



Dynamical behaviour of a prey-predator system in a destructive environment incorporating prey refuge

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Abstract. Degradation of habitat is a direct outcome of anthropogenic activities, which includes urbanization, mining, the emission of industrial waste, and many others. Many living organisms experience severe surviving challenges as a result of habitat degradation. Here, we have studied the impact of habitat destruction caused by human activities on the dynamics of a prey-predator interaction with prey refuge. Our analyses reveal that a higher rate of habitat destruction than the habitat regeneration rate is always detrimental to the survival of predators. Predator species may still be threatened with extinction even if the rate of habitat degradation is slightly lower than the habitat recovery rate. So, in order to maintain biodiversity, we must appropriately step up our efforts to slow down the rate of habitat degradation as well as accelerate the habitat restoration. Further, our investigation suggests that in order to achieve cohabitation, we should effectively control the habitat deterioration caused by human activity, rather than artificially introducing or eliminating the hiding places of prey species.

1. Introduction

The anthropogenic activities, these days, include urbanization, deforestation, industrialization, mining, etc., and these processes cause habitat loss on a global aspect. The habitat degradation creates serious trouble for many living organisms as it decreases the survival rate causing a drop in the biomass of a population. Acres of land are cleared because of grazing, mining, farming, etc., and almost 70%-80% of wildlife creatures are affected by these processes. It is quite surprising that approximately 15 billion trees have been cut down each year which has become a threat to the survival of many animals. An experiment regarding tree density on a global scale reveals that the number of trees has fallen by 46% since the start of human civilization [28]. Several forests, lakes, and other habitats disappear continuously because of agriculture, industrialization, etc. Almost 70%-75% of deforested areas were used for agricultural purposes in 1990. Between 2000 and 2010, approximately 5.2 million hectares of forest have undergone per year, among which the tropical and subtropical woodlands are affected the most [10]. Moreover, the marine and coastal ecosystems get hampered because of human interference with terrestrial and marine natural resources. Kruess and Tschardtke (1994), in their experiment, have observed that the prey biomass increases

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rapidly in a system when their predators die out due to habitat loss [3]. It means habitat destruction leads to a crowding situation for some particular species. On the other hand, Schneider (2001) has stated in his work that landscape management makes a significant impact on predator strategy and frequency of predators on their prey [19].

Habitat loss makes a negative impact on biodiversity. Donovan and Flather, in 2002, have shown that habitat destruction reduces the growth rate of a population [4]. Also, it is shown in the experiment conducted by Fahrig (2003) that habitat destruction decreases the count of specialist predators by a large amount, reduces the breeding success rate and dispersal ability of a species, etc [15]. A huge number of birds, mammals, and amphibians are in danger because of agriculture and hunting. Let us take a real-life example where habitat destruction has detrimental effects on prey-predator interaction and holds most of the assumptions of the proposed model. Many habitats in Latin America, Africa, and Asia have been destructed due to the plantation of palm oil plants, and it has affected the wildlife of many species also [20, 31]. Acres of rainforest are cut down to grow this plant on a large scale. As a consequence, tigers, elephants, orangutans, and many other species face trouble while searching for food and shelter. This is the reason they come to the locality to search for foods in lack of natural habitats and often get killed (or, captured).

In recent years, a lot of ecological model have been studied to understand the population dynamics of an ecosystem in terms of the long-term survival of various interacting species by analyzing the corresponding system of differential equation, fractional differential equation, or delayed differential equation [2, 7, 8, 21, 25, 29]. A refuge is considered as a habitat that is a protected zone for prey species. In some literature, the effect of prey refuge on prey-predator interaction is studied [24, 27]. The common observation of all these works states that the increase of prey refuge leads to the declination of their predator biomass. The use of prey refuge is one of the relevant behavioral traits which decreases the predation rate. As there is no risk of predation in the refuge patches, the survival rate of prey species increases if they are provided with sufficient resources. Prey species can adjust their activities and change their suitable habitat as the foraging of predators is quite higher in an open patch. When a portion of a species moves into some areas which are out of reach from their predators, then only a fraction of prey species is left for consumption of the predator [5]. The prey who successfully avoids predation by moving to the protected zone is considered as a prey refuge. The predation strength of predators follows a declination (at low prey biomass) as they opt for an alternative food source in this case, and this switching gives the prey population enough time to hide at a safer place to save themselves from the verge of extinction [1, 6]. There are some articles discussing the stabilizing effect of a system in presence of prey refuge [23–25, 27]. Hoy (1985), in his work, has mentioned that the hotspots of high spider mite densities set off outbreaks in almond orchards, as the predator cannot control the prey population in those areas [18]. The places with high mite density are unfit for the predator and can be considered as refugia because predator species no longer enable to survive there. On the other hand, Du and Shi (2006) have observed the dynamical nature of a diffusive prey-predator model with the consideration of a reserved zone for prey population [32]. Lv et al. (2013), in their experiment, have considered a two-patch prey-predator model where the second patch is considered as a reserved zone for the prey [33].

There are some works that are dealt with the impact of habitat destruction on population dynamics [9, 11, 17, 22, 26, 30]. In this work, we have focused on the effect of biodiversity degradation due to population expansion on a prey-predator system along with the incorporation of prey refuge. The paper is organized as follows: In Section 2, a prey-predator interaction is formulated, where the growth and death of the species are affected due to anthropogenic activities. The biological existence of the system is shown in Section 3 by showing the positivity and boundedness of the system variables. The feasibility of the equilibrium points is analyzed in Section 4. The next section displays the local dynamical behavior of equilibrium points of the proposed system. The persistency of the system is checked in Section 6. The change of stability of the equilibrium points via transcritical and Hopf bifurcations is proved in Section 7. The effect of habitat destruction on the system dynamics is observed through numerical figures in Section 8, and the work ends with a concise conclusion in Section 9.

2. Mathematical Model

Anthropogenic activities including urbanization and industrialization have resulted in an environmental crisis. In this work, we have proposed a prey-predator model in such destroyed environment. Let H be the human population that causes the anthropogenic activities, resulting in habitat loss as these activities pollute environments with a high quantity of organic pollutants, and D denotes the amount of habitat which is destroyed in those processes. The prey and predator population are denoted by N and P respectively. The human population is assumed to grow according to the logistic law, where r is the growth rate of humans and b is the rate of crowding effect because of the large size of the human population. It is considered that habitat loss takes place at a rate γ only when the size of the human population crosses a threshold value, i.e., $H > H_c$. The prey-predator interaction follows the Rosenzweig–MacArthur model with prey refuge until the human population reaches the critical value H_c . The rate of reduction in habitat degradation, when proper efforts are applied, is denoted by η . Now, the habitat loss reduces the growth rate of prey population $r(D)$, and is given as $r(D) = \frac{r_0}{1+r_1D}$, where r_0 is the intrinsic growth rate of prey population and r_1 is the rate of decrease in growth per unit habitat destruction. Also, when the habitat is destroyed, the carrying capacity of prey ($K(D)$) drops, and the amount becomes $K(D) = K_0e^{-K_1D}$, where K_0 is the natural carrying capacity for prey population and K_1 is the rate of loss in carrying capacity for habitat destruction. The predator consumes the prey population with coefficient of rate of consumption as c , whereas a^{-1} is the half-saturation constant in absence of destruction. The parameter e is the predator's biomass conversion efficiency. Prey refuge is the portion of the prey population who hide themselves to avoid high predation, and habitat destruction affects the portion of refuge also. If the surrounding environment becomes polluted due to urbanization, etc., then there will be space constraints for the prey species for hiding which makes the refuge term as $m(D) = \frac{m_0}{1+m_1D} \in [0, 1]$, where $m_0 \in [0, 1]$ is the portion of prey hiding in the natural environment and m_1 is the measure of decrease of the natural availability of prey refuge due to habitat destruction. So, in a destroyed environment, only $\left(1 - \frac{m_0}{1+m_1D}\right)$ portion of prey will be available for predation. Moreover, the death rate of predator also increases in this case for which the death rate function becomes $d(D) = (d_0 + d_1D)$, where d_0 is the natural death rate of predator population and d_1 is the death rate per unit habitat destruction. Assuming all these conditions, we propose the system as follows:

$$\begin{aligned} \frac{dH(t)}{dt} &= rH - bH^2, \\ \frac{dD(t)}{dt} &= \gamma(H - H_c) - \eta D, \\ \frac{dN(t)}{dt} &= \frac{r_0N}{1+r_1D} \left(1 - \frac{N}{K_0e^{-K_1D}}\right) - \frac{c\left(1 - \frac{m_0}{1+m_1D}\right)NP}{1+a\left(1 - \frac{m_0}{1+m_1D}\right)N}, \\ \frac{dP(t)}{dt} &= \frac{ec\left(1 - \frac{m_0}{1+m_1D}\right)NP}{1+a\left(1 - \frac{m_0}{1+m_1D}\right)N} - (d_0 + d_1D)P, \end{aligned} \quad (1)$$

with positive initial conditions: $H(0) > 0$, $D(0) > 0$, $N(0) > 0$ and $P(0) > 0$.

Now, $\lim_{D \rightarrow \infty} K(D) = 0$, which means that the population will go extinct with time for very large destruction due to lack of their habitat. So, for the biological existence of such models, it is considered that the proportion of habitat destruction lies in a finite range, and can not increase boundlessly. Moreover, $\frac{dD}{dt}$ may be positive only when $H > H_c$ which supports the assumption that habitat destruction takes place only when human population goes beyond a certain limit H_c . And, the parameters, in this model, are assumed to be non-negative due to biological restrictions. Description of the model parameters are summarized in Table 1.

Table 1: Description of the system parameters

Parameter	Description
r	Growth rate of human population
b	Measure of crowding effect due to large size of human population
γ	Rate of habitat destruction due to $H > H_c$
η	Rate of habitat restoration
r_0	Coefficient of intrinsic growth rate of prey population
r_1	Rate of decrease in prey population per unit habitat destruction
K_0	Natural carrying capacity for prey population
K_1	Rate of loss in carrying capacity from its natural level due to habitat destruction
c	Consumption rate of predator
m_0	Natural available refuge protecting prey population
m_1	Rate of loss in naturally available prey refuge due to habitat destruction
a^{-1}	Half saturation constant
e ($0 < e < 1$)	Biomass conversion rate
d_0	Natural death rate of predator population
d_1	Death rate of predator population per unit habitat destruction

3. Positivity and Boundedness

The following two theorems in this section demonstrate the well-posedness of the system, i.e., every positive solution remains positive and uniformly bounded over time. In the context of ecology, these indicate that the interacting species in our model behave in a manner that is compatible with the environment and the abundance of each species is constrained by limited resources.

Theorem 3.1. *The solutions of system (1), starting from \mathbb{R}_+^4 , are positive.*

Proof. The right hand side of system (1) are continuous locally Lipschitzian functions, which indicates the existence of an unique solution $(H(t), D(t), N(t), P(t))$ with positive initial conditions on $[0, \kappa)$, where $0 < \kappa \leq +\infty$ [14]. The first equation of (1) gives

$$\begin{aligned} \frac{dH}{dt} &= H(r - bH), \\ \Rightarrow H(t) &= H(0) \exp \left[\int_0^t (r - bH(s)) ds \right] > 0, \text{ for } H(0) > 0. \end{aligned}$$

Similarly, the third and fourth equations give

$$\begin{aligned} N(t) &= N(0) \exp \left(\int_0^t \phi_1(D(s), N(s), P(s)) ds \right) > 0, \text{ for } N(0) > 0, \\ P(t) &= P(0) \exp \left(\int_0^t \phi_2(D(s), N(s)) ds \right) > 0, \text{ for } P(0) > 0, \end{aligned}$$

where, $\phi_1(D(s), N(s), P(s)) = \frac{r_0}{1+r_1D(s)} \left(1 - \frac{N(s)}{K_0 e^{-K_1 D(s)}} \right) - \frac{c \left(1 - \frac{m_0}{1+m_1 D(s)} \right) P(s)}{1+a \left(1 - \frac{m_0}{1+m_1 D(s)} \right) N(s)}$

and $\phi_2(D(s), N(s)) = \frac{ec \left(1 - \frac{m_0}{1+m_1 D(s)} \right) N(s)}{1+a \left(1 - \frac{m_0}{1+m_1 D(s)} \right) N(s)} - (d_0 + d_1 D(s))$.

Next, we want to show that $D(t) \geq 0, \forall t \in [0, \kappa)$. If the statement does not hold, then $\exists t_1 \in (0, \kappa)$ such that $D(t_1) = 0, \dot{D}(t_1) < 0$ and $D(t) \geq 0, \forall t \in [0, t_1)$. From the second equation we get,

$$\begin{aligned} \left. \frac{dD}{dt} \right|_{t=t_1} &= \gamma \{H(t_1) - H_c\} - \eta D(t_1) = \gamma \{H(t_1) - H_c\} \\ &\geq 0, \text{ (as, } H(t) > H_c, \forall t > 0 \text{ holds for the persistence of the system)} \end{aligned}$$

which is a contradiction to $\dot{D}(t_1) < 0$. So, $D(t) \geq 0, \forall t \in [0, \kappa)$, where $0 < \kappa \leq \infty$. So, the solutions of the system (1) are feasible with time when $H(t) > H_c$. \square

Theorem 3.2. *The solutions of system (1) in \mathbb{R}_+^4 are bounded with time.*

Proof. The first equation of (1) is $\frac{dH}{dt} = H(r - bH)$, which is a logistic equation. So, $\limsup_{t \rightarrow \infty} H(t) \leq \frac{r}{b}$. Also, from the second equation of the system, we have

$$\begin{aligned} \frac{dD}{dt} &= \gamma\{H - H_c\} - \eta D \leq \frac{\gamma r}{b} - \eta D. \\ \text{Then, } D(t) &\leq D(0)e^{-\eta t} + \frac{\gamma r}{b\eta}(1 - e^{-\eta t}) \\ &\Rightarrow \limsup_{t \rightarrow \infty} D(t) \leq \frac{\gamma r}{b\eta}. \end{aligned}$$

From the third equation of (1), we have

$$\begin{aligned} \frac{dN(t)}{dt} &= \frac{r_0 N}{1 + r_1 D} \left(1 - \frac{N}{K_0 e^{-K_1 D}}\right) - \frac{c \left(1 - \frac{m_0}{1 + m_1 D}\right) NP}{1 + a \left(1 - \frac{m_0}{1 + m_1 D}\right) N} \\ &\leq \frac{r_0 N}{1 + r_1 D} \left\{1 - \frac{N}{\left(\frac{K_0}{1 + K_1 D}\right)}\right\} \leq r_0 N \left\{1 - \frac{N}{\left(\frac{K_0 b \eta}{b \eta + \gamma r r_1}\right)}\right\} \\ &\Rightarrow \limsup_{t \rightarrow \infty} N(t) \leq \frac{K_0 b \eta}{b \eta + \gamma r r_1} < K_0. \end{aligned}$$

Let us consider, $W(t) = eN(t) + P(t)$.

$$\begin{aligned} \text{Then, } \frac{dW}{dt} &= \frac{er_0 N}{1 + r_1 D} \left(1 - \frac{N}{K_0 e^{-K_1 D}}\right) - (d_0 + d_1 D)P \\ &\leq \frac{er_0 N}{1 + r_1 D} - d_0 P \\ &\leq e(r_0 + 1)N - \tau W \quad \text{where } \tau = \min\{1, d_0\} \\ &\leq \frac{e(r_0 + 1)K_0 b \eta}{(b \eta + \gamma r r_1)} - \tau W \quad (\text{for large time } t), \end{aligned}$$

then $W(t) \leq \frac{e(r_0 + 1)K_0 b \eta}{\tau(b \eta + \gamma r r_1)}(1 - \exp(-\tau t)) + W(H(0), D(0), N(0), P(0)) \exp(-\tau t)$;

As $t \rightarrow \infty, 0 < W(t) \leq \frac{eK_0 b \eta}{\tau} \left(\frac{r_0 + 1}{b \eta + \gamma r r_1}\right)$. Henceforth, the solutions of the proposed system eventually enter into the region:

$$\begin{aligned} \Omega &= \left\{ (H(t), D(t), N(t), P(t)) \in \mathbb{R}_+^4 : 0 < H(t) \leq \frac{r}{b} + \epsilon; 0 < D(t) \leq \frac{\gamma r}{b\eta} + \epsilon; 0 < N(t) \leq \frac{K_0 b \eta}{b \eta + \gamma r r_1} + \epsilon; \right. \\ &\quad \left. 0 < W(t) \leq \frac{eK_0 b \eta}{\tau} \left(\frac{r_0 + 1}{b \eta + \gamma r r_1}\right) + \epsilon, \text{ for any } \epsilon > 0 \right\}. \end{aligned}$$

This proves the theorem. \square

4. Equilibrium Points

In system (1), it is considered that the habitat can be destroyed only when the human population, H exceeds a threshold value H_c . And the human population grows according to logistic law, so, $H(t) \neq 0$ is taken into account here. Moreover, system (1) gives the carrying capacity of H as $\frac{r}{b}$.

Let us consider the case $D = 0$. Then we have $H = H_c = \frac{r}{b}$, and the prey-predator interaction turns into a Rosenzweig–MacArthur model with prey refuge. The equilibrium points become: $E_1(H_c, 0, 0, 0)$, $E_2(H_c, 0, K_0, 0)$ and $E_3(H_c, 0, N_3, P_3)$, where $N_3 = \frac{d_0}{(1-m_0)(ec-ad_0)}$ and $P_3 = \frac{r_0}{c(1-m_0)} \left(1 - \frac{N_3}{K_0}\right) \{1 + a(1 - m_0)N_3\}$. So, the equilibrium point E_3 exists when $ec > ad_0$.

Next we consider the case when $D \neq 0$. We get the equilibrium points as $E_4(H_4, D_4, 0, 0) \equiv \left(\frac{r}{b}, \frac{\gamma(r-bH_c)}{b\eta}, 0, 0\right)$ and $E_5 = (H_5, D_5, N_5, 0) \equiv \left(\frac{r}{b}, \frac{\gamma(r-bH_c)}{b\eta}, K_0 e^{-\left\{\frac{K_1\gamma(r-bH_c)}{b\eta}\right\}}, 0\right)$. The feasibility of E_4 and E_5 exist when $r > bH_c$ holds.

The model system (1) has a unique interior equilibrium point $E^*(H^*, D^*, N^*, P^*)$, where

$$H^* = \frac{r}{b}, D^* = \frac{\gamma(r-bH_c)}{b\eta}, N^* = \frac{(d_0+d_1D^*)}{\left(1-\frac{m_0}{1+m_1D^*}\right)\{ec-a(d_0+d_1D^*)\}}, \text{ and}$$

$P^* = \frac{\left(\frac{r_0}{1+r_1D^*}\right)}{c\left(1-\frac{m_0}{1+m_1D^*}\right)} \left(1 - \frac{N^*}{K_0 e^{-\left\{\frac{K_1\gamma}{b\eta}\right\}}}\right) \{1 + a\left(1 - \frac{m_0}{1+m_1D^*}\right)N^*\}$. So, the existence of the coexisting state depend on the following conditions: (i) $b\eta(ec - ad_0) > ad_1\gamma(r - bH_c) > 0$, and (ii) $N^* < K_0 e^{-\frac{K_1\gamma}{b\eta}(r-bH_c)}$.

5. Local Stability Analysis

This section contains the local stability criterion of the equilibrium points which can be determined by analyzing the eigenvalues of corresponding Jacobian matrices. The Jacobian matrix of system (1) is

$$J = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}, \tag{2}$$

where $a_{11} = r - 2bH$, $a_{21} = \gamma$, $a_{22} = -\eta$, $a_{32} = -\frac{r_0N}{(1+r_1D)^2} \left\{r_1 + \frac{N\{K_1(1+r_1D)-r_1\}}{K_0 e^{-K_1D}}\right\} - \frac{cm_0m_1NP}{(1+m_1D)^2 \left\{1+a\left(1-\frac{m_0}{1+m_1D}\right)N\right\}^2}$, $a_{33} = \frac{r_0}{1+r_1D} \left(1 - \frac{2N}{K_0 e^{-K_1D}}\right) - \frac{c\left(1-\frac{m_0}{1+m_1D}\right)P}{\left\{1+a\left(1-\frac{m_0}{1+m_1D}\right)N\right\}^2}$, $a_{34} = -\frac{c\left(1-\frac{m_0}{1+m_1D}\right)N}{1+a\left(1-\frac{m_0}{1+m_1D}\right)N}$, $a_{42} = \frac{ecm_0m_1NP}{(1+m_1D)^2 \left\{1+a\left(1-\frac{m_0}{1+m_1D}\right)N\right\}^2} - d_1P$, $a_{43} = \frac{ec\left(1-\frac{m_0}{1+m_1D}\right)P}{\left\{1+a\left(1-\frac{m_0}{1+m_1D}\right)N\right\}^2}$ and $a_{44} = \frac{ec\left(1-\frac{m_0}{1+m_1D}\right)N}{1+a\left(1-\frac{m_0}{1+m_1D}\right)N} - (d_0 + d_1D)$.

Theorem 5.1. E_1 is an unstable equilibrium point.

Proof.

$$\text{For } E_1(H_c, 0, 0, 0) : J|_{E_1} = \begin{pmatrix} -r & 0 & 0 & 0 \\ \gamma & -\eta & 0 & 0 \\ 0 & 0 & r_0 & 0 \\ 0 & 0 & 0 & -d_0 \end{pmatrix}.$$

So, $\lambda_1 = -r < 0$, $\lambda_2 = -\eta < 0$, $\lambda_3 = r_0 > 0$ and $\lambda_4 = -d_0 < 0$. Positive sign of one eigenvalue proves that E_1 is an unstable equilibrium point. \square

Theorem 5.2. E_2 is locally asymptotically stable (LAS) when $(ec - ad_0)(1 - m_0)K_0 < d_0$ holds.

Proof.

$$\text{For } E_2(H_c, 0, K_0, 0) : J|_{E_2} = \begin{pmatrix} -r & 0 & 0 & 0 \\ \gamma & -\eta & 0 & 0 \\ 0 & -r_0 K_0 K_1 & -r_0 & -\frac{c(1-m_0)K_0}{1+a(1-m_0)K_0} \\ 0 & 0 & 0 & \frac{ec(1-m_0)K_0}{1+a(1-m_0)K_0} - d_0 \end{pmatrix}.$$

So, $\lambda_1 = -r$, $\lambda_2 = -\eta$, $\lambda_3 = -r_0$ and $\lambda_4 = \frac{ec(1-m_0)K_0}{1+a(1-m_0)K_0} - d_0$. Here all the eigenvalues are always negative except λ_4 and so, E_2 is locally asymptotically stable only when $\lambda_4 < 0$, i.e., when $\frac{ec(1-m_0)K_0}{1+a(1-m_0)K_0} < d_0$ holds. \square

Theorem 5.3. E_3 is locally asymptotically stable when $(ec - ad_0)(1 - m_0)aK_0^2 < (ec - aK_0d_0)$ holds.

Proof.

$$\text{For } E_3(H_c, 0, N_3, P_3) : J|_{E_3} = \begin{pmatrix} -r & 0 & 0 & 0 \\ \gamma & -\eta & 0 & 0 \\ 0 & a_{32} & a_{33} & -\frac{c(1-m_0)N_3}{1+a(1-m_0)N_3} \\ 0 & a_{42} & a_{43} & 0 \end{pmatrix},$$

where $a_{32} = -\frac{r_0 K_1 N_3^2}{K_0} - \frac{r_1 c(-m_0) N_3 P_3}{1+a(1-m_0)N_3} - \frac{cm_0 m_1 N_3 P_3}{\{1+a(1-m_0)N_3\}^2}$, $a_{33} = -\frac{r_0 N_3}{K_0} + \frac{ca(1-m_0)^2 N_3 P_3}{\{1+a(1-m_0)N_3\}^2}$, $a_{42} = \frac{ecm_0 m_1 N_3 P_3}{\{1+a(1-m_0)N_3\}^2} - d_1 P_3$, $a_{43} = \frac{ec(1-m_0)P_3}{\{1+a(1-m_0)N_3\}^2}$. Two eigenvalues corresponding to $J|_{E_3}$ are given by So, $\lambda_1 = -r$, $\lambda_2 = -\eta$ which are always negative, and other two eigenvalues are roots of the equation $\lambda^2 + A_1\lambda + A_2 = 0$, where $A_1 = -a_{33}$ and $A_2 = -a_{34}a_{43} > 0$. According to Routh-Hurwitz criterion, E_3 is LAS when the equation has roots with negative real parts, and this happens if $a_{33} < 0$, i.e., $aK_0^2(ec - ad_0)(1 - m_0) < (ec - aK_0d_0)$. \square

Theorem 5.4. E_4 is an unstable equilibrium point.

Proof.

$$\text{For } E_4(H_4, D_4, 0, 0) : J|_{E_4} = \begin{pmatrix} -r & 0 & 0 & 0 \\ \gamma & -\eta & 0 & 0 \\ 0 & 0 & \frac{r_0}{1+r_1 D_4} & 0 \\ 0 & 0 & 0 & -(d_0 + d_1 D_4) \end{pmatrix},$$

The eigenvalues are $\lambda_1 = -r < 0$, $\lambda_2 = -\eta < 0$, $\lambda_3 = \frac{r_0}{1+r_1 D_4} > 0$ and $\lambda_4 = -(d_0 + d_1 D_4) < 0$. As one of the eigenvalues is positive, so, E_4 is an unstable equilibrium point. \square

Theorem 5.5. E_5 is locally asymptotically stable when $\{b\eta(ec - ad_0) - ad_1\gamma(r - bH_c)\}\{b\eta(1 - m_0) + m_1\gamma(r - bH_c)\}K_0 e^{-K_1 D_5} < \{d_0 b\eta + d_1\gamma(r - bH_c)\}\{b\eta + m_1\gamma(r - bH_c)\}$ holds.

Proof.

$$\text{For } E_5(H_5, D_5, N_5, 0) : J|_{E_5} = \begin{pmatrix} -r & 0 & 0 & 0 \\ \gamma & -\eta & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix},$$

where $a_{32} = -\frac{r_0 N_5}{(1+r_1 D_5)^2} \left\{ r_1 \left(1 - \frac{N_5}{K_0 e^{-K_1 D_5}} \right) + \frac{N_5 K_1 (1+r_1 D_5)}{K_0 e^{-K_1 D_5}} \right\}$, $a_{33} = -\frac{r_0 N_5}{(1+r_1 D_5) K_0 e^{-K_1 D_5}}$, $a_{34} = -\frac{c \left(1 - \frac{m_0}{1+m_1 D_5} \right) N_5}{1+a \left(1 - \frac{m_0}{1+m_1 D_5} \right) N_5}$, and $a_{44} = \frac{ec \left(1 - \frac{m_0}{1+m_1 D_5} \right) N_5}{1+a \left(1 - \frac{m_0}{1+m_1 D_5} \right) N_5} - (d_0 + d_1 D_5)$. The eigenvalues corresponding to $J|_{E_5}$ are given by $\lambda_1 = -r$, $\lambda_2 = -\eta$, $\lambda_3 = -\frac{r_0 N_5}{(1+r_1 D_5) K_0 e^{-K_1 D_5}}$ and $\lambda_4 = \frac{ec \left(1 - \frac{m_0}{1+m_1 D_5} \right) N_5}{1+a \left(1 - \frac{m_0}{1+m_1 D_5} \right) N_5} - (d_0 + d_1 D_5)$. By Routh-Hurwitz criterion, E_5 is LAS when $\lambda_4 < 0$, i.e., $\{b\eta(ec - ad_0) - ad_1\gamma(r - bH_c)\}\{b\eta(1 - m_0) + m_1\gamma(r - bH_c)\}K_0 e^{-K_1 D_5} < \{d_0 b\eta + d_1\gamma(r - bH_c)\}\{b\eta + m_1\gamma(r - bH_c)\}$. \square

Theorem 5.6. $E^*(H^*, D^*, N^*, P^*)$ is locally asymptotically stable when $a\left(1 - \frac{m_0}{1+m_1D^*}\right)K_0e^{-K_1D^*} < \frac{ec+a(d_0+d_1D^*)}{ec-a(d_0+d_1D^*)}$ holds.

Proof.

$$\text{For } E^*(H^*, D^*, N^*, P^*) : J|_{E^*} = \begin{pmatrix} -r & 0 & 0 & 0 \\ \gamma & -\eta & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & 0 \end{pmatrix},$$

where $a_{32} = -\frac{r_0N^*}{(1+r_1D^*)^2} \left\{r_1 + \frac{N^*[K_1(1+r_1D^*)-r_1]}{K_0e^{-K_1D^*}}\right\} - \frac{cm_0m_1N^*P^*}{(1+m_1D^*)^2\left\{1+a\left(1-\frac{m_0}{1+m_1D^*}\right)N^*\right\}^2}$, $a_{33} = -\frac{r_0N^*}{(1+r_1D^*)K_0e^{-K_1D^*}} + \frac{ac\left(1-\frac{m_0}{1+m_1D^*}\right)^2N^*P^*}{\left\{1+a\left(1-\frac{m_0}{1+m_1D^*}\right)N^*\right\}^2}$, $a_{34} = -\frac{c\left(1-\frac{m_0}{1+m_1D^*}\right)N^*}{1+a\left(1-\frac{m_0}{1+m_1D^*}\right)N^*}$, $a_{42} = \frac{ecm_0m_1N^*P^*}{(1+m_1D^*)^2\left\{1+a\left(1-\frac{m_0}{1+m_1D^*}\right)N^*\right\}^2} - d_1P^*$ and $a_{43} = \frac{ec\left(1-\frac{m_0}{1+m_1D^*}\right)P^*}{\left\{1+a\left(1-\frac{m_0}{1+m_1D^*}\right)N^*\right\}^2}$. Now, $J|_{E^*}$ has two negative eigenvalues $\lambda_1 = -r$ and $\lambda_2 = -\eta$, and other two eigenvalues are the roots of the equation:

$$\lambda^2 + B_1\lambda + B_2 = 0, \tag{3}$$

where $B_1 = -a_{33}$ and $B_2 = -a_{34}a_{43} > 0$. So, the equation has roots with negative real parts if $B_1 > 0$ which gives $a_{33} < 0$, i.e., $a\left(1 - \frac{m_0}{1+m_1D^*}\right)K_0e^{-K_1D^*} < \frac{ec+a(d_0+d_1D^*)}{ec-a(d_0+d_1D^*)}$. \square

6. Persistence

From a mathematical viewpoint, persistence of a system ensures that its solutions are always away from zero. Ecologically, it refers to the sustainability of all interacting species over the long term period irrespective of any initial population size. We use the average Lyapunov function to verify the persistence of the proposed system.

Theorem 6.1. Prey-predator subsystem of system (1) is persistent if $(ec - ad_0)(1 - m_0)K_0 > d_0$ holds when $D = 0$ and $\{b\eta(ec - ad_0) - ad_1\gamma(r - bH_c)\}\{b\eta(1 - m_0) + m_1\gamma(r - bH_c)\}K_0e^{-K_1D^*} > \{d_0b\eta + d_1\gamma(r - bH_c)\}\{b\eta + m_1\gamma(r - bH_c)\}$ holds when $D \neq 0$.

Proof. Consider the situation when the habitat is not destroyed ($D = 0$). The average Lyapunov function is considered as $V_0(H, 0, N, P) = H^{\beta_1}N^{\beta_2}P^{\beta_3}$, where $\beta_i > 0$ for $i = 1, 2, 3$.

$$\begin{aligned} \text{Then we have, } \frac{\dot{V}_0}{V_0} &= \psi(H, 0, N, P) = \beta_1[r - bH] + \beta_2 \left[r_0 \left(1 - \frac{N}{K_0} \right) - \frac{c(1 - m_0)P}{1 + a(1 - m_0)N} \right] \\ &\quad + \beta_3 \left[\frac{ec(1 - m_0)N}{1 + a(1 - m_0)N} - d_0 \right] \end{aligned}$$

Here we get, $\psi(E_1) = \psi(H_c, 0, 0, 0) = \beta_2r_0 - \beta_3d_0$,

$$\text{and } \psi(E_2) = \psi(H_c, 0, K_0, 0) = \beta_3 \left[\frac{ec(1 - m_0)K_0}{1 + a(1 - m_0)K_0} - d_0 \right].$$

So, $\psi(E_1)$ is positive for some positive β_i , $i = 2, 3$. And, $\psi(E_2) > 0$ if $(ec - ad_0)(1 - m_0)K_0 > d_0$ for some $\beta_3 > 0$, and system (1), in absence of destruction, is persistent under this condition.

Now, the average Lyapunov function, in presence of habitat destruction ($D \neq 0$), is considered as $V(H, D, N, P) = H^{\theta_1} D^{\theta_2} N^{\theta_3} P^{\theta_4}$, where $\theta_i > 0$ for $i = 1, 2, 3, 4$. In the interior of \mathbb{R}_+^4 , we get

$$\begin{aligned} \frac{\dot{V}}{V} &= \phi(H, D, N, P) \\ &= \theta_1[r - bH] + \frac{\theta_2}{D}[\gamma(H - H_c) - \eta D] + \theta_3 \left[\frac{r_0}{1 + r_1 D} \left(1 - \frac{N}{K_0 e^{-K_1 D}} \right) - \frac{c \left(1 - \frac{m_0}{1 + m_1 D} \right) P}{1 + a \left(1 - \frac{m_0}{1 + m_1 D} \right) N} \right] \\ &\quad + \theta_4 \left[\frac{ec \left(1 - \frac{m_0}{1 + m_1 D} \right) N}{1 + a \left(1 - \frac{m_0}{1 + m_1 D} \right) N} - (d_0 + d_1 D) \right] \end{aligned}$$

Then we have, $\phi(E_4) = \phi(H_4, D_4, 0, 0) = \theta_3 \left(\frac{r_0}{1 + r_1 D_4} \right) - \theta_4 (d_0 + d_1 D_4)$

and $\phi(E_5) = \phi(H_5, D_5, N_5, 0) = \theta_4 \left[\frac{ec \left(1 - \frac{m_0}{1 + m_1 D_5} \right) N_5}{1 + a \left(1 - \frac{m_0}{1 + m_1 D_5} \right) N_5} - (d_0 + d_1 D_5) \right]$.

So, $\phi(E_4)$ is positive for some positive θ_i , $i = 3, 4$. Also, $\phi(E_5) > 0$ if $\{b\eta(ec - ad_0) - ad_1\gamma(r - bH_c)\}\{b\eta(1 - m_0) + m_1\gamma(r - bH_c)\}K_0 e^{-K_1 D_5} > \{d_0 b\eta + d_1\gamma(r - bH_c)\}\{b\eta + m_1\gamma(r - bH_c)\}$ for some $\theta_4 > 0$, and system (1) is persistent [12] under this condition. \square

Remark 6.2. Persistence of prey-predator subsystem implies the instability of the boundary equilibrium points E_2 and E_5 respectively.

7. Bifurcation Analysis

The local bifurcations around the equilibrium points are analysed mainly with the help of Sotomayor’s theorem [16] and Hopf-bifurcation theorem [13].

Let system (1) is written as $\dot{X}(t) = f(X(t))$, where $X(t) = (H(t), D(t), N(t), P(t))^T$ and $f = (f_1, f_2, f_3, f_4)^T$ with $f_1 = H(r - bH)$, $f_2 = \gamma(H - H_c) - \eta D$, $f_3 = \frac{r_0 N}{1 + r_1 D} \left(1 - \frac{N}{K_0 e^{-K_1 D}} \right) - \frac{c \left(1 - \frac{m_0}{1 + m_1 D} \right) NP}{1 + a \left(1 - \frac{m_0}{1 + m_1 D} \right) N}$ and $f_4 = \frac{ec \left(1 - \frac{m_0}{1 + m_1 D} \right) NP}{1 + a \left(1 - \frac{m_0}{1 + m_1 D} \right) N} - (d_0 + d_1 D)P$.

Theorem 7.1. System (1) exhibits a transcritical bifurcation around $E_5(H_5, D_5, N_5, 0)$ taking d_1 as a bifurcation parameter, when $\{b\eta(ec - ad_0) - ad_1\gamma(r - bH_c)\}\{b\eta(1 - m_0) + m_1\gamma(r - bH_c)\}K_0 e^{-K_1 D_5} = \{d_0 b\eta + d_1\gamma(r - bH_c)\}\{b\eta + m_1\gamma(r - bH_c)\}$ holds good.

Proof.

$$J|_{E_5} = \begin{pmatrix} -r & 0 & 0 & 0 \\ \gamma & -\eta & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix},$$

where $a_{32} = -\frac{r_0 N_5}{(1 + r_1 D_5)^2} \left\{ r_1 \left(1 - \frac{N_5}{K_0 e^{-K_1 D_5}} \right) + \frac{N_5 K_1 (1 + r_1 D_5)}{K_0 e^{-K_1 D_5}} \right\}$, $a_{33} = -\frac{r_0 N_5}{(1 + r_1 D_5) K_0 e^{-K_1 D_5}}$, $a_{34} = -\frac{c \left(1 - \frac{m_0}{1 + m_1 D_5} \right) N_5}{1 + a \left(1 - \frac{m_0}{1 + m_1 D_5} \right) N_5}$, and $a_{44} = \frac{ec \left(1 - \frac{m_0}{1 + m_1 D_5} \right) N_5}{1 + a \left(1 - \frac{m_0}{1 + m_1 D_5} \right) N_5} - (d_0 + d_1 D_5)$. The eigenvalues are $\lambda_1 = -r$, $\lambda_2 = -\eta$, $\lambda_3 = -\frac{r_0 N_5}{(1 + r_1 D_5) K_0 e^{-K_1 D_5}}$ and $\lambda_4 =$

$\frac{ec\left(1-\frac{m_0}{1+m_1D_5}\right)N_5}{1+a\left(1-\frac{m_0}{1+m_1D_5}\right)N_5} - (d_0 + d_1D_5)$. Let, $J|_{E_5}$ has a simple zero eigenvalue at the threshold $d_1 = d_1^{[Tc]}$.

So, at $d_1 = d_1^{[Tc]} = \frac{b\eta[K_0e^{-K_1D_5}(ec - ad_0)\{b\eta(1 - m_0) + m_1\gamma(r - bH_c)\} - d_0\{b\eta + m_1\gamma(r - bH_c)\}]}{\gamma(r - bH_c)[b\eta\{1 + a(1 - m_0)\} + m_1\gamma(r - bH_c)(1 + K_0e^{-K_1D_5})]}$:

$$J|_{E_5} = \begin{pmatrix} -r & 0 & 0 & 0 \\ \gamma & -\eta & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and $\lambda_4 = 0$ in this case. Moreover, at zero eigenvalue, the right eigenvector of $J|_{E_5}$ is $V = (v_1, v_2, v_3, v_4)^T = (0, 0, -a_{34}, a_{43})^T$, whereas the left eigenvector is $W = (w_1, w_2, w_3, w_4)^T = (0, 0, 0, 1)^T$. Hence,

$$\Omega_1 = W^T \cdot f_{d_1}(E_5, d_1^{[Tc]}) = -(DP)_{E_5} = 0,$$

$$\Omega_2 = W^T [Df_{d_1}(E_5, d_1^{[Tc]})V] = -a_{33}D_5 \neq 0,$$

$$\Omega_3 = W^T [D^2f(E_5, d_1^{[Tc]})(V, V)] = -\frac{2ec\left(1-\frac{m_0}{1+m_1D_5}\right)a_{33}a_{34}}{\left\{1+a\left(1-\frac{m_0}{1+m_1D_5}\right)N_5\right\}^2} \neq 0.$$

By Sotomayor’s Theorem, the system undergoes a transcritical bifurcation around E_5 at $d_1 = d_1^{[Tc]}$. \square

Theorem 7.2. *If E^* exists with the feasibility conditions, then a simple Hopf-bifurcation occurs at unique $d_1 = d_1^{[H]}$, where $d_1^{[H]}$ is the positive root of $B_1(d_1) = 0$ (stated in equation (3)).*

Proof. From the Jacobian matrix corresponding to E^* ($J|_{E^*}$), we get the eigenvalues as $\lambda_1 = -r$ and $\lambda_2 = -\eta$, and other two eigenvalues are the roots of the equation $\lambda^2 + B_1\lambda + B_2 = 0$, where $B_1 = -\frac{r_0N^*}{(1+r_1D^*)K_0e^{-K_1D^*}} +$

$\frac{ac\left(1-\frac{m_0}{1+m_1D^*}\right)^2N^*P^*}{\left\{1+a\left(1-\frac{m_0}{1+m_1D^*}\right)N^*\right\}^2}$ and $B_2 = \frac{ec^2\left(1-\frac{m_0}{1+m_1D^*}\right)^2N^*P^*}{\left\{1+a\left(1-\frac{m_0}{1+m_1D^*}\right)N^*\right\}^3} > 0$. Now,

$$\begin{aligned} Tr(J|_{E^*}) &= \frac{\left(\frac{r_0N^*}{1+r_1D^*}\right)}{\left\{1+a\left(1-\frac{m_0}{1+m_1D^*}\right)N^*\right\}} \left[a\left(1-\frac{m_0}{1+m_1D^*}\right)\left(1-\frac{N^*}{K_0e^{-K_1D^*}}\right) - \frac{\left\{1+a\left(1-\frac{m_0}{1+m_1D^*}\right)N^*\right\}}{K_0e^{-K_1D^*}} \right] \\ &= \frac{r_0N^*X}{(1+r_1D^*)\left\{1+a\left(1-\frac{m_0}{1+m_1D^*}\right)N^*\right\}}, \end{aligned}$$

where $X = a\left(1-\frac{m_0}{1+m_1D^*}\right) - \frac{1}{K_0e^{-K_1D^*}}\left\{1+2a\left(1-\frac{m_0}{1+m_1D^*}\right)N^*\right\}$.

At $d_1 = d_1^{[H]} = \frac{(ec - ad_0)aK_0e^{-K_1D^*}\left(1-\frac{m_0}{1+m_1D^*}\right) - (ec + ad_0)}{aD^*\left[1+aK_0e^{-K_1D^*}\left(1-\frac{m_0}{1+m_1D^*}\right)\right]}$, we get $X(d_1^{[H]}) = 0$. So, the characteristic equation

has a pair of purely imaginary roots $\lambda_{1,2} = \pm i\sqrt{B_2}$ at $d_1 = d_1^{[H]}$ as $B_1 = Tr(J|_{E^*})|_{d_1^{[H]}} = 0$ and $B_2(d_1)$ is continuous function of d_1 .

Now, the transversality condition for Hopf-bifurcation is $\left(\frac{d}{dd_1}[B_1(d_1)]\right)\Big|_{d_1=d_1^{[H]}} \neq 0$, for $i = 1, 2$. Here,

the calculation gives $\left(\frac{d}{dd_1}[Tr(J|_{E^*})]\right)\Big|_{d_1=d_1^{[H]}} = -\frac{2ar_0N^*\left(1-\frac{m_0}{1+m_1D^*}\right)\frac{dN^*}{dd_1}}{K_0e^{-K_1D^*}(1+r_1D^*)\left\{1+a\left(1-\frac{m_0}{1+m_1D^*}\right)N^*\right\}} \neq 0$, where $\frac{dN^*}{dd_1} =$

$\frac{ecD^*}{\left(1-\frac{m_0}{1+m_1D^*}\right)\{ec-a(d_0+d_1^{[H]}D^*)\}^2}$. Hence it is proved. \square

8. Numerical Simulation

It has been already demonstrated that whenever the human population H does not exceed a threshold level H_c , the prey-predator interaction evolves through a Rosenzweig-MacArthur model with constant prey refuge. The dynamical behaviour of such type of Rosenzweig-MacArthur model with the impact of constant prey refuge are well studied. Therefore, in this section, we primarily focus on some numerical simulations under the consideration of $H(0) > H_c$, i.e., the human population exceeds the threshold level H_c and that includes the effect of habitat destruction in our proposed system. The system has three non-trivial steady states, identified as $E_4(H_4, D_4, 0, 0)$, $E_5(H_5, D_5, N_5, 0)$, and $E^*(H^*, D^*, N^*, P^*)$, each of which describes a variety of rich predator-prey dynamics. The equilibrium state E_4 is always unstable while the other two equilibrium points E_5 and E^* are locally asymptotically stable under some parametric restrictions stated in Theorem 5.5 and Theorem 5.6, respectively. This claims that even if the habitat is being destroyed rapidly, both prey and predator species cannot go extinct together. We have recognized some fundamental thresholds in the system while exploring the various kinds of phases around each stationary point whose in-depth analyses are presented numerically in the subsequent paragraphs. The default value of the different parameters are taken from Table 2.

Parametric Values															
r	b	γ	H_c	η	r_0	r_1	K_0	K_1	c	m_0	m_1	a	e	d_0	d_1
1.5	0.02	0.06	1.0	0.1	5	0.05	200	0.01	0.8	0.4	0.1	0.1	0.5	0.2	0.075

Table 2: Parameter values used for numerical simulation of system (1)

For this set of parametric values, the system exhibits a stable behaviour around a unique interior equilibrium point $E^*(75, 44.4, 81.07, 6.56)$ (Fig.1(b)). However, if the death rate of predator due to habitat destruction (d_1) is slightly increased to $d_1 = 0.08$, predator species goes extinct and the system settles into a predator-free steady state $E_5(75, 44.4, 128.3, 0)$ (Fig.1(a)). Therefore, it follows immediately that a transcritical bifurcation occurs at a certain value of $d_1 = d_1^{[Tc]} \in (0.075, 0.08)$ through which the system exchanges its stability from E^* to E_5 or vice-versa.

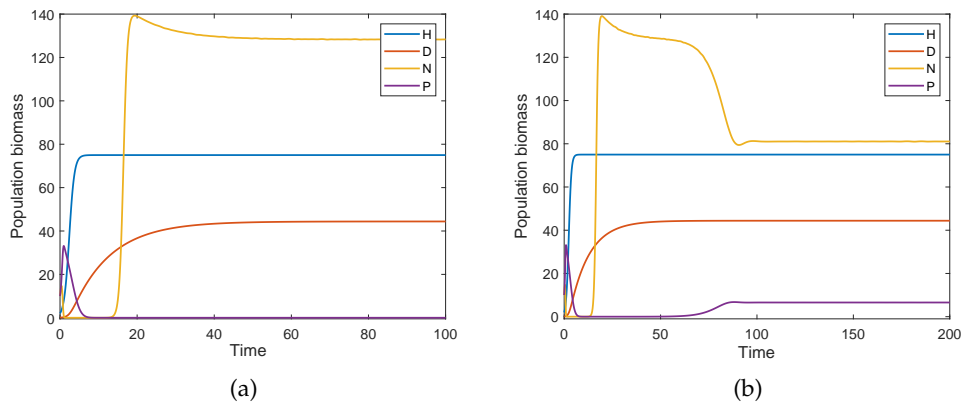


Figure 1: Time series of population biomass and the proportion of habitat destruction for system (1) describing various types of dynamical behaviour. (a): Predator-free equilibrium point $E_5(75, 44.4, 128.3, 0)$ is locally asymptotically stable for $d_1 = 0.08$. (b): Interior equilibrium point $E^*(75, 44.4, 81.07, 6.56)$ is locally asymptotically stable for $d_1 = 0.075$. Other parameter values are taken from Table 2.

Next we investigate the impact of habitat destruction caused by H on the qualitative dynamics of the proposed system. For the parametric set in Table 2, it is observed that the system exhibits a stable behaviour around a unique interior equilibrium point $E^*(75, 44.4, 81.07, 6.56)$. Now if the rate of habitat destruction γ is approximately increased to $\gamma = 0.063$, the system experiences a transcritical bifurcation at which the interior

equilibrium point E^* trades its stability with E_5 . However, if the value of γ is decreased to $\gamma = 0.057$, a Hopf-bifurcation occurs and the system starts to exhibit oscillating behaviour around the interior equilibrium point E^* . Since the first Lyapunov coefficient is negative, the resulting Hopf-bifurcation is supercritical. The pictorial description of these bifurcation analyses with respect to γ are depicted in Figs.2(a,b). Time series of the state variables are also plotted in Figs.2(c,d,e) for different representative values of γ ($0 < \gamma < 0.057$, $0.057 < \gamma < 0.063$ and $0.063 < \gamma$) to illustrate the dynamical behaviour of the system. Since increasing (decreasing) γ have similar impact on the system dynamics with decreasing (increasing) η , the bifurcation diagram with respect to η is qualitatively comparable with γ moving in opposite direction (see Fig.3). That means if γ is increased from lower value to higher value, the system first undergoes a Hopf-bifurcation and then a transcritical bifurcation, but for the case of increasing η , the system first experiences a transcritical bifurcation and then a Hopf-bifurcation. A more precise depiction of the system dynamics in terms of stability region in $\gamma - \eta$ parametric plane is shown in Fig.4. A Hopf-bifurcation curve and a transcritical bifurcation curve divide the entire parametric plane into three regions: (i) region above the Hopf-bifurcation curve where the system exhibits oscillatory behaviour around E^* , (ii) region in between the Hopf-bifurcation curve and the transcritical bifurcation curve where the interior equilibrium point E^* is stable, and (iii) region below the transcritical bifurcation curve where the predator-free equilibrium point E_5 is stable. Ecologically, this means that the higher rate of habitat destruction (γ) than the rate of reduction in habitat degradation (η) is always detrimental to the survival of predators. But the most surprising fact is that even if the value of η is slightly higher than the value of γ , the predator species may still face extinction possibility. Therefore, we must appropriately boost our efforts to increase the rate of habitat restoration in order to sustain biodiversity. Mean prey and predator biomass with changing γ and η are depicted in Fig.5 which demonstrates that higher rate of habitat degradation always drives the predator species to extinction. Moreover, for greater values of γ , a larger value of η is still not sufficient for the survival of predators. For instance, if we take $\gamma = 0.15$, survival of predator species is not possible even if we take $\eta = 0.2$. This suggests that in addition to accelerating habitat regeneration, we also need to slow down the rate of habitat deterioration in order to maintain the biodiversity.

Now we explore the dynamical behaviour of the system with the variation of the rate of loss in carrying capacity due to habitat destruction (K_1). From Fig.6, we observe that lower value of K_1 causes the oscillatory behaviour around E^* which becomes asymptotically stable via supercritical Hopf-bifurcation at $K_1^{[H]} \approx 0.00328$. Further increasing of K_1 may remove the existence of E^* and stabilize the system in E_5 through a transcritical bifurcation at $K_1^{[Tc]} \approx 0.0203$. From the two parametric bifurcation diagram in Fig.7, one can conclude that a higher value of K_1 may drive the predator species to extinction. However, this extinction possibility can be avoided by increasing η or decreasing γ .

In Fig.8, we have depicted the variation of prey and predator equilibrium biomass with increasing or decreasing the parametric values of m_0 and m_1 which are related to the refuge term $m(D) = \frac{m_0}{1+m_1D}$. Enhancing the proportion of natural available refuge (m_0) is beneficial for uplifting the prey equilibrium biomass N^* . In this scenario, prey species can locate sufficient hiding places to protect against the unexpected attack of predator species. Then the absence of sufficient prey biomass in open habitat can be expected to result in a fall in predator biomass. However, as seen in Fig.8(a), the predator equilibrium biomass remains almost same with the variation of m_0 . This occurs as a result of the rising of prey biomass (by reproduction), which fills the gap left by the lack of prey species (by hiding from predator). On the other hand, increasing of m_1 that measures the decrease of natural availability of prey refuge due to habitat destruction has negative impact on prey equilibrium biomass and positive impact on predator equilibrium biomass. According to Fig.8(b), growing m_1 from a very low level causes a sustained drop in N^* and a steady increase in P^* . However further increasing of m_1 has no significant impact on the population biomass. Therefore, a rapid steady decline in prey species caused by habitat loss may not be a warning sign for population extinction in long time run.

In Fig.9, we have plotted two-parametric bifurcation diagrams in $\gamma - m_0$ and $\gamma - m_1$ parametric planes. The vertical Hopf-bifurcation and transcritical bifurcation curves in Fig.9(a) indicate that a certain dynamical behaviour of the system for some fixed γ can not be changed only by varying the value of m_0 . Therefore, once a predator species begins to go extinct for a fixed γ , it cannot be prevented from going extinct by

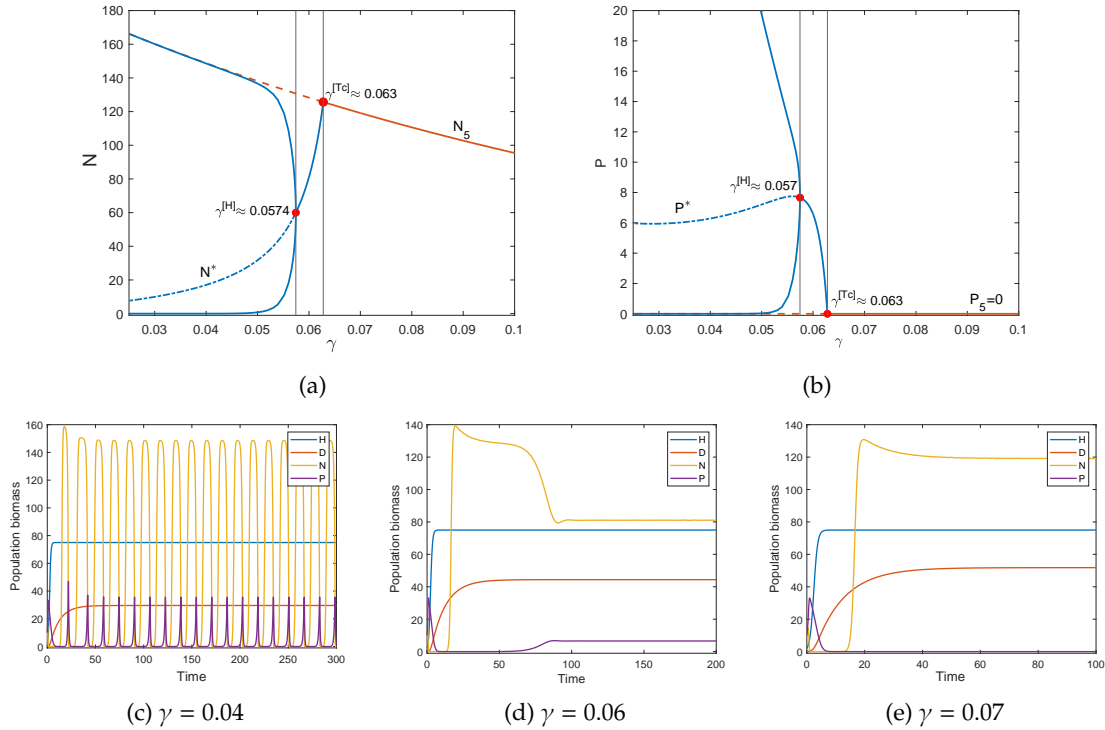


Figure 2: (a,b): Bifurcation diagram with respect to the bifurcation parameter γ . The system exhibits a transcritical bifurcation at $\gamma^{[Tc]} = 0.063$ and a supercritical Hopf-bifurcation at $\gamma^{[H]} = 0.057$. (c,d,e): Time series of the state variables for different representative values of γ . For $\gamma = 0.04 < \gamma^{[H]}$, oscillations around E^* occurs (Fig.c); for $\gamma = 0.06 \in (\gamma^{[H]}, \gamma^{[Tc]})$, E^* is stable (Fig.d) and for $\gamma = 0.07 > \gamma^{[Tc]}$, E_5 is stable (Fig.e). Other parameter values are taken from Table 2.

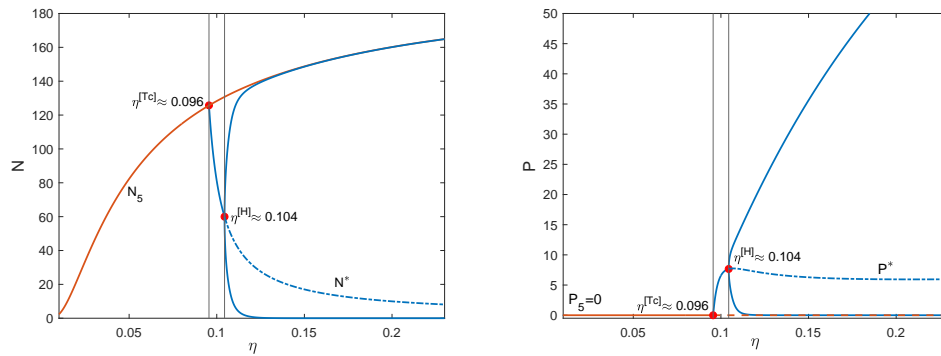


Figure 3: Bifurcation diagram with respect to the bifurcation parameter η . The system exhibits a transcritical bifurcation at $\eta^{[Tc]} = 0.096$ and a supercritical Hopf-bifurcation at $\eta^{[H]} = 0.104$. Other parameter values are taken from Table 2.

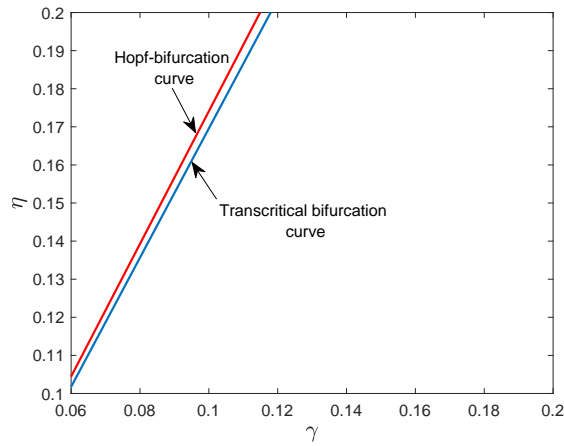


Figure 4: Two parametric bifurcation plane ($\gamma - \eta$) consisting of a Hopf-bifurcation curve and a transcritical bifurcation curve. In the region above the Hopf-bifurcation curve, the system exhibits oscillatory behaviour around E^* . In the region in between the Hopf-bifurcation curve and the transcritical bifurcation curve, the interior equilibrium point E^* is stable. In the region below the transcritical bifurcation curve, the predator-free equilibrium point E_5 is stable. Other parameter values are taken from Table 2.

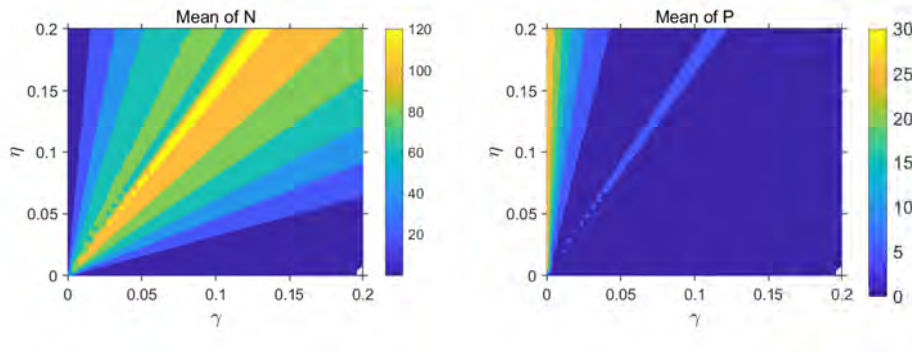


Figure 5: Mean prey and predator biomass with changing γ and η , demonstrates that a slightly higher value of η than γ leads to prey outbreak and predator extinction. Higher value of γ is always detrimental to the survival of predators. Other parameter values are taken from Table 2.

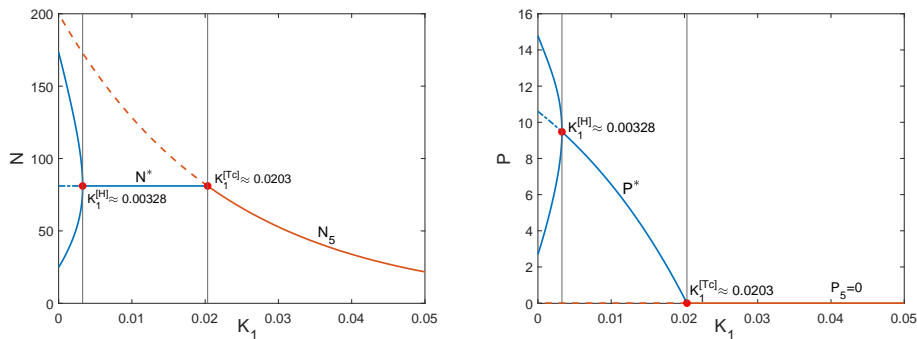


Figure 6: Bifurcation diagram with respect to the bifurcation parameter K_1 . The system exhibits a transcritical bifurcation at $K_1^{Tc} = 0.0203$ and a supercritical Hopf-bifurcation at $K_1^{H} = 0.00328$. Other parameter values are taken from Table 2.

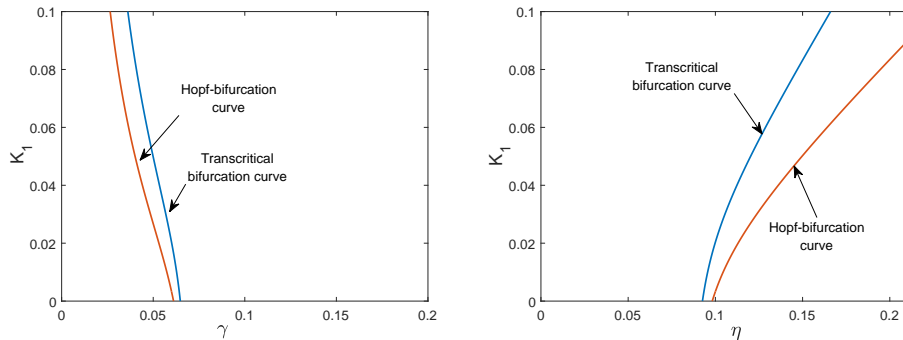


Figure 7: Two parametric bifurcation diagram in $\gamma - K_1$ and $\eta - K_1$ parametric plane consisting of a Hopf-bifurcation curve and a transcritical bifurcation curve. Other parameter values are taken from Table 2.

artificially building or eliminating the natural hiding places of prey species. However, in Fig.9(b), we can observe a tiny parametric interval of γ , where the dynamics of the system undergoes a transition from stable E^* to oscillatory E^* via Hopf-bifurcation (Fig.11) or stable E^* to stable E_5 via transcritical bifurcation (Fig.10) with the variation of m_1 .

Fig.12 depicts the variation of equilibrium level of population biomass and the bifurcation points with respect to the bifurcating parameter d_1 . For lower values of d_1 , in particular $d_1 < d_1^{[H]} = 0.0716$, the system exhibits oscillatory behaviour around E^* . In a tiny intermediate interval of d_1 ($0.0716 < d_1 < 0.0786$), E^* is stable and for $d_1 > d_1^{[Tc]} = 0.0786$, E^* exchanges its stability with E_5 , causing the extinction of predator species.

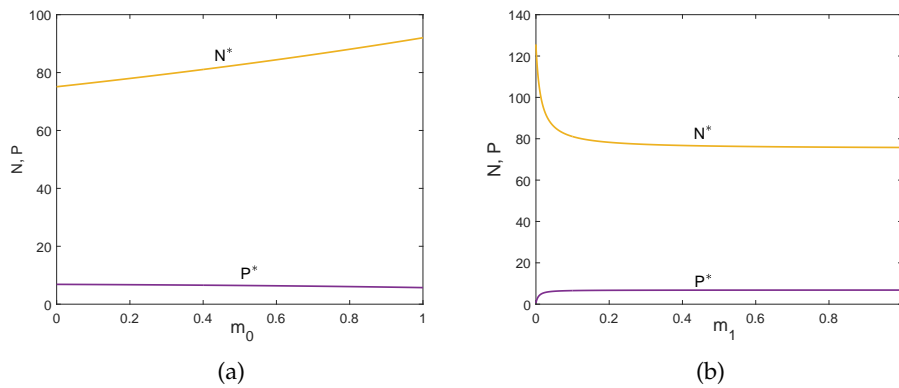


Figure 8: Equilibrium biomass of N and P varies with (a) m_0 and (b) m_1 . Other parameter values are taken from Table 2.

Now, we compare our proposed model with the dynamics of a Rosenzweig–MacArthur prey-predator system with constant prey refuge (without habitat destruction) in order to determine the role of habitat degradation on long-term population biomass. From Fig.13, it is observed that equilibrium biomass of predator species greatly declines in habitat-destructive environment which in turn enhances the equilibrium level of prey biomass. As illustrated in Fig.14(a), increasing the amount of refuge up to a certain level can raise the population biomass and cause a population outbreak for the model without habitat degradation. However, once the refuge parameter (m_0) reaches a certain critical level ($m_0 = 0.9974$), the predator biomass declines to the point of extinction and the prey species reaches its carrying capacity, i.e., one can observe a transition from the coexistence equilibrium state to a predator-free equilibrium state at $m_0 = 0.9974$. In contrast, population explosions and the transitions between equilibrium states do not arise in the model with habitat degradation as m_0 increases. In this case, raising m_0 does not significantly change the equilibrium

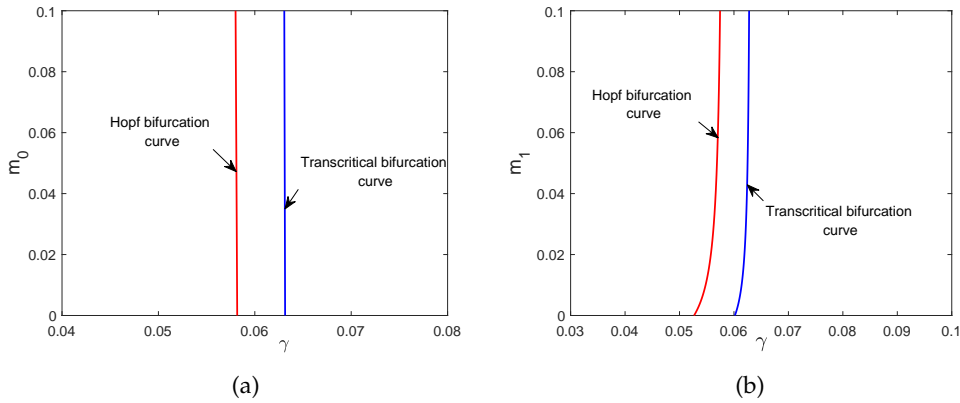


Figure 9: Two parametric bifurcation diagram in (a) $\gamma - m_0$ and (b) $\gamma - m_1$ parametric planes consisting of a Hopf-bifurcation curve and a transcritical bifurcation curve. Other parameter values are taken from Table 2.

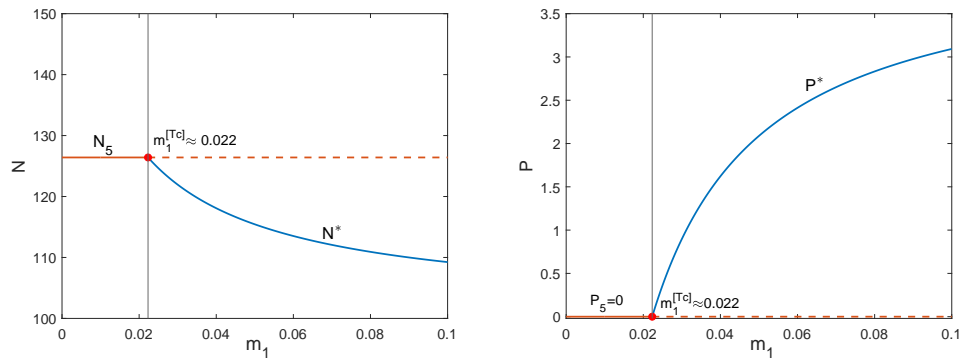


Figure 10: Bifurcation diagram with respect to the bifurcation parameter m_1 and fixed $\gamma = 0.062$. The system exhibits a transcritical bifurcation at $m_1^{[Tc]} = 0.022$. Other parameter values are taken from Table 2.

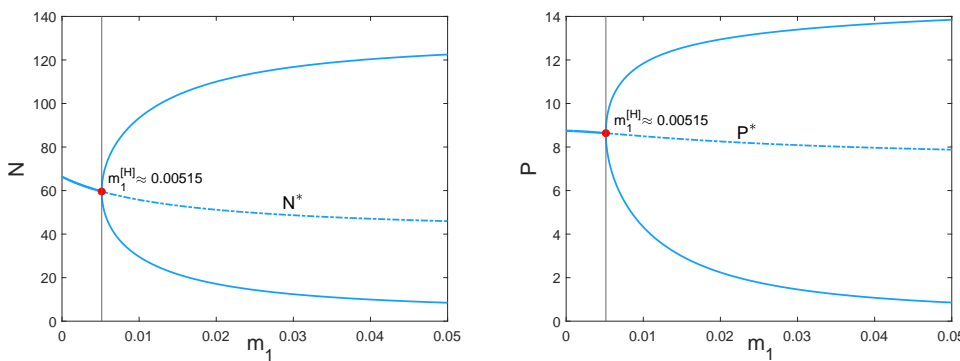


Figure 11: Bifurcation diagram with respect to the bifurcation parameter m_1 and fixed $\gamma = 0.054$. The system exhibits a supercritical Hopf-bifurcation at $m_1^{[H]} = 0.00515$. Other parameter values are taken from Table 2.

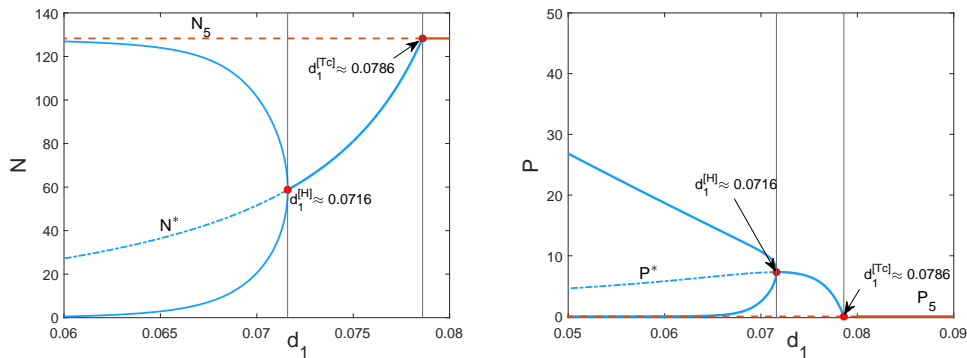


Figure 12: Bifurcation diagram with respect to the bifurcation parameter d_1 . The system exhibits a supercritical Hopf-bifurcation at $d_1^{[H]} = 0.0716$ and a transcritical bifurcation at $d_1^{[T]} = 0.0786$. Other parameter values are taken from Table 2.

level of predator biomass (Fig.14(b)). In Fig.15 and Fig.16, we have plotted the mean population biomass of prey and predator in $m_0 - d_0$ parametric plane for the model without habitat destruction and with habitat destruction respectively.

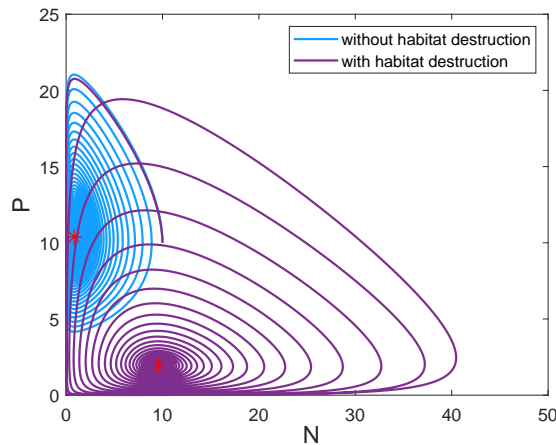


Figure 13: Phase portrait of prey and predator population with or without habitat degradation demonstrates the fluctuation in equilibrium biomass. Here $a = 0.001$ and other parameter values are taken from Table 2.

9. Discussion

Anthropogenic activities which include deforestation, urbanization, mining, release of industrial waste, and many more, have a great impact on climate change and hence result in habitat degradation. It affects living species in the habitat by decreasing their survival and dispersal ability, breeding mechanisms, and many more. This study describes how the dynamics of a prey-predator community living in a habitat is being affected due to destruction. We, therefore, develop a predator-prey model that takes into account the effect of habitat degradation caused by human activities associated with their growth. Additionally, it is assumed that the impact of habitat degradation occurs when the human population reaches a certain threshold H_c , below which the prey-predator interaction behaves according to the Rosenzweig-MacArthur model with constant prey refuge. The well-posedness of our proposed model is verified by establishing the positivity and boundedness of the solutions in \mathbb{R}_+^4 . In absence of habitat destruction, the corresponding system with Rosenzweig–MacArthur prey-predator interaction incorporating constant prey refuge possesses

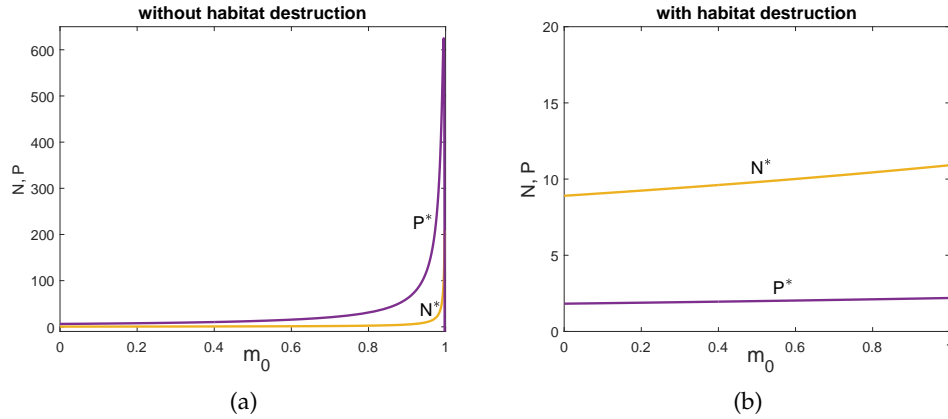


Figure 14: Equilibrium level of prey and predator biomass with the variation of m_0 . Here $a = 0.001$ and other parameter values are taken from Table 2.

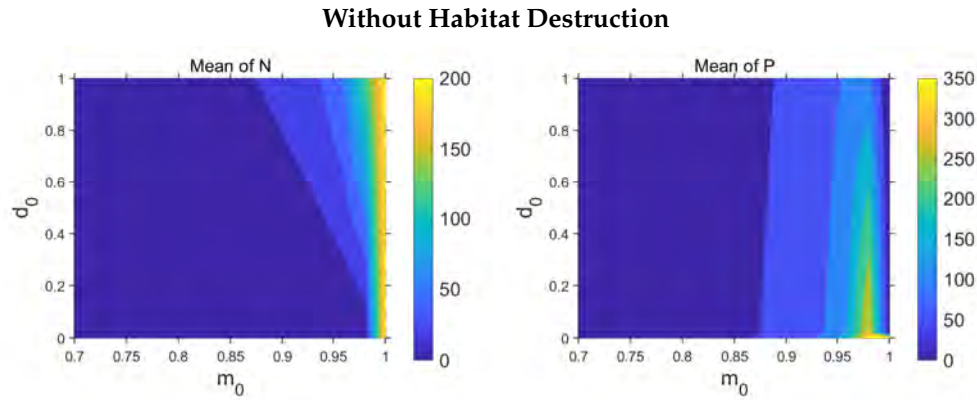


Figure 15: Mean prey and predator biomass in $m_0 - d_0$ parametric plane for the model without habitat destruction. The expansion of the prey equilibrium biomass benefits from increasing m_0 , whereas predator species goes extinct after a certain critical value of m_0 . Here $a = 0.001$ and other parameter values are taken from Table 2.

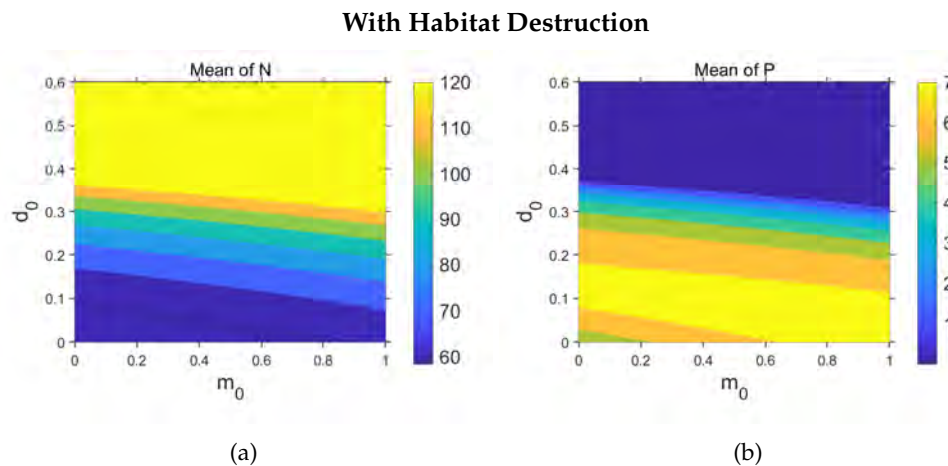


Figure 16: Mean prey and predator biomass in $m_0 - d_0$ parametric plane for the model with habitat destruction. A higher predator mortality rate is always advantageous for the expansion of prey biomass and detrimental to the biomass of the predator for any amount of prey refuge. Here $a = 0.001$ and other parameter values are taken from Table 2.

three equilibrium states, $E_1(H_c, 0, 0, 0)$, $E_2(H_c, 0, K_0, 0)$ and $E_3(H_c, 0, N_3, P_3)$ whose stability conditions are stated in Theorem 5.1, 5.2 and 5.3, respectively. Since the dynamical behaviour of such type of equilibrium states with the impact of prey refuge is well studied, we, therefore concentrate our study only on the effect of habitat destruction. By assuming $H(0) > H_c$ and incorporating the effect of habitat destruction, the resulting system possesses three non-trivial steady states, identified as $E_4(H_4, D_4, 0, 0)$, $E_5(H_5, D_5, N_5, 0)$, and $E^*(H^*, D^*, N^*, P^*)$, each of which describes a variety of rich predator-prey dynamics. The equilibrium state E_4 is always unstable, which suggests that neither prey nor predator can go extinct simultaneously. According to the stability analysis of equilibrium state E_5 , predator populations go extinct at higher rates of habitat degradation, and prey populations will eventually approach some equilibrium level (under some parametric restriction) that is lower than the natural carrying capacity K_0 . The coexistence of prey-predator populations is ensured by the stability of equilibrium point E^* . The system is persistent, meaning that over a long period of time, regardless of the initial population size, all of the species (which are present initially) in the system survive and it is generated under some parametric restriction stated in Theorem 6.1. Sotomayor's theorem [16] and the Hopf-bifurcation theorem [13] are the two basic theorems used to analyse the local bifurcations near the equilibrium points.

We observe the effect of habitat loss on the qualitative behaviour of the system dynamics by varying the parameters γ (rate of habitat destruction) and η (rate of habitat regeneration). In the set of parametric values for stable interior equilibrium point, if only the value of γ is increased to a certain value $\gamma^{[Tc]}$, the system experiences a transcritical bifurcation at which the interior equilibrium point E^* trades its stability with predator-free steady state E_5 . Now, if that γ is decreased at $\gamma^{[H]}$, we observe that the stable coexisting behavior is transformed to coexisting oscillatory behavior (see Fig.2). The two parametric bifurcation analysis in the $\gamma - \eta$ parametric plane reveals that a higher rate of habitat destruction than that of habitat restoration is always harmful for the survival of predators (see Fig.4). Surprisingly, even if the value of η is slightly larger than the value of γ , the predator species may still be in danger of going extinct. Therefore, we must appropriately boost our efforts to increase the rate of habitat restoration in order to sustain biodiversity. Therefore, in order to maintain biodiversity, we must suitably step up our efforts to accelerate the rate of habitat restoration. However, no efforts to restore the habitat will be successful in ensuring the survival of predators if the rate of habitat degradation is too high. This suggests that in addition to accelerating habitat regeneration, we also need to slow down the rate of habitat deterioration in order to maintain the biodiversity.

Increasing the amount of naturally available refuge (m_0) produces enough hiding places to protect against unexpected predators' attacks, which raise the equilibrium biomass of prey species. As a result, it is anticipated that the absence of sufficient prey biomass in open habitat will lead to a fall in the equilibrium biomass of predators. The predator equilibrium biomass, however, remains almost unchanged with the alteration of m_0 (see Fig.8(a)). This occurs as a result of the increase in prey biomass (through reproduction), which fills the gap left by the lack of prey species (by hiding from predator). We also observe that increasing of m_1 (the coefficient measuring the decline in the natural availability of prey refuge on account of habitat destruction) from a very low level results in a sustained drop in prey equilibrium biomass. On the other hand, predator equilibrium biomass increases steadily due to more availability of prey because of the decrease in the prey refuge. However further increasing of m_1 has no significant impact on the population biomass. Therefore, a rapid steady decline in prey species caused by habitat loss may not be a warning sign for population extinction in long time run. We have also demonstrated that changing the value of m_0 alone cannot modify a system's dynamical behaviour for a given fixed γ . Therefore, once a predator species begins to go extinct for a fixed γ , it cannot be prevented from going extinct by artificially building or eliminating the natural hiding places of prey species.

In a model without habitat destruction, if the amount of prey refuge is increased to a certain extent, it can increase both prey and predator equilibrium biomass and lead to a population explosion. However, once the refuge parameter reaches a certain critical level, the predator biomass steadily declines to the point of extinction and the prey species approaches its carrying capacity, i.e., one can observe a transition from the coexistence equilibrium state to a predator-free equilibrium state for some $m_0 \in (0, 1)$. On the other hand, our proposed system with habitat destruction does not exhibit any population explosions or the transitions between equilibrium states for varying m_0 .

The consumption of prey is not an instantaneous process; the predator needs some time to digest the food before continuing to hunt. Therefore, we also recommend another potential modification of our model for future work, which is based on the gestation delay. The model can be made more realistic by accounting for the Allee effect in the prey growth rate. It will also be intriguing to investigate the system in stochastic environment.

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The data used to support the findings of the study are available within the article.

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