}{\varrho_2 - \varrho_1}$$

$$+\frac{1}{(n-1)!}\int_{\varrho_1}^{\varrho_2} R_n(\mathrm{z},r)\,\mathfrak{f}^{(n)}(r)\,dr, \quad (3)$$

where

$$R_n(\mathrm{z},r)=\begin{cases} -\frac{(\mathrm{z}-r)^n}{n(\varrho_2-\varrho_1)}+\frac{\mathrm{z}-\varrho_1}{\varrho_2-\varrho_1}(\mathrm{z}-r)^{n-1}, & \varrho_1\le r\le \mathrm{z},\\ -\frac{(\mathrm{z}-r)^n}{n(\varrho_2-\varrho_1)}+\frac{\mathrm{z}-\varrho_2}{\varrho_2-\varrho_1}(\mathrm{z}-r)^{n-1}, & \mathrm{z}<r\le \varrho_2. \end{cases} \quad (4)$$

Butt et al. gave in [11] the following generalized identity for Popoviciu inequality under the effect of Montgomery identity:

Theorem 1.4. *Let all the assumptions of Theorem 1.3 be valid and let* $\mathfrak{m},\mathfrak{r}\in\mathbb{N}$, $\mathfrak{m}\ge 3$, $2\le\mathfrak{r}\le\mathfrak{m}-1$, $[\varrho_1,\varrho_2]\subset\mathbb{R}$, $\mathrm{z}=(\mathrm{z}_1,\mathrm{z}_2,\dots,\mathrm{z}_\mathfrak{m})\in[\varrho_1,\varrho_2]^\mathfrak{m}$ $\mathfrak{q}=(\mathfrak{q}_1,\mathfrak{q}_2,\dots,\mathfrak{q}_\mathfrak{m})$ *be a real* $\mathfrak{m}$*-tuple such that* $\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\neq 0$ *for any* $1\le i_1<\dots<i_\mathfrak{r}\le\mathfrak{m}$ *and* $\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i=1$. *Also let* $\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}\in[\varrho_1,\varrho_2]$ *with* $R_n(\mathrm{z},r)$ *be the same as defined in* (4) *then, we have the following identity:*

$$\mathbf{F}\Big(\mathrm{z},\mathfrak{q};\mathfrak{f}(\mathrm{z})\Big)=\frac{1}{(\varrho_2-\varrho_1)}\sum_{w=0}^{n-2}\left(\frac{1}{w!(w+2)}\right)\times$$
$$\Big\{\mathfrak{f}^{(w+1)}(\varrho_1)\,\mathbf{F}\Big(\mathrm{z},\mathfrak{q};(\mathrm{z}-\varrho_1)^{w+2}\Big)-\mathfrak{f}^{(w+1)}(\varrho_2)\,\mathbf{F}\Big(\mathrm{z},\mathfrak{q};(\mathrm{z}-\varrho_2)^{w+2}\Big)\Big\}$$
$$+\frac{1}{(n-1)!}\int_{\varrho_1}^{\varrho_2}\mathbf{F}\Big(\mathrm{z},\mathfrak{q};R_n(\mathrm{z},r)\Big)\mathfrak{f}^{(n)}(r)\,dr. \quad (5)$$

The following is another version of Montgomery identity which gives approximations at single point:

Theorem 1.5. *Let* $n\in\mathbb{N}$, $\mathfrak{f}:[\varrho_1,\varrho_2]\to\mathbb{R}$ *be such that* $\mathfrak{f}^{(n-1)}$ *is absolutely continuous,* $I\subset\mathbb{R}$ *an open interval,* $\varrho_1,\varrho_2\in I$, $\varrho_1<\varrho_2$. *Then the following identity holds*

$$\mathfrak{f}(\mathrm{z})=\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\mathfrak{f}(t)\,dt+\sum_{w=0}^{n-2}\mathfrak{f}^{(w+1)}(\mathrm{z})\,\frac{(\varrho_1-\mathrm{z})^{w+2}-(\varrho_2-\mathrm{z})^{w+2}}{(w+2)!(\varrho_2-\varrho_1)}$$
$$+\frac{1}{(n-1)!}\int_{\varrho_1}^{\varrho_2}\widehat{R}_n(\mathrm{z},r)\,\mathfrak{f}^{(n)}(r)\,dr, \quad (6)$$

where

$$\widehat{R}_n(\mathrm{z},r)=\begin{cases} \frac{-1}{n(\varrho_2-\varrho_1)}(\varrho_1-r)^n, & \varrho_1\le r\le \mathrm{z},\\ \frac{-1}{n(\varrho_2-\varrho_1)}(\varrho_2-r)^n, & \mathrm{z}<r\le\varrho_2. \end{cases} \quad (7)$$

Butt et al. in [11] formulate the following identity with help of generalized Montgomery identity (6).

Theorem 1.6. *Let all the assumptions of Theorem 1.5 and Theorem 1.4 are valid with* $\widehat{R}_n(\mathrm{z},r)$ *be the same as defined in* (7). *Then, we have the following identity:*

$$\mathbf{F}\Big(\mathrm{z},\mathfrak{q};\mathfrak{f}(\mathrm{z})\Big)-\frac{1}{(\varrho_2-\varrho_1)}\sum_{w=0}^{n-2}\left(\frac{1}{(w+2)!}\right)\times$$

$$\left\{\mathbf{F}\Big(z,\mathfrak{q};\mathfrak{f}^{(w+1)}(z)(\varrho_1-z)^{w+2}\Big)-\mathbf{F}\Big(z,\mathfrak{q};\mathfrak{f}^{(w+1)}(z)(\varrho_2-z)^{w+2}\Big)\right\}$$
$$=\frac{1}{(n-1)!}\int_{\varrho_1}^{\varrho_2}\mathbf{F}\Big(z,\mathfrak{q};\widehat{R}_n(z,r)\Big)\mathfrak{f}^{(n)}(r)\,dr. \quad (8)$$

Throughout the paper we will utilize the following weighted Hermite-Hadamard inequality [16] given as:

Theorem 1.7. *If $\mathfrak{f}:[\varrho_1,\varrho_2]\to\mathbb{R}$ is convex and $p:[\varrho_1,\varrho_2]\to\mathbb{R}$ is of a constant sign on $[\varrho_1,\varrho_2]$, then we have*

$$\mathfrak{f}(\lambda)\le\frac{1}{P(\varrho_2)}\int_{\varrho_1}^{\varrho_2}p(t)\mathfrak{f}(t)dt\le\frac{\varrho_2-\lambda}{\varrho_2-\varrho_1}\mathfrak{f}(\varrho_1)+\frac{\lambda-\varrho_1}{\varrho_2-\varrho_1}\mathfrak{f}(\varrho_2),$$

where

$$P=P(\varrho_2)=\int_{\varrho_1}^{\varrho_2}p(t)dt, \quad (9)$$

and

$$\lambda=\frac{1}{P(\varrho_2)}\int_{\varrho_1}^{\varrho_2}tp(t)dt. \quad (10)$$

We can write above inequality as

$$P\mathfrak{f}(\lambda)\le\int_{\varrho_1}^{\varrho_2}p(t)\mathfrak{f}(t)dt\le P\left(\frac{\varrho_2-\lambda}{\varrho_2-\varrho_1}\mathfrak{f}(\varrho_1)+\frac{\lambda-\varrho_1}{\varrho_2-\varrho_1}\mathfrak{f}(\varrho_2)\right). \quad (11)$$

In this research article, we will formulate upper and lower bounds for Popoviciu's difference by utilizing Montgomery identity along with weighted Hermite Hadamard inequality. We will also give results for a particular case when $n=2$, then for obtained results we will give applications in information theory.

2. Main results

Theorem 2.1. *Under the assumptions of Theorem 1.4, let for all $n\ge2$, $\mathfrak{f}^{(n)}(\cdot)$ (n-th derivative of $\mathfrak{f}$) is convex and*

$$\mathbf{F}\Big(z,\mathfrak{q};R_n(z,r)\Big)>0,\qquad r\in[\varrho_1,\varrho_2] \quad (12)$$

then the following inequalities are valid

$$\frac{P_1(n)}{(n-1)!}\mathfrak{f}^{(n)}(\lambda_1(n))\le\mathbf{F}\Big(z,\mathfrak{q};\mathfrak{f}(z)\Big)-\frac{1}{(\varrho_2-\varrho_1)}\sum_{w=0}^{n-2}\left(\frac{1}{w!(w+2)}\right)\times$$
$$\left\{\mathfrak{f}^{(w+1)}(\varrho_1)\mathbf{F}\Big(z,\mathfrak{q};(z-\varrho_1)^{w+2}\Big)-\mathfrak{f}^{(w+1)}(\varrho_2)\mathbf{F}\Big(z,\mathfrak{q};(z-\varrho_2)^{w+2}\Big)\right\}$$
$$\le\frac{P_1(n)}{(n-1)!}\left(\frac{\varrho_2-\lambda_1(n)}{\varrho_2-\varrho_1}\mathfrak{f}^{(n)}(\varrho_1)+\frac{\lambda_1(n)-\varrho_1}{\varrho_2-\varrho_1}\mathfrak{f}^{(n)}(\varrho_2)\right) \quad (13)$$

where

$$P_1(n)=\int_{\varrho_1}^{\varrho_2}\mathbf{F}\Big(z,\mathfrak{q};R_n(z,r)\Big)dr=\mathbf{F}\left(z,\mathfrak{q};\left(\frac{(z-\varrho_1)^{n+1}-(z-\varrho_2)^{n+1}}{(n+1)(\varrho_2-\varrho_1)}\right)\right),$$

or

$$P_1(n) = (n-1)!\Big[\mathbf{F}\Big(z, \mathfrak{q}; \frac{z^n}{n!}\Big) - \frac{1}{n!(\varrho_2-\varrho_1)}\sum_{w=0}^{n-2}\Big(\frac{n}{(w+2)}\Big)\binom{n-1}{w}\times \Big\{\varrho_1^{(n-w-1)}\mathbf{F}\Big(z, \mathfrak{q}; (z-\varrho_1)^{w+2}\Big) - \varrho_2^{(n-w-1)}\mathbf{F}\Big(z, \mathfrak{q}; (z-\varrho_2)^{w+2}\Big)\Big\}\Big], \quad (14)$$

and

$$\begin{aligned}\lambda_1(n) &= \frac{\int_{\varrho_1}^{\varrho_2} r\mathbf{F}\Big(z, \mathfrak{q}; R_n(z,r)\Big)dr}{\int_{\varrho_1}^{\varrho_2} \mathbf{F}\Big(z, \mathfrak{q}; R_n(z,r)\Big)dr} \\ &= \mathbf{F}\Bigg(z, \mathfrak{q}; \Bigg(\frac{\varrho_1(z-\varrho_1)^{n+1} - \varrho_2(z-\varrho_2)^{n+1}}{(n+1)(\varrho_2-\varrho_1)P_1(n)} + \frac{(z-\varrho_1)^{n+2} - (z-\varrho_2)^{n+2}}{n(n+2)(\varrho_2-\varrho_1)P_1(n)}\Bigg)\Bigg),\end{aligned}$$

or

$$\lambda_1(n) = \frac{(n-1)!}{P_1(n)}\Big[\mathbf{F}\Big(z, \mathfrak{q}; \frac{z^{n+1}}{(n+1)!}\Big) - \frac{1}{(n+1)!(\varrho_2-\varrho_1)}\sum_{w=0}^{n-2}\Big(\frac{n(n+1)}{w(w+2)}\Big)\times \binom{n-1}{w-1}\Big\{\varrho_1^{(n-w)}\mathbf{F}\Big(z, \mathfrak{q}; (z-\varrho_1)^{w+2}\Big) - \varrho_2^{(n-w)}\mathbf{F}\Big(z, \mathfrak{q}; (z-\varrho_2)^{w+2}\Big)\Big\}\Big].$$

Proof. As $\mathfrak{f}^{(n)}$ is assumed to be convex for all $n \geq 2$, thus by applying weighted Hermite Hadamard inequality (11) for the convex function $\mathfrak{f}^{(n)}$ with positive weight $p(r) = \mathbf{F}\Big(z, \mathfrak{q}; R_n(z,r)\Big)$, we get

$$\begin{aligned}&\frac{P_1(n)}{(n-1)!}\mathfrak{f}^{(n)}(\lambda_1(n)) \leq \frac{1}{(n-1)!}\int_{\varrho_1}^{\varrho_2}\mathbf{F}\Big(z, \mathfrak{q}; R_n(z,r)\Big)\mathfrak{f}^{(n)}(r)\,dr \\ &\leq \frac{P_1(n)}{(n-1)!}\Bigg(\frac{\varrho_2-\lambda_1(n)}{\varrho_2-\varrho_1}\mathfrak{f}^{(n)}(\varrho_1) + \frac{\lambda_1(n)-\varrho_1}{\varrho_2-\varrho_1}\mathfrak{f}^{(n)}(\varrho_2)\Bigg). \quad (15)\end{aligned}$$

Thus, by substituting the value of

$$\frac{1}{(n-1)!}\int_{\varrho_1}^{\varrho_2}\mathbf{F}\Big(z, \mathfrak{q}; R_n(z,r)\Big)\mathfrak{f}^{(n)}(r)\,dr,$$

from (5) in (15), we get required result.
Now, we find the value of $P_1(n)$ by utilizing the kernel function (4) defined in the remainder term of Montgomery identity. By replacing z with z_i in (4) we get,

$$R_n(z_\iota, r) = \begin{cases} -\frac{(z_i-r)^n}{n(\varrho_2-\varrho_1)} + \frac{z_i-\varrho_1}{\varrho_2-\varrho_1}(z_i-r)^{n-1}, & \varrho_1 \leq r \leq z_i, \\ -\frac{(z_i-r)^n}{n(\varrho_2-\varrho_1)} + \frac{z_i-\varrho_2}{\varrho_2-\varrho_1}(z_i-r)^{n-1}, & z_i < r \leq \varrho_2. \end{cases}$$

Now we calculate $\int_{\varrho_1}^{\varrho_2}\mathbf{F}\Big(z, \mathfrak{q}; R_n(z,r)\Big)dr$ as by comparing (9), we have

$$\begin{aligned} P_1(n) &= \int_{\varrho_1}^{\varrho_2} \mathbf{F}\Big(\mathrm{z}, \mathrm{q}; R_n(\mathrm{z}, r)\Big) dr \\ &= \int_{\varrho_1}^{\mathrm{z}} \mathbf{F}\Big(\mathrm{z}, \mathrm{q}; -\frac{(\mathrm{z}-r)^n}{n(\varrho_2-\varrho_1)} + \frac{\mathrm{z}-\varrho_1}{\varrho_2-\varrho_1}(\mathrm{z}-r)^{n-1}\Big) dr \\ &+ \int_{\mathrm{z}}^{\varrho_2} \mathbf{F}\Big(\mathrm{z}, \mathrm{q}; -\frac{(\mathrm{z}-r)^n}{n(\varrho_2-\varrho_1)} + \frac{\mathrm{z}-\varrho_2}{\varrho_2-\varrho_1}(\mathrm{z}-r)^{n-1}\Big) dr \\ &= I_1 + I_2 - I_3. \end{aligned} \tag{16}$$

It is pertinent to mention that the difference in last equality is obtained by using the explicit form of our functional as given in (2), where

$$I_1 = \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}} \mathrm{q}_i\Big[\int_{\varrho_1}^{\mathrm{z}_i}\Big(-\frac{(\mathrm{z}_i-r)^n}{n(\varrho_2-\varrho_1)} + \frac{\mathrm{z}_i-\varrho_1}{\varrho_2-\varrho_1}(\mathrm{z}_i-r)^{n-1}\Big)dr$$
$$+\int_{\mathrm{z}_i}^{\varrho_2}\Big(-\frac{(\mathrm{z}_i-r)^n}{n(\varrho_2-\varrho_1)} + \frac{\mathrm{z}_i-\varrho_2}{\varrho_2-\varrho_1}(\mathrm{z}_i-r)^{n-1}\Big)dr\Big],$$

$$I_2 = \frac{\mathfrak{r}-1}{\mathfrak{m}-1}\Big[\int_{\varrho_1}^{\sum_{i=1}^{\mathfrak{m}}\mathrm{q}_i\mathrm{z}_i}\left(-\frac{\left(\sum_{i=1}^{\mathfrak{m}}\mathrm{q}_i\mathrm{z}_i - r\right)^n}{n(\varrho_2-\varrho_1)} + \frac{\sum_{i=1}^{\mathfrak{m}}\mathrm{q}_i\mathrm{z}_i-\varrho_1}{\varrho_2-\varrho_1}\left(\sum_{i=1}^{\mathfrak{m}}\mathrm{q}_i\mathrm{z}_i - r\right)^{n-1}\right)dr$$
$$+\int_{\sum_{i=1}^{\mathfrak{m}}\mathrm{q}_i\mathrm{z}_i}^{\varrho_2}\left(-\frac{\left(\sum_{i=1}^{\mathfrak{m}}\mathrm{q}_i\mathrm{z}_i - r\right)^n}{n(\varrho_2-\varrho_1)} + \frac{\sum_{i=1}^{\mathfrak{m}}\mathrm{q}_i\mathrm{z}_i-\varrho_2}{\varrho_2-\varrho_1}\left(\sum_{i=1}^{\mathfrak{m}}\mathrm{q}_i\mathrm{z}_i - r\right)^{n-1}\right)dr\Big],$$

and

$$I_3 = \frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\le i_1<\ldots<i_{\mathfrak{r}}\le\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}\right)\times$$
$$\Big[\int_{\varrho_1}^{\frac{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}}}\left(-\frac{\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}} - r\right)^n}{n(\varrho_2-\varrho_1)} + \frac{\frac{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}}-\varrho_1}{\varrho_2-\varrho_1}\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}} - r\right)^{n-1}\right)dr$$
$$+\int_{\frac{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}}}^{\varrho_2}\left(-\frac{\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}} - r\right)^n}{n(\varrho_2-\varrho_1)} + \frac{\frac{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}}-\varrho_2}{\varrho_2-\varrho_1}\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathrm{q}_{i_j}} - r\right)^{n-1}\right)dr\Big].$$

First we calculate I_1 as,

$$I_1 = \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}}\mathrm{q}_i\Big[\int_{\varrho_1}^{\mathrm{z}_i}\Big(-\frac{(\mathrm{z}_i-r)^n}{n(\varrho_2-\varrho_1)} + \frac{(\mathrm{z}_i-\varrho_1)}{\varrho_2-\varrho_1}(\mathrm{z}_i-r)^{n-1}\Big)dr$$

$$+ \int_{z_i}^{\varrho_2}\left(-\frac{(z_i-r)^n}{n(\varrho_2-\varrho_1)}+\frac{z_i-\varrho_2}{\varrho_2-\varrho_1}(z_i-r)^{n-1}\right)dr\Bigg]$$
$$= \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\left[\frac{(z_i-\varrho_1)^{n+1}-(z_i-\varrho_2)^{n+1}}{(n+1)(\varrho_2-\varrho_1)}\right].$$

Similarly, we can evaluate

$$I_2=\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\left[\frac{\left(\sum\limits_{i=1}^{\mathfrak{m}}\mathfrak{q}_iz_i-\varrho_1\right)^{n+1}-\left(\sum\limits_{i=1}^{\mathfrak{m}}\mathfrak{q}_iz_i-\varrho_2\right)^{n+1}}{(n+1)(\varrho_2-\varrho_1)}\right],$$

and

$$I_3=\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\le i_1<\ldots<i_\mathfrak{r}\le\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\left[\frac{\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}z_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}-\varrho_1\right)^{n+1}-\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}z_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}-\varrho_2\right)^{n+1}}{(n+1)(\varrho_2-\varrho_1)}\right].$$

By taking difference $I_1+I_2-I_3$ and rewriting it in compact functional form, we get

$$I_1+I_2-I_3=\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\left[\frac{(z_i-\varrho_1)^{n+1}-(z_i-\varrho_2)^{n+1}}{(n+1)(\varrho_2-\varrho_1)}\right]$$
$$+\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\left[\frac{\left(\sum\limits_{i=1}^{\mathfrak{m}}\mathfrak{q}_iz_i-\varrho_1\right)^{n+1}-\left(\sum\limits_{i=1}^{\mathfrak{m}}\mathfrak{q}_iz_i-\varrho_2\right)^{n+1}}{(n+1)(\varrho_2-\varrho_1)}\right]$$
$$-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\le i_1<\ldots<i_\mathfrak{r}\le\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\left[\frac{\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}z_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}-\varrho_1\right)^{n+1}-\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}z_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}-\varrho_2\right)^{n+1}}{(n+1)(\varrho_2-\varrho_1)}\right]$$
$$=\mathbf{F}\left(z,\mathfrak{q};\left(\frac{(z-\varrho_1)^{n+1}-(z-\varrho_2)^{n+1}}{(n+1)(\varrho_2-\varrho_1)}\right)\right).$$

Method 2 **for computing** $P_1(n)$
If we choose $f(z)=\frac{z^n}{n!}$ in (5), then we obtain

$$\mathbf{F}\left(z,\mathfrak{q};\frac{z^n}{n!}\right)-\frac{1}{n!(\varrho_2-\varrho_1)}\sum_{w=0}^{n-2}\left(\frac{1}{w!(w+2)}\right)\Big(n(n-1)\cdots(n-w)\Big)\times$$
$$\left\{\varrho_1^{(n-w-1)}\mathbf{F}\left(z,\mathfrak{q};(z-\varrho_1)^{w+2}\right)-\varrho_2^{(n-w-1)}\mathbf{F}\left(z,\mathfrak{q};(z-\varrho_2)^{w+2}\right)\right\}$$
$$=\frac{1}{(n-1)!}\int_{\varrho_1}^{\varrho_2}\mathbf{F}\Big(z,\mathfrak{q};R_n(z,r)\Big)dr. \tag{17}$$

Comparing (16) and (17) leads to

$$\frac{P_1(n)}{(n-1)!} = \mathbf{F}\Big(\mathrm{z}, \mathfrak{q}; \frac{\mathrm{z}^n}{n!}\Big) - \frac{1}{n!(\varrho_2-\varrho_1)} \sum_{w=0}^{n-2} \Big(\frac{1}{w!(w+2)}\Big)\Big(n(n-1)\cdots(n-w)\Big)\times$$
$$\Big\{\varrho_1^{(n-w-1)}\mathbf{F}\Big(\mathrm{z}, \mathfrak{q}; (\mathrm{z}-\varrho_1)^{w+2}\Big) - \varrho_2^{(n-w-1)}\mathbf{F}\Big(\mathrm{z}, \mathfrak{q}; (\mathrm{z}-\varrho_2)^{w+2}\Big)\Big\}. \tag{18}$$

Now, by multiplying above result with $(n-1)!$ and after simple calculations, we get

$$P_1(n) = (n-1)!\Big[\mathbf{F}\Big(\mathrm{z}, \mathfrak{q}; \frac{\mathrm{z}^n}{n!}\Big) - \frac{1}{n!(\varrho_2-\varrho_1)} \sum_{w=2}^{n-1} \Big(\frac{n}{(w+2)}\Big)\binom{n-1}{w}\times$$
$$\Big\{\varrho_1^{(n-w-1)}\mathbf{F}\Big(\mathrm{z}, \mathfrak{q}; (\mathrm{z}-\varrho_1)^{w+2}\Big) - \varrho_2^{(n-w-1)}\mathbf{F}\Big(\mathrm{z}, \mathfrak{q}; (\mathrm{z}-\varrho_2)^{w+2}\Big)\Big\}\Big].$$

Here, we use the fact that

$$\frac{n(n-1)\cdots(n-w)}{w!(w+2)} = \frac{n}{(w+2)}\binom{n-1}{w}.$$

Now, we have to calculate the value of $\lambda_1(n)$, which can be obtained directly in our case by comparing with (10) as

$$\lambda_1(n) = \frac{\int_{\varrho_1}^{\varrho_2} r\mathbf{F}\Big(\mathrm{z}, \mathfrak{q}; R_n(\mathrm{z}, r)\Big)dr}{\int_{\varrho_1}^{\varrho_2} \mathbf{F}\Big(\mathrm{z}, \mathfrak{q}; R_n(\mathrm{z}, r)\Big)dr} = \frac{1}{P_1(n)} \int_{\varrho_1}^{\varrho_2} r\mathbf{F}\Big(\mathrm{z}, \mathfrak{q}; R_n(\mathrm{z}, r)\Big)dr.$$

For this, we first calculate $\int_{\varrho_1}^{\varrho_2} r\mathbf{F}\Big(\mathrm{z}, \mathfrak{q}; R_n(\mathrm{z}, r)\Big)dr$ as under by using integration by parts:

$$\int_{\varrho_1}^{\varrho_2} r\mathbf{F}\Big(\mathrm{z}, \mathfrak{q}; R_n(\mathrm{z}, r)\Big)dr \tag{19}$$
$$= \int_{\varrho_1}^{\mathrm{z}} r\mathbf{F}\Big(\mathrm{z}, \mathfrak{q}; -\frac{(\mathrm{z}-r)^n}{n(\varrho_2-\varrho_1)} + \frac{\mathrm{z}-\varrho_1}{\varrho_2-\varrho_1}(\mathrm{z}-r)^{n-1}\Big)dr$$
$$+ \int_{\mathrm{z}}^{\varrho_2} r\mathbf{F}\Big(\mathrm{z}, \mathfrak{q}; -\frac{(\mathrm{z}-r)^n}{n(\varrho_2-\varrho_1)} + \frac{\mathrm{z}-\varrho_2}{\varrho_2-\varrho_1}(\mathrm{z}-r)^{n-1}\Big)dr$$
$$= I_4 + I_5 - I_6.$$

Again, we use the explicit form of our functional as given in (2) and for simplifications of integrals, we use the following notations

$$I_4 = \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i\Big[\int_{\varrho_1}^{\mathrm{z}_i} \Big(\frac{-r(\mathrm{z}_i-r)^n}{n(\varrho_2-\varrho_1)} + \frac{r(\mathrm{z}_i-\varrho_1)}{\varrho_2-\varrho_1}(\mathrm{z}_i-r)^{n-1}\Big)dr$$
$$+ \int_{\mathrm{z}_i}^{\varrho_2} \Big(\frac{-r(\mathrm{z}_i-r)^n}{n(\varrho_2-\varrho_1)} + \frac{r(\mathrm{z}_i-\varrho_2)}{\varrho_2-\varrho_1}(\mathrm{z}_i-r)^{n-1}\Big)dr\Big],$$

$$I_5 = \frac{\mathfrak{r}-1}{\mathfrak{m}-1}\Big[\int_{\varrho_1}^{\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \mathrm{z}_i} \left(\frac{-r\left(\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \mathrm{z}_i - r\right)^n}{n\,(\varrho_2-\varrho_1)} + \frac{r\left(\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \mathrm{z}_i - \varrho_1\right)}{\varrho_2-\varrho_1}\left(\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \mathrm{z}_i - r\right)^{n-1}\right) dr$$

$$+ \int_{\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \mathrm{z}_i}^{\varrho_2} \left(\frac{-r\left(\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \mathrm{z}_i - r\right)^n}{n\,(\varrho_2-\varrho_1)} + \frac{r\left(\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \mathrm{z}_i - \varrho_2\right)}{\varrho_2-\varrho_1}\left(\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \mathrm{z}_i - r\right)^{n-1}\right) dr\Big],$$

and

$$I_6 = \frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}} \sum_{1\le i_1<\ldots<i_{\mathfrak{r}}\le \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}\right)\times$$

$$\Big[\int_{\varrho_1}^{\frac{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}}} \left(\frac{-r\left(\frac{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}} - r\right)^n}{n\,(\varrho_2-\varrho_1)} + \frac{r\left(\frac{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}} - \varrho_1\right)}{\varrho_2-\varrho_1}\left(\frac{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}} - r\right)^{n-1}\right) dr$$

$$+ \int_{\frac{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}}}^{\varrho_2} \left(\frac{-r\left(\frac{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}} - r\right)^n}{n\,(\varrho_2-\varrho_1)} + \frac{r\left(\frac{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}} - \varrho_2\right)}{\varrho_2-\varrho_1}\left(\frac{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}} - r\right)^{n-1}\right) dr\Big]$$

$$= I_4 + I_5 - I_6.$$

First we calculate the integrals in I_4 as

$$\int_{\varrho_1}^{\mathrm{z}_i} \left(\frac{-r\,(\mathrm{z}_i - r)^n}{n\,(\varrho_2-\varrho_1)} + \frac{r(\mathrm{z}_i-\varrho_1)}{\varrho_2-\varrho_1}\,(\mathrm{z}_i - r)^{n-1}\right) dr = \frac{\varrho_1(\mathrm{z}_i-\varrho_1)^{n+1}}{(n+1)(\varrho_2-\varrho_1)} + \frac{(\mathrm{z}_i-\varrho_1)^{n+2}}{n(n+2)(\varrho_2-\varrho_1)},$$

and

$$\int_{\mathrm{z}_i}^{\varrho_2} \left(\frac{-r\,(\mathrm{z}_i - r)^n}{n\,(\varrho_2-\varrho_1)} + \frac{r(\mathrm{z}_i-\varrho_2)}{\varrho_2-\varrho_1}\,(\mathrm{z}_i - r)^{n-1}\right) dr = \frac{-\varrho_2(\mathrm{z}_i-\varrho_2)^{n+1}}{(n+1)(\varrho_2-\varrho_1)} - \frac{(\mathrm{z}_i-\varrho_2)^{n+2}}{n(n+2)(\varrho_2-\varrho_1)}.$$

Hence, we can get

$$I_4 = \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \Big[\frac{\varrho_1(\mathrm{z}_i-\varrho_1)^{n+1} - \varrho_2(\mathrm{z}_i-\varrho_2)^{n+1}}{(n+1)(\varrho_2-\varrho_1)P_1(n)} + \frac{(\mathrm{z}_i-\varrho_1)^{n+2} - (\mathrm{z}_i-\varrho_2)^{n+2}}{n(n+2)(\varrho_2-\varrho_1)P_1(n)}\Big].$$

Similarly, we have

$$I_5 = \frac{\mathfrak{r}-1}{\mathfrak{m}-1}\Big[\frac{\varrho_1\left(\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \mathrm{z}_i - \varrho_1\right)^{n+1} - \varrho_2\left(\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \mathrm{z}_i - \varrho_2\right)^{n+1}}{(n+1)(\varrho_2-\varrho_1)P_1(n)} + \frac{\left(\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \mathrm{z}_i - \varrho_1\right)^{n+2} - \left(\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \mathrm{z}_i - \varrho_2\right)^{n+2}}{n(n+2)(\varrho_2-\varrho_1)P_1(n)}\Big],$$

and

$$I_6=\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\le i_1<\ldots<i_\mathfrak{r}\le\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\Bigg[\frac{\varrho_1\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathfrak{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}-\varrho_1\right)^{n+1}-\varrho_2\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathfrak{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}-\varrho_2\right)^{n+1}}{(n+1)(\varrho_2-\varrho_1)P_1(n)}$$
$$+\frac{\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathfrak{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}-\varrho_1\right)^{n+2}-\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathfrak{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}-\varrho_2\right)^{n+2}}{n(n+2)(\varrho_2-\varrho_1)P_1(n)}\Bigg].$$

Now, by taking the difference $I_4+I_5-I_6$, we get

$$I_4+I_5-I_6=\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\Bigg[\frac{\varrho_1(\mathfrak{z}_i-\varrho_1)^{n+1}-\varrho_2(\mathfrak{z}_i-\varrho_2)^{n+1}}{(n+1)(\varrho_2-\varrho_1)P_1(n)}+\frac{(\mathfrak{z}_i-\varrho_1)^{n+2}-(\mathfrak{z}_i-\varrho_2)^{n+2}}{n(n+2)(\varrho_2-\varrho_1)P_1(n)}\Bigg]$$
$$+\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\Bigg[\frac{\varrho_1\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\mathfrak{z}_i-\varrho_1\right)^{n+1}-\varrho_2\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\mathfrak{z}_i-\varrho_2\right)^{n+1}}{(n+1)(\varrho_2-\varrho_1)P_1(n)}$$
$$+\frac{\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\mathfrak{z}_i-\varrho_1\right)^{n+2}-\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\mathfrak{z}_i-\varrho_2\right)^{n+2}}{n(n+2)(\varrho_2-\varrho_1)P_1(n)}\Bigg]$$
$$-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\le i_1<\ldots<i_\mathfrak{r}\le\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\Bigg[\frac{\varrho_1\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathfrak{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}-\varrho_1\right)^{n+1}-\varrho_2\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathfrak{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}-\varrho_2\right)^{n+1}}{(n+1)(\varrho_2-\varrho_1)P_1(n)}$$
$$+\frac{\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathfrak{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}-\varrho_1\right)^{n+2}-\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathfrak{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}-\varrho_2\right)^{n+2}}{n(n+2)(\varrho_2-\varrho_1)P_1(n)}\Bigg]$$
$$=\mathbf{F}\left(\mathfrak{z},\mathfrak{q};\left(\frac{\varrho_1(\mathfrak{z}-\varrho_1)^{n+1}-\varrho_2(\mathfrak{z}-\varrho_2)^{n+1}}{(n+1)(\varrho_2-\varrho_1)P_1(n)}+\frac{(\mathfrak{z}-\varrho_1)^{n+2}-(\mathfrak{z}-\varrho_2)^{n+2}}{n(n+2)(\varrho_2-\varrho_1)P_1(n)}\right)\right).$$

Method 2 for computing $\lambda_1(n)$
Now, we will find the value of $\lambda_1(n)$ by employing identity (5) for $f(\mathfrak{z})=\frac{\mathfrak{z}^{n+1}}{(n+1)!}$ as:

$$\mathbf{F}\left(\mathfrak{z},\mathfrak{q};\frac{\mathfrak{z}^{n+1}}{(n+1)!}\right)-\frac{1}{(n+1)!(\varrho_2-\varrho_1)}\sum_{w=0}^{n-2}\left(\frac{1}{w!(w+2)}\right)\Big((n+1)n(n-1)\cdots(n-w+1)\Big)\times$$
$$\left\{\varrho_1^{(n-w)}\mathbf{F}\left(\mathfrak{z},\mathfrak{q};(\mathfrak{z}-\varrho_1)^{w+2}\right)-\varrho_2^{(n-w)}\mathbf{F}\left(\mathfrak{z},\mathfrak{q};(\mathfrak{z}-\varrho_2)^{w+2}\right)\right\}$$
$$=\frac{1}{(n-1)!}\int_{\varrho_1}^{\varrho_2}r\mathbf{F}\Big(\mathfrak{z},\mathfrak{q};R_n(\mathfrak{z},r)\Big)dr. \tag{20}$$

By comparing (19) and (20), we get

$$\frac{P_1(n)}{(n-1)!}\lambda_1(n)=\mathbf{F}\left(\mathfrak{z},\mathfrak{q};\frac{\mathfrak{z}^{n+1}}{(n+1)!}\right)-$$

$$\frac{1}{(n+1)!(\varrho_2-\varrho_1)}\sum_{w=0}^{n-2}\left(\frac{1}{w!(w+2)}\right)\Big((n+1)n(n-1)\cdots(n-w+1)\Big)\times$$
$$\left\{\varrho_1^{(n-w)}\mathbf{F}\Big(z,\mathfrak{q};(z-\varrho_1)^{w+2}\Big)-\varrho_2^{(n-w)}\mathbf{F}\Big(z,\mathfrak{q};(z-\varrho_2)^{w+2}\Big)\right\},$$

after simple calculations, we get

$$\lambda_1(n)=\frac{(n-1)!}{P_1(n)}\left[\mathbf{F}\left(z,\mathfrak{q};\frac{z^{n+1}}{(n+1)!}\right)-\frac{1}{(n+1)!(\varrho_2-\varrho_1)}\sum_{w=0}^{n-2}\left(\frac{n(n+1)}{w(w+2)}\right)\times\right.$$
$$\left.\binom{n-1}{w-1}\left\{\varrho_1^{(n-w)}\mathbf{F}\Big(z,\mathfrak{q};(z-\varrho_1)^{w+2}\Big)-\varrho_2^{(n-w)}\mathbf{F}\Big(z,\mathfrak{q};(z-\varrho_2)^{w+2}\Big)\right\}\right].$$

Here, we use the fact that

$$\frac{(n+1)n(n-1)\cdots(n-w+1)}{w!(w+2)}=\frac{n(n+1)}{w(w+2)}\binom{n-1}{w-1}.$$

□

Now we give another version of Theorem 2.1 by utilizing generalized Montgomery identity (8):

Theorem 2.2. *Under the assumptions of Theorem 1.6, let for all $n\geq 2$, $\mathfrak{f}^{(n)}(\cdot)$(n-th derivative of $\mathfrak{f}$) is convex and*

$$\mathbf{F}\Big(z,\mathfrak{q};\widehat{R}_n(z,r)\Big)>0,\qquad r\in[\varrho_1,\varrho_2],\tag{21}$$

then the following inequalities are valid

$$\frac{\widehat{P}_1(n)}{(n-1)!}\mathfrak{f}^{(n)}(\widehat{\lambda}_1(n))\leq\mathbf{F}\Big(z,\mathfrak{q};\mathfrak{f}(z)\Big)-\frac{1}{(\varrho_2-\varrho_1)}\sum_{w=0}^{n-2}\left(\frac{1}{w!(w+2)}\right)\times$$
$$\left\{\mathbf{F}\Big(z,\mathfrak{q};\mathfrak{f}^{(w+1)}(z)(\varrho_1-z)^{w+2}\Big)-\mathbf{F}\Big(z,\mathfrak{q};\mathfrak{f}^{(w+1)}(z)(\varrho_2-z)^{w+2}\Big)\right\}$$
$$\leq\frac{\widehat{P}_1(n)}{(n-1)!}\left(\frac{\varrho_2-\widehat{\lambda}_1(n)}{\varrho_2-\varrho_1}\mathfrak{f}^{(n)}(\varrho_1)+\frac{\widehat{\lambda}_1(n)-\varrho_1}{\varrho_2-\varrho_1}\mathfrak{f}^{(n)}(\varrho_2)\right),\tag{22}$$

where

$$\widehat{P}_1(n)=\int_{\varrho_1}^{\varrho_2}\mathbf{F}\Big(z,\mathfrak{q};\widehat{R}_n(z,r)\Big)dr=\mathbf{F}\left(z,\mathfrak{q};\left(\frac{(\varrho_1-z)^{n+1}-(\varrho_2-z)^{n+1}}{n(n+1)(\varrho_2-\varrho_1)}\right)\right),\tag{23}$$

or

$$\widehat{P}_1(n)=(n-1)!\left[\mathbf{F}\left(z,\mathfrak{q};\frac{z^n}{n!}\right)-\frac{1}{(\varrho_2-\varrho_1)}\sum_{w=0}^{n-2}\left(\frac{1}{(w+2)!}\right)\times\right.$$
$$\left.\left\{\mathbf{F}\left(z,\mathfrak{q};\frac{d^{w+1}}{dz^{w+1}}\left(\frac{z^n}{n!}\right)(\varrho_1-z)^{w+2}\right)-\mathbf{F}\left(z,\mathfrak{q};\frac{d^{w+1}}{dz^{w+1}}\left(\frac{z^n}{n!}\right)(\varrho_2-z)^{w+2}\right)\right\}\right],\tag{24}$$

and

$$\widehat{\lambda}_1(n)=\frac{\int_{\varrho_1}^{\varrho_2}r\mathbf{F}\Big(z,\mathfrak{q};\widehat{R}_n(z,r)\Big)dr}{\int_{\varrho_1}^{\varrho_2}\mathbf{F}\Big(z,\mathfrak{q};\widehat{R}_n(z,r)\Big)dr}$$

$$= \mathbf{F}\left(\mathrm{z}, \mathfrak{q}; \left(\frac{\mathrm{z}(\varrho_1 - \mathrm{z})^{n+1} - \mathrm{z}(\varrho_2 - \mathrm{z})^{n+1}}{n(n+1)(\varrho_2 - \varrho_1)P_1(n)} + \frac{(\varrho_1 - \mathrm{z})^{n+2} - (\varrho_2 - \mathrm{z})^{n+2}}{n(n+1)(n+2)(\varrho_2 - \varrho_1)P_1(n)}\right)\right), \quad (25)$$

or

$$\widehat{\lambda}_1(n) = \frac{(n-1)!}{P_1(n)}\Bigg[\mathbf{F}\left(\mathrm{z}, \mathfrak{q}; \frac{\mathrm{z}^{n+1}}{(n+1)!}\right) - \frac{1}{(\varrho_2 - \varrho_1)} \sum_{w=0}^{n-2} \left(\frac{1}{(w+2)!}\right) \times$$
$$\left\{\mathbf{F}\left(\mathrm{z}, \mathfrak{q}; \frac{d^{w+1}}{d\mathrm{z}^{w+1}}\left(\frac{\mathrm{z}^{n+1}}{(n+1)!}\right)(\varrho_1 - \mathrm{z})^{w+2}\right) - \mathbf{F}\left(\mathrm{z}, \mathfrak{q}; \frac{d^{w+1}}{d\mathrm{z}^{w+1}}\left(\frac{\mathrm{z}^{n+1}}{(n+1)!}\right)(\varrho_2 - \mathrm{z})^{w+2}\right)\right\}\Bigg]. \quad (26)$$

Proof. Proof is similar as Theorem 2.1. □

The following corollary is the special case of Theorem 2.1

Corollary 2.3. *Considering the assumptions of Theorem 2.1, then for a particular case when $n = 2$, if $\mathfrak{f}''(\cdot)$ (twice derivative of $\mathfrak{f}$) is convex and the weight function condition* (12) *is satisfied, then the following result holds:*

$$P_1(2)\mathfrak{f}''(\lambda_1(2)) \le \mathbf{F}\left(\mathrm{z}, \mathfrak{q}; \mathfrak{f}(\mathrm{z})\right) - \frac{1}{2(\varrho_2 - \varrho_1)} \times$$
$$\left\{\mathfrak{f}'(\varrho_1)\mathbf{F}\left(\mathrm{z}, \mathfrak{q}; (\mathrm{z} - \varrho_1)^2\right) - \mathfrak{f}'(\varrho_2)\mathbf{F}\left(\mathrm{z}, \mathfrak{q}; (\mathrm{z} - \varrho_2)^2\right)\right\}$$
$$\le P_1(2)\left(\frac{\varrho_2 - \lambda_1(2)}{\varrho_2 - \varrho_1}\mathfrak{f}''(\varrho_1) + \frac{\lambda_1(2) - \varrho_1}{\varrho_2 - \varrho_1}\mathfrak{f}''(\varrho_2)\right), \quad (27)$$

where

$$P_1(2) = \int_{\varrho_1}^{\varrho_2} \mathbf{F}\left(\mathrm{z}, \mathfrak{q}; R_2(\mathrm{z}, r)\right) dr = \mathbf{F}\left(\mathrm{z}, \mathfrak{q}; \left(\frac{(\mathrm{z} - \varrho_1)^3 - (\mathrm{z} - \varrho_2)^3}{3(\varrho_2 - \varrho_1)}\right)\right),$$

or

$$P_1(2) = \mathbf{F}\left(\mathrm{z}, \mathfrak{q}; \frac{\mathrm{z}^2}{2!}\right) - \frac{1}{2!(\varrho_2 - \varrho_1)}\left\{\varrho_1\mathbf{F}\left(\mathrm{z}, \mathfrak{q}; (\mathrm{z} - \varrho_1)^2\right) - \varrho_2\mathbf{F}\left(\mathrm{z}, \mathfrak{q}; (\mathrm{z} - \varrho_2)^2\right)\right\},$$

and

$$\lambda_1(2) = \frac{\int_{\varrho_1}^{\varrho_2} r\mathbf{F}\left(\mathrm{z}, \mathfrak{q}; R_2(\mathrm{z}, r)\right) dr}{\int_{\varrho_1}^{\varrho_2} \mathbf{F}\left(\mathrm{z}, \mathfrak{q}; R_2(\mathrm{z}, r)\right) dr}$$
$$= \mathbf{F}\left(\mathrm{z}, \mathfrak{q}; \left(\frac{\varrho_1(\mathrm{z} - \varrho_1)^3 - \varrho_2(\mathrm{z} - \varrho_2)^3}{3P_1(2)(\varrho_2 - \varrho_1)} + \frac{(\mathrm{z} - \varrho_1)^4 - (\mathrm{z} - \varrho_2)^4}{8P_1(2)(\varrho_2 - \varrho_1)}\right)\right),$$

or

$$\lambda_1(2) = \frac{1}{P_1(2)}\left[\mathbf{F}\left(\mathrm{z}, \mathfrak{q}; \frac{\mathrm{z}^3}{3!}\right) - \frac{5}{12(\varrho_2 - \varrho_1)}\left\{{\varrho_1}^4\mathbf{F}\left(\mathrm{z}, \mathfrak{q}; (\mathrm{z} - \varrho_1)^2\right) - \varrho_2^4\mathbf{F}\left(\mathrm{z}, \mathfrak{q}; (\mathrm{z} - \varrho_2)^2\right)\right\}\right].$$

The followings are the examples related to our main results given in Theorem 2.1:

Example 2.4. *For even ($n \ge 4$) the function $g(\mathrm{z}) = R_n(\mathrm{z}, r)$ in Montgomery identity* (5) *is convex as given in [11]. We start with the case when $n = 4$, first we satisfy the following condition:*

$$\mathbf{F}\left(\mathrm{z}, \mathfrak{q}; R_4(\mathrm{z}, r)\right) > 0, \qquad r \in [\varrho_1, \varrho_2]$$

Now, taking $[\varrho_1, \varrho_2] = [0.5, 2]$ *and* $z_1 = 1.3, z_2 = 1.4, z_3 = 1.5$ *and* $r = 1$. *Since it is true for any positive* q_i, *here we choose* $q_1 = q_2 = q_3 = \frac{1}{3}$.
We can rewrite the above functional for $m = 3$ *and* $r = 2$ *as:*

$$\begin{aligned}
&\mathbf{F}\Big(z, q; R_4(z, r)\Big) = q_1\left(-\frac{(z_1 - r)^4}{4(\varrho_2 - \varrho_1)} + \frac{z_1 - \varrho_1}{\varrho_2 - \varrho_1}(z_1 - r)^3\right) \\
&+ q_2\left(-\frac{(z_2 - r)^4}{4(\varrho_2 - \varrho_1)} + \frac{z_2 - \varrho_1}{\varrho_2 - \varrho_1}(z_2 - r)^3\right) + q_3\left(-\frac{(z_3 - r)^4}{4(\varrho_2 - \varrho_1)} + \frac{z_3 - \varrho_1}{\varrho_2 - \varrho_1}(z_3 - r)^3\right) \\
&+ (q_1 + q_2 + q_3)\left(-\frac{\left(\frac{q_1z_1+q_2z_2+q_3z_3}{q_1+q_2+q_3} - r\right)^4}{4(\varrho_2 - \varrho_1)} + \frac{\frac{q_1z_1+q_2z_2+q_3z_3}{q_1+q_2+q_3} - \varrho_1}{\varrho_2 - \varrho_1}\left(\frac{q_1z_1 + q_2z_2 + q_3z_3}{q_1 + q_2 + q_3} - r\right)^3\right) \\
&- \Bigg[(q_2 + q_3)\left(-\frac{\left(\frac{q_1z_1+q_2z_2}{q_1+q_2} - r\right)^4}{4(\varrho_2 - \varrho_1)} + \frac{\frac{q_1z_1+q_2z_2}{q_1+q_2} - \varrho_1}{\varrho_2 - \varrho_1}\left(\frac{q_1z_1 + q_2z_2}{q_1 + q_2} - r\right)^3\right) \\
&+ (q_3 + q_1)\left(-\frac{\left(\frac{q_3z_3+q_1z_1}{q_3+q_1} - r\right)^4}{4(\varrho_2 - \varrho_1)} + \frac{\frac{q_3z_3+q_1z_1}{q_3+q_1} - \varrho_1}{\varrho_2 - \varrho_1}\left(\frac{q_3z_3 + q_1z_1}{q_3 + q_1} - r\right)^3\right) \\
&\qquad + (q_1 + q_2)\left(-\frac{\left(\frac{q_1z_1+q_2z_2}{q_1+q_2} - r\right)^4}{4(\varrho_2 - \varrho_1)} + \frac{\frac{q_1z_1+q_2z_2}{q_1+q_2} - \varrho_1}{\varrho_2 - \varrho_1}\left(\frac{q_1z_1 + q_2z_2}{q_1 + q_2} - r\right)^3\right)\Bigg].
\end{aligned}$$

By substituting the values and after simple calculations we get

$$\mathbf{F}\Big(z, q; R_4(z, r)\Big) = 0.002963 > 0$$

For the function $f(z) = z^6$ *where* $f'(z) = 6z^5$, $f''(z) = 30z^4$, $f'''(z) = 120z^3$, $f''''(z) = 360z^2$. *Now by writing* (13) *for* $n = 4$, *we get*

$$\begin{aligned}
&\frac{P_1(4)}{3!}f^{(4)}(\lambda_1(4)) \leq \mathbf{F}\Big(z, q; f(z)\Big) \\
&- \frac{1}{3}\left\{f'(0.5)\,\mathbf{F}\Big(z, q; (z - 0.5)^2\Big) - f'(2)\,\mathbf{F}\Big(z, q; (z - 2)^2\Big)\right\} \\
&- \frac{1}{4.5}\left\{f''(0.5)\,\mathbf{F}\Big(z, q; (z - 0.5)^3\Big) - f''(2)\,\mathbf{F}\Big(z, q; (z - 2)^3\Big)\right\} \\
&- \frac{1}{12}\left\{f'''(0.5)\,\mathbf{F}\Big(z, q; (z - 0.5)^4\Big) - f'''(2)\,\mathbf{F}\Big(z, q; (z - 2)^4\Big)\right\} \\
&\qquad \leq \frac{P_1(4)}{3!}\left(\frac{2 - \lambda_1(4)}{1.5}f^{(4)}(0.5) + \frac{\lambda_1(4) - 0.5}{1.5}f^{(4)}(2)\right), \quad (28)
\end{aligned}$$

now by substituting the values, we get

$$0.266171 \leq 0.323514 \leq 0.396453,$$

where

$$P_1(4) = \mathbf{F}\left(z, q; \left(\frac{(z - 0.5)^5 - (z - 2)^5}{7.5}\right)\right) = 0.004258,$$

and

$$\lambda_1(4) = \mathbf{F}\Big(z, q; \Big(\frac{0.5(z-0.5)^5 - 2(z-2)^5}{0.031938} + \frac{(z-0.5)^6 - (z-2)^6}{0.153300}\Big)\Big) = 1.020670.$$

The following example is same as above example but we calculate $P_1(n)$ and $\lambda_1(n)$ with second method which give us same results.

Example 2.5. *Assuming all the values taken in Example 1 but here we calculate $P_1(n)$ and $\lambda_1(n)$ as given below:*

$$\begin{aligned} P_1(4) = 3!\Big[&\mathbf{F}\Big(z, q; \frac{z^4}{4!}\Big) \\ &- \frac{1}{72}\Big\{\frac{d}{dz}(z^4)|_{z=0.5}\mathbf{F}\Big(z, q; (z-0.5)^2\Big) - \frac{d}{dz}(z^4)|_{z=2}\mathbf{F}\Big(z, q; (z-2)^2\Big)\Big\} \\ &- \frac{1}{108}\Big\{\frac{d^2}{dz^2}(z^4)|_{z=0.5}\mathbf{F}\Big(z, q; (z-0.5)^3\Big) - \frac{d^2}{dz^2}(z^4)|_{z=2}\mathbf{F}\Big(z, q; (z-2)^3\Big)\Big\} \\ &- \frac{1}{288}\Big\{\frac{d^3}{dz^3}(z^4)|_{z=0.5}\mathbf{F}\Big(z, q; (z-0.5)^4\Big) - \frac{d^3}{dz^3}(z^4)|_{z=2}\mathbf{F}\Big(z, q; (z-2)^4\Big)\Big\}\Big] \end{aligned}$$

$$P_1(4) = 0.004258,$$

and

$$\begin{aligned} \lambda_1(4) = \frac{3!}{P_1(4)}\Big[&\mathbf{F}\Big(z, q; \frac{z^5}{5!}\Big) \\ &- \frac{1}{360}\Big\{\frac{d}{dz}(z^5)|_{z=0.5}\mathbf{F}\Big(z, q; (z-0.5)^2\Big) - \frac{d}{dz}(z^5)|_{z=2}\mathbf{F}\Big(z, q; (z-2)^2\Big)\Big\} \\ &- \frac{1}{540}\Big\{\frac{d^2}{dz^2}(z^5)|_{z=0.5}\mathbf{F}\Big(z, q; (z-0.5)^3\Big) - \frac{d^2}{dz^2}(z^5)|_{z=2}\mathbf{F}\Big(z, q; (z-2)^3\Big)\Big\} \\ &- \frac{1}{1440}\Big\{\frac{d^3}{dz^3}(z^5)|_{z=0.5}\mathbf{F}\Big(z, q; (z-0.5)^4\Big) - \frac{d^3}{dz^3}(z^5)|_{z=2}\mathbf{F}\Big(z, q; (z-2)^4\Big)\Big\}\Big] \end{aligned}$$

$$\lambda_1(4) = 1.020675.$$

Now, by substituting $P_1(4)$, $\lambda_1(4)$ and all other values in (28)*, we get*

$$0.266173 \le 0.323514 \le 0.396454.$$

3. Applications in information theory

Mathematically, the techniques and ideas that control message transmission through communication systems are covered by information theory. When Claude Shannon, an American electrical engineer, published a ground breaking article that transformed communication, the field received a significant boost. Recently, Butt et al. in [17] have provided some interesting bounds for Shannon, relative, and Mandelbrot entropies.

First, we present some fundamental definitions and results concerned with this section. Csiszár in [18] presented the following divergence functional:

Definition 3.1. *(**Csiszár divergence**). Let $I=[\varrho_1,\varrho_2]\subset R$ and $\mathfrak{f}:I\to R$ be a function, then for $\mathfrak{q}=(\mathfrak{q}_1,...,\mathfrak{q}_\mathfrak{m})\in R^\mathfrak{m}$ and $\mathfrak{p}=(\mathfrak{p}_1,...,\mathfrak{p}_\mathfrak{m})\in R_+^\mathfrak{m}$ such that $\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\in I(i=1,...,\mathfrak{m})$, the Csiszár f-divergence functional is defined as:*

$$\bar{D}_c(\mathfrak{q},\mathfrak{p})=\sum_{i=1}^{\mathfrak{m}}\mathfrak{p}_i\mathfrak{f}\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\right)$$

Theorem 3.2. *Let the assumptions of Theorem 2.1 are valid and also let $\mathfrak{q}=(\mathfrak{q}_1,...,\mathfrak{q}_\mathfrak{m})\in R^\mathfrak{m}$ and $\mathfrak{p}=(\mathfrak{p}_1,...,\mathfrak{p}_\mathfrak{m})\in R_+^\mathfrak{m}$ such that $\frac{\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i}{\sum_{i=1}^{\mathfrak{m}}\mathfrak{p}_i}$, $\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\in[\varrho_1,\varrho_2]$ for $i=1,...,\mathfrak{m}$, then*

$$\begin{aligned}
&\frac{P_1(n)}{(n-1)!}\mathfrak{f}^{(n)}(\lambda_1(n))\le\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\bar{D}_c(\mathfrak{q},\mathfrak{p})+\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\mathfrak{f}\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\right)\\
&-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\le i_1<...<i_\mathfrak{r}\le\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}\right)\mathfrak{f}\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}}\right)-\frac{1}{(\varrho_2-\varrho_1)}\sum_{w=0}^{n-2}\left(\frac{1}{w!(w+2)}\right)\times\\
&\left\{\mathfrak{f}^{(w+1)}(\varrho_1)\mathbf{F}\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i},\mathfrak{q};(\mathrm{z}-\varrho_1)^{\mathbf{w+2}}\right)-\mathfrak{f}^{(w+1)}(\varrho_2)\mathbf{F}\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i},\mathfrak{q};(\mathrm{z}-\varrho_2)^{\mathbf{w+2}}\right)\right\}\\
&\le\frac{P_1(n)}{(n-1)!}\left(\frac{\varrho_2-\lambda_1(n)}{\varrho_2-\varrho_1}\mathfrak{f}^{(n)}(\varrho_1)+\frac{\lambda_1(n)-\varrho_1}{\varrho_2-\varrho_1}\mathfrak{f}^{(n)}(\varrho_2)\right).
\end{aligned}\tag{29}$$

Proof. First we rewrite the equivalent form of (13) as

$$\begin{aligned}
&\frac{P_1(n)}{(n-1)!}\mathfrak{f}^{(n)}(\lambda_1(n))\le\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\mathfrak{f}(\mathrm{z}_i)+\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\mathfrak{f}\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\mathrm{z}_i\right)\\
&-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\le i_1<...<i_\mathfrak{r}\le\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\mathfrak{f}\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\mathrm{z}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}\right)-\frac{1}{(\varrho_2-\varrho_1)}\sum_{w=0}^{n-2}\left(\frac{1}{w!(w+2)}\right)\times\\
&\left\{\mathfrak{f}^{(w+1)}(\varrho_1)\mathbf{F}\left(\mathrm{z}_i,\mathfrak{q};(\mathrm{z}-\varrho_1)^{w+2}\right)-\mathfrak{f}^{(w+1)}(\varrho_2)\mathbf{F}\left(\mathrm{z}_i,\mathfrak{q};(\mathrm{z}-\varrho_2)^{w+2}\right)\right\}\\
&\le\frac{P_1(n)}{(n-1)!}\left(\frac{\varrho_2-\lambda_1(n)}{\varrho_2-\varrho_1}\mathfrak{f}^{(n)}(\varrho_1)+\frac{\lambda_1(n)-\varrho_1}{\varrho_2-\varrho_1}\mathfrak{f}^{(n)}(\varrho_2)\right),
\end{aligned}$$

now replace z_i by $\frac{\mathfrak{q}_i}{\mathfrak{p}_i}$ and $\mathfrak{q}_i$ by $\mathfrak{p}_i$ after simple calculations, we get the required result. □

Remark 3.3. *Under the assumptions of Theorem 3.2, then for $n=2$ we have the following result*

$$\begin{aligned}
&P_1(2)\mathfrak{f}''(\lambda_1(2))\le\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\bar{D}_c(\mathfrak{q},\mathfrak{p})+\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\mathfrak{f}\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\right)\\
&-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\le i_1<...<i_\mathfrak{r}\le\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}\right)\mathfrak{f}\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}}\right)-\frac{1}{2(\varrho_2-\varrho_1)}\times\\
&\left\{\mathfrak{f}'(\varrho_1)\mathbf{F}\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i},\mathfrak{q};(\mathrm{z}-\varrho_1)^2\right)-\mathfrak{f}'(\varrho_2)\mathbf{F}\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i},\mathfrak{q};(\mathrm{z}-\varrho_2)^2\right)\right\}
\end{aligned}$$

$$\leq P_1(2)\left(\frac{\varrho_2-\lambda_1(2)}{\varrho_2-\varrho_1}\mathfrak{f}''(\varrho_1)+\frac{\lambda_1(2)-\varrho_1}{\varrho_2-\varrho_1}\mathfrak{f}''(\varrho_2)\right). \tag{30}$$

Definition 3.4. *(***Kullback-Leibler divergence)**. *Let $\mathfrak{q}$ and $\mathfrak{p}$ be two p.p.d, then Kullback-Leibler divergence is defined as*

$$D_{kl}(\mathfrak{q},\mathfrak{p})=\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\ln\frac{\mathfrak{q}_i}{\mathfrak{p}_i}.$$

Corollary 3.5. *Let $[\varrho_1,\varrho_2]$ be a subset of the set of positive real numbers and $\mathfrak{q}$, $\mathfrak{p}$ be two p.p.d such that $\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\in[\varrho_1,\varrho_2]$ for $i=1,...,\mathfrak{m}$, then*

$$P_1(2)\mathfrak{f}''(\lambda_1(2))\leq\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}D_{kl}(\mathfrak{q},\mathfrak{p})+\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\ln\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\right)$$
$$-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\leq i_1<...<i_{\mathfrak{r}}\leq\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\ln\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}}\right)-\frac{1}{2(\varrho_2-\varrho_1)}\times$$
$$\left\{(1+\ln\varrho_1)\mathbf{F}\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i},\mathfrak{q};(\mathrm{z}-\varrho_1)^2\right)-(1+\ln\varrho_2)\mathbf{F}\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i},\mathfrak{q};(\mathrm{z}-\varrho_2)^2\right)\right\}$$
$$\leq P_1(2)\left(\frac{\varrho_2-\lambda_1(2)}{\varrho_2-\varrho_1}\mathfrak{f}''(\varrho_1)+\frac{\lambda_1(2)-\varrho_1}{\varrho_2-\varrho_1}\mathfrak{f}''(\varrho_2)\right). \tag{31}$$

Proof. Let $\mathfrak{f}(\mathrm{z})=\mathrm{z}\ln\mathrm{z}$, $\mathrm{z}\in[\varrho_1,\varrho_2]$ then $\mathfrak{f}''(\mathrm{z})=\frac{1}{\mathrm{z}}$. It is easy to veirfy that $\mathfrak{f}''$ is convex for $\mathrm{z}>0$, therefore using (29) for $\mathfrak{f}(\mathrm{z})=\mathrm{z}\ln\mathrm{z}$ and for $n=2$, we derive (31). □

Definition 3.6. (Rényi-divergence). *Let $\mathfrak{q}$ and $\mathfrak{p}$ be two positive probability distributions (p.p.d) and μ be a real number which is nonnegative such that $\mu\neq1$, the Rényi-divergence is defined as*

$$D_{re}(\mathfrak{q},\mathfrak{p})=\frac{1}{\mu-1}\ln\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i^{\mu}\mathfrak{p}_i^{(1-\mu)}\right).$$

Corollary 3.7. *Let $[\varrho_1,\varrho_2]\subseteq\mathbb{R}^+$. Also let $\mathfrak{q}$, $\mathfrak{p}$ be two p.p.d and $\mu>1$ such that $\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i(\frac{\mathfrak{q}_i}{\mathfrak{p}_i})^{\mu}$, $(\frac{\mathfrak{q}_i}{\mathfrak{p}_i})^{\mu-1}\in[\varrho_1,\varrho_2]$ for $i=1,...,\mathfrak{m}$, then*

$$\frac{(-1)^nP_1(n)}{(\mu-1)(\lambda_1(n))^n}\leq\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\left(\frac{-1}{\mu-1}\ln\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\right)^{\mu-1}\right)-\frac{\mathfrak{r}-1}{\mathfrak{m}-1}D_{re}(\mathfrak{q},\mathfrak{p})$$
$$+\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\leq i_1<...<i_{\mathfrak{r}}\leq\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\ln\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}^{\mu}\mathfrak{p}_{i_j}{}^{1-\mu}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}\right)-\frac{1}{(\mu-1)(\varrho_2-\varrho_1)}\sum_{w=0}^{n-2}\left(\frac{1}{(w+2)}\right)\times$$
$$\left\{\left(\frac{(-1)^{w+1}}{\varrho_1^{w+1}}\right)\mathbf{F}\left(\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\right)^{\mu-1},\mathfrak{q};(\mathrm{z}-\varrho_1)^{w+2}\right)-\left(\frac{(-1)^{w+1}}{\varrho_2^{w+1}}\right)\mathbf{F}\left(\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\right)^{\mu-1},\mathfrak{q};(\mathrm{z}-\varrho_2)^{w+2}\right)\right\}$$
$$\leq\frac{P_1(n)}{(\mu-1)}\left(\frac{(-1)^n(\varrho_2-\lambda_1(n))}{\varrho_1^n(\varrho_2-\varrho_1)}+\frac{(-1)^n(\lambda_1(n)-\varrho_1)}{\varrho_2^n(\varrho_2-\varrho_1)}\right). \tag{32}$$

Proof. Let $\mathrm{z} \in [\varrho_1, \varrho_2]$, $\mathfrak{f}(\mathrm{z}) = -\frac{1}{\mu-1}\ln(\mathrm{z})$ then for even n, $\mathfrak{f}^{(n)}(\mathrm{z}) = (-1)^n \frac{(n-1)!}{(\mu-1)\mathrm{z}^n} > 0$ and $\mathfrak{f}^{(n+2)}(\mathrm{z}) = (-1)^{(n+2)} \frac{(n+1)!}{(\mu-1)\mathrm{z}^{n+2}} > 0$. Which guarantee that the function $\mathfrak{f}^{(n)}$ is convex, now using (13) for $\mathfrak{f}(\mathrm{z}) = -\frac{1}{\mu-1}\ln(\mathrm{z})$ and $\mathrm{z}_i = (\frac{\mathfrak{q}_i}{\mathfrak{p}_i})^{\mu-1}$, we derive (32). □

Remark 3.8. *Under the assumptions of Corollary 3.7, then for $n = 2$ we have the following result*

$$\frac{P_1(2)}{(\mu-1)(\lambda_1(2))^2} \leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \left(\frac{-1}{\mu-1}\ln\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\right)^{\mu-1}\right) - \frac{\mathfrak{r}-1}{\mathfrak{m}-1} D_{re}(\mathfrak{q},\mathfrak{p})$$
$$+ \frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}} \sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}\right) \ln\left(\frac{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}^{\mu} \mathfrak{p}_{i_j}{}^{1-\mu}}{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}}\right) - \frac{1}{2(\mu-1)(\varrho_2-\varrho_1)} \times$$
$$\left\{ \left(\frac{-1}{\varrho_1}\right) \mathbf{F}\left(\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\right)^{\mu-1}, \mathfrak{q}; (\mathrm{z}-\varrho_1)^2\right) - \left(\frac{-1}{\varrho_2}\right) \mathbf{F}\left(\left(\frac{\mathfrak{q}_i}{\mathfrak{p}_i}\right)^{\mu-1}, \mathfrak{q}; (\mathrm{z}-\varrho_2)^2\right) \right\}$$
$$\leq \frac{P_1(n)}{(\mu-1)} \left(\frac{(\varrho_2 - \lambda_1(2))}{\varrho_1^2(\varrho_2-\varrho_1)} + \frac{(\lambda_1(2)-\varrho_1)}{\varrho_2^2(\varrho_2-\varrho_1)} \right).$$

Definition 3.9. (**Shannon-entropy**). *The Shannon-entropy (information divergence) for a p.p.d $\mathfrak{p}$ is given as:*

$$S(\mathfrak{p}) = -\sum_{i=1}^{\mathfrak{m}} \mathfrak{p}_i \ln \mathfrak{p}_i. \tag{33}$$

Corollary 3.10. *Let $[\varrho_1, \varrho_2] \subseteq \mathbb{R}^+$ and $\mathfrak{p}$ be a p.p.d such that $\frac{1}{\mathfrak{p}_i} \in [\varrho_1, \varrho_2]$ for $i = 1, \ldots, \mathfrak{m}$, then*

$$\frac{(-1)^n P_1(n)}{(\lambda_1(n))^n} \leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} S(\mathfrak{p}) - \frac{\mathfrak{r}-1}{\mathfrak{m}-1}\ln(\mathfrak{m})$$
$$- \frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}} \sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j}\right) \ln\left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j}\right) - \sum_{w=0}^{n-2} \left(\frac{1}{(w+2)(\varrho_2-\varrho_1)}\right) \times$$
$$\left\{ \left(\frac{(-1)^{w+1}}{\varrho_1^{w+1}}\right) \mathbf{F}\left(\frac{1}{\mathfrak{p}_i}, \mathfrak{q}; (\mathrm{z}-\varrho_1)^{w+2}\right) - \left(\frac{(-1)^{w+1}}{\varrho_2^{w+1}}\right) \mathbf{F}\left(\frac{1}{\mathfrak{p}_i}, \mathfrak{q}; (\mathrm{z}-\varrho_2)^{w+2}\right) \right\}$$
$$\leq P_1(n) \left(\frac{(-1)^n(\varrho_2 - \lambda_1(n))}{\varrho_1^n(\varrho_2-\varrho_1)} + \frac{(-1)^n(\lambda_1(n)-\varrho_1)}{\varrho_2^n(\varrho_2-\varrho_1)} \right). \tag{34}$$

Proof. Let $\mathfrak{f}(\mathrm{z}) = -\ln(\mathrm{z})$, $\mathrm{z} \in [\varrho_1, \varrho_2]$ then for even n, $\mathfrak{f}^{(n)}(\mathrm{z}) = (-1)^n \frac{(n-1)!}{\mathrm{z}^n} > 0$ and $\mathfrak{f}^{(n+2)}(\mathrm{z}) = (-1)^{n+2} \frac{(n+1)!}{\mathrm{z}^{n+2}} > 0$. It follows that the function $\mathfrak{f}^{(n)}$ is convex, therefore using (29) for $\mathfrak{f}(\mathrm{z}) = -\ln(\mathrm{z})$, and $(\mathfrak{q}_1, \ldots, \mathfrak{q}_{\mathfrak{m}}) = (1, 1, \ldots, 1)$, we derive (34). □

Remark 3.11. *Under the assumptions of Corollary 3.10 then for $n = 2$ we have the following result*

$$\frac{P_1(2)}{(\lambda_1(2))^2} \leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} S(\mathfrak{p}) - \frac{\mathfrak{r}-1}{\mathfrak{m}-1}\ln(\mathfrak{m})$$
$$- \frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}} \sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j}\right) \ln\left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j}\right) - \frac{1}{2(w+2)(\varrho_2-\varrho_1)}$$

$$\left\{\left(\frac{-1}{\varrho_1}\right)\mathbf{F}\left(\frac{1}{\mathfrak{p}_i},\mathfrak{q};(\mathrm{z}-\varrho_1)^2\right)-\left(\frac{-1}{\varrho_2}\right)\mathbf{F}\left(\frac{1}{\mathfrak{p}_i},\mathfrak{q};(\mathrm{z}-\varrho_2)^2\right)\right\}$$

$$\leq P_1(2)\left(\frac{(\varrho_2-\lambda_1(2))}{\varrho_1^2(\varrho_2-\varrho_1)}+\frac{(\lambda_1(2)-\varrho_1)}{\varrho_2^2(\varrho_2-\varrho_1)}\right).$$

4. Results for Zipf and hybrid Zipf-Mandelbrot entropy

One of the basic laws in information science is Zipf's law (see [19, 20]). It has many interesting applications in linguistics. Let c, d be the real numbers such that $c \in [0,\infty)$ and $d \in (0,\infty)$ and N be natural numbers, then the Zipf-Mandelbrot entropy can be given as:

$$Z_M(H,c,d)=\frac{d}{H_{c,d}^N}\sum_{i=1}^{N}\frac{\ln(i+c)}{(i+c)^d}+\ln(H_{c,d}^N) \tag{35}$$

where

$$H_{c,d}^N=\sum_{\sigma=1}^{N}\frac{1}{(\sigma+c)^d}.$$

Consider

$$\mathfrak{p}_i=\mathfrak{f}(i;N,c,d)=\frac{1}{((i+c)^d H_{c,d}^N)}, \tag{36}$$

where (36) is a discrete probability distribution known as Zipf-Mandelbrot law. There are many applications of Zipf-Mandelbrot law in linguistic and information sciences. One can find recent results about Zipf-Mandelbrot in (see [21–23]).

Theorem 4.1. *Let $\mathfrak{p}$ be a discrete probability distribution given in* (36) *with parameters $0 \leq c < \infty$, $0 < d < \infty$ and N be natural numbers, we have the following result*

$$\frac{(-1)^n P_1(n)}{(\lambda_1(n))^n}\leq\frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}Z_M(H,c,d)=S(\mathfrak{p})-\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\ln(\mathfrak{m})$$
$$-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq\mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}\right)\ln\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}\right)-\frac{1}{(\varrho_2-\varrho_1)}\sum_{w=0}^{n-2}\left(\frac{1}{(w+2)}\right)\times$$
$$\left\{\left(\frac{(-1)^{w+1}}{\varrho_1^{w+1}}\right)\mathbf{F}\left(((i+c)^d H_{c,d}^N),\mathfrak{q},;(\mathrm{z}-\varrho_1)^{w+2}\right)\right.$$
$$\left.-\left(\frac{(-1)^{w+1}}{\varrho_2^{w+1}}\right)\mathbf{F}\left(((i+c)^d H_{c,d}^N),\mathfrak{q};(\mathrm{z}-\varrho_1)^{w+2}\right)\right\}$$
$$\leq P_1(n)\left(\frac{(-1)^n(\varrho_2-\lambda_1(n))}{\varrho_1^n(\varrho_2-\varrho_1)}+\frac{(-1)^n(\lambda_1(n)-\varrho_1)}{\varrho_2^n(\varrho_2-\varrho_1)}\right).$$

Proof. We can obtain Mandelbrot entropy (35) from Shannon entropy by substituting $\mathfrak{p}_i$ in (33).

$$S(\mathfrak{p}) = -\sum_{i=1}^{N}\mathfrak{p}_i\ln\mathfrak{p}_i$$

$$= \quad -\sum_{i=1}^{N} \frac{1}{((i+c)^d H_{c,d}^N)} \ln \frac{1}{((i+c)^d H_{c,d}^N)}$$

$$= \quad \frac{d}{(H_{c,d}^N)} \sum_{i=1}^{N} \frac{\ln(i+c)}{(i+c)^d} + \ln(H_{c,d}^N).$$

Finally, substituting this $\mathfrak{p}_i = \frac{1}{((i+c)^d H_{c,d}^N)}$ in (34), we get the required result. □

Remark 4.2. *Under the assumptions of Theorem 4.1, then for $n = 2$ we have the following result*

$$\frac{P_1(2)}{(\lambda_1(2))^2} \leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} Z_M(H,c,d) = S(\mathfrak{p})$$

$$-\frac{\mathfrak{r}-1}{\mathfrak{m}-1}\ln(\mathfrak{m}) - \frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}} \sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j}\right) \ln\left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j}\right) - \frac{1}{2(\varrho_2-\varrho_1)}$$

$$\left\{\left(\frac{-1}{\varrho_1}\right)\mathbf{F}\Big(((i+c)^d H_{c,d}^N), \mathfrak{q}; (\mathrm{z}-\varrho_1)^{w+2}\Big) - \left(\frac{-1}{\varrho_2}\right)\mathbf{F}\Big(((i+c)^d H_{c,d}^N), \mathfrak{q}; (\mathrm{z}-\varrho_2)^{w+2}\Big)\right\}$$

$$\leq P_1(2)\left(\frac{(\varrho_2-\lambda_1(2))}{\varrho_1^2(\varrho_2-\varrho_1)} + \frac{(\lambda_1(2)-\varrho_1)}{\varrho_2^2(\varrho_2-\varrho_1)}\right).$$

Corollary 4.3. *Let Zipf-Mandelbrot law defined as $\mathfrak{q}$ and $\mathfrak{p}$ with parameters $c_1, c_2 \geq 0$, $d_1, d_2 > 0$, let $H_{c_1,d_1}^N = \sum_{\sigma=1}^{m} \frac{1}{(\sigma+c_1)^{d_1}}$ and $H_{c_2,d_2}^N = \sum_{\sigma=1}^{m} \frac{1}{(\sigma+c_2)^{d_2}}$. Now using $\mathfrak{p}_i = \dfrac{1}{(i+c_1)^{d_1} H_{c_1,d_1}^N}$ and $\mathfrak{q}_i = \dfrac{1}{(i+c_2)^{d_2} H_{c_2,d_2}^N}$ in (31), then the following holds*

$$P_1(2)\mathfrak{f}''(\lambda_1(2)) \leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} D_{kl}(\mathfrak{q},\mathfrak{p}) + \frac{\mathfrak{r}-1}{\mathfrak{m}-1} \sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i \ln\left(\sum_{i=1}^{\mathfrak{m}} \mathfrak{q}_i\right)$$

$$-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}} \sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}\right) \ln\left(\frac{\sum_{j=1}^{\mathfrak{r}} \mathfrak{q}_{i_j}}{\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j}}\right) - \frac{1}{2(\varrho_2-\varrho_1)} \times$$

$$\left\{(1+\ln\varrho_1)\mathbf{F}\Big(\frac{(i+c_1)^{d_1} H_{c_1,d_1}^N}{(i+c_2)^{d_2} H_{c_2,d_2}^N}, \mathfrak{q}; (\mathrm{z}-\varrho_1)^2\Big) - (1+\ln\varrho_2)\mathbf{F}\Big(\frac{(i+c_1)^{d_1} H_{c_1,d_1}^N}{(i+c_2)^{d_2} H_{c_2,d_2}^N}, \mathfrak{q}; (\mathrm{z}-\varrho_2)^2\Big)\right\}$$

$$\leq P_1(2)\left(\frac{\varrho_2-\lambda_1(2)}{\varrho_2-\varrho_1}\mathfrak{f}''(\varrho_1) + \frac{\lambda_1(2)-\varrho_1}{\varrho_2-\varrho_1}\mathfrak{f}''(\varrho_2)\right).$$

Now we will give result for Hybrid Zipf-Mandelbrot entropy.

Hybrid Zipf-Mandelbrot entropy is the generalization of Zipf-Mandelbrot entropy. Consider N be natural numbers, $0 \leq c < \infty$ and $0 < d, \omega < \infty$.
Hybrid Zipf-Mandelbrot entropy is defined as:

$$\widehat{Z}_M(H^*, c, d, \omega) = \frac{1}{H_{c,d,\omega}^*} \sum_{i=1}^{N} \frac{\omega^i}{(i+c)^s} \ln\left(\frac{(i+c)^s}{\omega^i}\right) + \ln(H_{c,d,\omega}^*), \tag{37}$$

where

$$H_{c,d,\omega}^* = \sum_{i=1}^{N} \frac{\omega^i}{(i+c)^d}.$$

Consider

$$\mathfrak{p}_i = \mathfrak{f}(i; N, c, d, \omega) = \frac{\omega^i}{(i+c)^d H^*_{c,d,\omega}}, \tag{38}$$

is known as Hybrid Zipf-Mandelbrot law.

Theorem 4.4. *Let* $\mathfrak{p}$ *be a discrete probability distribution given in* (38) *with parameters* $0 \le c < \infty$, $0 < d, \omega < \infty$ *and* N *be natural numbers, then we have*

$$\begin{aligned}
&\frac{(-1)^n P_1(n)}{(\lambda_1(n))^n} \le \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} S(\mathfrak{p}) - \frac{\mathfrak{r}-1}{\mathfrak{m}-1} \ln(\mathfrak{m}) \\
&- \frac{1}{C^{\mathfrak{m}-1}_{\mathfrak{r}-1}} \sum_{1 \le i_1 < \ldots < i_{\mathfrak{r}} \le \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j} \right) \ln \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j} \right) - \frac{1}{(\varrho_2 - \varrho_1)} \sum_{w=0}^{n-2} \left(\frac{1}{(w+2)} \right) \times \\
&\left\{ \left(\frac{(-1)^{w+1}}{\varrho_1^{w+1}} \right) \mathbf{F}\left(\frac{(i+c)^d H^*_{c,d,\omega}}{\omega^i}, \mathfrak{q}; (\mathrm{z} - \varrho_1)^{w+2} \right) \right. \\
&\left. - \left(\frac{(-1)^{w+1}}{\varrho_2^{w+1}} \right) \mathbf{F}\left(\frac{(i+c)^d H^*_{c,d,\omega}}{\omega^i}, \mathfrak{q}; (\mathrm{z} - \varrho_2)^{w+2} \right) \right\} \\
&\le P_1(n) \left(\frac{(-1)^n (\varrho_2 - \lambda_1(n))}{\varrho_1^n (\varrho_2 - \varrho_1)} + \frac{(-1)^n (\lambda_1(n) - \varrho_1)}{\varrho_2^n (\varrho_2 - \varrho_1)} \right).
\end{aligned} \tag{39}$$

Proof. We can get Hybrid Zipf-Mandelbrot entropy (37) by substituting $\mathfrak{p}_i$ defined in (38) from Shannon entropy (33).

$$\begin{aligned}
S(\mathfrak{p}) &= -\sum_{i=1}^{N} \mathfrak{p}_i \ln \mathfrak{p}_i \\
&= -\sum_{i=1}^{N} \frac{\omega^i}{(i+c)^d H^*_{c,d,\omega}} \ln \frac{\omega^i}{(i+c)^d H^*_{c,d,\omega}} \\
&= \frac{-1}{H^*_{c,d,\omega}} \sum_{i=1}^{N} \frac{\omega^i}{(i+c)^d} \left[\ln\left(\frac{\omega^i}{(i+c)^d} \right) + \ln\left(\frac{1}{H^*_{c,d,\omega}} \right) \right] \\
&= \frac{1}{H^*_{c,d,\omega}} \sum_{i=1}^{N} \frac{\omega^i}{(i+c)^d} \left[\ln\left(\frac{(i+c)^d}{\omega^i} \right) + \ln\left(H^*_{c,d,\omega} \right) \right] \\
&= \frac{1}{H^*_{c,d,\omega}} \sum_{i=1}^{N} \frac{\omega^i}{(i+c)^d} \ln\left(\frac{(i+c)^d}{\omega^i} \right) + \ln\left(H^*_{c,d,\omega} \right).
\end{aligned}$$

Finally, substituting this $\mathfrak{p}_i = \frac{\omega^i}{(i+c)^d H^*_{c,d,\omega}}$ in (34), after simple calculations we get (39). □

Remark 4.5. *Under the assumption of Theorem 4.4, then for* $n = 2$ *we have the following result*

$$\begin{aligned}
&\frac{P_1(2)}{(\lambda_1(2))} \le \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1} S(\mathfrak{p}) - \frac{\mathfrak{r}-1}{\mathfrak{m}-1} \ln(\mathfrak{m}) \\
&- \frac{1}{C^{\mathfrak{m}-1}_{\mathfrak{r}-1}} \sum_{1 \le i_1 < \ldots < i_{\mathfrak{r}} \le \mathfrak{m}} \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j} \right) \ln \left(\sum_{j=1}^{\mathfrak{r}} \mathfrak{p}_{i_j} \right) - \frac{1}{2(\varrho_2 - \varrho_1)} \\
&\left\{ \left(\frac{-1}{\varrho_1} \right) \mathbf{F}\left(\frac{(i+c)^d H^*_{c,d,\omega}}{\omega^i}, \mathfrak{q}; (\mathrm{z} - \varrho_1)^2 \right) - \left(\frac{-1}{\varrho_2} \right) \mathbf{F}\left(\frac{(i+c)^d H^*_{c,d,\omega}}{\omega^i}, \mathfrak{q}; (\mathrm{z} - \varrho_2)^2 \right) \right\}
\end{aligned}$$

$$\leq P_1(n)\left(\frac{(\varrho_2-\lambda_1(2))}{\varrho_1^2(\varrho_2-\varrho_1)}+\frac{(\lambda_1(2)-\varrho_1)}{\varrho_2^2(\varrho_2-\varrho_1)}\right).$$

Corollary 4.6. *Let Hybrid Zipf-Mandelbrot law defined as* $\mathfrak{q}$ *and* $\mathfrak{p}$ *with parameters* $c_1, c_2 \geq 0$, $\omega_1, \omega_2, d_1, d_2 > 0$. *Now using* $\mathfrak{p}_i = \dfrac{\omega_1^i}{(i+c_1)^{d_1} H^*_{c_1,d_1,\omega_1}}$ *and* $\mathfrak{q}_i = \dfrac{\omega_2^i}{(i_2+c_2)^{d_2} H^*_{c_2,d_2,\omega_2}}$ *in* (31), *then the following holds*

$$P_1(2)\mathfrak{f}''(\lambda_1(2)) \leq \frac{\mathfrak{m}-\mathfrak{r}}{\mathfrak{m}-1}D_{kl}(\mathfrak{q},\mathfrak{p}) + \frac{\mathfrak{r}-1}{\mathfrak{m}-1}\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i \ln\left(\sum_{i=1}^{\mathfrak{m}}\mathfrak{q}_i\right)$$

$$-\frac{1}{C_{\mathfrak{r}-1}^{\mathfrak{m}-1}}\sum_{1\leq i_1<\ldots<i_{\mathfrak{r}}\leq \mathfrak{m}}\left(\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}\right)\ln\left(\frac{\sum_{j=1}^{\mathfrak{r}}\mathfrak{q}_{i_j}}{\sum_{j=1}^{\mathfrak{r}}\mathfrak{p}_{i_j}}\right)-\frac{1}{2(\varrho_2-\varrho_1)}\times$$

$$\left\{(1+\ln\varrho_1)\mathbf{F}\left(\frac{\omega_2^J(J_1+c_1)^{d_1}H^*_{c_1,d_1,\omega_1}}{\omega_1^J(J_2+c_2)^{d_2}H^*_{c_2,d_2,\omega_2}},\mathfrak{q};(\mathrm{z}-\varrho_1)^2\right)\right.$$

$$\left.-(1+\ln\varrho_2)\mathbf{F}\left(\frac{\omega_2^J(J_1+c_1)^{d_1}H^*_{c_1,d_1,\omega_1}}{\omega_1^J(J_2+c_2)^{d_2}H^*_{c_2,d_2,\omega_2}},\mathfrak{q};(\mathrm{z}-\varrho_2)^2\right)\right\}$$

$$\leq P_1(2)\left(\frac{\varrho_2-\lambda_1(2)}{\varrho_2-\varrho_1}\mathfrak{f}''(\varrho_1)+\frac{\lambda_1(2)-\varrho_1}{\varrho_2-\varrho_1}\mathfrak{f}''(\varrho_2)\right). \tag{40}$$

5. Conclusion

In this research article we give new bounds for Popoviciu inequality by using weighted Hermite Hadamard inequality along with the approximations of Montgomery two point formula. We also give bounds of Popoviciu inequality by employing weighted Hermite Hadamard inequality along with the approximations of Montgomery one point formula. We testify this scenario by utilizing the theory of n-times differentiable convex functions. Our results hold for all $n \geq 2$ and we provides explicit examples to show the correctness of the bounds obtained for special cases. As an application in information theory we give new divergence functionals i.e. Csiszár divergence, Rényi-divergence, Shannon-entropy, Kullback-Leibler divergence, Zipf and Hybrid Zipf-Mandelbrot entropy and investigate their properties. It is interesting to give such bounds by considering Taylor, Abel-Gontscharoff and Hermite interpolating polynomials, whose remainder terms contains n-times differentiable functions. In future, we are also working to apply similar idea as in this paper but on Jensen and cyclic Jensen differences functionals.

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