Filomat 37:22 (2023), 7607–7622
https://doi.org/10.2298/FIL2322607S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Stancu-type integral generalization of modified Jain operators

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Abstract. In this paper, we introduce a Stancu-type integral generalization of modified Lupaş-Jain operators. First, we discuss some auxiliary results and then using them we represent a Korovkin-type theorem for these operators. Next, we establish a Voronovskaja-type asymptotic result and then find a quantitative estimation for the defined operators. Also, we examine their rate of convergence with the help of modulus of continuity and the Peetre's K-functional and analyze a convergence result for the Lipschitz-type class of functions. Lastly, we provide some graphical examples to show the relevance of our generalization.

1. Introduction

A revolution came into the field of Approximation theory, when Weierstrass contributed a crucial result about approximating continuous functions by polynomials in 1885. Later on, among all mathematicians who provided proofs for Weierstrass theorem, the one from Bernstein, gave birth to the theory of positive linear operators. This became more significant after the famous Bohman–Korovkin theorem [13, 24]. In the following years, Baskakov operators [11], Stancu operators [40], Szász-Mirakyan operators [30, 41] etc. made contributions to the approximation theory of positive linear operators and then their modifications as well as some generalizations were also introduced in the works of [6, 16, 18, 21–23, 25–27, 29, 32, 38].

Başcanbaz-Tunca et al. [10], have introduced a generalization of Jain operators [22] as

$$L_n^{\alpha}(u)(x) = \sum_{k=0}^{\infty} \frac{nx(nx+1+k\alpha)_{k-1}}{2^k k!} 2^{-(nx+k\alpha)} u\left(\frac{k}{n}\right), \quad x \in (0,\infty) \tag{1}$$

where $u : [0,\infty) \to \mathbb{R}$, $n \in \mathbb{N}$ and $\alpha \in [0,1)$ with α depending only on n. Also, $L_n^{\alpha}(u)(0) = u(0)$.

The notation $(a)_n$, $a \neq 0$ is known as Pochhammer symbol and is defined as

$$(a)_n = \begin{cases} a(a+1)...(a+n-1), & n \geq 1 \\ 1, & n = 0, \end{cases}$$

2020 *Mathematics Subject Classification*. 41A36, 41A35, 41A25.
Keywords. Weighted approximation process; Rate of convergence; Modulus of continuity; Voronovskaja-type theorem.
Received: 19 May 2022; Revised: 23 January 2023; Accepted: 25 March 2023
Communicated by Miodrag Spalević
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from which we also have

$$(a)_{-n} = \frac{1}{(a-1)(a-2)(a-3)\cdots(a-n)} = \frac{1}{(a-n)_n} = \frac{(-1)^n}{(1-a)_n}$$

when $a \neq 1, 2, 3, \cdots, n$ (see, e.g., p.5 of [19]).

Recently, Patel and Bodur [37] have defined a Durrmeyer-type integral generalization of the operators (1) as

$$D_n^{\alpha}(u;x) = n\sum_{k=0}^{\infty} l_{n,k}^{\alpha}(x)\int_0^{\infty} s_{n,k}(t)u(t)dt,\ x \in (0,\infty), \tag{2}$$

where

$$l_{n,k}^{\alpha}(x) = \frac{nx(nx+1+k\alpha)_{k-1}}{2^k k!}2^{-(nx+k\alpha)}, \tag{3}$$

$$s_{n,k}(t) = e^{-nt}\frac{(nt)^k}{k!}, \tag{4}$$

and $u : [0,\infty) \to \mathbb{R}$ is an integrable function such that $D_n^{\alpha}(u;x)$ exists.

In 1968, Stancu [40] generalized the Bernstein polynomials by introducing the positive linear operators $P_n^{(\mu,\beta)} : C[0,1] \to C[0,1]$ defined by

$$P_n^{(\mu,\beta)}(u;x) = \sum_{k=0}^{\infty} b_{n,k}(x)u\left(\frac{k+\mu}{n+\beta}\right),\ x \in [0,1]$$

where $b_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ is the Bernstein basis function. Also, $0 \leq \mu \leq \beta$ and $C[0,1]$ represents the set of all continuous functions on [0,1]. An important advantage of this type of generalization is their modeling flexibility due to the newly introduced parameters. In the recent years, Stancu-type generalizations of several operators have been proposed by many researchers. For more details, we refer to the readers [2–4, 7, 12, 14, 28, 31, 33–35].

Motivated by the above works, in this paper we introduce a Stancu-type generalization of the operators (2) as follows:

$$A_n^{(\xi,\psi,\alpha)}(u;x) := A_n^{(\xi,\psi,\alpha)}(u)(x) = n\sum_{k=0}^{\infty} l_{n,k}^{\alpha}(x)\int_0^{\infty} s_{n,k}(t)u\left(\frac{nt+\xi}{n+\psi}\right)dt, \tag{5}$$

where $l_{n,k}^{\alpha}(x)$ and $s_{n,k}(t)$ are given by (3) and (4) respectively and $u : [0,\infty) \to \mathbb{R}$ is an integrable function such that $A_n^{(\xi,\psi,\alpha)}(u;x)$ exists. Here, $x \in (0,\infty)$, $\alpha \in [0,1)$, $0 \leq \xi \leq \psi$ and $A_n^{(\xi,\psi,\alpha)}(u;0) = u(0)$. It can be easily seen that the operators (5) are positive and linear on $[0,\infty)$.

Remark 1.1. *If we put $\xi = \psi = 0$ in* (5)*, then we get the operators defined in* (2)*.*

Throughout the paper, by $\mathcal{W}[0,\infty)$, we denote the space of all real valued bounded continuous functions on $[0,\infty)$ and by $\widetilde{\mathcal{W}}[0,\infty)$, we denote the space of all real valued bounded and uniformly continuous functions on $[0,\infty)$. Both the spaces are endowed with the norm:

$$\|u\| = \sup\{|u(x)| : x \in [0,\infty)\}.$$

2. Auxiliary Results

In this section, we shall discuss some basic lemmas that will serve as supports for the main results.

Lemma 2.1. *[37] Let $e_i(t) := t^i$ for $i = 0, 1, 2, 3$. Then, for the operators defined by* (2), *we have*

$$\begin{aligned} D_n^{\alpha}(e_0;x) =&1,\\ D_n^{\alpha}(e_1;x) =&\frac{x}{1-\alpha}+\frac{1}{n},\\ D_n^{\alpha}(e_2;x) =&\frac{x^2}{(1-\alpha)^2}+\frac{x(3\alpha^2-6\alpha+5)}{n(1-\alpha)^3}+\frac{2}{n^2},\\ D_n^{\alpha}(e_3;x) =&\frac{x^3}{(1-\alpha)^3}+\frac{6x^2(\alpha^2-2\alpha+2)}{n(1-\alpha)^4}+\frac{x^2(11\alpha^4-44\alpha^3+78\alpha^2-62\alpha+29)}{n^2(1-\alpha)^5}+\frac{6}{n^3},\\ D_n^{\alpha}(e_4;x) =&\frac{x^4}{(1-\alpha)^4}+\frac{2x^3(5\alpha^2-10\alpha+11)}{n(1-\alpha)^5}+\frac{x^2(35\alpha^4-140\alpha^3+270\alpha^2-236\alpha+131)}{n^2(1-\alpha)^6}\\ &+\frac{2x(25\alpha^6-150\alpha^5+410\alpha^4-610\alpha^3+568\alpha^2-286\alpha+103)}{n^3(1-\alpha)^7}+\frac{24}{n^4}. \end{aligned}$$

Depending upon the above lemma, we derive following moments for the operator defined by (5).

Lemma 2.2. *The moments for the operator* (5) *are calculated to be*

$$\begin{aligned} A_n^{(\xi,\psi,\alpha)}(e_0;x) =&1,\\ A_n^{(\xi,\psi,\alpha)}(e_1;x) =&\frac{1}{(n+\psi)}\left[\frac{nx}{1-\alpha}+(1+\xi)\right],\\ A_n^{(\xi,\psi,\alpha)}(e_2;x) =&\frac{1}{(n+\psi)^2}\left[\frac{n^2x^2}{(1-\alpha)^2}\right.\\ &\left.+\frac{nx\{\alpha^2(2\xi+3)-2\alpha(2\xi+3)+(2\xi+5)\}}{(1-\alpha)^3}+(\xi^2+2\xi+2)\right],\\ A_n^{(\xi,\psi,\alpha)}(e_3;x) =&\frac{1}{(n+\psi)^3}\left[\frac{n^3x^3}{(1-\alpha)^3}+\frac{3n^2x^2\{\alpha^2(\xi+2)-2\alpha(\xi+2)+(\xi+4)\}}{(1-\alpha)^4}\right.\\ &+\frac{nx\{\alpha^4(3\xi^2+9\xi+11)-\alpha^3(12\xi^2+36\xi+44)+\alpha^2(18\xi^2+60\xi+78)\}}{(1-\alpha)^5}\\ &\left.-\frac{nx\{\alpha(12\xi^2+48\xi+62)-(3\xi^2+15\xi+29)\}}{(1-\alpha)^5}+(\xi^3+3\xi^2+6\xi+6)\right]. \end{aligned}$$

Proof. Here, we'll directly use the Lemma 2.1 and the facts:

$$\begin{aligned} A_n^{(\xi,\psi,\alpha)}(e_0;x) =&D_n^{\alpha}(e_0;x),\\ A_n^{(\xi,\psi,\alpha)}(e_1;x) =&\frac{1}{(n+\psi)}\left[nD_n^{\alpha}(e_1;x)+\xi D_n^{\alpha}(e_0;x)\right],\\ A_n^{(\xi,\psi,\alpha)}(e_2;x) =&\frac{1}{(n+\psi)^2}\left[n^2D_n^{\alpha}(e_2;x)+2n\xi D_n^{\alpha}(e_1;x)+\xi^2D_n^{\alpha}(e_0;x)\right],\\ A_n^{(\xi,\psi,\alpha)}(e_3;x) =&\frac{1}{(n+\psi)^3}\left[n^3D_n^{\alpha}(e_3;x)+3n^2\xi D_n^{\alpha}(e_2;x)+3n\xi^2D_n^{\alpha}(e_1;x)+\xi^3D_n^{\alpha}(e_0;x)\right] \end{aligned}$$

to conclude our desired results. □

As a consequence of the above lemma, in the next lemma, we define the central moments of the operators (5).

Lemma 2.3. *The central moments for the operators (5) are*

$$A_n^{(\xi,\psi,\alpha)}((t-x);x) = \frac{1}{n+\psi}\left[\frac{x(n\alpha-\psi+\psi\alpha)}{(1-\alpha)} + (\xi+1)\right],$$

$$\begin{aligned} A_n^{(\xi,\psi,\alpha)}((t-x)^2;x) = & \frac{1}{(n+\psi)^2}\Bigg[\frac{x^2\{\alpha^2(n^2+2n\psi+\psi^2)-2\alpha\psi(n+\psi)+\psi^2\}}{(1-\alpha)^2} \\ & + \frac{1}{(1-\alpha)^3}\{x\{2\alpha^3(n+\psi)(1+\xi)-\alpha^2(3n+4n\xi+6\psi+6\psi\xi) \\ & + 2\alpha(n\xi+3\psi\xi+3\psi)+(3n-2\psi-2\psi\xi)\}\}+(\xi^2+2\xi+2)\Bigg] = \lambda_n^{(\xi,\psi,\alpha)}(x). \end{aligned} \tag{6}$$

Remark 2.4. *For the operators* (2)*, the third central moment is calculated to be:*

$$\begin{aligned} D_n^{\alpha}((t-x)^3;x) = & \frac{x^3\alpha^3}{1-\alpha^3} + \frac{3x^2\alpha(\alpha^3-\alpha^2-\alpha+11)}{n(1-\alpha)^4} \\ & + \frac{x(6\alpha^5-19\alpha^4+16\alpha^3+8\alpha^2-32\alpha+23)}{n^2(1-\alpha)^5} + \frac{6}{n^3}. \end{aligned}$$

Also, if $\{\alpha_n\}_{n\geq 1}$ *be a sequence such that* $\alpha_n \in [0,1)\ \forall n$, $\lim\limits_{n\to\infty}\alpha_n = 0$ *and* $\lim\limits_{n\to\infty} n\alpha_n = 0$, *then*

$$\lim_{n\to\infty} n^2\left(D_n^{\alpha_n}((t-x)^3;x)\right) = 23x. \tag{7}$$

Remark 2.5. *Let* $\{\alpha_n\}_{n\geq 1}$ *be a sequence such that* $\alpha_n \in [0,1)\ \forall n$, $\lim\limits_{n\to\infty}\alpha_n = 0$ *and* $\lim\limits_{n\to\infty} n\alpha_n = 0$. *Then it is observe that for the operators defined by* (5)*, the central moments satisfy*

$$\begin{aligned} &\lim_{n\to\infty} n\left(A_n^{(\xi,\psi,\alpha_n)}((t-x);x)\right) = 1, \\ &\lim_{n\to\infty} n\left(A_n^{(\xi,\psi,\alpha_n)}((t-x)^2;x)\right) = 3x. \end{aligned}$$

It is also noticed that

$$\begin{aligned} A_n^{(\xi,\psi,\alpha_n)}((t-x)^4;x) = & \frac{n^4}{(n+\psi)^4}\left(D_n^{\alpha_n}((t-x)^4;x)\right) \\ & + \frac{4n^3(\xi-\psi x)}{(n+\psi)^4}\left(D_n^{\alpha_n}((t-x)^3;x)\right) + \frac{6n^2(\xi-\psi x)^2}{(n+\psi)^4}\left(D_n^{\alpha_n}((t-x)^2;x)\right) \\ & + \frac{4n(\xi-\psi x)^3}{(n+\psi)^4}\left(D_n^{\alpha_n}(t-x;x)\right) + \frac{(\xi-\psi x)^4}{(n+\psi)^4}. \end{aligned} \tag{8}$$

Using (7) *and Remark 2.5 of [37] in* (8)*, we have*

$$\lim_{n\to\infty} n^2\left(A_n^{(\xi,\psi,\alpha_n)}((t-x)^4;x)\right) = 27x^2. \tag{9}$$

Remark 2.6. *For $\alpha \in [0,1)$, we notice the following*

$$\begin{aligned}
A_n^{(\xi,\psi,\alpha)}(t^2+1;x) =& \frac{1}{(n+\psi)^2}\left[\frac{n^2x^2}{(1-\alpha)^2}\right.\\
&+\left.\frac{nx\{\alpha^2(2\xi+3)-2\alpha(2\xi+3)+(2\xi+5)\}}{(1-\alpha)^3}+(\xi^2+2\xi+2)\right]+1\\
=&\frac{1}{(n+\psi)^2}\left[\frac{n^2x^2}{(1-\alpha)^2}\right.\\
&+\left.\frac{nx\{(2\xi\alpha^2-4\xi\alpha+2\xi)+(3\alpha^2-6\alpha+5)\}}{(1-\alpha)^3}+(\xi^2+2\xi+2)\right]+1\\
\leq&\frac{(1+x^2)}{(n+\psi)^2}\left[\frac{n^2x^2}{(1-\alpha)^2(1+x^2)}+\frac{nx(2\xi+5)}{(1-\alpha)^3(1+x^2)}+\frac{(\xi^2+2\xi+2)}{(1+x^2)}\right]+1\\
\leq&\frac{(1+x^2)}{(n+\psi)^2}\left[\frac{n^2}{(1-\alpha)^2}+\frac{n(2\xi+5)}{(1-\alpha)^3}+(\xi^2+2\xi+2)\right]+(1+x^2)\\
\leq&(1+x^2)\left[\frac{n^2+n(2\xi+5)+(\xi^2+2\xi+2)}{(n+\psi)^2(1-\alpha)^3}+1\right]\\
=&M_n(1+x^2)
\end{aligned}$$

where $M_n = \frac{n^2+n(2\xi+5)+(\xi^2+2\xi+2)}{(n+\psi)^2(1-\alpha)^3}+1$.

Lemma 2.7. *If $u \in \mathcal{W}[0,\infty)$, then* $\left\|A_n^{(\xi,\psi,\alpha)}(u)\right\| \leq \|u\|$.

Proof. Using Lemma 2.2 and the norm defined for $\mathcal{W}[0,\infty)$, the result follows immediately. □

Theorem 2.8. *Let $u \in \mathcal{W}[0,\infty)$ and $\{\alpha_n\}_{n\geq 1}$ be a sequence such that $\alpha_n \in [0,1)$, $\forall n$ and $\lim_{n\to\infty}\alpha_n = 0$. Then $\lim_{n\to\infty}A_n^{(\xi,\psi,\alpha_n)}(u;x) = u(x)$ uniformly on each compact subset of $[0,\infty)$.*

Proof. We notice that from Lemma 2.2

$$\lim_{n\to\infty} A_n^{(\xi,\psi,\alpha_n)}(e_i;x) = e_i, \text{ for i=0,1,2 as } \lim_{n\to\infty}\alpha_n = 0.$$

The above convergence is true in each compact subset of $[0,\infty)$. Hence the required result follows from Korovkin's theorem (see[24]). □

3. Transferring the Asymptotic Formula

After observing the convergence of the operators (5), the most significant question comes to mind is the speed of approximation of these operators to the function u. To answer this, we present a Voronovskaya type asymptotic formula for the operators $A_n^{(\xi,\psi,\alpha_n)}$.

Theorem 3.1. *Let u be a bounded integrable function on $[0,\infty)$ and u', u'' exist at a point $x \in [0,\infty)$. Also, let $\{\alpha_n\}_{n\geq 1}$ be a sequence with $\alpha_n \in [0,1)$, $\forall n$ such that $\lim_{n\to\infty}\alpha_n = 0$, and $\lim_{n\to\infty} n\alpha_n = 0$. Then*

$$\lim_{n\to\infty} n\left(A_n^{(\xi,\psi,\alpha_n)}(u;x)-u\right) = u'(x)+\frac{3x}{2}u''(x).$$

Proof. By the well-known Taylor's expansion,

$$u(t) = u(x)+u'(x)(t-x)+\frac{1}{2}u''(x)(t-x)^2+\Lambda(t,x)(t-x)^2. \tag{10}$$

Here $\Lambda(t,x) \in \mathcal{W}[0,\infty)$ and satisfies $\lim_{t\to x} \Lambda(t,x) = 0$. Next, we apply the operators $A_n^{(\xi,\psi,\alpha_n)}$ and then multiply n on both sides of (10) to get

$$n\left(A_n^{(\xi,\psi,\alpha_n)}(u;x) - u\right) = nA_n^{(\xi,\psi,\alpha_n)}(t-x;x)u'(x) + \frac{n}{2}A_n^{(\xi,\psi,\alpha_n)}((t-x)^2;x)u''(x) + nA_n^{(\xi,\psi,\alpha_n)}(\Lambda(t,x)(t-x)^2;x). \tag{11}$$

Applying Cauchy-Schwarz inequality to the third term of the right hand side of (11), we find

$$n\left|A_n^{(\xi,\psi,\alpha_n)}(\Lambda(t,x)(t-x)^2;x)\right| \le \left(n^2 A_n^{(\xi,\psi,\alpha_n)}((t-x)^4;x)\right)^{1/2}\left(A_n^{(\xi,\psi,\alpha_n)}(\Lambda^2(t,x);x)\right)^{1/2},$$

which yields

$$\lim_{n\to\infty} n\left(A_n^{(\xi,\psi,\alpha_n)}(u;x) - u\right) \le \lim_{n\to\infty} nA_n^{(\xi,\psi,\alpha_n)}(t-x;x)u'(x) + \lim_{n\to\infty}\frac{n}{2}A_n^{(\xi,\psi,\alpha_n)}((t-x)^2;x)u''(x) + \lim_{n\to\infty}\left(n^2A_n^{(\xi,\psi,\alpha_n)}((t-x)^4;x)\right)^{1/2}\lim_{n\to\infty}\left(A_n^{(\xi,\psi,\alpha_n)}(\Lambda^2(t,x);x)\right)^{1/2}.$$

But using Remark 2.5, we have

$$\lim_{n\to\infty} n\left(A_n^{(\xi,\psi,\alpha_n)}(u;x) - u\right) \le u'(x) + \frac{3x}{2}u''(x) + 3\sqrt{3}x\lim_{n\to\infty}\left(A_n^{(\xi,\psi,\alpha_n)}(\Lambda^2(t,x);x)\right)^{1/2}. \tag{12}$$

Also, we notice, $\Lambda^2(t,x) \in \mathcal{W}[0,\infty)$ and $\Lambda^2(x,x) = 0$. Hence using Theorem 2.8, we can have $\lim_{n\to\infty}\left(A_n^{(\xi,\psi,\alpha_n)}(\Lambda^2(t,x);x)\right)^{1/2} = 0$ and hence (12) concludes,

$$\lim_{n\to\infty} n\left(A_n^{(\xi,\psi,\alpha_n)}(u;x) - u\right) = u'(x) + \frac{3x}{2}u''(x).$$

This completes the proof. □

4. Weighted Approximation

In this section, we provide a Korovkin type theorem for the weighted approximation of the operators (5). In this context, we use the following notations:
Setting the weight function $\sigma(x) = 1 + x^2$, we define the space

$$B_\sigma[0,\infty) = \{u : [0,\infty) \to \mathbb{R} : |u(x)| \le M_u\sigma(x)\},$$

where M_u is a constant depending on u. The space $B_\sigma[0,\infty)$ is a normed space with the weighted norm $\|u\|_\sigma = \sup_{x\in\mathbb{R}^+}\frac{|u(x)|}{\sigma(x)}$.
Now, we define the following sub-spaces of $B_\sigma[0,\infty)$ as follows:

$$C_\sigma[0,\infty) = \{u \in B_\sigma[0,\infty) : u \text{ is continuous}\},$$
$$\widetilde{C}_\sigma[0,\infty) = \left\{u \in C_\sigma[0,\infty) : \lim_{x\to\infty}\frac{u(x)}{\sigma(x)} = k_u\right\},$$

where k_u is a constant depending upon u. It is clear that, $\widetilde{C}_\sigma[0,\infty) \subset C_\sigma[0,\infty) \subset B_\sigma[0,\infty)$. For contributions of researchers on approximation of functions in such type of space we refer to [1, 5, 8, 9, 20, 39].

We have the following result from Gadjiev [17]:

Lemma 4.1. *[17] The positive linear operators J_n, $n \ge 1$, act from $C_\sigma[0,\infty)$ to $B_\sigma[0,\infty)$ if and only if*

$$|J_n(\sigma)(x)| \le K_n\sigma(x),$$

where K_n is a positive constant.

We provide a weighted uniform approximation result for the newly defined operators as follows:

Theorem 4.2. *Let* $\{\alpha_n\}_{n\geq 1}$ *be a sequence with* $\alpha_n \in [0,1)$, $\forall n$ *such that* $\lim_{n\to\infty} \alpha_n = 0$. *Then for each* $u \in \widetilde{C}_\sigma[0,\infty)$, *we have*

$$\lim_{n\to\infty} \left\| A_n^{(\xi,\psi,\alpha_n)}(u) - u \right\|_\sigma = 0.$$

Proof. By Lemma 4.1 and Remark 2.6, it is easy to see that each term of the sequence of operators $\left\{A_n^{(\xi,\psi,\alpha_n)}\right\}_{n\geq 1}$ acts from $C_\sigma[0,\infty)$ to $B_\sigma[0,\infty)$.
To prove our result, it is sufficient to show that

$$\lim_{n\to\infty} \left\| A_n^{(\xi,\psi,\alpha_n)}(e_j) - e_j \right\|_\sigma = 0, \text{ for } j = 0,1,2.$$

For $j = 0$, using Lemma 2.2 we have

$$\lim_{n\to\infty} \left\| A_n^{(\xi,\psi,\alpha_n)}(e_0) - e_0 \right\|_\sigma = 0.$$

Next we obtain

$$\begin{aligned}
\left\| A_n^{(\xi,\psi,\alpha_n)}(e_1) - e_1 \right\|_\sigma &\leq \sup_{x\in\mathbb{R}^+} \frac{\left|A_n^{(\xi,\psi,\alpha_n)}(e_1;x) - e_1\right|}{1+x^2} = \sup_{x\in\mathbb{R}^+} \frac{\left|\frac{1}{(n+\psi)}\left\{\frac{nx}{1-\alpha_n} + (1+\xi)\right\} - x\right|}{1+x^2} \\
&= \sup_{x\in\mathbb{R}^+} \left| \frac{n\alpha_n x - \psi x + \psi\alpha_n x}{(n+\psi)(1-\alpha_n)(1+x^2)} + \frac{(1+\xi)}{(n+\psi)(1+x^2)} \right| \\
&\leq \sup_{x\in\mathbb{R}^+} \left[\frac{n\alpha_n + \psi + \psi\alpha_n}{(n+\psi)(1-\alpha_n)} + \frac{(1+\xi)}{(n+\psi)} \right].
\end{aligned}$$

The above implies that

$$\lim_{n\to\infty} \left\| A_n^{(\xi,\psi,\alpha_n)}(e_1) - e_1 \right\|_\sigma = 0.$$

Also,

$$\begin{aligned}
\left\| A_n^{(\xi,\psi,\alpha_n)}(e_2) - e_2 \right\|_\sigma &\leq \sup_{x\in\mathbb{R}^+} \frac{\left|A_n^{(\xi,\psi,\alpha_n)}(e_2) - e_2\right|}{1+x^2} \\
&= \sup_{x\in\mathbb{R}^+} \left| \frac{-x^2\{n^2(\alpha_n^2 - 2\alpha_n) + (\psi^2 + 2n\psi)(1-\alpha_n)^2\}}{(n+\psi)^2(1-\alpha_n)^2(1+x^2)} \right. \\
&\quad \left. + \frac{1}{(n+\psi)^2}\left[\frac{nx\{\alpha_n^2(2\xi+3) - 2\alpha_n(2\xi+3) + (2\xi+5)\}}{(1-\alpha_n)^3(1+x^2)} + \frac{(\xi^2+2\xi+2)}{(1+x^2)} \right] \right| \\
&\leq \sup_{x\in\mathbb{R}^+} \left[\frac{n^2(\alpha_n^2 + 2\alpha_n) + (\psi^2 + 2n\psi)(1-\alpha_n)^2}{(n+\psi)^2(1-\alpha_n)^2} \right. \\
&\quad \left. + \frac{1}{(n+\psi)^2}\left\{ \frac{n\{\alpha_n^2(2\xi+3) - 2\alpha_n(2\xi+3) + (2\xi+5)\}}{(1-\alpha_n)^3} + (\xi^2+2\xi+2) \right\} \right].
\end{aligned}$$

Hence, from the above discussion we must have

$$\lim_{n\to\infty} \left\| A_n^{(\xi,\psi,\alpha_n)}(e_2) - e_2 \right\|_\sigma = 0.$$

Thus, the proof is completed. □

5. Quantitative Estimation

Next, we will try to find a quantitative estimation for the operators (5) using the weighted modulus of continuity.
Let $\phi : [0,\infty) \to [0,\infty)$ be the unbounded strictly increasing continuous function such that $\exists\, M > 0$ and $\alpha \in (0,1]$ with the property

$$|x-y| \leq M|\phi(x)-\phi(y)|^{\alpha} \text{ for every } x,y \geq 0.$$

For each $u \in C_\sigma[0,\infty)$ and $x \in [0,\infty)$, the following weighted modulus of continuity is defined in [20] as

$$\omega_\phi(u,\delta) := \sup_{\substack{x,t\in[0,\infty)\\ |\phi(t)-\phi(x)|\leq\delta}} \frac{|u(t)-u(x)|}{\sigma(t)+\sigma(x)}, \forall \quad \delta \geq 0,$$

where $\omega_\phi(u,\delta)$ is a nonnegative and increasing function with respect to δ for $u \in C_\sigma[0,\infty)$.
Also, $\lim_{\delta\to 0} \omega_\phi(u,\delta) = 0$, for all $u \in C_\sigma[0,\infty)$.
Let us define a subspace of $C_\sigma[0,\infty)$ as

$$\widetilde{C^*}_\sigma[0,\infty) = \left\{u \in C_\sigma[0,\infty) : \frac{u}{\sigma} \text{ is uniformly continuous}\right\}.$$

Using the properties of $\omega_\phi(u,\delta)$, Holhoş [20] provided the following theorem and remark:

Theorem 5.1. *[20] Let $\{J_n\}_{n\geq 1}$ be a sequence of positive linear operators mapping from $C_\sigma[0,\infty)$ into $B_\sigma[0,\infty)$ with*

$$\left\|J_n(\phi^0)-\phi^0\right\|_{\sigma^0} = p_n,$$
$$\left\|J_n(\phi)-\phi\right\|_{\sigma^{1/2}} = q_n,$$
$$\left\|J_n(\phi^2)-\phi^2\right\|_{\sigma} = r_n,$$
$$\left\|J_n(\phi^3)-\phi^3\right\|_{\sigma^{3/2}} = s_n,$$

where p_n, q_n, r_n, s_n, tend to zero as n tends to infinity. Then

$$\left\|J_n(u)-u\right\|_{\sigma^{3/2}} \leq (7+4p_n+2r_n)\omega_\phi(u;\delta_n) + \|u\|_\sigma\, p_n$$

for all $u \in C_\sigma[0,\infty)$, where

$$\delta_n = 2\sqrt{(p_n+2q_n+r_n)(1+p_n)} + p_n + 3q_n + 3r_n + s_n.$$

Remark 5.2. *[20] Under the condition of the Theorem 5.1 and using the fact that $\lim_{\delta_n\to 0}\omega_\phi(u;\delta_n) = 0$, we can conclude that*

$$\forall u \in \widetilde{C^*}_{\sigma^{3/2}}[0,\infty), \quad \lim_{n\to\infty}\left\|J_n(u)-(u)\right\|_{\sigma^{3/2}} = 0.$$

Theorem 5.3. *Let $\phi(x) = x$ and $\sigma(x) = 1+x^2$. Let $\{\alpha_n\}_{n\geq 1}$ be a sequence with $\alpha_n \in [0,1)$, $\forall n$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\left\{A_n^{(\xi,\psi,\alpha_n)}\right\}_{n\geq 1}$ be a sequence of positive linear operators. Then $\forall\ u \in C_\sigma^k[0,\infty)$, we have*

$$\begin{aligned}\left\|A_n^{(\xi,\psi,\alpha_n)}(u)-u\right\|_{\sigma^{3/2}} \leq \Bigg[7 &+ \frac{2n^2(\alpha_n^2+2\alpha_n)+(2\psi^2+4n\psi)(1-\alpha_n)^2}{(n+\psi)^2(1-\alpha_n)^2} \\ &+ \frac{2n\{\alpha_n^2(2\xi+3)-2\alpha_n(2\xi+3)+(2\xi+5)\}}{(n+\psi)^2(1-\alpha_n)^3} \\ &+ \frac{(2\xi^2+4\xi+4)}{(n+\psi)^2}\Bigg]\omega_\phi(u;\delta_n)\end{aligned}$$

where

$$\begin{aligned}
\delta := \delta_n =& 2\left[\frac{2n\alpha_n + 2\psi + 2\psi\alpha_n}{(n+\psi)(1-\alpha_n)} + \frac{2+2\xi}{(n+\psi)} + \frac{n^2(\alpha_n^2 + 2\alpha_n) + (\psi^2 + 2n\psi)(1-\alpha_n)^2}{(n+\psi)^2(1-\alpha_n)^2}\right.\\
&\left. + \frac{n\{\alpha_n^2(2\xi+3) - 2\alpha_n(2\xi+3) + (2\xi+5)\}}{(n+\psi)^2(1-\alpha_n)^3} + \frac{(\xi^2+2\xi+2)}{(n+\psi)^2}\right]^{1/2}\\
&+ \frac{3n\alpha_n + 3\psi + 3\psi\alpha_n}{(n+\psi)(1-\alpha_n)} + \frac{3+3\xi}{(n+\psi)} + \frac{3n^2(\alpha_n^2 + 2\alpha_n) + (3\psi^2 + 6n\psi)(1-\alpha_n)^2}{(n+\psi)^2(1-\alpha_n)^2}\\
&+ \frac{3n\{\alpha_n^2(2\xi+3) - 2\alpha_n(2\xi+3) + (2\xi+5)\}}{(n+\psi)^2(1-\alpha_n)^3} + \frac{(3\xi^2+6\xi+6)}{(n+\psi)^2}\\
&+ \frac{n^3(\alpha_n^3 - 3\alpha_n^2 + 3\alpha_n) + (\psi^3 + 3n^2\psi + 3\psi^2 n)(1-\alpha_n)^3}{(n+\psi)^3(1-\alpha_n)^3}\\
&+ \frac{3n^2\{\alpha_n^2(\xi+2) - 2\alpha_n(\xi+2) + (\xi+4)\}}{(n+\psi)^3(1-\alpha_n)^4}\\
&+ \frac{n\{\alpha_n^4(3\xi^2+9\xi+11) + \alpha_n^3(12\xi^2+36\xi+44)\}}{(n+\psi)^3(1-\alpha_n)^5}\\
&+ \frac{n\{\alpha_n^2(18\xi^2+60\xi+78) + \alpha_n(12\xi^2+48\xi+62)\}}{(n+\psi)^3(1-\alpha_n)^5}\\
&+ \frac{n(\xi^3+6\xi+6)}{(n+\psi)^3}.
\end{aligned}$$

Proof. In view of Lemma 2.2, we have the followings

$$\begin{aligned}
p_n &= \left\|A_n^{(\xi,\psi,\alpha_n)}(\phi^0) - (\phi^0)\right\|_{\sigma^0} = 0,\\
q_n &= \left\|A_n^{(\xi,\psi,\alpha_n)}(\phi) - (\phi)\right\|_{\sigma^{1/2}} \le \frac{n\alpha_n + \psi + \psi\alpha_n}{(n+\psi)(1-\alpha_n)} + \frac{1+\xi}{(n+\psi)},\\
r_n &= \left\|A_n^{(\xi,\psi,\alpha_n)}(\phi^2) - (\phi^2)\right\|_{\sigma} \le \frac{n^2(\alpha_n^2 + 2\alpha_n) + (\psi^2 + 2n\psi)(1-\alpha_n)^2}{(n+\psi)^2(1-\alpha_n)^2}\\
&\qquad + \frac{1}{(n+\psi)^2}\left[\frac{n\{\alpha_n^2(2\xi+3) - 2\alpha_n(2\xi+3) + (2\xi+5)\}}{(1-\alpha_n)^3}\right.\\
&\qquad \left. + (\xi^2+2\xi+2)\right],\\
s_n &= \left\|A_n^{(\xi,\psi,\alpha_n)}(\phi^3) - (\phi^3)\right\|_{\sigma^{3/2}} \le \frac{n^3(\alpha_n^3 - 3\alpha_n^2 + 3\alpha_n) + (\psi^3 + 3n^2\psi + 3\psi^2 n)(1-\alpha_n)^3}{(n+\psi)^3(1-\alpha_n)^3}\\
&\qquad + \frac{3n^2\{\alpha_n^2(\xi+2) - 2\alpha_n(\xi+2) + (\xi+4)\}}{(n+\psi)^3(1-\alpha_n)^4}\\
&\qquad + \frac{n\{\alpha_n^4(3\xi^2+9\xi+11) + \alpha_n^3(12\xi^2+36\xi+44)\}}{(n+\psi)^3(1-\alpha_n)^5}\\
&\qquad + \frac{n\{\alpha_n^2(18\xi^2+60\xi+78) + \alpha_n(12\xi^2+48\xi+62)\}}{(n+\psi)^3(1-\alpha_n)^5}\\
&\qquad + \frac{n(\xi^3+6\xi+6)}{(n+\psi)^3}.
\end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, it is clear that p_n, q_n, r_n and s_n tends to 0 as n goes to infinity.

Finally, selecting $\delta := \delta_n$, we obtain

$$\left\|A_n^{(\xi,\psi,\alpha_n)}(u) - u\right\|_{\sigma^{3/2}} \leq \Bigg[7 + \frac{2n^2(\alpha_n^2 + 2\alpha_n) + (2\psi^2 + 4n\psi)(1-\alpha_n)^2}{(n+\psi)^2(1-\alpha_n)^2} + \frac{2n\{\alpha_n^2(2\xi+3) - 2\alpha_n(2\xi+3) + (2\xi+5)\}}{(n+\psi)^2(1-\alpha_n)^3} + \frac{(2\xi^2+4\xi+4)}{(n+\psi)^2}\Bigg]\omega_\phi(u;\delta_n).$$

Which completes the desired proof. □

Remark 5.4. *Depending upon all the conditions of Theorem 5.3 and in view of Remark 5.2 we can conclude: if* $\lim_{\delta_n \to 0} \omega_\phi(u;\delta_n) = 0$, *then*

$$\forall u \in \widetilde{C^*}_{\sigma^{3/2}}[0,\infty), \quad \lim_{n\to\infty} \|A_n^{(\xi,\psi,\alpha_n)}(u) - (u)\|_{\sigma^{3/2}} = 0.$$

6. Rate of Convergence

In this section, we estimate the rate of convergence of the newly defined operators (5) in terms of the modulus of continuity.
For $u \in \widetilde{\mathcal{W}}[0,\infty)$ and $\delta > 0$, the modulus of continuity defined by

$$\omega(u;\delta) = \sup_{0\leq h\leq\delta} \sup_{x\in[0,\infty)} |u(x+h) - u(x)|.$$

Also, the 2nd order modulus of continuity is defined by

$$\omega_2(u;\delta) = \sup_{0\leq h\leq\delta} \sup_{x\in[0,\infty)} |u(x+2h) - 2u(x+h) + u(x)|.$$

The Peetre's K-functional of the function is defined by

$$K_2(u,\delta) := \inf_{v\in\mathcal{W}^2} \{\|u - v\| + \delta\|v''\|\},$$

where

$$\mathcal{W}^2 := \{v \in \mathcal{W}[0,\infty) : v', v'' \in \mathcal{W}[0,\infty)\}.$$

The relationship between $K_2(u,\delta)$ and $\omega_2(u,\delta)$ is described in [15] as

$$K_2(u,\delta) \leq M\omega_2(u,\sqrt{\delta}), \tag{13}$$

where $\delta > 0$ and the positive constant M is independent of u and δ.

Theorem 6.1. *let* $u \in \widetilde{\mathcal{W}}[0,\infty)$, $n \in \mathbb{N}$, $\alpha \in [0,1)$ *and* $x \in [0,\infty)$. *Then the operators* $A_n^{(\xi,\psi,\alpha)}$ *satisfy*

$$\left|A_n^{(\xi,\psi,\alpha)}(u;x) - u(x)\right| \leq 2\omega(u,\delta),$$

where $\delta := \sqrt{\lambda_n^{(\xi,\psi,\alpha)}(x)}$ *and* $\lambda_n^{(\xi,\psi,\alpha)}(x)$ *is defined in* (6).

Proof. Using the following property of modulus of continuity

$$|u(t) - u(x)| \leq \omega(u, \delta)\left(\frac{(t-x)^2}{\delta^2} + 1\right)$$

and then applying the newly defined operators on both sides of the above inequality, we get

$$\left|A_n^{(\xi,\psi,\alpha)}(u;x) - u(x)\right| \leq A_n^{(\xi,\psi,\alpha)}(|u(t) - u(x)|;x) \leq \omega(u,\delta)\left(1 + \frac{1}{\delta^2}A_n^{(\xi,\psi,\alpha)}((t-x)^2 : x)\right).$$

Next, we choose $\delta := \sqrt{\lambda_n^{(\xi,\psi,\alpha)}(x)}$ to obtain

$$\left|A_n^{(\xi,\psi,\alpha)}(u;x) - u(x)\right| \leq 2\omega(u,\delta).$$

This completes the proof. □

Theorem 6.2. *Let $u \in \widetilde{\mathcal{W}}[0,\infty)$ and $n > 1$, Then there exists $C > 0$ such that*

$$\left|A_n^{(\xi,\psi,\alpha)}(u;x) - u(x)\right| \leq C\omega_2\left(u, \sqrt{\delta_n^{(\xi,\psi,\alpha)}(x)}\right) + \omega\left(u, \frac{nx}{(1-\alpha)(n+\psi)} + \frac{(1+\xi)}{(n+\psi)}\right),$$

where

$$\begin{aligned}\delta_n^{(\xi,\psi,\alpha)}(x) =& \frac{1}{(n+\psi)^2}\Bigg[\frac{x^2\{\alpha^2(n^2+2n\psi+\psi^2) - 2\alpha\psi(n+\psi) + \psi^2\}}{(1-\alpha)^2} + \frac{2x\alpha^3(n+\psi)(1+\xi)}{(1-\alpha)^3} \\ &+ \frac{x\{-\alpha^2(7n+8n\xi+12\psi+12\psi\xi) + 2\alpha(2n\xi+6\psi\xi+6\psi+n) + (3n-4\psi-4\psi\xi)\}}{2(1-\alpha)^3} \\ &+ \left(\xi^2 + 2\xi + \frac{3}{2}\right)\Bigg]\end{aligned} \tag{14}$$

Proof. We introduce an auxiliary operator $\tilde{A}_n^{(\xi,\psi,\alpha)} : \widetilde{\mathcal{W}}[0,\infty) \to \widetilde{\mathcal{W}}[0,\infty)$ as follows

$$\tilde{A}_n^{(\xi,\psi,\alpha)}(u;x) = A_n^{(\xi,\psi,\alpha)}(u;x) - u\left(\frac{nx}{(1-\alpha)(n+\psi)} + \frac{(1+\xi)}{(n+\psi)}\right) + u(x). \tag{15}$$

Clearly, the above defined auxiliary operator preserves linear as well as constant functions.
Let $v \in \mathcal{W}^2$ and $x, t \in [0,\infty)$. Then by Taylor's expansion, we have

$$v(t) = v(x) + (t-x)v'(x) + \int_x^t (t-z)v''(z)dz. \tag{16}$$

Now, applying the auxiliary operators on both sides on (16), we obtain

$$\begin{aligned}
\left|\tilde{A}_n^{(\xi,\psi,\alpha)}(v;x)-v(x)\right| \leq& \tilde{A}_n^{(\xi,\psi,\alpha)}\left(\left|\int_x^t (t-z)v''(z)dz, x\right|\right)\\
\leq& A_n^{(\xi,\psi,\alpha)}\left(\left|\int_x^t (t-z)v''(z)dz, x\right|\right)\\
&+\left|\int_x^{\frac{nx}{(1-\alpha)(n+\psi)}+\frac{(1+\xi)}{(n+\psi)}}\left(\frac{nx}{(1-\alpha)(n+\psi)}+\frac{(1+\xi)}{(n+\psi)}-z\right)v''(z)dz\right|\\
\leq& \frac{1}{2}\left[A_n^{(\xi,\psi,\alpha)}((t-x)^2;x)\|v''\|+\left(\frac{nx}{(1-\alpha)(n+\psi)}+\frac{(1+\xi)}{(n+\psi)}-x\right)^2\|v''\|\right]\\
=& \frac{1}{(n+\psi)^2}\left[\frac{x^2\{\alpha^2(n^2+2n\psi+\psi^2)-2\alpha\psi(n+\psi)+\psi^2\}}{(1-\alpha)^2}+\frac{2x\alpha^3(n+\psi)(1+\xi)}{(1-\alpha)^3}\right.\\
&+\frac{x\{-\alpha^2(7n+8n\xi+12\psi+12\psi\xi)+2\alpha(2n\xi+6\psi\xi+6\psi+n)+(3n-4\psi-4\psi\xi)\}}{2(1-\alpha)^3}\\
&\left.+\left(\xi^2+2\xi+\frac{3}{2}\right)\right]\|v''\|\\
=& \,\delta_n^{(\xi,\psi,\alpha)}(x)\|v''\|,
\end{aligned}$$

where $\delta_n^{(\xi,\psi,\alpha)}(x)$ is as given in (14).
On the other hand by Lemma 2.7 for $u \in \widetilde{\mathcal{W}}[0,\infty)$ and $x \in [0,\infty)$, we have

$$\left\|A_n^{(\xi,\psi,\alpha)}(u)\right\| \leq \|u\|.$$

Now applying triangle inequality on (15), we obtain

$$\left\|\tilde{A}_n^{(\xi,\psi,\alpha)}(u;x)\right\| \leq \left\|A_n^{(\xi,\psi,\alpha)}(u;x)\right\| + 2\|u\| \ \leq \ 3\|u\|.$$

Using all the above inequalities, we can write

$$\begin{aligned}
\left|A_n^{(\xi,\psi,\alpha)}(u;x)-u(x)\right| \leq& \left|\tilde{A}_n^{(\xi,\psi,\alpha)}(u-v;x)-(u-v)(x)\right|+\left|\tilde{A}_n^{(\xi,\psi,\alpha)}(v;x)-v(x)\right|\\
&+\left|u(x)-u\left(\frac{nx}{(1-\alpha)(n+\psi)}+\frac{(1+\xi)}{(n+\psi)}\right)\right|\\
\leq& 4\|u-v\|+\delta_n^{(\xi,\psi,\alpha)}(x)\|v''\|+\omega\left(u,\frac{nx}{(1-\alpha)(n+\psi)}+\frac{(1+\xi)}{(n+\psi)}\right)\\
\leq& 4\left\{\|u-v\|+\delta_n^{(\xi,\psi,\alpha)}(x)\|v''\|\right\}+\omega\left(u,\frac{nx}{(1-\alpha)(n+\psi)}+\frac{(1+\xi)}{(n+\psi)}\right).
\end{aligned}$$

Now, taking infimum over all $v \in \mathcal{W}^2$ on right hand side of the above equation and using (13), we get the desired assertion. □

We conclude this section by establishing a convergence result for the Lipschitz-type class of functions.

A Lipschitz-type space with two parameters ζ_1 and ζ_2 is defined in [36] as

$$Lip_M^{\zeta_1,\zeta_2}(\gamma) := \left\{u \in \mathcal{W}[0,\infty) : |u(t)-u(x)| \leq M\frac{|t-x|^\gamma}{(t+\zeta_1x^2+\zeta_2x)^{\gamma/2}};\ x,t \in (0,\infty)\right\}$$

where $M > 0$ and $\gamma \in (0,1]$.

Theorem 6.3. *If $u \in Lip_M^{\zeta_1,\zeta_2}(\eta)$, Then*

$$\left|A_n^{(\xi,\psi,\alpha)}(u;x) - u(x)\right| \le M\left(\frac{\lambda_n^{(\xi,\psi,\alpha)}(x)}{\zeta_1 x^2 + \zeta_2 x}\right)^{\gamma/2}.$$

Proof. First of all we notice: if $\gamma = 1$, then

$$\left|A_n^{(\xi,\psi,\alpha)}(u;x) - u(x)\right| \le A_n^{(\xi,\psi,\alpha)}(|u(t) - u(x)|;x) \le M\left[A_n^{(\xi,\psi,\alpha)}\left(\frac{|t-x|}{\sqrt{t + \zeta_1 x^2 + \zeta_2 x}};x\right)\right].$$

Using the fact: $\frac{1}{\sqrt{\frac{nt+\xi}{n+\psi} + \zeta_1 x^2 + \zeta_2 x}} \le \frac{1}{\sqrt{\zeta_1 x^2 + \zeta_2 x}}$ and the well-known Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\left|A_n^{(\xi,\psi,\alpha)}(u;x) - u(x)\right| \le & \frac{M}{\sqrt{\zeta_1 x^2 + \zeta_2 x}} A_n^{(\xi,\psi,\alpha)}\left(|t-x|;x\right) \\
\le & \frac{M}{\sqrt{\zeta_1 x^2 + \zeta_2 x}}\left(A_n^{(\xi,\psi,\alpha)}((t-x)^2;x)\right)^{1/2} = M\left(\frac{\lambda_n^{(\xi,\psi,\alpha)}(x)}{\zeta_1 x^2 + \zeta_2 x}\right)^{1/2},
\end{aligned}$$

where $\lambda_n^{(\xi,\psi,\alpha)}(x)$ is defined by the right hand side of (6).
Hence the result holds for $\gamma = 1$.
Next we consider the case when $\gamma \in (0,1)$.
Clearly,

$$\left|A_n^{(\xi,\psi,\alpha)}(u;x) - u(x)\right| \le n \sum_{k=0}^{\infty} l_{n,k}^{\alpha}(x) \int_0^{\infty} s_{n,k}(t)\left|u\left(\frac{nt+\xi}{n+\psi}\right) - u(x)\right| dt.$$

Let $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$ and using the Hölder inequality, we obtain

$$\begin{aligned}
\left|A_n^{(\xi,\psi,\alpha)}(u;x) - u(x)\right| \le & \left[n \sum_{k=0}^{\infty} l_{n,k}^{\alpha}(x) \int_0^{\infty} s_{n,k}(t)\left|u\left(\frac{nt+\xi}{n+\psi}\right) - u(x)\right|^{2/\gamma} dt\right]^{\gamma/2} \\
& \times\left[n \sum_{k=0}^{\infty} l_{n,k}^{\alpha}(x) \int_0^{\infty} s_{n,k}(t) dt\right]^{(2-\gamma)/2} \\
\le & M\left[n \sum_{k=0}^{\infty} l_{n,k}^{\alpha}(x) \int_0^{\infty} s_{n,k}(t) \frac{\left(\frac{nt+\xi}{n+\psi} - x\right)^2}{\left(\frac{nt+\xi}{n+\psi} + \zeta_1 x^2 + \zeta_2 x\right)} dt\right]^{\gamma/2} \\
\le & \frac{M}{(\zeta_1 x^2 + \zeta_2 x)^{\gamma/2}}\left[A_n^{(\xi,\psi,\alpha)}((t-x)^2 : x)\right]^{\gamma/2} \\
= & M\left(\frac{\lambda_n^{(\xi,\psi,\alpha)}(x)}{\zeta_1 x^2 + \zeta_2 x}\right)^{\gamma/2}.
\end{aligned}$$

This completes the proof. □

Remark 6.4. *One of the crucial remark is that we have investigated the local approximation results from Theorem 6.1 to Theorem 6.3. Therefore, we indicate that $\lambda_n^{(\xi,\psi,\alpha)}$ and $\delta_n^{(\xi,\psi,\alpha)}$ tend to zero while taking $\alpha = \alpha_n$ with $\alpha_n \in [0,1), \forall n$ and $\lim_{n\to\infty} \alpha_n = 0$. Otherwise, these results just stand an inequality or an estimation.*

7. Graphical Examples

This section provides a few numerical examples for the operators (5) to justify their approximation properties as well as the relevance of the generalization.

In both the Figures 1 and 2, we can see that the operator $A_n^{(1,2,\alpha_n)}(u;x)$ converges faster than the operator $D_n^{\alpha_n}(u;x)\Big(\text{ i.e., } A_n^{(0,0,\alpha_n)}(u;x)\Big)$ to the considered test functions $u(x) = e^{-\frac{x}{2}}(5x^8 + x^4 + 1)$ and $u(x) = x^2 sin(\pi x)$ respectively. Here we assumed $n = 500$, $x \in [0,1]$ and $\{\alpha_n\}_{n\geq 1}$ is the sequence $\{\frac{1}{n^2}\}_{n\geq 1}$ so that the hypothesis $\lim\limits_{n\to\infty} \alpha_n = 0$ is satisfied.

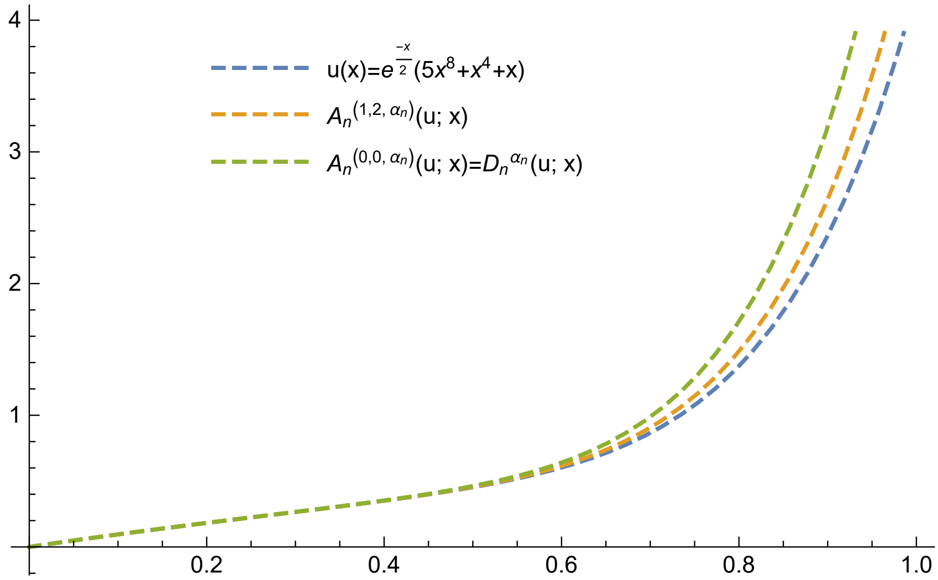

Figure 1: Convergence of the considered operators $A_n^{(\xi,\psi,\alpha)}(u;x)$ to the test function $u(x) = e^{-\frac{x}{2}}(5x^8 + x^4 + 1)$ for $n = 500$, $x \in [0,1]$.

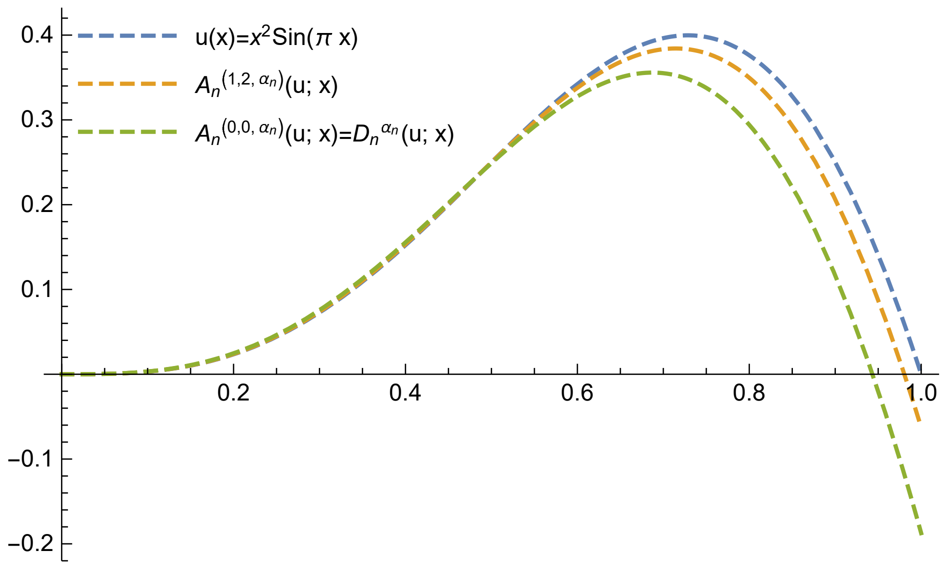

Figure 2: Convergence of the considered operators $A_n^{(\xi,\psi,\alpha)}(u;x)$ to the test function $u(x) = x^2 \sin(\pi x)$ for $n = 500$, $x \in [0,1]$.

8. Conclusion

The generalized Stancu variant of the Modified Jain operators has been introduced and numerically tested. We have established some of the approximation properties of newly defined operators, such as Voronovskaya type theorems, weighted approximation and degree of local approximation. Finally, we added some numerical experiments to show the faster convergence rate of the newly defined operators than the previous ones.

Acknowledgment

The first author is thankful to the Council of Scientific and Industrial Research (CSIR), India, Grant Code:-09/1217(0083)/2020-EMR-I for the financial support to pursue his research work. Furthermore, we would like to express our gratitude to the reviewers for their valuable suggestions to enhance the betterment of the paper. We also thank the handling editors for submitting the reports on time.

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