



Nonlinear contractions and Caputo tempered implicit fractional differential equations in b -metric spaces with infinite delay

Salim Krim^a, Abdelkrim Salim^{a,b,*}, Mouffak Benchohra^a

^aLaboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbes, P.O. Box 89, Sidi Bel-Abbes 22000, Algeria

^bFaculty of Technology, Hassiba Benbouali University of Chlef, P.O. Box 151 Chlef 02000, Algeria

Abstract. This paper deals with some existence and uniqueness results for a class of problems for nonlinear Caputo tempered implicit fractional differential equations in b -Metric spaces with initial condition and infinite delay. The results are based on the $\omega - \psi$ -Geraghty type contraction, the F -contraction and the fixed point theory. Furthermore, an two illustrations are presented to demonstrate the plausibility of our results.

1. Introduction

In recent years, fractional calculus has shown to be a very useful tool for tackling the complexity structures seen in several disciplines of research. It is concerned with the extension of integer order differentiation and integration of a function to non-integer order, and its theory and application are substantial. We refer the reader to the monographs [1–3, 36, 39] and the papers [24, 25, 27, 31, 33, 37]. Many papers and monographs have lately been published in which the authors studied a wide range of results for various forms of fractional differential equations, inclusions with different types of conditions. One may see the papers [1, 23, 32], and the references therein.

In [12, 13], Czerwik introduced the notion of b -metric. Following these early studies, the existence fixed point for various classes of operators in the context of b -metric spaces has been intensively researched; see [4–6, 11, 16, 17, 21, 30] for more details on the concept of b -metric and contractions.

Wardowski [38] has asserted a novel inequality using auxiliary functions to ensure the existence and uniqueness of a particular mapping in the setting of standard metric space. This inequality is referred to as F -contraction. For more details on the $\omega - \psi$ -Geraghty type contraction and the F -contraction, we refer the reader to the recent papers [8, 9, 18, 20, 22].

Tempered fractional calculus can be considered as the extension of fractional calculus. Buschman's earlier work [10] was the first to disclose the definitions of fractional integration with weak singular and

2020 *Mathematics Subject Classification.* 26A33, 34A08, 34K37.

Keywords. Fixed point; Implicit differential equations; Existence; Uniqueness; Delay; Tempered fractional derivative; $\omega - \psi$ -Geraghty contraction; F -contraction; Nonlocal condition.

Received: 21 September 2022; Accepted: 25 March 2023

Communicated by Erdal Karapınar

* Corresponding author: Abdelkrim Salim

Email addresses: salimsalimkrim@gmail.com (Salim Krim), salim.abdelkrim@yahoo.com, a.salim@univ-chlef.dz (Abdelkrim Salim), benchohra@yahoo.com (Mouffak Benchohra)

exponential kernels. See the papers [7, 15, 26, 28, 29, 34, 35] and references therein for more details and results on the tempered fractional calculus.

In [21], the authors considered the following conformable impulsive problem:

$$\begin{cases} \mathcal{T}_{\zeta_j}^{\vartheta} \chi(\zeta) = \mathfrak{N}(\zeta, \chi_{\zeta}, \mathcal{T}_j^{\vartheta} \chi(\zeta)), & \zeta \in \Omega_j; j = 0, 1, \dots, \beta, \\ \Delta \chi|_{\zeta=\zeta_j} = \Upsilon_j(\chi_{\zeta_j}), & j = 1, 2, \dots, \beta, \\ \chi(\zeta) = \mu(\zeta), & \zeta \in (-\infty, \kappa], \end{cases}$$

where $0 \leq \kappa = \zeta_0 < \zeta_1 < \dots < \zeta_{\beta} < \zeta_{\beta+1} = \bar{\kappa} < \infty$, $\mathcal{T}_{\zeta_j}^{\vartheta} \chi(\zeta)$ is the conformable fractional derivative of order $0 < \vartheta < 1$, $\mathfrak{N} : \Omega \times \mathcal{Q} \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function, $\Omega := [\kappa, \bar{\kappa}]$, $\Omega_0 := [\kappa, \zeta_1]$, $\Omega_j := (\zeta_j, \zeta_{j+1}]$; $j = 1, 2, \dots, \beta$, $\mu : (-\infty, \kappa] \rightarrow \mathbb{R}$ and $\Upsilon_j : \mathcal{Q} \rightarrow \mathbb{R}$ are given continuous functions, and \mathcal{Q} is called a phase space.

Taking inspiration of the previous mentioned publications, in this paper, we study the existence and uniqueness of solutions for the implicit problem with nonlinear fractional differential equation involving the Caputo tempered fractional derivative:

$$({}_0^C \mathcal{D}_{\zeta}^{\vartheta, \ell} \chi)(\zeta) = \mathfrak{N}(\zeta, \chi_{\zeta}, ({}_0^C \mathcal{D}_{\zeta}^{\vartheta, \ell} \chi)(\zeta)); \zeta \in \Omega := [0, \kappa], \tag{1}$$

$$\chi(\zeta) = \mu(\zeta), \zeta \in (-\infty, 0], \tag{2}$$

where $0 < \vartheta < 1$, $\ell \geq 0$, ${}_0^C \mathcal{D}_{\zeta}^{\vartheta, \ell}$ is the Caputo tempered fractional derivative, $\mathfrak{N} : \Omega \times \mathcal{Q} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\mu : (-\infty, 0] \rightarrow \mathbb{R}$ are given functions where $\mu(0) = \chi_0$. For any $\zeta \in \Omega$, we define $\chi_{\zeta} \in \mathcal{Q}$ by

$$\chi_{\zeta}(\sigma) = \chi(\zeta + \sigma); \text{ for } \sigma \in (-\infty, 0].$$

To the best of our knowledge, there are no existing publications in the literature that address the implicit Caputo tempered fractional problems with infinite delay. The limited number of published works on tempered fractional calculus highlights the need for further exploration and development. Therefore, our objective is to advance the field by exploring various problems with novel conditions that have not been previously studied. Furthermore, our study intends to incorporate different techniques such as the F -contraction method, which distinguishes it from prior research such as [21].

The study of implicit differential equations using the Caputo tempered fractional derivative in b -metric spaces is initiated in this paper. It is organized as follows: Section 2 introduces some preliminaries, definitions, lemmas and auxiliary results about the tempered fractional derivative. In section 3, we give some existence and uniqueness results for the problem (1)-(2) that are based on the $\omega - \psi$ -Geraghty type contraction, F -contraction and the fixed point theory. Finally we present an example to show the validity of our results.

2. Preliminaries

First, we give the definitions and the notations that we will use throughout this paper. We denote by $C(\Omega, \mathbb{R})$ the Banach space of all continuous functions from Ω into \mathbb{R} with the following norm

$$\|\mathfrak{N}\|_{\infty} = \sup_{\zeta \in \Omega} \{|\mathfrak{N}(\zeta)|\}.$$

As usual, $AC(\Omega)$ denotes the space of absolutely continuous functions from Ω into \mathbb{R} . For any $n \in \mathbb{N}$, we denote by $AC^n(\Omega)$ the space defined by

$$AC^n(\Omega) := \left\{ \mathfrak{N} : \Omega \rightarrow \mathbb{R} : \frac{d^n}{d\zeta^n} \mathfrak{N}(\zeta) \in AC(\Omega) \right\}.$$

Consider the space $X_b^p(\kappa_1, \kappa_2)$, ($b \in \mathbb{R}$, $1 \leq p \leq \infty$) of those complex-valued Lebesgue measurable functions f on $[\kappa_1, \kappa_2]$ for which $\|\mathfrak{N}\|_{X_b^p} < \infty$, where the norm is defined by:

$$\|\mathfrak{N}\|_{X_b^p} = \left(\int_{\kappa_1}^{\kappa_2} |\zeta^b \mathfrak{N}(\zeta)|^p \frac{d\zeta}{\zeta} \right)^{\frac{1}{p}}, \quad (1 \leq p < \infty, b \in \mathbb{R}).$$

Definition 2.1 (The Riemann-Liouville tempered fractional integral [15, 34, 35]). Suppose that the real function f is piecewise continuous on $[\kappa_1, \kappa_2]$ and $f \in X_b^p(\kappa_1, \kappa_2)$, $\ell > 0$. Then, the Riemann-Liouville tempered fractional integral of order ϑ is defined by

$${}_{\kappa_1} I_{\zeta}^{\vartheta, \ell} \mathfrak{N}(\zeta) = e^{-\ell \zeta} {}_{\kappa_1} I_{\zeta}^{\vartheta} (e^{\ell \zeta} \mathfrak{N}(\zeta)) = \frac{1}{\Gamma(\vartheta)} \int_{\kappa_1}^{\zeta} \frac{e^{-\ell(\zeta-\tau)} \mathfrak{N}(\tau)}{(\zeta-\tau)^{1-\vartheta}} d\tau, \tag{3}$$

where ${}_{\kappa_1} I_{\zeta}^{\vartheta}$ denotes the Riemann-Liouville fractional integral [19], defined by

$${}_{\kappa_1} I_{\zeta}^{\vartheta} \mathfrak{N}(\zeta) = \frac{1}{\Gamma(\vartheta)} \int_{\kappa_1}^{\zeta} \frac{\mathfrak{N}(\tau)}{(\zeta-\tau)^{1-\vartheta}} d\tau. \tag{4}$$

Obviously, the tempered fractional integral (3) reduces to the Riemann-Liouville fractional integral (4) if $\ell = 0$.

Definition 2.2 (The Riemann-Liouville tempered fractional derivative [15, 34]). For $n - 1 < \vartheta < n$; $n \in \mathbb{N}$, $\ell \geq 0$. The Riemann-Liouville tempered fractional derivative is defined by

$${}_{\kappa_1} \mathfrak{D}_{\zeta}^{\vartheta, \ell} \mathfrak{N}(\zeta) = e^{-\ell \zeta} {}_{\kappa_1} \mathfrak{D}_{\zeta}^{\vartheta} (e^{\ell \zeta} \mathfrak{N}(\zeta)) = \frac{e^{-\ell \zeta}}{\Gamma(n-\vartheta)} \frac{d^n}{d\zeta^n} \int_{\kappa_1}^{\zeta} \frac{e^{\ell \tau} f(\tau)}{(\zeta-\tau)^{\vartheta-n+1}} d\zeta,$$

where ${}_{\kappa_1} \mathfrak{D}_{\zeta}^{\vartheta}$ denotes the Riemann-Liouville fractional derivative [19], given by

$${}_{\kappa_1} \mathfrak{D}_{\zeta}^{\vartheta} (e^{\ell \zeta} \mathfrak{N}(\zeta)) = \frac{d^n}{d\zeta^n} ({}_{\kappa_1} I_{\zeta}^{n-\vartheta} (e^{\ell \zeta} \mathfrak{N}(\zeta))) = \frac{1}{\Gamma(n-\vartheta)} \frac{d^n}{d\zeta^n} \int_{\kappa_1}^{\zeta} \frac{(e^{\ell \tau} \mathfrak{N}(\tau))}{(\zeta-\tau)^{\vartheta-n+1}} d\tau.$$

Definition 2.3 (The Caputo tempered fractional derivative [15, 35]). For $n - 1 < \vartheta < n$; $n \in \mathbb{N}^+$, $\ell \geq 0$. The Caputo tempered fractional derivative is defined as

$${}_{\kappa_1}^C \mathfrak{D}_{\zeta}^{\vartheta, \ell} \mathfrak{N}(\zeta) = e^{-\ell \zeta} {}_{\kappa_1}^C \mathfrak{D}_{\zeta}^{\vartheta} (e^{\ell \zeta} \mathfrak{N}(\zeta)) = \frac{e^{-\ell \zeta}}{\Gamma(n-\vartheta)} \int_{\kappa_1}^{\zeta} \frac{1}{(\zeta-\tau)^{\vartheta-n+1}} \frac{d^n (e^{\ell \tau} \mathfrak{N}(\tau))}{d\tau^n} d\tau,$$

where ${}_{\kappa_1}^C \mathfrak{D}_{\zeta}^{\vartheta, \ell}$ denotes the Caputo fractional derivative [19], given by

$${}_{\kappa_1}^C \mathfrak{D}_{\zeta}^{\vartheta} (e^{\ell \zeta} \mathfrak{N}(\zeta)) = \frac{1}{\Gamma(n-\vartheta)} \int_{\kappa_1}^{\zeta} \frac{1}{(\zeta-\tau)^{\vartheta-n+1}} \frac{d^n (e^{\ell \tau} \mathfrak{N}(\tau))}{d\tau^n} d\tau.$$

Lemma 2.4 ([15]). For a constant C ,

$${}_{\kappa_1} \mathfrak{D}_{\zeta}^{\vartheta, \ell} C = C e^{-\ell \zeta} {}_{\kappa_1} \mathfrak{D}_{\zeta}^{\vartheta} e^{\ell \zeta}, \quad {}_{\kappa_1}^C \mathfrak{D}_{\zeta}^{\vartheta, \ell} C = C e^{-\ell \zeta} {}_{\kappa_1}^C \mathfrak{D}_{\zeta}^{\vartheta} e^{\ell \zeta}.$$

Obviously, ${}_{\kappa_1} \mathfrak{D}_{\zeta}^{\vartheta, \ell} (C) \neq {}_{\kappa_1}^C \mathfrak{D}_{\zeta}^{\vartheta, \ell} (C)$. And, ${}_{\kappa_1}^C \mathfrak{D}_{\zeta}^{\vartheta, \ell} (C)$ is no longer equal to zero, being different from ${}_{\kappa_1}^C \mathfrak{D}_{\zeta}^{\vartheta} (C) \equiv 0$.

Lemma 2.5 ([15, 35]). Let $f(\zeta) \in AC^n[\kappa_1, \kappa_2]$, $\ell \geq 0$ and $n - 1 < \vartheta < n$. Then the Caputo tempered fractional derivative and the Riemann-Liouville tempered fractional integral have the following composite properties:

$${}_{\kappa_1} I_{\zeta}^{\vartheta, \ell} [{}_{\kappa_1}^C \mathfrak{D}_{\zeta}^{\vartheta, \ell} f(\zeta)] = f(\zeta) - \sum_{j=0}^{n-1} e^{-\ell \zeta} \frac{(\zeta - \kappa_1)^j}{j!} \left[\frac{d^j (e^{\ell \zeta} f(\zeta))}{d\zeta^j} \Big|_{\zeta=\kappa_1} \right],$$

and

$${}_{\kappa_1}^C \mathfrak{D}_{\zeta}^{\vartheta, \ell} [{}_a I_{\zeta}^{\vartheta, \ell} f(\zeta)] = f(\zeta), \text{ for } \vartheta \in (0, 1).$$

Lemma 2.6. Let $\bar{\mathfrak{N}} \in L^1(\Omega)$ and $0 < \vartheta \leq 1$. Then the initial value problem

$$\begin{cases} ({}^C_0\mathfrak{D}_\zeta^{\vartheta,\ell}\chi)(\zeta) = \bar{\mathfrak{N}}(\zeta); \zeta \in \Omega := [0, \kappa], \\ \chi(0) = \chi_0, \end{cases} \quad (5)$$

has a unique solution defined by

$$\chi(\zeta) = \chi_0 e^{-\ell\zeta} + \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} \bar{\mathfrak{N}}(\tau) d\tau. \quad (6)$$

Proof. Applying the Riemann-Liouville tempered fractional integral of order ϑ to

$$({}^C_0\mathfrak{D}_\zeta^{\vartheta,\ell}\chi)(\zeta) = \bar{\mathfrak{N}}(\zeta),$$

and by employing Lemma 2.5 and if $\zeta \in \Omega$, we obtain

$$\chi(\zeta) - \chi(0)e^{-\ell\zeta} = \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} \bar{\mathfrak{N}}(\tau) d\tau.$$

From the initial conditions, we get

$$\chi(\zeta) = \chi_0 e^{-\ell\zeta} + \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} \bar{\mathfrak{N}}(\tau) d\tau.$$

Conversely, by Lemma 2.4 and Lemma 2.5, we deduce that if χ verifies equation (6), then it satisfied the problem (5). \square

As a consequence of Lemma 2.6, we give the following result.

Lemma 2.7. A function χ is a solution of problem (1)-(2) if and only if χ satisfies the following:

$$\chi(\zeta) = \begin{cases} \chi_0 e^{-\ell\zeta} + \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} \bar{\mathfrak{N}}(\tau) d\tau, & \zeta \in \Omega, \\ \mu(\zeta), & \zeta \in (-\infty, 0], \end{cases} \quad (7)$$

where $\bar{\mathfrak{N}} \in C(\Omega)$ such that $\bar{\mathfrak{N}}(\zeta) = \mathfrak{N}(\zeta, \chi_\zeta, \bar{\mathfrak{N}}(\zeta))$.

Definition 2.8 ([4]). Let \mathfrak{Q} be a set and $\varepsilon \geq 1$. A distance function $\omega : \mathfrak{Q} \times \mathfrak{Q} \rightarrow (0, \infty)$ is a *b-metric* if the following requirements hold for all $\chi_1, \chi_2, \chi_3 \in \mathfrak{Q}$:

- (1) $\omega(\chi_1, \chi_2) = 0$ if and only if $\chi_1 = \chi_2$,
- (2) $\omega(\chi_1, \chi_2) = \omega(\chi_2, \chi_1)$,
- (3) $\omega(\chi_1, \chi_2) \leq \varepsilon[\omega(\chi_1, \chi_3) + \omega(\chi_3, \chi_2)]$.

Then, $(\mathfrak{Q}, \omega, \varepsilon)$ is called a *b-metric space with parameter ε* .

Let Λ be the set of all increasing and continuous function $\psi : (0, \infty) \rightarrow (0, \infty)$ satisfying: $\psi(\varepsilon\chi) \leq \varepsilon\psi(\chi) \leq \varepsilon\chi$, for $\varepsilon > 1$ and $\psi(0) = 0$. We denote by Θ the family of all nondecreasing functions $\eta : (0, \infty) \rightarrow [0, \frac{1}{\varepsilon^2})$ for some $\varepsilon \geq 1$.

Definition 2.9 ([4]). Let $(\mathfrak{L}, \omega, \varepsilon)$ be a b -metric space, $\mathfrak{S} : \mathfrak{L} \rightarrow \mathfrak{L}$ is said to be a generalized ω - ψ -Geraghty mapping whenever there exists $\omega : \mathfrak{L} \times \mathfrak{L} \rightarrow (0, \infty)$ such that

$$\omega(\chi_1, \chi_2) \psi(\varepsilon^3 d(\mathfrak{S}(\chi_1), \mathfrak{S}(\chi_2))) \leq \eta(\psi(\omega(\chi_1, \chi_2)) \psi(\omega(\chi_1, \chi_2))),$$

for $\chi_1, \chi_2 \in \mathfrak{L}$, where $\eta \in \Theta$.

Definition 2.10 ([4]). Let \mathfrak{L} be a non empty set, $\mathfrak{S} : \mathfrak{L} \rightarrow \mathfrak{L}$ and $\omega : \mathfrak{L} \times \mathfrak{L} \rightarrow (0, \infty)$ be given mappings. The operator \mathfrak{S} is orbital ω -admissible if for $\chi \in \mathfrak{L}$, we have

$$\omega(\chi, \mathfrak{S}(\chi)) \geq 1 \Rightarrow \omega(\mathfrak{S}(\chi), \mathfrak{S}^2(\chi)) \geq 1.$$

Definition 2.11 ([8]). A mapping $\Psi : \mathfrak{L} \rightarrow \mathfrak{L}$ is said to be a generalized nonlinear F -contraction if there exist the functions $F : (0, \infty) \rightarrow \mathbb{R}$ and $\wp : (0, \infty) \rightarrow (0, \infty)$ such that for all $\chi, \mathfrak{J} \in \mathfrak{L}$ such that $\Psi\chi \neq \Psi\mathfrak{J}$,

$$\wp(\omega(\chi, \mathfrak{J})) + F(\bar{\omega}(\Psi\chi, \Psi\mathfrak{J})) \leq F(A^{\varepsilon\omega}(\chi, \mathfrak{J})), \quad (8)$$

where $\bar{\omega} > 1$, and

$$A^{\varepsilon\omega}(\chi, \mathfrak{J}) = \max \left\{ \omega(\chi, \mathfrak{J}), \omega(\chi, \Psi\chi), \omega(\mathfrak{J}, \Psi\mathfrak{J}), \frac{\beta}{2\varepsilon} [\omega(\mathfrak{J}, \Psi\chi) + \omega(\chi, \Psi\mathfrak{J})] \right\}, \beta \in [0, 1].$$

Theorem 2.12 ([4]). Let (\mathfrak{L}, ω) be a complete b -metric space and $\Psi : \mathfrak{L} \rightarrow \mathfrak{L}$ be a generalized ω - ψ -Geraghty mapping where

- (a) Ψ is ω -admissible;
- (b) there exists $\chi_0 \in \mathfrak{L}$ where $\omega(\chi_0, \Psi(\chi_0)) \geq 1$;
- (c) If $(\chi_n)_{n \in \mathbb{N}} \subset \mathfrak{L}$ with $\chi_n \rightarrow \chi$ and $\omega(\chi_n, \chi_{n+1}) \geq 1$, then $\omega(\chi_n, \chi) \geq 1$.

Then Ψ has a fixed point. Moreover, if

- (d) for all fixed points χ, χ' of Ψ , either

$$\omega(\chi, \chi') \geq 1 \text{ or } \omega(\chi', \chi) \geq 1,$$

then Ψ has a unique fixed point.

Theorem 2.13 ([8]). Let $(\mathfrak{L}, \omega, \varepsilon)$ be a complete b -metric space. A generalized nonlinear F -contraction Ψ has a fixed point if the following statements are true:

- (1) F is strictly increasing, that is, if $a < b$, then $F(a) < F(b)$, for all $a, b \in (0, \infty)$;
- (2) $\beta < 1$;
- (3) $\frac{\varepsilon}{\beta} < 1$;
- (4) $\liminf_{\chi \rightarrow \zeta^+} \wp(\chi) > 0$, for any $\zeta \geq 0$.

3. Main Results

In this section, we establish some existence results for problem (1)-(2).

Let $(C(\Omega), \omega, 2)$ be the complete b -metric space with $\varepsilon = 2$, such that $\omega : C(\Omega) \times C(\Omega) \rightarrow (0, \infty)$, is given by:

$$\omega(\chi, \mathfrak{J}) = \|\chi - \mathfrak{J}\|_{\infty}^2 := \sup_{\zeta \in \Omega} |\chi(\zeta) - \mathfrak{J}(\zeta)|^2.$$

Let the space $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into \mathbb{R} , and verifying the following axioms which were derived from Hale and Kato's originals [14]:

(Ax₁) If $\chi : (-\infty, 0] \rightarrow \mathbb{R}$, and $\chi_0 \in \mathcal{Q}$, then there exist constants $\xi_1, \xi_2, \xi_3 > 0$, such that for each $\zeta \in \Omega$; we have:

- (i) χ_ζ is in \mathcal{Q} ,
- (ii) $\|\chi_\zeta\|_{\mathcal{Q}} \leq \xi_1 \|\chi_1\|_{\mathcal{Q}} + \xi_2 \sup_{\sigma \in [0, \zeta]} |\chi(\sigma)|$,
- (iii) $|\chi(\zeta)| \leq \xi_3 \|\chi_\zeta\|_{\mathcal{Q}}$.

(Ax₂) For the function $\chi(\cdot)$ in (Ax₁), y_ζ is a \mathcal{Q} - valued continuous function on Ω .

(Ax₃) The space \mathcal{Q} is complete.

Consider the space

$$\Theta = \{\chi : (-\infty, \kappa] \rightarrow \mathbb{R}, \chi|_{(-\infty, 0]} \in \mathcal{Q}, \chi|_{\Omega} \in C([0, \kappa], \mathbb{R})\}.$$

The hypotheses:

(H₁) There exist continuous functions $\bar{p} : \Omega \rightarrow (0, \infty)$ and $\bar{q} : \Omega \rightarrow (0, 1)$ such that for each $\chi, \chi_1 \in \mathcal{Q}$, $\mathfrak{J}, \mathfrak{J}_1 \in \mathbb{R}$ and $\zeta \in \Omega$

$$|\mathfrak{N}(\zeta, \chi, \mathfrak{J}) - \mathfrak{N}(\zeta, \chi_1, \mathfrak{J}_1)| \leq \bar{p}(\zeta) \|\chi - \chi_1\|_{\mathcal{Q}} + \bar{q}(\zeta) |\mathfrak{J} - \mathfrak{J}_1|$$

with

$$\left\| \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} \frac{\bar{p}(\tau)}{1 - \bar{q}^*} d\tau \right\|_\infty^2 \leq \psi(\|(\chi - \chi_1)^2\|_\infty).$$

(H₂) There exist $\psi \in \Lambda$ and $\bar{\ell}_0 \in C(\Omega)$ and a function $\xi : \mathbb{R}^2 \rightarrow \mathbb{R}$, such that

$$\xi \left(\bar{\ell}_0(\zeta), \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} \bar{\mathfrak{N}}(\tau) d\tau \right) \geq 0$$

where $\bar{\mathfrak{N}} \in C(\Omega)$ such that $\bar{\mathfrak{N}}(\zeta) = \mathfrak{N}(\zeta, \bar{\ell}_0(\zeta), \bar{\mathfrak{N}}(\zeta))$.

(H₃) For each $\zeta \in \Omega$, and $\chi, \mathfrak{J} \in C(\Omega)$, we have that

$$\xi(\chi(\zeta), \mathfrak{J}(\zeta)) \geq 0$$

implies

$$\xi \left(\frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} \bar{\mathfrak{N}}(\tau) d\tau, \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} \bar{\mathfrak{N}}'(\tau) d\tau \right) \geq 0,$$

where $\bar{\mathfrak{N}}, \bar{\mathfrak{N}}' \in C(\Omega)$ such that

$$\bar{\mathfrak{N}}(\zeta) = \mathfrak{N}(\zeta, \chi_\zeta, \bar{\mathfrak{N}}(\zeta))$$

and

$$\bar{\mathfrak{N}}'(\zeta) = \mathfrak{N}(\zeta, \mathfrak{J}_\zeta, \bar{\mathfrak{N}}'(\zeta)).$$

(H₄) If $(\chi_n)_{n \in \mathbb{N}} \subset C(\Omega)$ with $\chi_n \rightarrow \chi$ and $\xi(\chi_n(\zeta), \chi_{n+1}(\zeta)) \geq 1$, then

$$\xi(\chi_n(\zeta), \chi(\zeta)) \geq 1.$$

(H₅) For all fixed solutions χ, χ' of problem (1)-(2), either

$$\xi(\chi(\zeta), \chi'(\zeta)) \geq 0,$$

or

$$\xi(\chi'(\zeta), \chi(\zeta)) \geq 0.$$

First, we prove the existence and uniqueness results by utilizing the ω - ψ -Geraghty type contraction and the fixed point theory.

Theorem 3.1. *Assume that the hypotheses (H_1) – (H_4) hold. Then the problem (1)–(2) has at least one solution defined on Ω . Moreover, if (H_5) holds, then we get a unique solution.*

Proof. Consider the operator $\Psi : \Theta \rightarrow \Theta$ defined by:

$$(\Psi\chi)(\zeta) = \begin{cases} \mu(0)e^{-\ell\zeta} + \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)}(\zeta-\tau)^{(\vartheta-1)}\bar{\mathfrak{N}}(\tau)d\tau, & \zeta \in \Omega, \\ \mu(\zeta), & \zeta \in (-\infty, 0], \end{cases} \quad (9)$$

where $\bar{\mathfrak{N}} \in C(\Omega)$ such that $\bar{\mathfrak{N}}(\zeta) = \mathfrak{N}(\zeta, \chi_\zeta, \bar{\mathfrak{N}}(\zeta))$.

Let $w : (-\infty, \kappa] \rightarrow \mathbb{R}$ be a function given by

$$w(\zeta) = \begin{cases} \mu(\zeta); & \zeta \in (-\infty, 0], \\ \mu(0)e^{-\ell\zeta} & \zeta \in \Omega. \end{cases}$$

Then $w_0 = \mu$. For each $z \in C(\Omega)$, with $z(0) = 0$, we denote by \bar{z} the function defined by

$$\bar{z} = \begin{cases} 0, & \zeta \in (-\infty, 0], \\ z(\zeta), & \zeta \in \Omega. \end{cases}$$

If $\chi(\cdot)$ satisfies the integral equation

$$\chi(\zeta) = \mu(0)e^{-\ell\zeta} + \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)}(\zeta-\tau)^{(\vartheta-1)}\bar{\mathfrak{N}}(\tau)d\tau,$$

we can decompose $\chi(\cdot)$ as $\chi(\zeta) = \bar{z}(\zeta) + w(\zeta)$; for $\zeta \in \Omega$, which implies that $\chi_\zeta = \bar{z}_\zeta + w_\zeta$ for every $\zeta \in \Omega$, and the function $z(\cdot)$ satisfies

$$z(\zeta) = \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)}(\zeta-\tau)^{(\vartheta-1)}\bar{\mathfrak{N}}(\tau)d\tau,$$

where

$$\bar{\mathfrak{N}}(\zeta) = \mathfrak{N}(\zeta, \bar{z}_\zeta + w_\zeta, \bar{\mathfrak{N}}(\zeta)); \zeta \in \Omega.$$

Set

$$\mathcal{D}_0 = \{z \in C(\Omega); z_0 = 0\},$$

and let $\|\cdot\|_\kappa$ be the norm in \mathcal{D}_0 defined by

$$\|z\|_\kappa = \|z_0\|_Q + \sup_{\zeta \in \Omega} |z(\zeta)| = \sup_{\zeta \in \Omega} |z(\zeta)|; z \in \mathcal{D}_0,$$

where \mathcal{D}_0 is a Banach space with norm $\|\cdot\|_\kappa$. Define the operator $\mathcal{W} : \mathcal{D}_0 \rightarrow \mathcal{D}_0$ by

$$(\mathcal{W}z)(\zeta) = \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)}(\zeta-\tau)^{(\vartheta-1)}\bar{\mathfrak{N}}(\tau)d\tau, \quad (10)$$

where

$$\bar{\mathfrak{N}}(\zeta) = \mathfrak{N}(\zeta, \bar{z}_\zeta + w_\zeta, \bar{\mathfrak{N}}(\zeta)); \zeta \in \Omega.$$

The function $\omega : C(\Omega) \times C(\Omega) \rightarrow (0, \infty)$ is given by:

$$\begin{cases} \omega(z, y) = 1; & \text{if } \xi(z(\zeta), y(\zeta)) \geq 0, \zeta \in \Omega, \\ \omega(z, y) = 0; & \text{else.} \end{cases}$$

First, we prove that \mathcal{W} is a generalized ω - ψ -Geraghty operator:
 Let $z, y \in \mathcal{D}_0$. Then, for each $\zeta \in \Omega$, we have

$$|(\mathcal{W}z)(\zeta) - (\mathcal{W}y)(\zeta)| \leq \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} |\bar{\mathfrak{S}}(\sigma) - \bar{\mathfrak{S}}'(\sigma)| d\tau,$$

where $\bar{\mathfrak{S}}, \bar{\mathfrak{S}}' \in C(\Omega)$ such that

$$\bar{\mathfrak{S}}(\zeta) = \mathfrak{S}(\zeta, \bar{z}_\zeta + w_\zeta, \bar{\mathfrak{S}}(\zeta)) \text{ and } \bar{\mathfrak{S}}'(\zeta) = \mathfrak{S}(\zeta, \bar{y}_\zeta + w_\zeta, \bar{\mathfrak{S}}'(\zeta)).$$

From (H_1) we have

$$\|\bar{\mathfrak{S}} - \bar{\mathfrak{S}}'\|_\infty \leq \frac{\bar{p}(\zeta)}{1 - \bar{q}^*} \|(z - y)^2\|_\infty^{\frac{1}{2}},$$

where $\bar{q}^* = \sup_{\zeta \in \Omega} |\bar{q}(\zeta)|$. Next, we have

$$|(\mathcal{W}z)(\zeta) - (\mathcal{W}y)(\zeta)| \leq \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} \frac{\bar{p}(\tau)}{1 - \bar{q}^*} \|(z - y)^2\|_\infty^{\frac{1}{2}} d\tau.$$

Thus,

$$\begin{aligned} \omega(z, y) |(\mathcal{W}z)(\zeta) - (\mathcal{W}y)(\zeta)|^2 &\leq \|(z - y)^2\|_\infty \omega(z, y) \left\| \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} \frac{\bar{p}(\tau)}{1 - \bar{q}^*} d\tau \right\|_\infty^2 \\ &\leq \|(z - y)^2\|_\infty \psi(\|(z - y)^2\|_\infty). \end{aligned}$$

Hence,

$$\omega(z, y) \psi(2^3 d(\mathcal{W}z, \mathcal{W}y)) \leq \eta(\psi(\omega(z, y))) \psi(\omega(z, y)),$$

where $\eta \in \Theta$, $\psi \in \Lambda$, with $\eta(\zeta) = \frac{1}{8}\zeta$, and $\psi(\zeta) = \zeta$. So, \mathcal{W} is generalized ω - ψ -Geraghty operator.

Let $z, y \in C(\Omega)$ such that

$$\omega(z, y) \geq 1.$$

Thus, for each $\zeta \in \Omega$, we have

$$\xi(\mathcal{X}_{2\zeta}, \mathcal{X}'_{2\zeta}) \geq 0.$$

This implies from (H_3) that

$$\xi(\mathcal{W}z(\zeta), \mathcal{W}y(\zeta)) \geq 0,$$

which gives

$$\omega(\mathcal{W}(z), \mathcal{W}(y)) \geq 1.$$

Hence, \mathcal{W} is a ω -admissible.

Now, by (H_2) , there exists $\bar{\ell}_0 \in C(\Omega)$ such that

$$\omega(\bar{\ell}_0, \mathcal{W}(\bar{\ell}_0)) \geq 1.$$

Thus, by (H_4) , if $(\bar{\ell}_n)_{n \in \mathbb{N}} \subset \mathcal{Q}$ with $\bar{\ell}_n \rightarrow \bar{\ell}$ and $\omega(\bar{\ell}_n, \bar{\ell}_{n+1}) \geq 1$, then

$$\omega(\bar{\ell}_n, \bar{\ell}) \geq 1.$$

From an application of Theorem 2.12, we conclude that \mathcal{W} has a fixed point. Consequently, Ψ has a fixed point which is the solution of problem.

Moreover, (H_5) implies that if z and y are fixed points of \mathcal{W} , then either

$$\xi(z, y) \geq 0 \text{ or } \xi(y, z) \geq 0.$$

This implies that either

$$\omega(z, y) \geq 1 \text{ or } \omega(y, z) \geq 1.$$

Hence, problem (1) has a unique solution.

□

Now, we prove an existence and uniqueness result by using the F -contraction fixed point theorem.

Theorem 3.2. Assume there exist constants $\lambda, \widehat{\lambda} > 0$, where $\bar{\lambda} = \lambda(1 - \widehat{\lambda}) > \sqrt{2}$ such that for each $\chi_\zeta, \widehat{\chi}_\zeta \in \mathcal{Q}$, $\mathfrak{Y}, \widehat{\mathfrak{Y}} \in \mathbb{R}$ and $\zeta \in \Omega$

$$|\mathfrak{N}(\zeta, \chi_\zeta, \mathfrak{Y}) - \mathfrak{N}(\zeta, \widehat{\chi}_\zeta, \widehat{\mathfrak{Y}})| \leq \frac{\Gamma(\vartheta)}{2\lambda\kappa^{(\vartheta-1)} \left[1 + \sup_{\zeta \in \Omega} |\chi(\zeta)| + \sup_{\zeta \in \Omega} |\widehat{\chi}(\zeta)| \right]} |\chi(\zeta) - \widehat{\chi}(\zeta)| + \widehat{\lambda} |\mathfrak{Y} - \widehat{\mathfrak{Y}}|. \tag{11}$$

Then the problem (1)-(2) has a unique solution defined on Ω .

Proof. Let $\mathcal{W} : \mathcal{D}_0 \rightarrow \mathcal{D}_0$ defined as in (10), For any $z, y \in \mathcal{D}_0$. For each $\zeta \in \Omega$ we have

$$|(\mathcal{W}z)(\zeta) - (\mathcal{W}y)(\zeta)|^2 \leq \left\{ \frac{1}{\Gamma(\vartheta)} \int_0^\zeta e^{-\ell(\zeta-\tau)} (\zeta - \tau)^{(\vartheta-1)} |\bar{\mathfrak{S}}(\tau) - \bar{\mathfrak{S}}'(\tau)| d\tau \right\}^2,$$

where $\bar{\mathfrak{S}}, \bar{\mathfrak{S}}' \in C(\Omega)$ such that

$$\bar{\mathfrak{S}}(\zeta) = \mathfrak{N}(\zeta, \bar{z}_\zeta + w_\zeta, \bar{\mathfrak{S}}(\zeta)) \text{ and } \bar{\mathfrak{S}}'(\zeta) = \mathfrak{N}(\zeta, \bar{y}_\zeta + w_\zeta, \bar{\mathfrak{S}}'(\zeta)).$$

Since, for each $\zeta \in \Omega$, we have

$$\|\bar{\mathfrak{S}}(\zeta) - \bar{\mathfrak{S}}'(\zeta)\| \leq \frac{\Gamma(\vartheta)}{2(1 - \widehat{\lambda})\lambda\kappa^{(\vartheta-1)} \left[1 + \sup_{\zeta \in \Omega} |z(\zeta)| + \sup_{\zeta \in \Omega} |y(\zeta)| \right]} |z(\zeta) - y(\zeta)|.$$

Then, we get

$$\begin{aligned} |(\mathcal{W}z)(\zeta) - (\mathcal{W}y)(\zeta)|^2 &\leq \left\{ \frac{1}{\bar{\lambda} \left[2 + 2 \sup_{\zeta \in \Omega} |z(\zeta)| + 2 \sup_{\zeta \in \Omega} |y(\zeta)| \right]} |z(\zeta) - y(\zeta)| \right\}^2 \\ &\leq \frac{1}{\bar{\lambda}^2 \left[2 + 2 \sup_{\zeta \in \Omega} |z(\zeta)| + 2 \sup_{\zeta \in \Omega} |y(\zeta)| \right]^2} \left\{ \sqrt{|z(\zeta) - y(\zeta)|^2} \right\}^2 \\ &\leq \frac{1}{\bar{\lambda}^2 \left[2 + 2 \sup_{\zeta \in \Omega} |z(\zeta)| + 2 \sup_{\zeta \in \Omega} |y(\zeta)| \right]^2} \left\{ \sqrt{\sup_{\zeta \in \Omega} |z(\zeta) - y(\zeta)|^2} \right\}^2 \\ &\leq \frac{1}{\bar{\lambda}^2 \left[2 + \sup_{\zeta \in \Omega} |z(\zeta) - y(\zeta)|^2 \right]} \left\{ \sqrt{\sup_{\zeta \in \Omega} |z(\zeta) - y(\zeta)|^2} \right\}^2. \end{aligned}$$

Consequently, we get

$$\bar{\lambda}^{-2} \omega(\mathcal{W}z, \mathcal{W}y) \leq \frac{\omega(z, y)}{2 + \omega(z, y)}.$$

Now, applying natural logarithm on the previous inequality, we obtain

$$\ln(2 + \omega(z, y)) + \ln(\bar{\lambda}^{-2} \omega(\mathcal{W}z, \mathcal{W}y)) \leq \ln(\omega(z, y)) \leq \ln(A^{\varepsilon\omega}(z, y)),$$

where

$$A^{\epsilon\omega}(z, y) = \max \left\{ \omega(z, y), \omega(z, \mathcal{W}z), \omega(y, \mathcal{W}y), \frac{\beta}{2\epsilon} [\omega(y, \mathcal{W}z) + \omega(z, \mathcal{W}y)] \right\}, \beta < \frac{1}{2}.$$

If we choose $F(\zeta) = \ln(\zeta)$ and $\wp(\zeta) = \ln(2 + \zeta)$ we see that all the conditions of Theorem 2.13 are satisfied, so that \mathcal{W} has a unique fixed point. Consequently, Ψ has a unique fixed point which is the solution of problem.

□

4. Some Examples

Example 4.1. Consider the following problem which is an example of problem (1)-(2):

$$\begin{cases} \left({}_0^C \mathfrak{D}_{\zeta}^{\frac{1}{2}, \ell} \chi \right) (\zeta) = \frac{\arctan(\|\chi_{\zeta}\|_{\mathcal{Q}})}{380(1 + \|\chi_{\zeta}\|_{\mathcal{Q}})} + \frac{1}{380 \left(1 + \left| \left({}_0^C \mathfrak{D}_{\zeta}^{\frac{1}{2}, \ell} \chi \right) (\zeta) \right| \right)}; \zeta \in \Omega := [0, 1], \\ \chi(\zeta) = \zeta + 1; \zeta \in (-\infty, 0]. \end{cases} \quad (12)$$

Let γ be a positive real constant and

$$B_{\gamma} = \{ \chi \in C((-\infty, 1], \mathbb{R}), : \lim_{\tau \rightarrow -\infty} e^{\gamma\tau} \chi(\tau) \text{ exists in } \mathbb{R} \}. \quad (13)$$

The norm of B_{γ} is given by

$$\|\chi\|_{\gamma} = \sup_{\tau \in (-\infty, 1]} e^{\gamma\tau} |\chi(\tau)|.$$

Let $\chi : (-\infty, 0] \rightarrow \mathbb{R}$ be such that $\chi_0 \in B_{\gamma}$. Then

$$\begin{aligned} \lim_{\tau \rightarrow -\infty} e^{\gamma\tau} \chi_{\zeta}(\tau) &= \lim_{\tau \rightarrow -\infty} e^{\gamma\tau} \chi(\zeta + \tau - 1) = \lim_{\tau \rightarrow -\infty} e^{\gamma(\tau - \zeta + 1)} \chi(\tau) \\ &= e^{\gamma(-\zeta + 1)} \lim_{\tau \rightarrow -\infty} e^{\gamma\tau} \chi_1(\tau) < \infty. \end{aligned}$$

Hence $\chi_{\zeta} \in B_{\gamma}$. Finally we prove that

$$\|\chi_{\zeta}\|_{\gamma} \leq \xi_1 \|\chi_1\|_{\gamma} + \xi_2 \sup_{\sigma \in [0, \zeta]} |\chi(\sigma)|,$$

where $\xi_1 = \xi_2 = 1$ and $\xi_3 = 1$. We have

$$|\chi_{\zeta}(\tau)| = |\chi(\zeta + \tau)|.$$

If $\zeta + \tau \leq 1$, we get

$$|\chi_{\zeta}(\xi)| \leq \sup_{\sigma \in (-\infty, 0]} |\chi(\sigma)|.$$

For $\zeta + \tau \geq 0$, then we have

$$|\chi_{\zeta}(\xi)| \leq \sup_{\sigma \in [0, \zeta]} |\chi(\sigma)|.$$

Thus for all $\zeta + \tau \in \Omega$, we get

$$|\chi_{\zeta}(\xi)| \leq \sup_{\sigma \in (-\infty, 0]} |\chi(\sigma)| + \sup_{\sigma \in [0, \zeta]} |\chi(\sigma)|.$$

Then

$$\|\chi_{\zeta}\|_{\gamma} \leq \|\chi_0\|_{\gamma} + \sup_{\sigma \in [0, \zeta]} |\chi(\sigma)|.$$

It is clear that $(B_{\gamma}, \|\cdot\|)$ is a Banach space. We can conclude that B_{γ} a phase space.

Set

$$\mathfrak{N}(\zeta, \chi, \mathfrak{J}) = \frac{\arctan(\|\chi\|_{\mathcal{Q}})}{380(1 + \|\chi\|_{\mathcal{Q}})} + \frac{1}{380(1 + |\mathfrak{J}|)},$$

where $\zeta \in \Omega$, $\chi \in \mathcal{Q}$, $\mathfrak{J} \in \mathbb{R}$.

Let $(C(\Omega), \omega, 2)$ be the complete b -metric space with $\varepsilon = 2$, such that $\omega : C(\Omega) \times C(\Omega) \rightarrow (0, \infty)$, is given by:

$$\omega(\chi, \mathfrak{J}) = \|(\chi - \mathfrak{J})^2\|_{\infty} := \sup_{\zeta \in \Omega} |\chi(\zeta) - \mathfrak{J}(\zeta)|^2.$$

For any $\chi, \mathfrak{J} \in \mathcal{Q}$, $\bar{\chi}, \bar{\mathfrak{J}} \in \mathbb{R}$ and $\zeta \in \Omega$, we have

$$|\mathfrak{N}(\zeta, \chi, \bar{\chi}) - \mathfrak{N}(\zeta, \mathfrak{J}, \bar{\mathfrak{J}})| \leq \frac{\|\chi - \mathfrak{J}\|_{\mathcal{Q}}}{380} + \frac{|\bar{\chi} - \bar{\mathfrak{J}}|}{380}.$$

Thus, hypothesis (H_1) is satisfied with

$$p(\zeta) = q(\zeta) = \frac{1}{380}.$$

Define the functions $\eta(\zeta) = \frac{1}{8}\zeta$, $\phi(\zeta) = \zeta$, $\vartheta : C(\Omega) \times C(\Omega) \rightarrow \mathbb{R}_+^*$ with

$$\begin{cases} \vartheta(\chi, \mathfrak{J}) = 1; & \text{if } \omega(\chi(\zeta), \mathfrak{J}(\zeta)) \geq 0, \zeta \in \Omega, \\ \vartheta(\chi, \mathfrak{J}) = 0; & \text{else,} \end{cases}$$

and $\omega : C(\Omega) \times C(\Omega) \rightarrow \mathbb{R}$ with $\omega(\chi, \mathfrak{J}) = \|\chi - \mathfrak{J}\|_{\infty}$.

Hypothesis (H_2) is satisfied with $\bar{\ell}_0(\zeta) = \chi_0$. Also, (H_3) holds from the definition of the function ω .

Simple computations show that all conditions of Theorem 3.1 are satisfied. Hence, we get the existence and the uniqueness of solutions for problem (12).

Example 4.2. Next, consider the following problem:

$$\begin{cases} \left({}_0^C \mathfrak{D}_{\zeta}^{\frac{1}{2}, \ell} \chi \right) (\zeta) = \frac{\Gamma(\frac{1}{2})}{4(1 + \sup_{\zeta \in \Omega} |\chi(\zeta)|)} + \frac{1}{20 \left(1 + \left| \left({}_0^C \mathfrak{D}_{\zeta}^{\frac{1}{2}, \ell} \chi \right) (\zeta) \right| \right)}; & \zeta \in \Omega, \\ \chi(\zeta) = 2\zeta + 4; & \zeta \in (-\infty, 0]. \end{cases} \quad (14)$$

Set

$$\mathfrak{N}(\zeta, \chi_{\zeta}, \mathfrak{J}) = \frac{\Gamma(\frac{1}{2})}{4(1 + \sup_{\zeta \in \Omega} |\chi(\zeta)|)} + \frac{1}{20(1 + |\mathfrak{J}|)},$$

where $\zeta \in \Omega$, $\chi \in C(\Omega)$, $\mathfrak{J} \in \mathbb{R}$.

Let $(C(\Omega), \omega, 2)$ be a complete b -metric space with $\varepsilon = 2$, such that $\omega : C(\Omega) \times C(\Omega) \rightarrow (0, \infty)$, is given by:

$$\omega(\chi, \mathfrak{J}) = \|(\chi - \mathfrak{J})^2\|_{\infty} := \sup_{\zeta \in \Omega} |\chi(\zeta) - \mathfrak{J}(\zeta)|^2.$$

For any $\chi, \mathfrak{J} \in C(\Omega)$, $\bar{\chi}, \bar{\mathfrak{J}} \in \mathbb{R}$ and $\zeta \in \Omega$, we have

$$|\mathfrak{N}(\zeta, \chi_{\zeta}, \bar{\chi}) - \mathfrak{N}(\zeta, \mathfrak{J}_{\zeta}, \bar{\mathfrak{J}})| \leq \frac{\Gamma(\frac{1}{2})|\chi(\zeta) - \mathfrak{J}(\zeta)|}{4(1 + \sup_{\zeta \in \Omega} |\chi(\zeta)| + \sup_{\zeta \in \Omega} |\mathfrak{J}(\zeta)|)} + \frac{1}{20}|\bar{\chi} - \bar{\mathfrak{J}}|.$$

Then, hypothesis (11) is satisfied with

$$\lambda = 2, \widehat{\lambda} = \frac{1}{20} \text{ and } \bar{\lambda} = \frac{19}{10} > \sqrt{2}.$$

Since all requirements of Theorem 2.13 are verified, we conclude the existence the uniqueness of solutions and for problem (14).

Declarations

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Competing interests It is declared that authors has no competing interests.

Author’s contributions The study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

Funding Not available.

Availability of data and materials Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

References

- [1] S. Abbas, M. Benchohra, J.R. Graef and J. Henderson, *Implicit Fractional Differential and Integral Equations: Existence and Stability*, De Gruyter, Berlin, 2018.
- [2] S. Abbas, M. Benchohra and G. M. N’Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, 2012.
- [3] S. Abbas, M. Benchohra and G. M. N’Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.
- [4] H. Afshari, H. Aydi, E. Karapinar, On generalized $\alpha - \psi$ -Geraghty contractions on b -metric spaces, *Georgian Math. J.* **27**(1) (2020), 9-21.
- [5] H. Afshari and E. Karapinar, A solution of the fractional differential equations in the setting of b -metric space, *Carpathian Math. Publ.* **13** (2021), 764-774. <https://doi.org/10.15330/cmp.13.3.764-774>
- [6] H. Afshari and E. Karapinar, A discussion on the existence of positive solutions of the boundary value problems via ψ -Hilfer fractional derivative on b -metric spaces, *Adv. Difference Equ.* (2020), 616.
- [7] R. Almeida, M. L. Morgado, Analysis and numerical approximation of tempered fractional calculus of variations problems, *J. Comput. Appl. Math.* **361** (2019), 1-12.
- [8] B. Alqahtani, A. Fulga, F. Jarad and E. Karapinar, Nonlinear F -contractions on b -metric spaces and differential equations in the frame of fractional derivatives with Mittag-Leffler kernel, *Chaos, Solitons & Fractals* **128** (2019), 349-354.
- [9] M. Bota, A. Molnar, and C. Varga, On Ekeland’s variational principle in b -metric spaces, *Fixed Point Theory.* **12** (2011), 21-28.
- [10] R. G. Buschman, Decomposition of an integral operator by use of Mikusinski calculus, *SIAM J. Math. Anal.* **3** (1972), 83-85.
- [11] S. Cobzas and S. Czerwik. The completion of generalized b -metric spaces and fixed points, *Fixed Point Theory* **21** (1) (2020), 133-150.
- [12] S. Czerwik, Nonlinear set-valued contraction mappings in b -metric spaces, *Atti Semin. Mat. Fis. Univ. Modena.* **46** (2) 1998, 263-276.
- [13] S. Czerwik, Contraction mappings in b -metric spaces, *Acta Math. Inf. Univ. Ostrav.* **1** (1993), 5-11.
- [14] J. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Funk. Ekvac.* **21** (1978), 11-41.
- [15] C. Li, W. Deng and L. Zhao, Well-posedness and numerical algorithm for the tempered fractional differential equations, *Disc. Contin. Dyn. Syst. Ser. B.* **24** (2019), 1989-2015.
- [16] E. Karapinar, C. Chifu, Results in wt-distance over b -metric spaces, *Math.* **8** (2020).
- [17] E. Karapinar, A. Fulga, A. Petrusel, On Istratescu type contractions in b -metric spaces, *Math.* **8** (2020). <https://doi.org/10.3390/math8030388>
- [18] E. Karapinar, A. Fulga, Fixed point on convex b -metric space via admissible mappings, *TWMS J. Pure Appl. Math.* **12** (2021), no. 2, 254–264.
- [19] A.A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, Amsterdam, 2006.
- [20] S. Krim, S. Abbas, M. Benchohra and E. Karapinar, Terminal value problem for implicit Katugampola fractional differential equations in b -metric spaces, *J. Funct. Spaces* **2021** (2021), 7 pp.
- [21] S. Krim, A. Salim, S. Abbas and M. Benchohra, On implicit impulsive conformable fractional differential equations with infinite delay in b -metric spaces. *Rend. Circ. Mat. Palermo (2)* (2022), 1-14. <https://doi.org/10.1007/s12215-022-00818-8>.
- [22] P. S. Kumari, O. Alqahtani and E. Karapinar, Some fixed-point theorems in b -dislocated metric space and applications. *Symmetry* **10** (2018), 691. <https://doi.org/10.3390/sym10120691>
- [23] N. Laledj, A. Salim, J. E. Lazreg, S. Abbas, B. Ahmad and M. Benchohra, On implicit fractional q -difference equations: Analysis and stability. *Math. Methods Appl. Sci.* **45** (2022), 10775-10797. <https://doi.org/10.1002/mma.8417>
- [24] L. D. Long, H. D. Binh, D. Kumar, N. H. Luc, N. H. Can, Stability of fractional order of time nonlinear fractional diffusion equation with Riemann-Liouville derivative, *Math. Methods Appl. Sci.* **45** (2022), 6194-6216. <https://doi.org/10.1002/mma.8166>
- [25] L. D. Long, N. H. Luc, S. Tatar, et al. An inverse source problem for pseudo-parabolic equation with Caputo derivative, *J. Appl. Math. Comput.* **68** (2022), 739-765. <https://doi.org/10.1007/s12190-021-01546-5>

- [26] M. Medved and E. Brestovanska, Differential equations with tempered ψ -Caputo fractional derivative, *Math. Model. Anal.* **26** (2021), 631-650.
- [27] H. T. Nguyen, H. C. Nguyen, R. Wang and Y. Zhou, Initial value problem for fractional Volterra integro-differential equations with Caputo derivative, *Discr. Contin. Dyn. Syst., Ser. B.* **26** (2021), 6483-6510. <https://doi.org/10.3934/dcdsb.2021030>
- [28] N. A. Obeidat, D. E. Benteil, New theories and applications of tempered fractional differential equations, *Nonlinear Dyn.* **105** (2021), 1689-1702.
- [29] M. D. Ortigueira, G. Bengochea and J. T. Machado, Substantial, tempered, and shifted fractional derivatives: Three faces of a tetrahedron, *Math. Methods Appl. Sci.* **44** (2021), 9191-9209.
- [30] S. G. Ozyurt, On some α -admissible contraction mappings on Branciari b -metric spaces, *Adv. Theo. Nonl. Anal. Appl.* **1** (2017), 1-13.
- [31] A. Salim, S. Abbas, M. Benchohra and E. Karapinar, Global stability results for Volterra-Hadamard random partial fractional integral equations. *Rend. Circ. Mat. Palermo (2)*. (2022), 1-13. <https://doi.org/10.1007/s12215-022-00770-7>.
- [32] A. Salim, M. Benchohra, J. R. Graef and J. E. Lazreg, Initial value problem for hybrid ψ -Hilfer fractional implicit differential equations. *J. Fixed Point Theory Appl.* **24** (2022), 14 pp. <https://doi.org/10.1007/s11784-021-00920-x>.
- [33] A. Salim, J. E. Lazreg, B. Ahmad, M. Benchohra and J. J. Nieto, A Study on k -Generalized ψ -Hilfer Derivative Operator, *Vietnam J. Math.* (2022). <https://doi.org/10.1007/s10013-022-00561-8>
- [34] F. Sabzikar, M. M. Meerschaert and J. Chen, Tempered fractional calculus, *J. Comput. Phys.* **293** (2015), 14-28.
- [35] B. Shiri, G. Wu and D. Baleanu, Collocation methods for terminal value problems of tempered fractional differential equations, *Appl. Numer. Math.* **156** (2020), 385-395.
- [36] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg; Higher Education Press, Beijing, 2010.
- [37] R. Wang, N. H. Can, A. T. Nguyen, N. H. Tuan, Local and global existence of solutions to a time-fractional wave equation with an exponential growth, *Commun. Nonlinear Sci. Numer. Simul.* **118** (2023), 107050. <https://doi.org/10.1016/j.cnsns.2022.107050>
- [38] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2012** (2012), 1-6. <https://doi.org/10.1186/1687-1812-2012-94>.
- [39] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.