



## $b_3$ -subbalancing and $b_3$ -Lucas subbalancing numbers

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**Abstract.** In this paper, we derive some new algebraic identities on  $b_3$ -subbalancing numbers and obtain the functions that generate  $b_3$ -subbalancing numbers. We also introduce  $b_3$ -Lucas subbalancing numbers and derive some algebraic identities between  $b_3$ -Lucas subbalancing numbers and  $b_3$ -subbalancing numbers. Further, we give the Binet formulas for  $b_3$ -subbalancing and  $b_3$ -Lucas subbalancing numbers.

### 1. Introduction

Behera and Panda [1] introduced balancing numbers  $n$  as solutions of the Diophantine equation

$$1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r) \quad (1)$$

for some positive integers  $r$ . The positive integer  $r$  in (1) is called the balancer of  $n$ . For example, 6, 35, 204 and 1189 are balancing numbers with balancers which are 2, 14, 84 and 492, respectively. The  $n^{\text{th}}$  balancing number is denoted by  $B_n$ .

If  $n$  is a balancing number with balancer  $r$ , then from (1)

$$n^2 = \frac{(n+r)(n+r+1)}{2} \quad \text{and} \quad r = \frac{-(2n+1) + \sqrt{8n^2+1}}{2} \quad (2)$$

It follows from (2) that  $n$  is a balancing number if and only if  $8n^2 + 1$  is a perfect square. Since  $8n^2 + 1$  is a perfect square for  $n = 1$ , 1 is accepted as a balancing number.

Balancing numbers satisfy the following recurrence relation

$$B_{n+1} = 6B_n - B_{n-1} \quad , \quad (n \geq 2)$$

with initial terms  $B_1 = 1$ ,  $B_2 = 6$ .

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By modifying (1) slightly, Panda and Ray [2] introduced cobalancing numbers  $n$  as solutions of the Diophantine equation

$$1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r) \quad (3)$$

for some positive integers  $r$ . The positive integer  $r$  in (3) is called the cobalancer of  $n$ . For example, 2, 14, 84 are cobalancing numbers with cobalancers which are 1, 6, 35, respectively. The  $n^{\text{th}}$  cobalancing number is denoted by  $b_n$ .

Cobalancing numbers satisfy the following recurrence relation

$$b_{n+1} = 6b_n - b_{n-1} + 2, \quad (n \geq 2)$$

with initial terms  $b_1 = 0$ ,  $b_2 = 2$ .

If  $n$  is a cobalancing number with cobalancer  $r$ , then from (3)

$$n(n+1) = \frac{(n+r)(n+r+1)}{2} \quad \text{and} \quad r = \frac{-(2n+1) + \sqrt{8n^2 + 8n + 1}}{2} \quad (4)$$

It follows from (4) that  $n$  is a cobalancing number if and only if  $8n^2 + 8n + 1$  is a perfect square.

The numbers  $C_n = \sqrt{8b_n^2 + 1}$  and  $c_n = \sqrt{8b_n^2 + 8b_n + 1}$  are called the  $n^{\text{th}}$  Lucas-balancing number and the  $n^{\text{th}}$  Lucas-cobalancing number, respectively ([3, 4]). For example, 3, 17, 99 are Lucas-balancing numbers and 1, 7, 41 are Lucas-cobalancing numbers.

Lucas-balancing and Lucas-cobalancing numbers satisfy the following recurrence relations

$$C_{n+1} = 6C_n - C_{n-1}, \quad (n \geq 2)$$

$$c_{n+1} = 6c_n - c_{n-1}, \quad (n \geq 2)$$

with initial terms  $C_1 = 3$ ,  $C_2 = 17$  and  $c_1 = 1$ ,  $c_2 = 7$ , respectively.

Further in [3], the Binet formulas for balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers are given as:

$$B_n = \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} \quad b_n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2}$$

$$C_n = \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2} \quad c_n = \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2}$$

where  $\alpha_1 = 1 + \sqrt{2}$  and  $\alpha_2 = 1 - \sqrt{2}$ .

The concept of balancing numbers, cobalancing numbers, Lucas-balancing numbers and Lucas-cobalancing numbers has been extended and generalized by many authors. In [5], Panda and Ray observed relations between balancing and cobalancing numbers with Pell and Associated Pell numbers. Later in [6], relations between balancing numbers and Pell, Pell-Lucas and oblong numbers were obtained. Further in [7], some identities on the sums, divisibility properties, perfect squares, Pythagorean triples involving Pell, Pell-Lucas and balancing numbers were derived.

There have been studies about investigating balancing numbers in some sequences. Liptai [8] proved that there is no Fibonacci balancing number except 1. Later in [9], he derived that there is no balancing number in the terms of the Lucas sequence. In [10], Szalay also obtained the same result using different

method.

Panda and Rout [11], introduced gap balancing numbers and studied specifically on 2-gap balancing numbers. Further in [12], some algebraic identities of  $k$ -gap balancing numbers for arbitrary  $k$  were given. In [13], Panda and Panda defined almost balancing numbers and obtained some algebraic equations related to these numbers. Later in [14], Tekcan derived algebraic relations between almost balancing numbers and triangular and square triangular numbers. Further in [15], he obtained some algebraic identities on sums of almost balancing numbers.

In [16], Panda generalized the concept of balancing and cobalancing numbers to an arbitrary sequence. Later in [17],  $(k, l)$ -balancing numbers were defined and some effective and ineffective finiteness results were proved for  $(k, l)$ -balancing numbers. Some properties of generalized balancing numbers were obtained in [18]. Further in [19], a survey about balancing, cobalancing and generalized balancing numbers was given.

Davala and Panda [20] defined supercobalancing numbers. Later, in [21] they generalized the concept of almost balancing numbers by defining  $D$ -subbalancing numbers  $n$  as solutions of the Diophantine equation

$$1 + 2 + \cdots + (n - 1) + D = (n + 1) + (n + 2) + \cdots + (n + r) \quad (5)$$

for some positive integer  $r$ , where  $D$  is a fixed positive integer. The positive integer  $r$  in (5) is called  $D$ -subbalancer of  $n$ . They observed that the choice of the positive integer  $D$  is important for obtaining  $D$ -subbalancing numbers and there is no positive integer  $n$  that satisfies the equation (5), for every  $D$ . They showed that it is possible to find a positive integer  $n$  that satisfies the equation (5) when  $D$  is chosen as cobalancing numbers. They obtained basic concepts and theorems about  $b_n$ -subbalancing numbers. Specifically, they examined  $b_3$ -subbalancing numbers and  $b_5$ -subbalancing numbers and showed that  $b_3$ -subbalancing numbers can be classified in two classes of the form  $11B_m + C_m$ ,  $11B_{m+1} - C_{m+1}$ ,  $m \geq 0$ .

In the present work, we consider  $b_3$ -subbalancing numbers and give some new algebraic identities provided by the  $b_3$ -subbalancing numbers and obtain the functions that generate  $b_3$ -subbalancing numbers. We also prove that any two consecutive even terms of the sequence of  $b_3$ -subbalancing numbers and any two consecutive odd terms of the sequence of  $b_3$ -subbalancing numbers are relatively prime. Further, similar to Lucas-balancing numbers, we introduce  $b_3$ -Lucas subbalancing numbers. We give the Binet formulas for  $b_3$ -subbalancing and  $b_3$ -Lucas subbalancing numbers. Later, we obtain some algebraic relations between  $b_3$ -Lucas subbalancing numbers and  $b_3$ -subbalancing numbers.

## 2. Preliminaries

Throughout this paper, we denote the  $m^{\text{th}}$   $b_3$ -subbalancing number and the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number as  $(Sb_3)_m$  and  $(CSb_3)_m$ , respectively. We get the following results regarding  $b_3$ -subbalancing numbers from [21].

**Theorem 2.1.** Let  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number, let  $B_m$  denote the  $m^{\text{th}}$  balancing number and let  $C_m$  denote the  $m^{\text{th}}$  Lucas-balancing number. Then

$$\begin{aligned} (Sb_3)_{2m} &= 11B_m + C_m \\ (Sb_3)_{2m+1} &= 11B_{m+1} - C_{m+1} \end{aligned}$$

for  $m \geq 0$ .

**Theorem 2.2.** Let  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number. Then, the recurrence relation for  $b_3$ -subbalancing numbers is

$$(Sb_3)_{m+2} = 6(Sb_3)_m - (Sb_3)_{m-2} \quad (m \geq 2)$$

where  $(Sb_3)_0 = 1, (Sb_3)_1 = 8, (Sb_3)_2 = 14, (Sb_3)_3 = 49$ .

**Corollary 2.1.**  $(Sb_3)_m$  is a  $b_3$ -subbalancing number if and only if  $8(Sb_3)_m^2 + 113$  is a perfect square.

**Theorem 2.3.** Let  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number and  $B_m$  denote the  $m^{\text{th}}$  balancing number. Then

$$B_m = \frac{11(Sb_3)_{2m} - \sqrt{8(Sb_3)_{2m}^2 + 113}}{113} \quad (m \geq 0)$$

$$B_m = \frac{11(Sb_3)_{2m-1} + \sqrt{8(Sb_3)_{2m-1}^2 + 113}}{113} \quad (m \geq 1).$$

### 3. Main Results

In this section we derive our main results.

#### 3.1. Algebraic Identities for $b_3$ -Subbalancing Numbers

In this subsection, we obtain some new algebraic identities between  $b_3$ -subbalancing numbers and balancing and Lucas-balancing numbers.

**Theorem 3.1.** Let  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number and  $B_m$  denote the  $m^{\text{th}}$  balancing number. Then

$$(Sb_3)_{2m+1} = 8(Sb_3)_{2m} - 63B_m \quad (m \geq 0)$$

and

$$(Sb_3)_{2m} = 22B_m - (Sb_3)_{2m-1} \quad (m \geq 1).$$

*Proof.* It follows from Theorem 2.1 that

$$(Sb_3)_{2m+1} = 8B_{m+1} + B_m \quad (6)$$

and

$$(Sb_3)_{2m} = 8B_m + B_{m+1} \quad (7)$$

From (6) and (7), we get

$$(Sb_3)_{2m+1} = 8(Sb_3)_{2m} - 63B_m.$$

Further from (6), we obtain

$$(Sb_3)_{2m-1} = 8B_m + B_{m-1} \quad (8)$$

From (7), we get

$$(Sb_3)_{2m} = 14B_m - B_{m-1} \quad (9)$$

From (8) and (9), we get

$$(Sb_3)_{2m} = 22B_m - (Sb_3)_{2m-1}.$$

**Remark 3.1.** We observe that all terms of the sequence of  $b_3$ -subbalancing numbers can be written in terms of the sequence of the balancing numbers without involving any other number sequence.

**Corollary 3.1.** *The sum of any even term of the sequence of  $b_3$ -subbalancing numbers and the preceding one is always even. Equivalently to this, the sum of any odd term of the sequence of  $b_3$ -subbalancing numbers and the next one is always even.*

**Corollary 3.2.** *Let  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number and  $B_m$  denote the  $m^{\text{th}}$  balancing number. Then*

$$4(Sb_3)_{2m} = 7(Sb_3)_{2m-1} - 11B_{m-1}$$

for  $m \geq 1$ .

*Proof.* Using (8) and (9), we obtain

$$\begin{aligned} 7(Sb_3)_{2m-1} &= 56B_m + 7B_{m-1} \\ 4(Sb_3)_{2m} &= 56B_m - 4B_{m-1}. \end{aligned}$$

Thus, we get

$$4(Sb_3)_{2m} = 7(Sb_3)_{2m-1} - 11B_{m-1}.$$

**Theorem 3.2.** *Let  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number. Then*

$$(Sb_3)_{2m} = \frac{113(Sb_3)_{2m-1} - 22(Sb_3)_{2m-2}}{63}$$

and

$$(Sb_3)_{2m+1} = \frac{113(Sb_3)_{2m} - 63(Sb_3)_{2m-1}}{22}$$

for  $m \geq 1$ .

*Proof.* From Theorem 3.1, we deduce that

$$8(Sb_3)_{2m-2} = (Sb_3)_{2m-1} + 63B_{m-1} \tag{10}$$

Using (10) and Corollary 3.2, we obtain

$$252(Sb_3)_{2m} + 88(Sb_3)_{2m-2} = 452(Sb_3)_{2m-1} \tag{11}$$

From (11), we get

$$63(Sb_3)_{2m} = 113(Sb_3)_{2m-1} - 22(Sb_3)_{2m-2}.$$

Thus, we get

$$(Sb_3)_{2m} = \frac{113(Sb_3)_{2m-1} - 22(Sb_3)_{2m-2}}{63}.$$

The other case can be proved similarly.

In the following two theorems the relations between  $b_3$ -subbalancing numbers and Lucas-balancing numbers are given.

**Theorem 3.3.** *Let  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number and  $C_m$  denote the  $m^{\text{th}}$  Lucas-balancing number. Then*

$$11(Sb_3)_{2m+1} = 25(Sb_3)_{2m} + 63C_m$$

for  $m \geq 0$ .

*Proof.* Using Theorem 2.1 and relations between balancing numbers and Lucas-balancing numbers, we obtain

$$(Sb_3)_{2m+1} = 25B_m + 8C_m.$$

From Theorem 2.1, we get

$$11(Sb_3)_{2m+1} = 25(Sb_3)_{2m} + 63C_m.$$

**Theorem 3.4.** Let  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number and  $C_m$  denote the  $m^{\text{th}}$  Lucas-balancing number. Then

$$(Sb_3)_{2m} = (Sb_3)_{2m-1} + 2C_m$$

for  $m \geq 1$ .

*Proof.* It follows from Theorem 2.1 that

$$(Sb_3)_{2m-1} = 11B_m - C_m.$$

Thus, we get

$$(Sb_3)_{2m} = (Sb_3)_{2m-1} + 2C_m.$$

**Theorem 3.5.** Let  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number. Then

$$(Sb_3)_m^2 = (Sb_3)_{m-2}(Sb_3)_{m+2} + 113$$

for  $m \geq 2$ .

*Proof.* This theorem is proved by induction. It is easily seen that the assertion is true for  $m = 2$ . Assuming the assertion is true for  $m \leq k$ , we have

$$\begin{aligned} (Sb_3)_{k+1}^2 &= (Sb_3)_{k+1}(Sb_3)_{k+1} \\ &= (6(Sb_3)_{k-1} - (Sb_3)_{k-3})(Sb_3)_{k+1} \\ &= 6(Sb_3)_{k-1}(Sb_3)_{k+1} - (Sb_3)_{k-1}^2 + 113 \\ &= (Sb_3)_{k-1}(6(Sb_3)_{k+1} - (Sb_3)_{k-1}) + 113 \\ &= (Sb_3)_{k-1}(Sb_3)_{k+3} + 113. \end{aligned}$$

Thus it is shown that the assertion is true for  $m = k + 1$ .

**Theorem 3.6.** Let  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number and  $B_m$  denote the  $m^{\text{th}}$  balancing number. Then

$$(Sb_3)_{m+1}(Sb_3)_m - (Sb_3)_{m-1}(Sb_3)_{m-2} = 113B_m$$

for  $m \geq 2$ .

*Proof.* This theorem is proved by induction. It is easily seen that the assertion is true for  $m = 2$ . Assuming the assertion is true for  $m \leq k$ , we have

$$\begin{aligned} (Sb_3)_{k+2}(Sb_3)_{k+1} - (Sb_3)_k(Sb_3)_{k-1} &= (6(Sb_3)_k - (Sb_3)_{k-2})(Sb_3)_{k+1} - (Sb_3)_k(Sb_3)_{k-1} \\ &= 6(Sb_3)_{k+1}(Sb_3)_k - (Sb_3)_{k+1}(Sb_3)_{k-2} - (Sb_3)_k(Sb_3)_{k-1} \\ &= 6(Sb_3)_{k+1}(Sb_3)_k - 6(Sb_3)_{k-1}(Sb_3)_{k-2} \\ &\quad + (Sb_3)_{k-2}(Sb_3)_{k-3} - (Sb_3)_k(Sb_3)_{k-1} \\ &= 6((Sb_3)_{k+1}(Sb_3)_k - (Sb_3)_{k-1}(Sb_3)_{k-2}) \\ &\quad - ((Sb_3)_k(Sb_3)_{k-1} - (Sb_3)_{k-2}(Sb_3)_{k-3}) \\ &= 6(113B_k) - 113B_{k-1} \\ &= 113B_{k+1}. \end{aligned}$$

Thus it is shown that the assertion is true for  $m = k + 1$ .

**Theorem 3.7.** Let  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number and  $B_n$  denote the  $n^{\text{th}}$  balancing number. Then

$$(Sb_3)_{2n+m-1} = (Sb_3)_{m-1}B_{n+1} - (Sb_3)_{m-3}B_n$$

for any two positive integers  $m$  and  $n$ .

*Proof.* This theorem is proved by induction. It is easily seen that the assertion is true for  $n = 1$ . Assuming the assertion is true for  $n \leq k$ , we have

$$\begin{aligned} (Sb_3)_{2k+m+1} &= 6(Sb_3)_{2k+m-1} - (Sb_3)_{2k+m-3} \\ &= 6(B_{k+1}(Sb_3)_{m-1} - B_k(Sb_3)_{m-3}) - (B_k(Sb_3)_{m-1} - B_{k-1}(Sb_3)_{m-3}) \\ &= (Sb_3)_{m-1}(6B_{k+1} - B_k) - (Sb_3)_{m-3}(6B_k - B_{k-1}) \\ &= B_{k+2}(Sb_3)_{m-1} - B_{k+1}(Sb_3)_{m-3}. \end{aligned}$$

Thus it is shown that the assertion is true for  $n = k + 1$ .

**Theorem 3.8.** Any two consecutive even terms of the sequence of  $b_3$ -subbalancing numbers and any two consecutive odd terms of the sequence of  $b_3$ -subbalancing numbers are relatively prime, that is

$$((Sb_3)_{2m}, (Sb_3)_{2m-2}) = 1 \quad \text{and} \quad ((Sb_3)_{2m+1}, (Sb_3)_{2m-1}) = 1.$$

*Proof.* Using the Euclidean Algorithm and Theorem 2.1, we deduce that

$$\begin{aligned} (Sb_3)_{2m} &= 6(Sb_3)_{2m-2} + [-(Sb_3)_{2m-2} + (Sb_3)_{2m-2}] - (Sb_3)_{2m-4} \\ (Sb_3)_{2m-2} &= [(Sb_3)_{2m-2} - (Sb_3)_{2m-4}] \times 1 + (Sb_3)_{2m-4} \\ (Sb_3)_{2m-2} - (Sb_3)_{2m-4} &= (6 - 2)(Sb_3)_{2m-4} + [(Sb_3)_{2m-4} - (Sb_3)_{2m-6}] \\ (Sb_3)_{2m-4} &= [(Sb_3)_{2m-4} - (Sb_3)_{2m-6}] \times 1 + (Sb_3)_{2m-6} \\ (Sb_3)_{2m-4} - (Sb_3)_{2m-6} &= (6 - 2)(Sb_3)_{2m-6} + [(Sb_3)_{2m-6} - (Sb_3)_{2m-8}] \\ &\vdots \\ (Sb_3)_4 - (Sb_3)_2 &= (6 - 2)(Sb_3)_2 + [(Sb_3)_2 - (Sb_3)_0] \\ (Sb_3)_2 &= [(Sb_3)_2 - (Sb_3)_0] \times 1 + (Sb_3)_0 \\ (Sb_3)_2 - (Sb_3)_0 &= [(Sb_3)_2 - (Sb_3)_0](Sb_3)_0 + 0. \end{aligned}$$

Since  $((Sb_3)_2 - (Sb_3)_0, (Sb_3)_0) = 1$ , we get  $((Sb_3)_{2m}, (Sb_3)_{2m-2}) = 1$ .

The other case can be proved similarly.

In the following theorem the Binet formula for  $b_3$ -subbalancing numbers is given.

**Theorem 3.9.** Let  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number. Then

$$(Sb_3)_{2m} = \frac{(11\sqrt{2} + 4)\alpha_1^m - (11\sqrt{2} - 4)\alpha_2^m}{8}$$

and

$$(Sb_3)_{2m+1} = \frac{(11\sqrt{2} - 4)\alpha_1^{m+1} - (11\sqrt{2} + 4)\alpha_2^{m+1}}{8}$$

where  $\alpha_1 = 3 + 2\sqrt{2}$ ,  $\alpha_2 = 3 - 2\sqrt{2}$ .

*Proof.* From the Binet formulas for balancing and Lucas-balancing numbers, we get

$$\begin{aligned}
(Sb_3)_{2m} &= 11 \left( \frac{\alpha_1^m - \alpha_2^m}{4\sqrt{2}} \right) + \left( \frac{\alpha_1^m + \alpha_2^m}{2} \right) \\
&= \frac{(4 + 11\sqrt{2})\alpha_1^m + (4 - 11\sqrt{2})\alpha_2^m}{8} \\
&= \frac{(11\sqrt{2} + 4)\alpha_1^m - (11\sqrt{2} - 4)\alpha_2^m}{8}.
\end{aligned}$$

The other case can be proved similarly.

### 3.2. Functions Generating $b_3$ -Subbalancing Numbers

In this subsection, we present some functions that generate  $b_3$ -subbalancing numbers.

**Theorem 3.10.** For any even term of the sequence of  $b_3$ -subbalancing numbers  $x$ ,  $g(x) = \frac{129x-22\sqrt{8x^2+113}}{113}$  is the  $b_3$ -subbalancing number just prior to it and  $\tilde{g}(x) = \frac{211x+63\sqrt{8x^2+113}}{113}$  is the  $b_3$ -subbalancing number next to it.

*Proof.* Since  $x$  even term of the sequence of  $b_3$ -subbalancing numbers,  $x = (Sb_3)_{2m}$

for some positive integer  $m$ .

From Theorem 2.3 and Theorem 3.1, we get

$$\begin{aligned}
(Sb_3)_{2m-1} &= 22 \left( \frac{11(Sb_3)_{2m} - \sqrt{8(Sb_3)_{2m}^2 + 113}}{113} \right) - (Sb_3)_{2m} \\
&= \frac{129(Sb_3)_{2m} - 22\sqrt{8(Sb_3)_{2m}^2 + 113}}{113}
\end{aligned}$$

which is equivalent to

$$(Sb_3)_{2m-1} = g((Sb_3)_{2m})$$

proving that the  $b_3$ -subbalancing number just prior to  $x$ .  
Similarly, we get

$$\begin{aligned}
(Sb_3)_{2m+1} &= 8(Sb_3)_{2m} - 63 \left( \frac{11(Sb_3)_{2m} - \sqrt{8(Sb_3)_{2m}^2 + 113}}{113} \right) \\
&= \frac{211(Sb_3)_{2m} + 63\sqrt{8(Sb_3)_{2m}^2 + 113}}{113}.
\end{aligned}$$

Thus the  $b_3$ -subbalancing number next to  $(Sb_3)_{2m}$  is

$$\tilde{g}((Sb_3)_{2m}) = (Sb_3)_{2m+1}.$$

**Theorem 3.11.** For any odd term of the sequence of  $b_3$ -subbalancing numbers  $x$ ,  $g(x) = \frac{211x-63\sqrt{8x^2+113}}{113}$  is the  $b_3$ -subbalancing number just prior to it and  $\tilde{g}(x) = \frac{129x+22\sqrt{8x^2+113}}{113}$  is the  $b_3$ -subbalancing number next to it.



*Proof.* Since  $x$  any odd term of the sequence of  $b_3$ -subbalancing numbers,  $x = (Sb_3)_{2m+1}$  for some positive integer  $m$ .

From (6) and Theorem 2.3, we get

$$\begin{aligned} B_m &= (Sb_3)_{2m+1} - 8 \left( \frac{11(Sb_3)_{2m+1} + \sqrt{8(Sb_3)_{2m+1}^2 + 113}}{113} \right) \\ &= \frac{25(Sb_3)_{2m+1} - 8\sqrt{8(Sb_3)_{2m+1}^2 + 113}}{113} \end{aligned} \tag{12}$$

It follows from (7) and (12) that

$$\begin{aligned} (Sb_3)_{2m} &= 8 \left( \frac{25(Sb_3)_{2m+1} - 8\sqrt{8(Sb_3)_{2m+1}^2 + 113}}{113} \right) + \left( \frac{11(Sb_3)_{2m+1} + \sqrt{8(Sb_3)_{2m+1}^2 + 113}}{113} \right) \\ &= \frac{211(Sb_3)_{2m+1} - 63\sqrt{8(Sb_3)_{2m+1}^2 + 113}}{113}. \end{aligned}$$

Thus the  $b_3$ -subbalancing number just prior to  $(Sb_3)_{2m+1}$  is

$$g((Sb_3)_{2m+1}) = (Sb_3)_{2m}.$$

Using Theorem 2.3 and Theorem 3.1, we get

$$\begin{aligned} (Sb_3)_{2m+2} &= 22 \left( \frac{11(Sb_3)_{2m+1} + \sqrt{8(Sb_3)_{2m+1}^2 + 113}}{113} \right) - (Sb_3)_{2m+1} \\ &= \frac{129(Sb_3)_{2m+1} + 22\sqrt{8(Sb_3)_{2m+1}^2 + 113}}{113}. \end{aligned}$$

Thus the  $b_3$ -subbalancing number next to  $(Sb_3)_{2m+1}$  is

$$\tilde{g}((Sb_3)_{2m+1}) = (Sb_3)_{2m+2}.$$

**Theorem 3.12.** *If  $x$  is a  $b_3$ -subbalancing number, then the functions  $f(x) = 3x + \sqrt{8x^2 + 113}$  and  $g(x) = 17x + 6\sqrt{8x^2 + 113}$  are also  $b_3$ -subbalancing numbers.*

*Proof.*

$$\begin{aligned} 8(f(x))^2 + 113 &= 8(3x + \sqrt{8x^2 + 113})^2 + 113 \\ &= 136x^2 + 48x\sqrt{8x^2 + 113} + 1017 \\ &= (8x + 3\sqrt{8x^2 + 113})^2 \end{aligned} \tag{13}$$

Since  $8(f(x))^2 + 113$  is a perfect square,  $f(x)$  is a  $b_3$ -subbalancing number.

$$\begin{aligned} g(x) &= 17x + 6\sqrt{8x^2 + 113} \\ &= 3(3x + \sqrt{8x^2 + 113}) + 8x + 3\sqrt{8x^2 + 113}. \end{aligned}$$

From (13), we get

$$\begin{aligned} g(x) &= 3(3x + \sqrt{8x^2 + 113}) + \sqrt{8(f(x))^2 + 113} \\ &= 3f(x) + \sqrt{8(f(x))^2 + 113} \\ &= f(f(x)). \end{aligned}$$

Since  $g(x) = f(f(x))$ ,  $g(x)$  is a  $b_3$ -subbalancing number.

**Remark 3.2.** We observe that when  $x$  is even  $f(x)$  generates odd  $b_3$ -subbalancing numbers while  $g(x)$  generates even  $b_3$ -subbalancing numbers. Similarly, when  $x$  is odd  $f(x)$  generates even  $b_3$ -subbalancing numbers while  $g(x)$  generates odd  $b_3$ -subbalancing numbers.

The following theorem shows that the function  $f(x)$  in Theorem 3.12 is equal to the  $(m + 2)^{\text{th}}$   $b_3$ -subbalancing number when  $x$  is equal to the  $m^{\text{th}}$   $b_3$ -subbalancing number.

**Theorem 3.13.** Let  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number and  $f(x) = 3x + \sqrt{8x^2 + 113}$ . Then

$$f((Sb_3)_m) = (Sb_3)_{m+2}$$

for  $m \geq 0$ .

*Proof.* Using the recurrence relation for  $b_3$ -subbalancing numbers, we get

$$\begin{aligned} (Sb_3)_{m+2} &= 6(Sb_3)_m - (Sb_3)_{m-2} \\ &= 3(Sb_3)_m + (3(Sb_3)_m - (Sb_3)_{m-2}) \end{aligned} \tag{14}$$

Then

$$\begin{aligned} (3(Sb_3)_m - (Sb_3)_{m-2})^2 &= 9(Sb_3)_m^2 + (Sb_3)_{m-2}^2 - ((Sb_3)_{m+2} + (Sb_3)_{m-2})(Sb_3)_{m-2} \\ &= 9(Sb_3)_{m-2} - (Sb_3)_{m+2}(Sb_3)_{m-2} \\ &= 8(Sb_3)_m^2 + 113. \end{aligned}$$

Hence

$$3(Sb_3)_m - (Sb_3)_{m-2} = \sqrt{8(Sb_3)_m^2 + 113} \tag{15}$$

From (14) and (15), we obtain

$$(Sb_3)_{m+2} = 3(Sb_3)_m + \sqrt{8(Sb_3)_m^2 + 113} \tag{16}$$

Thus, we get

$$f((Sb_3)_m) = (Sb_3)_{m+2}.$$

**Theorem 3.14.** Let  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number and  $\tilde{f}(x) = 3x - \sqrt{8x^2 + 113}$ . Then

$$\tilde{f}((Sb_3)_m) = (Sb_3)_{m-2}$$

for  $m \geq 2$ .

*Proof.* Since  $x$  any  $b_3$ -subbalancing number, then  $x = (Sb_3)_m$  for  $m \geq 2$ .

From (15), we get

$$(Sb_3)_{m-2} = 3(Sb_3)_m - \sqrt{8(Sb_3)_m^2 + 113} \tag{17}$$

Thus, we get

$$\tilde{f}((Sb_3)_m) = (Sb_3)_{m-2}.$$

### 3.3. $b_3$ -Lucas Subbalancing Numbers

In this subsection, first we introduce the concept of  $b_3$ -Lucas subbalancing numbers. Then, we deduce some algebraic identities between  $b_3$ -Lucas subbalancing numbers and balancing, Lucas-balancing numbers.

**Definition 3.1.** Let  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number. The square root of the number  $8(Sb_3)_m^2 + 113$  is called the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number and denoted by  $(CSb_3)_m$ . That is,

$$(CSb_3)_m = \sqrt{8(Sb_3)_m^2 + 113}.$$

For example,  $(CSb_3)_0 = 11$  and  $(CSb_3)_1 = 25$ , since  $(Sb_3)_0 = 1$  and  $(Sb_3)_1 = 8$ .

**Corollary 3.3.** Using definition of  $b_3$ -Lucas subbalancing numbers, (16) and (17), we get

$$\begin{aligned}(CSb_3)_m &= (Sb_3)_{m+2} - 3(Sb_3)_m \quad (m \geq 0) \\ (CSb_3)_m &= 3(Sb_3)_m - (Sb_3)_{m-2} \quad (m \geq 2).\end{aligned}$$

It is obvious that these identities shows the relations between  $b_3$ -Lucas subbalancing number and  $b_3$ -subbalancing number.

**Theorem 3.15.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number and  $(CSb_3)_0 = 11, (CSb_3)_2 = 41$

$$(CSb_3)_{2m} = 6(CSb_3)_{2m-2} - (CSb_3)_{2m-4}$$

for  $m \geq 2$ .

*Proof.* Using the equation (16), we get

$$\begin{aligned}(CSb_3)_{2m}^2 &= 8(Sb_3)_{2m}^2 + 113 \\ &= 8(3(Sb_3)_{2m-2} + (CSb_3)_{2m-2})^2 + 113 \\ &= (3(CSb_3)_{2m-2} + 8(Sb_3)_{2m-2})^2.\end{aligned}$$

Thus, we get

$$(CSb_3)_{2m} = 3(CSb_3)_{2m-2} + 8(Sb_3)_{2m-2} \tag{18}$$

Similarly, from the equation (17), we get

$$(CSb_3)_{2m-4} = 3(CSb_3)_{2m-2} - 8(Sb_3)_{2m-2} \tag{19}$$

From (18) and (19), we get

$$(CSb_3)_{2m} = 6(CSb_3)_{2m-2} - (CSb_3)_{2m-4}.$$

**Theorem 3.16.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number and  $(CSb_3)_1 = 25, (CSb_3)_3 = 139$

$$(CSb_3)_{2m+1} = 6(CSb_3)_{2m-1} - (CSb_3)_{2m-3}$$

for  $m \geq 2$ .

*Proof.* Using the equation (16), we get

$$\begin{aligned}(CSb_3)_{2m+1}^2 &= 8(Sb_3)_{2m+1}^2 + 113 \\ &= 8\left(3(Sb_3)_{2m-1} + \sqrt{8(Sb_3)_{2m-1}^2 + 113}\right)^2 + 113 \\ &= (3(CSb_3)_{2m-1} + 8(Sb_3)_{2m-1})^2.\end{aligned}$$

Thus, we get

$$(CSb_3)_{2m+1} = 3(CSb_3)_{2m-1} + 8(Sb_3)_{2m-1} \quad (20)$$

Similarly, from the equation (17), we get

$$(CSb_3)_{2m-3} = 3(CSb_3)_{2m-1} - 8(Sb_3)_{2m-1} \quad (21)$$

From (20) and (21), we get

$$(CSb_3)_{2m+1} = 6(CSb_3)_{2m-1} - (CSb_3)_{2m-3}.$$

The following theorem shows that the odd and even terms of the sequence of  $b_3$ -Lucas subbalancing numbers can be written depending on the balancing and Lucas-balancing numbers.

**Theorem 3.17.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number,  $B_m$  denote the  $m^{\text{th}}$  balancing number and  $C_m$  denote the  $m^{\text{th}}$  Lucas-balancing number. Then

$$\begin{aligned} (CSb_3)_{2m} &= 11C_m + 8B_m \\ (CSb_3)_{2m+1} &= 11C_{m+1} - 8B_{m+1} \end{aligned}$$

for  $m \geq 0$ .

*Proof.* This theorem is proved by induction. It is easily seen that the assertion is true for  $m = 0$ . Assuming the assertion is true for  $m \leq k$ , we have

$$\begin{aligned} (CSb_3)_{2k+2} &= 6(CSb_3)_{2k} - (CSb_3)_{2k-2} \\ &= 6(11C_k + 8B_k) - (11C_{k-1} + 8B_{k-1}) \\ &= 11(6C_k - C_{k-1}) + 8(6B_k - B_{k-1}) \\ &= 11C_{k+1} + 8B_{k+1}. \end{aligned}$$

Thus it is shown that the assertion is true for  $m = k + 1$ . The other case can be proved similarly.

From Theorem 3.17, we can give the following two results that show the connection between  $b_3$ -Lucas subbalancing numbers and Lucas-balancing and balancing numbers.

**Theorem 3.18.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number and  $C_m$  denote the  $m^{\text{th}}$  Lucas-balancing number. Then

$$\begin{aligned} (CSb_3)_{2m} &= 8C_m + C_{m+1} \\ (CSb_3)_{2m+1} &= C_m + 8C_{m+1} \end{aligned}$$

for  $m \geq 0$ .

**Theorem 3.19.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number and  $B_m$  denote the  $m^{\text{th}}$  balancing number. Then

$$\begin{aligned} (CSb_3)_{2m} &= 11B_{m+1} - 25B_m \\ (CSb_3)_{2m+1} &= 25B_{m+1} - 11B_m \end{aligned}$$

for  $m \geq 0$ .

**Theorem 3.20.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number and  $C_m$  denote the  $m^{\text{th}}$  Lucas-balancing number. Then

$$(CSb_3)_{2m+1} = 8(CSb_3)_{2m} - 63C_m$$

for  $m \geq 0$ .

*Proof.* From Theorem 3.18, we get

$$\begin{aligned}(CSb_3)_{2m+1} &= C_m + 8C_{m+1} \\ &= C_m + 8C_{m+1} + 8(CSb_3)_{2m} - 64C_m - 8C_{m+1} \\ &= 8(CSb_3)_{2m} - 63C_m.\end{aligned}$$

**Theorem 3.21.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number and  $C_m$  denote the  $m^{\text{th}}$  Lucas-balancing number. Then

$$(CSb_3)_{2m} = 22C_m - (CSb_3)_{2m-1}$$

for  $m \geq 1$ .

*Proof.* From Theorem 3.18, we obtain

$$(CSb_3)_{2m} = 14C_m - C_{m-1} \quad (22)$$

and

$$(CSb_3)_{2m-1} = 8C_m + C_{m-1} \quad (23)$$

From (22) and (23), we get

$$(CSb_3)_{2m} = 22C_m - (CSb_3)_{2m-1}.$$

**Corollary 3.4.** The sum of any even term of the sequence of  $b_3$ -Lucas subbalancing numbers and the preceding term of the sequence of  $b_3$ -Lucas subbalancing number is always even. Similarly, the sum of any odd term of the sequence of  $b_3$ -Lucas subbalancing numbers and the next term of the sequence of  $b_3$ -Lucas subbalancing number is always even.

**Theorem 3.22.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number. Then

$$22(CSb_3)_{2m+1} = 113(CSb_3)_{2m} - 63(CSb_3)_{2m-1}$$

for  $m \geq 1$ .

*Proof.* Using Theorem 3.20 and Theorem 3.21, we obtain

$$\begin{aligned}22(CSb_3)_{2m+1} &= 176(CSb_3)_{2m} - 1386C_m \\ 63(CSb_3)_{2m} &= 1386C_m - 63(CSb_3)_{2m-1}.\end{aligned}$$

Thus

$$22(CSb_3)_{2m+1} = 113(CSb_3)_{2m} - 63(CSb_3)_{2m-1}.$$

**Theorem 3.23.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number and  $C_m$  denote the  $m^{\text{th}}$  Lucas-balancing number. Then

$$4(CSb_3)_{2m} = 7(CSb_3)_{2m-1} - 11C_{m-1}$$

for  $m \geq 1$ .

*Proof.* Using (22) and (23), we get

$$7(CSb_3)_{2m-1} = 56C_m + 7C_{m-1}$$

and

$$4(CSb_3)_{2m} = 56C_m - 4C_{m-1}.$$

Then

$$4(CSb_3)_{2m} = 7(CSb_3)_{2m-1} - 11C_{m-1}.$$

**Theorem 3.24.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number. Then

$$63(CSb_3)_{2m} = 113(CSb_3)_{2m-1} - 22(CSb_3)_{2m-2}$$

for  $m \geq 1$ .

*Proof.* From Theorem 3.20 and Theorem 3.23, we obtain

$$8(CSb_3)_{2m-2} = (CSb_3)_{2m-1} + 63C_{m-1}$$

and

$$252(CSb_3)_{2m} + 88(CSb_3)_{2m-2} = 452(CSb_3)_{2m-1}.$$

Then

$$63(CSb_3)_{2m} = 113(CSb_3)_{2m-1} - 22(CSb_3)_{2m-2}.$$

In the following theorem the Binet formula for  $b_3$ -Lucas subbalancing numbers is given.

**Theorem 3.25.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number. Then

$$(CSb_3)_{2m} = \frac{(8\sqrt{2} + 44)\alpha_1^m - (8\sqrt{2} - 44)\alpha_2^m}{8}$$

and

$$(CSb_3)_{2m+1} = \frac{(-8\sqrt{2} + 44)\alpha_1^{m+1} + (8\sqrt{2} + 44)\alpha_2^{m+1}}{8}$$

where  $\alpha_1 = 3 + 2\sqrt{2}$ ,  $\alpha_2 = 3 - 2\sqrt{2}$ .

*Proof.* Using Theorem 3.17 and the Binet formulas for balancing and Lucas-balancing numbers, we obtain

$$\begin{aligned} (CSb_3)_{2m} &= 11\left(\frac{\alpha_1^m + \alpha_2^m}{2}\right) + 8\left(\frac{\alpha_1^m - \alpha_2^m}{4\sqrt{2}}\right) \\ &= \frac{(44 + 8\sqrt{2})\alpha_1^m + (44 - 8\sqrt{2})\alpha_2^m}{8} \\ &= \frac{(8\sqrt{2} + 44)\alpha_1^m + (44 - 8\sqrt{2})\alpha_2^m}{8}. \end{aligned}$$

The other case can be proved similarly.

**Theorem 3.26.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number. Then

$$(CSb_3)_m^2 = (CSb_3)_{m-2}(CSb_3)_{m+2} - 904$$

for  $m \geq 2$ .

*Proof.* This theorem is proved by induction. It is easily seen that the assertion is true for  $m = 2$ .

Assuming the assertion is true for  $m \leq k$ , we have

$$\begin{aligned} (CSb_3)_{k+1}^2 &= (CSb_3)_{k+1}(CSb_3)_{k+1} \\ &= (6(CSb_3)_{k-1} - (CSb_3)_{k-3})(CSb_3)_{k+1} \\ &= 6(CSb_3)_{k-1}(CSb_3)_{k+1} - (CSb_3)_{k-1}^2 - 904 \\ &= (CSb_3)_{k-1}(6(CSb_3)_{k+1} - (CSb_3)_{k-1}) - 904 \\ &= (CSb_3)_{k-1}(CSb_3)_{k+3} - 904. \end{aligned}$$

Thus it is shown that the assertion is true for  $m = k + 1$ .

**Theorem 3.27.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number and  $B_m$  denote the  $m^{\text{th}}$  balancing number. Then

$$(CSb_3)_{m+2}(CSb_3)_{m+1} - (CSb_3)_m(CSb_3)_{m-1} = 904B_{m+1}$$

for  $m \geq 1$ .

*Proof.* This theorem is proved by induction. It is easily seen that the assertion is true for  $m = 1$ . Assuming the assertion is true for  $m \leq k$  and using Theorem 3.6 and Corollary 3.3, we have

$$\begin{aligned} (CSb_3)_{k+3}(CSb_3)_{k+2} - (CSb_3)_{k+1}(CSb_3)_k &= (3(Sb_3)_{k+3} - (Sb_3)_{k+1})(3(Sb_3)_{k+2} - (Sb_3)_k) \\ &\quad - ((Sb_3)_{k+3} - 3(Sb_3)_{k+1})((Sb_3)_{k+2} - 3(Sb_3)_k) \\ &= 8((Sb_3)_{k+3}(Sb_3)_{k+2} - (Sb_3)_{k+1}(Sb_3)_k) \\ &= 904B_{m+1}. \end{aligned}$$

Thus it is shown that the assertion is true for  $m = k + 1$ .

#### 3.4. Relationship Between $b_3$ -Lucas Subbalancing Numbers and $b_3$ -Subbalancing Numbers

In this subsection, some relations between  $b_3$ -Lucas subbalancing numbers and  $b_3$ -subbalancing numbers are given.

**Theorem 3.28.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number,  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number and  $B_m$  denote the  $m^{\text{th}}$  balancing number. Then

$$(CSb_3)_m(Sb_3)_{m-1} + (CSb_3)_{m-1}(Sb_3)_m = 113B_m$$

for  $m \geq 1$ .

*Proof.* From Theorem 3.6 and Corollary 3.3, we get

$$\begin{aligned} (CSb_3)_m(Sb_3)_{m-1} + (CSb_3)_{m-1}(Sb_3)_m &= (3(Sb_3)_m - (Sb_3)_{m-2})(Sb_3)_{m-1} \\ &\quad + ((Sb_3)_{m+1} - 3(Sb_3)_{m-1})(Sb_3)_m \\ &= 3(Sb_3)_m(Sb_3)_{m-1} - (Sb_3)_{m-1}(Sb_3)_{m-2} \\ &\quad + (Sb_3)_{m+1}(Sb_3)_m - 3(Sb_3)_m(Sb_3)_{m-1} \\ &= (Sb_3)_{m+1}(Sb_3)_m - (Sb_3)_{m-1}(Sb_3)_{m-2} \\ &= 113B_m. \end{aligned}$$

**Theorem 3.29.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number and  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number. Then

$$(CSb_3)_m(Sb_3)_{m+2} - (CSb_3)_{m+2}(Sb_3)_m = 113$$

for  $m \geq 0$ .

*Proof.* If  $m = 0$ , we have

$$(CSb_3)_0(Sb_3)_2 - (CSb_3)_2(Sb_3)_0 = 113.$$

Thus the assertion is true for  $m = 0$ . If  $m = 1$ , we have

$$(CSb_3)_1(Sb_3)_3 - (CSb_3)_3(Sb_3)_1 = 113.$$

Thus the assertion is true for  $m = 1$ .

Assuming the assertion is true for  $m \leq k$ , we have

$$\begin{aligned} (CSb_3)_{k+1}(Sb_3)_{k+3} - (CSb_3)_{k+3}(Sb_3)_{k+1} &= (CSb_3)_{k+1}(6(Sb_3)_{k+1} - (Sb_3)_{k-1}) \\ &\quad - (Sb_3)_{k+1}(6(CSb_3)_{k+1} - (CSb_3)_{k-1}) \\ &= (Sb_3)_{k+1}(CSb_3)_{k-1} - (CSb_3)_{k+1}(Sb_3)_{k-1} \\ &= 113. \end{aligned}$$

Thus it is shown that the assertion is true for  $m = k + 1$ .

**Theorem 3.30.** Let  $(CSb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -Lucas subbalancing number and  $(Sb_3)_m$  denote the  $m^{\text{th}}$   $b_3$ -subbalancing number. Then

$$(CSb_3)_{2m}(Sb_3)_{2m+1} - (CSb_3)_{2m+1}(Sb_3)_{2m} = 63$$

and

$$(CSb_3)_{2m+1}(Sb_3)_{2m+2} - (CSb_3)_{2m+2}(Sb_3)_{2m+1} = 22$$

for  $m \geq 0$ .

*Proof.* If  $m = 0$ , we have

$$(CSb_3)_0(Sb_3)_1 - (CSb_3)_1(Sb_3)_0 = 63.$$

Thus the assertion is true for  $m = 0$ . If  $m = 1$ , we have

$$(CSb_3)_2(Sb_3)_3 - (CSb_3)_3(Sb_3)_2 = 63.$$

Thus the assertion is true for  $m = 1$ .

Assuming the assertion is true for  $m \leq k$  and using Theorem 3.18 and equations (6), (7), we get

$$\begin{aligned} (CSb_3)_{2k+2}(Sb_3)_{2k+3} - (CSb_3)_{2k+3}(Sb_3)_{2k+2} &= (8C_{k+1} + C_{k+2})(8B_{k+2} + B_{k+1}) \\ &\quad - (8C_{k+2} + C_{k+1})(8B_{k+1} + B_{k+2}) \\ &= 63B_{k+2}C_{k+1} - 63C_{k+2}B_{k+1} \\ &= 63(B_{k+2}C_{k+1} - C_{k+2}B_{k+1}) \\ &= 63. \end{aligned}$$

Thus it is shown that the assertion is true for  $m = k + 1$ . The other case can be proved similarly.

## References

- [1] A. Behera, G.K. Panda, On the square roots of triangular numbers, *Fibonacci Quarterly* 37(2) (1999) 98-105.
- [2] G.K. Panda, P.K. Ray, Cobalancing numbers and cobalancers, *International Journal of Mathematics and Mathematical Sciences* 8 (2005) 1189-2000.
- [3] P.K. Ray, Balancing and cobalancing numbers (Thesis(PhD)), Department of Mathematics National Institute of Technology 2009.
- [4] G.K. Panda, Some fascinating properties of balancing numbers, *Congressus Numerantium* 194 (2009) 185-189.
- [5] G.K. Panda, P.K. Ray, Some links of balancing and cobalancing numbers with Pell and associated Pell numbers, *Bulletin of the Institute of Mathematics Academia Sinica(New Series)* 6(1) (2011) 41-72.
- [6] G.K. Giçizeri, A. İzzetkoç, A. Tekcan, Some algebraic relations on balancing numbers, *Util. Math.* 103 (2017) 217-236.
- [7] G.K. Giçizeri, On Pell, Pell-Lucas, and balancing numbers, *Journal of Inequalities and Applications*, 3 (2018).
- [8] K. Liptai, Fibonacci balancing numbers, *Fibonacci Quarterly* 42(4) (2004) 330-340.
- [9] K. Liptai, Lucas balancing numbers, *Acta Math. Univ. Ostrav.* 14(1) (2006) 43-47.
- [10] L. Szalay, On the resolution of simultaneous Pell equations, *Annales Mathematicae et Informaticae* 34 (2007) 77-87.
- [11] G.K. Panda, S.S. Rout, Gap balancing numbers, *Fibonacci Quarterly* 51(3) (2013) 239-248.



- [12] S.S. Rout, G.K. Panda,  $k$ -Gap balancing numbers, *Periodica Mathematica Hungarica* 70(1) (2015) 109-121.
- [13] G.K. Panda, A.K. Panda, Almost balancing numbers, *Journal of the Indian Mathematical Society* 82(3-4) (2015) 147-156.
- [14] A. Tekcan, Almost balancing, triangular and square triangular numbers, *Notes on Number Theory and Discrete Mathematics* 25(1) (2019) 108-121.
- [15] A. Tekcan, Sums and spectral norms of all almost balancing numbers, *Creat. Math. Inform.* 28(2) (2019) 203-214.
- [16] G.K. Panda, Sequence balancing and cobalancing numbers, *Fibonacci Quarterly* 45(3) (2007) 265-271.
- [17] K. Liptai, F. Luca, A. Pinter, L. Szalay, Generalized balancing numbers, *Indagationes Math. N. S.* 20(1) (2009) 87-100.
- [18] A. Berczes, K. Liptai, I. Pink, On generalized balancing sequences, *Fibonacci Quarterly* 48(2) (2010) 121-128.
- [19] P. Olajos, Properties of balancing, cobalancing and generalized balancing numbers, *Annales Mathematicae et Informaticae* 37 (2010) 125-138.
- [20] R.K. Davala, G.K. Panda, Supercobalancing numbers, *Matematika* 32(1) (2016) 31-42.
- [21] R.K. Davala, G.K. Panda, Subbalancing numbers, *Matematika* 34(1) (2018) 163-172.