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Pointwise summability of Fourier–Laguerre series of integrable functions

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Abstract. We present an approximation version of the results of D. P. Gupta [J. of Approx. Theory, 7 (1973), 226-238] A. N. S. Singroura [Proc. Japan Acad., 39 (4) (1963), 208-210] and G. Szegö [Math. Z., 25 (1926), 87-115]. Some corollaries and examples will also be given.

1. Introduction

Let L_w be the class of all real–valued functions, integrable in the Lebesgue sense over $\mathbb{R}^+$ with the weight $w(t) = e^{-t}t^{\alpha}$ $(\alpha > -1)$, i.e.

$$\int_{\mathbb{R}^+} e^{-t}t^{\alpha} \mid f(t) \mid dt < \infty.$$

We will consider the Fourier–Laguerre series

$$S^{(\alpha)}f(x) := \sum_{\nu=0}^{\infty} a_{\nu}^{(\alpha)}(f) L_{\nu}^{(\alpha)}(x), \text{ with } \alpha > -1,$$

where

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}\left(x^{n+\alpha}e^{-x}\right) = \sum_{\nu=0}^{n} \frac{(-1)^{\nu}}{\nu!}\binom{n+\alpha}{n-\nu}x^{\nu}$$

and

$$a_{\nu}^{(\alpha)}(f) = \frac{1}{\Gamma(\alpha+1)A_n^{(\alpha)}}\int_0^{\infty} e^{-y}y^{\alpha}L_{\nu}^{(\alpha)}(y) f(y)\, dy, \text{ with } A_n^{(\alpha)} = \binom{n+\alpha}{n}.$$

Let define the (C,γ) - means of partial sums

$$S_k^{(\alpha)}f(x) = \sum_{\nu=0}^{n} a_{\nu}^{(\alpha)}(f) L_{\nu}^{(\alpha)}(x)$$

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of $S^{(\alpha)}f$ as follows

$$S_n^{(\gamma,\alpha)}f(x) = \frac{1}{A_n^{(\gamma)}} \sum_{k=0}^{n} A_{n-k}^{(\gamma-1)} S_k^{(\alpha)} f(x), \quad (n = 0, 1, 2, ...)$$

and let

$$\Delta_x f(t) = f(x+t) - f(x).$$

The deviation $S_n^{(\gamma,\alpha)}f(0) - f(0)$ was examined in the papers [5], [7] and [1] as follows:

Theorem A. [1, Theorem 1] *Let $f \in L_w$, $\alpha > -1$ and $\gamma > \alpha + \frac{1}{2}$. If a function f satisfies the conditions*

$$\int_0^u e^{-t} t^{\alpha} \left|\Delta_0 f(t)\right| dt = o\left(u^{\alpha+1}\right) \text{ as } u \to 0^+ \tag{1}$$

and

$$\int_1^{\infty} e^{-\frac{t}{2}} t^{\alpha-\gamma-\frac{1}{3}} \left|\Delta_0 f(t)\right| dt < \infty, \tag{2}$$

then

$$\left|S_n^{(\gamma,\alpha)}f(0) - f(0)\right| = o(1) \text{ as } n \to \infty.$$

Similar results in a case of norm approximation due of C. Markett and E. L. Poiani in papers [3] and [4] were obtained.

We will say that a nonnegative function ω is a function of the modulus of continuity type if it is nondecreasing continuous function on $[0, \infty)$ having the following conditions: $\omega(0) = 0$ and $\omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2)$ for any $\delta_1, \delta_2 \in [0, \infty)$.

In this paper, we will study the upper bound of the quantity $\left|S_n^{(\gamma,\alpha)}f(0) - f(0)\right|$ by some means of a function of the modulus of continuity type ω. From our result we will derive some corollaries, remarks and construct some examples.

2. Statement of the results

First we present the estimate of the quantity $\left|S_n^{(\gamma,\alpha)}f(0) - f(0)\right|$.

Theorem 2.1. *Let $f \in L_w$, $\alpha > -1$, $\gamma > \alpha + \frac{1}{2}$ and let a function ω of the modulus of continuity type satisfy the conditions:*

$$\frac{u^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^u e^{-\frac{t}{2}} t^{\alpha} \left|\Delta_0 f(t)\right| dt = O(\omega(u)) \quad (u > 0) \tag{3}$$

and

$$\frac{1}{\Gamma(\alpha+1)} \int_u^{\infty} e^{-\frac{t}{2}} t^{\alpha-\gamma-\frac{1}{3}} \left|\Delta_0 f(t)\right| dt = O(\omega(1/u)) \quad (u \ge 1). \tag{4}$$

Then

$$\left|S_n^{(\gamma,\alpha)}f(0) - f(0)\right| = O\left(n^{\eta+\frac{2(\alpha-\gamma)+1}{4}}\right) \sum_{k=1}^{n} \frac{\omega(1/k)}{k^{\eta+\frac{2(\alpha-\gamma)+1}{4}+1}} + O(\omega(1/n^{\eta}))$$

for $0 < \eta < -\frac{2(\alpha-\gamma)+1}{4}$.

Now, we formulate some corollaries and remarks.

Corollary 2.2. *Under the assumptions of the above theorem*

$$\left|S_n^{(\gamma,\alpha)} f(0) - f(0)\right| = o(1) \text{ as } n \to \infty.$$

This relation is an immediately consequence of Theorem 2.1, since (3) when $u \to 0$ becomes a condition (1) ((1.8) from [1]), (4) when $u = 1$ is becomes exactly (2) ((1.9) from [1]) and mean

$$n^{\eta+\frac{2(\alpha-\gamma)+1}{4}} \sum_{k=1}^{n} \frac{\omega(1/k)}{k^{\eta+\frac{2(\alpha-\gamma)+1}{4}+1}}$$

tends to 0 as $n \to 0$ for any $\eta \in \left(0, -\frac{2(\alpha-\gamma)+1}{4}\right)$ as well as $\omega(1/n^\eta)$ tends to 0 as $n \to 0$ for any $\eta > 0$ with any function ω of modulus of continuity type. Thus we have Theorem A (Theorem 1 from [1]) without any additional conditions and assumptions.

Remark 2.3. *Using Theorem 2.1 and Corollary 2.2 we obtain the result of D. P. Gupta from Theorem A.*

Remark 2.4. *Similar type of result was obtained by S.P.Yadav [8] for Hölder classes. There was considered Cesàro* (C,γ) *- means of the order* $\gamma \in \left(\alpha - \frac{1}{2}, \alpha + \frac{1}{2}\right)$ *but in our paper* $\gamma > \alpha + \frac{1}{2}$*. D.P.Gupta, S.M.Mazhar [2] also examined similar problems with partial sums (*$\gamma = 0$*) and* $\alpha \in \left(-\frac{1}{2}, 1\right)$ *but in our result we have Cesàro mean of the order* $\gamma > \alpha + \frac{1}{2}$ *with an assumption* $\alpha < -\frac{1}{2}$*.*

Corollary 2.5. *Analyzing the proof of Theorem 2.1 we can obtain, under the assumptions of this theorem, the following more precise estimate*

$$\left|S_n^{(\gamma,\alpha)} f(0) - f(0)\right| = O\left(n^{\frac{2(\alpha-\gamma)+1}{4}} \omega(n^\eta)\right) + O(\omega(1/n^\eta)) + O(\omega(1/n)),$$

when $\frac{2(\alpha-\gamma)+1}{4} + 1 < 0$.

In the special case, taking $\eta = -\frac{2(\alpha-\gamma)+1}{8}$ *, we have*

$$\left|S_n^{(\gamma,\alpha)} f(0) - f(0)\right| = O(\omega(1/n^\eta)),$$

when $\eta \leq 1$.

3. Examples

Let $f_1(t) = e^{-\frac{t}{2}}$ and $\omega_1(t) = t$ for $t \geqslant 0$.

It is clear that $f_1 \in L_w$. Moreover, applying the Lagrange mean value theorem we get that

$$|\Delta_0 f_1(t)| = |e^{-\frac{t}{2}} - 1| \leqslant \frac{t}{4}$$

for $t \geq 0$. Therefore, by elementary calculations we get

$$\frac{u^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^u e^{-\frac{t}{2}} t^\alpha |\Delta_0 f_1(t)| dt$$

$$\leqslant \frac{u^{-(\alpha+1)}}{2\Gamma(\alpha+1)} \int_0^u t^{\alpha+1} dt = \frac{1}{2\Gamma(\alpha+1)(\alpha+2)} \omega_1(u)$$

for $u > 0$ and

$$\frac{1}{\omega_1\left(\frac{1}{u}\right)\Gamma(\alpha+1)}\int_u^\infty e^{-\frac{t}{2}}t^{\alpha-\gamma-\frac{1}{3}}|\Delta_0 f_1(t)|dt$$

$$\leqslant \frac{u}{2\Gamma(\alpha+1)}\int_u^\infty e^{-\frac{t}{2}}t^{\alpha-\gamma+\frac{2}{3}}dt \leqslant \frac{1}{2\Gamma(\alpha+1)}\int_u^\infty e^{-\frac{t}{2}}t^{\alpha-\gamma+\frac{5}{3}}dt$$

$$\leqslant \frac{1}{2\Gamma(\alpha+1)}\int_0^\infty e^{-\frac{t}{2}}t^{\alpha-\gamma+\frac{5}{3}}dt = \frac{1}{2\Gamma(\alpha+1)}\int_0^\infty e^{-\frac{t}{2}}t^{-1+\alpha-\gamma+\frac{8}{3}}dt$$

$$= \frac{1}{\Gamma(\alpha+1)}2^{\alpha-\gamma+\frac{5}{3}}\Gamma\left(\alpha-\gamma+\frac{8}{3}\right) < \infty$$

for $u \geq 1$ and $\alpha - \gamma + \frac{8}{3} > 0$.

Hence the function f_1 satisfies the conditions (3) and (4). Using Theorem 2.1 we get the following estimate for $|S_n^{(\gamma,\alpha)} f_1(0) - f_1(0)|$:

Example 3.1. *Let* $\alpha > -1$, $\alpha + \frac{1}{2} < \gamma < \alpha + \frac{8}{3}$ *and* $0 < \eta < -\frac{2(\alpha-\gamma)+1}{4}$. *Then*

$$|S_n^{(\gamma,\alpha)} f_1(0) - f_1(0)| = O\left(n^{\eta+\frac{2(\alpha-\gamma)+1}{4}}\right)\sum_{k=1}^n \frac{1}{k^{\eta+\frac{2(\alpha-\gamma)+1}{4}+2}} + O\left(\frac{1}{n^\eta}\right).$$

Suppose $f_2(t) = t^\delta$ and $\omega_2(t) = t^\delta$ for $\delta \in (0, 1]$ and $t \geqslant 0$.

Obviously $f_2 \in L_w$. In addition, it is easy to show that

$$\frac{u^{-(\alpha+1)}}{\Gamma(\alpha+1)}\int_0^u e^{-\frac{t}{2}}t^\alpha|\Delta_0 f_2(t)|dt$$

$$\leqslant \frac{u^{-(\alpha+1)}}{\Gamma(\alpha+1)}\int_0^u t^{\alpha+\delta}dt = \frac{1}{\Gamma(\alpha+1)(\alpha+\delta+1)}\omega_2(u)$$

for $u > 0$ and

$$\frac{1}{\omega_2\left(\frac{1}{u}\right)\Gamma(\alpha+1)}\int_u^\infty e^{-\frac{t}{2}}t^{\alpha-\gamma-\frac{1}{3}}|\Delta_0 f_2(t)|dt$$

$$\leqslant \frac{u^\delta}{\Gamma(\alpha+1)}\int_u^\infty e^{-\frac{t}{2}}t^{\alpha-\gamma+\delta-\frac{1}{3}}dt \leqslant \frac{1}{\Gamma(\alpha+1)}\int_u^\infty e^{-\frac{t}{2}}t^{\alpha-\gamma+2\delta-\frac{1}{3}}dt$$

$$\leqslant \frac{1}{\Gamma(\alpha+1)}\int_0^\infty e^{-\frac{t}{2}}t^{\alpha-\gamma+2\delta-\frac{1}{3}}dt = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty e^{-\frac{t}{2}}t^{-1+\alpha-\gamma+2\delta+\frac{2}{3}}dt$$

$$= \frac{1}{\Gamma(\alpha+1)}2^{\alpha-\gamma+2\delta+\frac{2}{3}}\Gamma\left(\alpha-\gamma+2\delta+\frac{2}{3}\right) < \infty$$

for $u \geq 1$ and $\alpha - \gamma + 2\delta + \frac{2}{3} > 0$.

Therefore the function f_2 satisfies the conditions (3) and (4). Using Theorem 2.1 we get the following estimate for $|S_n^{(\gamma,\alpha)} f_2(0) - f_2(0)|$:

Example 3.2. *Let* $\alpha > -1$, $\delta \in (0, 1]$, $\alpha + \frac{1}{2} < \gamma < \alpha + 2\delta + \frac{2}{3}$ *and* $0 < \eta < -\frac{2(\alpha-\gamma)+1}{4}$. *Then*

$$|S_n^{(\gamma,\alpha)} f_2(0) - f_2(0)| = O\left(n^{\eta+\frac{2(\alpha-\gamma)+1}{4}}\right)\sum_{k=1}^n \frac{1}{k^{\eta+\frac{2(\alpha-\gamma)+1}{4}+1+\delta}} + O\left(\frac{1}{n^{\eta\delta}}\right).$$

4. Auxiliary results

We begin this section by some notations from [6] . We have

$$L_k^{(\alpha+1)}(y)=\sum_{\nu=0}^{k}L_\nu^{(\alpha)}(y)\,,\quad L_\nu^{(\alpha)}(0)=\binom{\nu+\alpha}{\nu}$$

and therefore

$$\begin{aligned} S_k^{(\alpha)}f(0) &= \frac{1}{\Gamma(\alpha+1)}\int_0^\infty e^{-y}y^\alpha L_k^{(\alpha+1)}(y)\,f(y)\,dy,\\ S_n^{(\gamma,\alpha)}f(0) &= \frac{1}{\Gamma(\alpha+1)\,A_n^{(\gamma)}}\int_0^\infty e^{-y}y^\alpha L_n^{(\alpha+\gamma+1)}(y)\,f(y)\,dy. \end{aligned}$$

Hence, by evidence equality

$$\frac{1}{\Gamma(\alpha+1)}\int_0^\infty e^{-y}y^\alpha L_\nu^{(\alpha+1)}(y)\,dy=\begin{cases}1 & \text{if } \nu=0,\\ 0 & \text{if } \nu\neq 0,\end{cases}$$

we have

$$\begin{aligned} S_k^{(\alpha)}f(0)-f(0) &= \frac{1}{\Gamma(\alpha+1)}\int_0^\infty e^{-y}y^\alpha L_k^{(\alpha+1)}(y)\,\Delta_0 f(y)\,dy,\\ S_n^{(\gamma,\alpha)}f(0)-f(0) &= \frac{1}{\Gamma(\alpha+1)\,A_n^{(\gamma)}}\int_0^\infty e^{-y}y^\alpha L_n^{(\alpha+\gamma+1)}(y)\,\Delta_0 f(y)\,dy. \end{aligned}$$

Next, we present the useful estimates:

Lemma 4.1. *[6, p. 172] Let β be an arbitrary real number, c and δ be fixed positive constants. Then*

$$\left|L_n^{(\beta)}(x)\right|=\begin{cases} O\left(n^\beta\right) & \text{if } 0\le x\le \frac{c}{n},\\ O\left(x^{-(2\beta+1)/4}n^{(2\beta-1)/4}\right) & \text{if } \frac{c}{n}\le x\le\delta.\end{cases}$$

Lemma 4.2. *[6, p. 235] Let β and λ be arbitrary real numbers, $\delta>0$ and $0<\theta<4$. Then*

$$\max_x e^{-x/2}x^\lambda\left|L_n^{(\beta)}(x)\right|=\begin{cases} O\left(n^{\max\left(\lambda-\frac12,\frac{\beta}{2}-\frac14\right)}\right) & \text{if } \delta\le x\le(4-\theta)\,n,\\ O\left(n^{\max\left(\lambda-\frac13,\frac{\beta}{2}-\frac14\right)}\right) & \text{if } \qquad x\ge\delta.\end{cases}$$

Lemma 4.3. *[9, Vol. I, (1.15) and Theorem 1.17] If $\gamma>-1$, then*

$$A_n^{(\gamma)}=\binom{n+\gamma}{n}\simeq O\left((n+1)^\gamma\right)$$

and $A_n^{(\gamma)}$ is positive for $\gamma>-1$ increasing (as a function of n) for $\gamma>0$ and decreasing for $-1<\gamma<0$.

5. Proof of Theorem 2.1

It is clear that if

$$\begin{aligned} &S_n^{(\gamma,\alpha)}f(0)-f(0)\\ &= \frac{1}{\Gamma(\alpha+1)\,A_n^{(\gamma)}}\int_0^\infty e^{-y}y^\alpha L_n^{(\alpha+\gamma+1)}(y)\,\Delta_0 f(y)\,dy\\ &= \left(\int_0^{1/n}+\int_{1/n}^{1}+\int_1^{n^\eta}+\int_{n^\eta}^{\infty}\right)=J_1+J_2+J_3+J_4\,, \end{aligned}$$

then

$$\left|S_n^{(\gamma,\alpha)} f(0) - f(0)\right| \le |J_1| + |J_2| + |J_3| + |J_4|.$$

By Lemma 4.1, Lemma 4.33 and (3)

$$\begin{aligned}
|J_1| &\le \frac{1}{\Gamma(\alpha+1) A_n^{(\gamma)}} \int_0^{1/n} e^{-\frac{y}{2}} y^\alpha \left|L_n^{(\alpha+\gamma+1)}(y)\right| \left|\Delta_0 f(y)\right| dy \\
&= \frac{O\left(n^{\alpha+\gamma+1}\right)}{\Gamma(\alpha+1) A_n^{(\gamma)}} \int_0^{1/n} e^{-\frac{y}{2}} y^\alpha \left|\Delta_0 f(y)\right| dy \\
&= \frac{O\left(n^{\alpha+1}\right)}{\Gamma(\alpha+1)} \int_0^{1/n} e^{-\frac{y}{2}} y^\alpha \left|\Delta_0 f(y)\right| dy = O(\omega(1/n)) \\
&\le O\left(n^{(2\alpha-2\gamma+1)/4}\right) \sum_{k=1}^{n} \frac{\omega(1/k)}{k^{(2\alpha-2\gamma+1)/4+1}} \\
&\le O\left(n^{\eta+\frac{2(\alpha-\gamma)+1}{4}}\right) \sum_{k=1}^{n} \frac{\omega(1/k)}{k^{\eta+\frac{2(\alpha-\gamma)+1}{4}+1}},
\end{aligned}$$

with $0 < \eta < -\frac{2(\alpha-\gamma)+1}{4}$.

Using Lemma 4.1, we get

$$\begin{aligned}
|J_2| &\le \frac{1}{\Gamma(\alpha+1) A_n^{(\gamma)}} \int_{1/n}^{1} e^{-y} y^\alpha \left|L_n^{(\alpha+\gamma+1)}(y)\right| \left|\Delta_0 f(y)\right| dy \\
&= \frac{O\left(n^{\frac{2(\alpha+\gamma)+1}{4}}\right)}{\Gamma(\alpha+1) A_n^{(\gamma)}} \int_{1/n}^{1} e^{-\frac{y}{2}} y^\alpha \left|\Delta_0 f(y)\right| y^{-\frac{2(\alpha+\gamma)+3}{4}} dy.
\end{aligned}$$

Let $F_\alpha(y) = \frac{y^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^y e^{-\frac{u}{2}} u^\alpha \left|\Delta_0 f(u)\right| du$. Applying Lemma 4.3 and integrating by parts with $\gamma > \alpha + \frac{1}{2}$ and $\alpha > -1$ we have

$$\begin{aligned}
|J_2| &= \frac{O\left(n^{(2\alpha+2\gamma+1)/4}\right)}{\Gamma(\alpha+1) A_n^{(\gamma)}} \int_{1/n}^{1} e^{-y/2} y^{(2\alpha-2\gamma-3)/4} \left|\Delta_0 f(y)\right| dy \\
&= O\left(n^{(2\alpha-2\gamma+1)/4}\right) \left\{ \left[F_\alpha(y)\, y^{\frac{2(\alpha-\gamma)+1}{4}}\right]_{y=1/n}^{1} \right. \\
&\quad \left. + \frac{2(\alpha+\gamma)+3}{4} \int_{1/n}^{1} F_\alpha(y)\, y^{\frac{2(\alpha-\gamma)+1}{4}-1} dy \right\} \\
&\le O\left(n^{(2\alpha-2\gamma+1)/4}\right) \left\{ F_\alpha(1) + \int_{1/n}^{1} F_\alpha(y)\, y^{\frac{2(\alpha-\gamma)+1}{4}-1} dy \right\}
\end{aligned}$$

$$\begin{aligned}
&= O\left(n^{(2\alpha-2\gamma+1)/4}\right) \left\{ F_\alpha(1) + \int_{1}^{n} F_\alpha(1/y)\, y^{-\frac{2(\alpha-\gamma)+1}{4}-1} dy \right\} \\
&= O\left(n^{(2\alpha-2\gamma+1)/4}\right) \left\{ F_\alpha(1) + \sum_{k=1}^{n-1} \int_{k}^{k+1} F_\alpha(1/y)\, y^{-\frac{2(\alpha-\gamma)+1}{4}-1} dy \right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq O\left(n^{(2\alpha-2\gamma+1)/4}\right)\left\{F_{\alpha}(1)+\sum_{k=1}^{n-1}F_{\alpha}(1/k)\,k^{-\frac{2(\alpha-\gamma)+1}{4}-1}\right\}\\
&\leq O\left(n^{(2\alpha-2\gamma+1)/4}\right)\sum_{k=1}^{n-1}2F_{\alpha}(1/k)\,k^{-\frac{2(\alpha-\gamma)+1}{4}-1}\\
&\leq O\left(n^{(2\alpha-2\gamma+1)/4}\right)\sum_{k=1}^{n}\frac{F_{\alpha}(1/k)}{k^{(2\alpha-2\gamma+1)/4+1}}.
\end{aligned}
$$

By (3) we obtain

$$
\begin{aligned}
|J_2| &= O\left(n^{(2\alpha-2\gamma+1)/4}\right)\sum_{k=1}^{n}\frac{\omega(1/k)}{k^{(2\alpha-2\gamma+1)/4+1}}\\
&\leq O\left(n^{\eta+\frac{2(\alpha-\gamma)+1}{4}}\right)\sum_{k=1}^{n}\frac{\omega(1/k)}{k^{\eta+\frac{2(\alpha-\gamma)+1}{4}+1}},
\end{aligned}
$$

with $0<\eta<-\frac{2(\alpha-\gamma)+1}{4}$.

Applying Lemma 4.2 with $\alpha+\gamma+1$ instead of β and $\lambda=\frac{2\alpha+2\gamma+3}{4}$ (since $\max\left(\lambda-\frac{1}{2},\frac{\alpha+\gamma+1}{2}-\frac{1}{4}\right)=\frac{2\alpha+2\gamma+1}{4}$) we have

$$
\begin{aligned}
&|J_3|\\
&\leq \frac{1}{\Gamma(\alpha+1)A_n^{(\gamma)}}\int_1^{n^{\eta}}e^{-y/2}y^{(2\alpha-2\gamma-3)/4}\left|\Delta_0 f(y)\right|e^{-y/2}y^{(2\alpha+2\gamma+3)/4}\left|L_n^{(\alpha+\gamma+1)}(y)\right|dy\\
&= \frac{O\left(n^{(2\alpha+2\gamma+1)/4}\right)}{\Gamma(\alpha+1)A_n^{(\gamma)}}\int_1^{n^{\eta}}e^{-y/2}y^{(2\alpha-2\gamma-3)/4}\left|\Delta_0 f(y)\right|dy.
\end{aligned}
$$

Using Lemma 4.3 and integrating by parts with $\gamma>\alpha+\frac{1}{2}$ and $\alpha>-1$, we get

$$
\begin{aligned}
|J_3| &= \frac{O\left(n^{(2\alpha-2\gamma+1)/4}\right)}{\Gamma(\alpha+1)}\int_1^{n^{\eta}}\left[e^{-y/2}y^{\alpha}\left|\Delta_0 f(y)\right|\right]y^{(-2\alpha-2\gamma-3)/4}dy\\
&= O\left(n^{(2\alpha-2\gamma+1)/4}\right)\int_1^{n^{\eta}}\frac{d}{dy}\left[\int_0^{y}\frac{e^{-u/2}u^{\alpha}\left|\Delta_0 f(u)\right|}{\Gamma(\alpha+1)}du\right]y^{(-2\alpha-2\gamma-3)/4}dy\\
&= O\left(n^{(2\alpha-2\gamma+1)/4}\right)\left\{\left[y^{(-2\alpha-2\gamma-3)/4}\int_0^{y}\frac{e^{-u/2}u^{\alpha}\left|\Delta_0 f(u)\right|}{\Gamma(\alpha+1)}du\right]_1^{n^{\eta}}\right.\\
&\quad\left.+\frac{2\alpha+2\gamma+3}{4}\int_1^{n^{\eta}}\frac{d}{dy}\left[\int_0^{y}\frac{e^{-u/2}u^{\alpha}\left|\Delta_0 f(u)\right|}{\Gamma(\alpha+1)}du\right]y^{\frac{-2\alpha-2\gamma-3}{4}-1}dy\right\}\\
&\leq O\left(n^{(2\alpha-2\gamma+1)/4}\right)\left\{F_{\alpha}(n^{\eta})\,n^{\eta\frac{2(\alpha-\gamma)+1}{4}}+\frac{2\alpha+2\gamma+3}{4}\int_1^{n^{\eta}}F_{\alpha}(y)\,y^{\frac{2(\alpha-\gamma)+1}{4}-1}dy\right\}.
\end{aligned}
$$

By (3) we obtain

$$
|J_3|\leq O\left(n^{(2\alpha-2\gamma+1)/4}\right)\left\{\omega(n^{\eta})\,n^{\eta\frac{2(\alpha-\gamma)+1}{4}}+\frac{2\alpha+2\gamma+3}{4}\int_1^{n^{\eta}}\omega(y)\,y^{\frac{2(\alpha-\gamma)+1}{4}-1}dy\right\}
$$

$$\leq O(1)\left\{n^{(2\alpha-2\gamma+1)/4}n^{\eta}\omega(1)\,n^{\eta\frac{2(\alpha-\gamma)+1}{4}}+n^{(2\alpha-2\gamma+1)/4}\omega(n^{\eta})\int_1^{n^{\eta}}y^{\frac{2(\alpha-\gamma)+1}{4}-1}dy\right\}$$

$$\leq O\left(n^{(2\alpha-2\gamma+1)/4}n^{\eta}n^{\eta\frac{2(\alpha-\gamma)+1}{4}}+n^{\frac{2(\alpha-\gamma)+1}{4}}n^{\eta}\right)\omega(1)\quad\leq O\left(n^{\frac{2(\alpha-\gamma)+1}{4}}n^{\eta}\right)\omega(1)$$

$$\leq O\left(n^{\eta+\frac{2(\alpha-\gamma)+1}{4}}\right)\sum_{k=1}^{n}\frac{\omega(1/k)}{k^{\eta+\frac{2(\alpha-\gamma)+1}{4}+1}},$$

with $0<\eta<-\frac{2(\alpha-\gamma)+1}{4}$.

If $\lambda=\gamma+\frac{1}{3}$ then $\lambda-\frac{1}{3}=\gamma>\frac{\alpha+\gamma+1}{2}-\frac{1}{4}$ since $\gamma>\alpha+\frac{1}{2}$. So, applying Lemma 4.2 with $\alpha+\gamma+1$ instead of β and $\lambda=\gamma+\frac{1}{3}$ we obtain

$$\begin{aligned}
&|J_4|\\
\leq\ & \frac{1}{\Gamma(\alpha+1)A_n^{(\gamma)}}\int_{n^{\eta}}^{\infty}e^{-y/2}y^{(3\alpha-3\gamma-1)/3}\left|\Delta_0 f(y)\right|e^{-y/2}y^{(3\gamma+1)/3}\left|L_n^{(\alpha+\gamma+1)}(y)\right|dy\\
=\ & \frac{O(n^{\gamma})}{\Gamma(\alpha+1)A_n^{(\gamma)}}\int_{n^{\eta}}^{\infty}e^{-y/2}y^{(3\alpha-3\gamma-1)/3}\left|\Delta_0 f(y)\right|dy.
\end{aligned}$$

Next, using Lemma 4.3 and (4) we get

$$|J_4|=\frac{O(1)}{\Gamma(\alpha+1)}\int_{n^{\eta}}^{\infty}e^{-y/2}y^{(3\alpha-3\gamma-1)/3}\left|\Delta_0 f(y)\right|dy=O\left(\omega\left(1/n^{\eta}\right)\right).$$

Finally, collecting the above estimates we have

$$\left|S_n^{(\gamma,\alpha)}f(0)-f(0)\right|=O\left(n^{\eta+\frac{2(\alpha-\gamma)+1}{4}}\right)\sum_{k=1}^{n}\frac{\omega(1/k)}{k^{\eta+\frac{2(\alpha-\gamma)+1}{4}+1}}+O\left(\omega\left(1/n^{\eta}\right)\right).$$

and our proof is completed. ■

6. Conclusion

We investigated pointwise approximation of real–valued functions, integrable in the Lebesgue sense over $\mathbb{R}^+$ with the weight $w(t)=e^{-t}t^{\alpha}$ $(\alpha>-1)$ by the (C,γ) - means of partial sums of their Fourier–Laguerre series. In particular, we estimated the deviation $\left|S_n^{(\gamma,\alpha)}f(0)-f(0)\right|$ by means of a function of the modulus of continuity type ω. From our result some corollaries were derived and some examples were constructed.

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