Filomat 37:22 (2023), 7479–7489
https://doi.org/10.2298/FIL2322479M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Modular matrix calculations in the finite topological spaces

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Abstract. When performing important calculations in a finite topological space (FTS), matrix calculation methods are more accurate and convenient than traditional methods. However, even when dealing with relatively small subsets involved in the calculations, all elements of the entire space are necessary. This leads to significant time and space waste in practical applications. Therefore, we introduce a modular calculation method as a crucial improvement.

Our motivation is as follows: the topological space being processed is divided into modules, ensuring that when any subset is involved in the calculations, only relevant modules are considered instead of the entire space, while ensuring the same result. In addition, the subsets are further divided into smaller subsets within the relevant modules for calculation, greatly reducing the calculation scope and improving the computational efficiency and accuracy. Based on the modularization of the topological space, we propose a modular matrix calculation method and conduct a detailed study of it. Finally, we provide some examples to demonstrate the modular calculation method and modular matrix calculation method.

1. Introduction

The finite topological spaces (i.e., the topological spaces for which the underlying set is finite) have many practical applications in various fields such as physics, graph theory, big data theory, etc. Characterizing the interior, closure or boundary of a set is the main calculations in a topological space.

In reference[5], an ideal matrix calculation method has been introduced to compute the Kuratowski 14 sets of a given set in a finite topological space. The matrix method is more accurate and efficient because it does not need logical analysis of related concepts, and thus it can be easily executed by computer programs. However, all elements of the whole space X must be needed to apply this method, even if the subset of X that is involved in calculations is very small, and this results in a huge waste of time and space in applications. So we propose the modular calculation method and the modular matrix calculation method for finite the topological spaces in this paper.

The first step is to modularize the topological space. Then, for a given subset involved in calculations, it is unnecessary to consider the whole space but the related modules, and we need only to divide the subset into smaller subsets within the related modules to perform the necessary calculations. Thus the advantages of modularization are to narrow down the scope of involvement and guarantee the same results. Based on

2020 *Mathematics Subject Classification*. 5454B.
Keywords. Kuratowski 14 sets; Base matrix; Boolean matrix; Modular matrix.
Received: 07 December 2022; Revised: 12 March 2023; Accepted: 25 March 2023
Communicated by Ljubiša Kočinac
Research supported by Beijing Natural Science Foundation (1222011).
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the modularization of the topological space, a module matrix calculation method is studied in detail, and some examples are given to illustrate those calculation methods.

This paper is arranged as follows. Some basic knowledge is given in Section 2. In Section 3, the modular method is formulated to calculate the Kuratowski 14 sets, and in Section 4, the modular matrix method is proposed to further explore the Kuratowski 14 sets. We conclude the paper in Section 5.

2. Preliminaries

Before going further, we first recall some basic knowledge that will be used in the following sections. Throughout this paper, finite topological space will be abbreviated as FTS for convenience.

Definition 2.1. [1, 2] Let (T, τ) be a topological space.

(1) If a collection $\theta \subset \tau$ satisfies

$$\forall t \in T, \forall U \in \tau[t \in U] \Rightarrow \exists V \in \theta[t \in V \subset U],$$

then θ is called a base for the topology τ.

(2) If a collection $\vartheta \subset \tau$ satisfies that the family $\{A_1 \cap A_2 \cdots \cap A_n : A_i \in \vartheta,\ i = 1, 2, \cdots, n,\ n = 1, 2, \cdots\}$ is a base for τ, then ϑ is called a subbase for the topology τ.

The next proposition is obvious.

Proposition 2.2. *Let (T, τ) be an FTS and $N(t)$ be the intersection of all open subsets of T containing t for each $t \in T$, then $N(t)$ is the smallest open set containing t and $\{N(t) : t \in T\}$ is a base for the topology τ.*

Definition 2.3. [3, 4] For each subset Y of a finite set $T = \{t_1, t_2, \cdots, t_n\}$, the characteristic vector of Y is defined as $\phi_Y = (y_1, y_2, \cdots, y_n)^T$, where

$$y_i = \begin{cases} 1, & t_i \in Y \\ 0, & t_i \notin Y \end{cases},\quad i = 1, 2, \cdots, n,$$

and the vector $(y_1, y_2, \cdots, y_n)^T$ is the transpose of the vector $(y_1, y_2, \cdots, y_n)$.

According to this proposition, we give the following definition.

Definition 2.4. [5] Let (T, τ) be an FTS, where $T = \{t_1, t_2, \cdots, t_n\}$. The $n \times n$ Boolean matrix B is called the base matrix for the topology τ, if for each $i = 1, 2, \cdots, n$, the transpose of ith row of B is the characteristic vector $\phi_{N(t_i)}$ of $N(t_i)$.

Proposition 2.5. [5] *Let (T, τ) be an FTS, where $T = \{t_1, t_2, \cdots, t_n\}$, and $B = (b_{ij})_{n\times n}$ be the base matrix for τ. Then the ith column of B is the characteristic vector $\phi_{\overline{\{t_i\}}}$ for $i = 1, 2, \cdots, n$.*

Definition 2.6. Let (T, τ) be an FTS, where $T = \{t_1, t_2, \cdots, t_n\}$, and $\vartheta = \{A_1, A_2, \cdots, A_k\}$ be a subbase for the topology τ. Then the Boolean matrix $D = (d_{ij})_{n\times k}$ is called a matrix representation for ϑ if

$$d_{ij} = \begin{cases} 1, & t_i \in A_j \\ 0, & t_i \notin A_j \end{cases},$$

for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, k$.

Definition 2.7. [3, 4] The products $P = (p_{ij})_{n\times m} = E \cdot F$, and $Q = (q_{ij})_{n\times m} = E * F$ of two Boolean matrices $E = (e_{ik})_{n\times k}$ and $F = (f_{kj})_{k\times m}$ are defined as follows:

$$p_{ij} = \vee_{l=1}^{k}(\, e_{il} \wedge f_{lj}\,),\quad q_{ij} = \wedge_{l=1}^{k}[(1 - e_{il}) \vee f_{lj}]$$

for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$, where the symbol $'\wedge'$ denotes the operation $'$ min$'$ and the symbol $'\vee'$ denotes the operation $'$ max$'$, respectively.

Proposition 2.8. [5] *Let (T, τ) be an FTS, where $T = \{t_1, t_2, \cdots, t_n\}$. If $\vartheta = \{A_1, A_2, \cdots, A_k\}$ is a subbase for the topology τ and D is a matrix representation for ϑ, then $D * D^T$ is a base matrix of the topology τ.*

Proposition 2.9. [5] *Suppose that (T, τ) and B are the same as those of Definition 2.4. Then for each subset $Y \subset T$, the interior Y° and closure $\overline{Y}$ of Y can be computed as:*

$$\phi_{Y^\circ} = B * \phi_Y, \quad \phi_{\overline{Y}} = B \cdot \phi_Y.$$

Theorem 2.10. [1, 5] *If Y is a subset of a topological space (T, τ), then performing any combination of the three operations of complement (denoted by "c"), closure (denoted by "-"), and interior (denoted by "$\circ$") on Y in any order and any number of times results in at most 14 distinct sets. These 14 sets are known as the Kuratowski 14 sets, and are given as follows:*

$$Y,\ Y^c,\quad Y^\circ,\ Y^{\circ c},\quad \overline{Y^\circ},\ \overline{Y^\circ}^c,\quad \overline{Y^\circ}^\circ,\ \overline{Y^\circ}^{\circ c},\quad \overline{Y},\ \overline{Y}^c,\quad \overline{Y}^\circ,\ \overline{Y}^{\circ c},\quad \overline{\overline{Y}^\circ},\ \overline{\overline{Y}^\circ}^c.$$

Moreover, if (T, τ) is an FTS, $\vartheta = \{A_1, A_2, \cdots, A_k\}$ is a subbase for the topology τ, D is the matrix representation for ϑ and $B = D * D^T$, then the Kuratowski 14 sets of a subset Y can be computed as follows:

$$\begin{array}{lll}
\phi_Y, & \phi_{Y^{\circ c}} = -B * \phi_Y, & \phi_{\overline{\overline{Y}^\circ}} = B \cdot [B * (B \cdot \phi_Y)], \\
\chi_{Y^c} = -\phi_Y, & \phi_{\overline{Y}^\circ} = B * (B \cdot \phi_Y), & \phi_{\overline{\overline{Y}^\circ}^c} = -B \cdot [B * (B \cdot \phi_Y)], \\
\phi_{\overline{Y}} = B \cdot \phi_Y, & \phi_{\overline{Y}^{\circ c}} = -B * (B \cdot \phi_Y), & \phi_{\overline{Y^\circ}^\circ} = B * [B \cdot (B * \phi_Y)], \\
\phi_{\overline{Y}^c} = -B \cdot \phi_Y, & \phi_{\overline{Y^\circ}} = B \cdot (B * \phi_Y), & \phi_{\overline{Y^\circ}^{\circ c}} = -B * [B \cdot (B * \phi_Y)], \\
\phi_{Y^\circ} = B * \phi_Y, & \phi_{\overline{Y^\circ}^c} = -B \cdot (B * \phi_Y), &
\end{array}$$

where $-B = (1 - b_{ij})_{n \times n}$ for $B = (b_{ij})_{n \times n}$.

According to Theorem 2.10, one can divide the above 14 sets into 7 pairs:

$$(Y, Y^c),\ (Y^\circ,\ Y^{\circ c}),\ (\overline{Y^\circ},\ \overline{Y^\circ}^c),\ (\overline{Y^\circ}^\circ,\ \overline{Y^\circ}^{\circ c}),\ (\overline{Y},\ \overline{Y}^c),\ (\overline{Y}^\circ,\ \overline{Y}^{\circ c}),\ (\overline{\overline{Y}^\circ},\ \overline{\overline{Y}^\circ}^c).$$

The two sets of each pair are complementary.

3. Modularity of finite topological spaces

Definition 3.1. [1] Let (T, τ) be a topological space and A and B be two subsets of T. If $\overline{A} \cap B = \emptyset$ and $\overline{B} \cap A = \emptyset$, then A and B are called separated. If a subset C of T is not the union of two non-empty separated subsets, then C is called a connected subset of T. A maximal connected subset of T is called a component of T.

Proposition 3.2. [1] *Let (T, τ) be a topological space. Then*

(1) the closure of a connected subset of T is also connected;
(2) for a family of connected subsets $\{A_\lambda : \lambda \in \Lambda\}$ of T in which no two members are separated, their union $\cup_{\lambda \in \Lambda} A_\lambda$ is also connected;
(3) each component is closed and every two distinct components are separated.

Based on this, we can easily obtain the following results in an FTS (T, τ). First of all, we specify some terminologies. For each $t \in T$, $M(t)$ denotes $\overline{\{t\}}$. For each $A \subset T$, $N(A)$ is the smallest open set containing A and $M(A)$ is the smallest closed set containing A. One can easily check that $N(A)$ and $M(A)$ can be formulated by the following formulas respectively.

$$N(A) = \{y : \exists t \in A, y \in N(t)\} = \bigcup_{t \in A} N(t), \quad M(A) = \{y : \exists t \in A, y \in M(t)\} = \bigcup_{t \in A} M(t).$$

Proposition 3.3. *Let (T, τ) be an FTS. Then*

(1) there are finite many components;
(2) a subset of T is a component if and only if it is both open and closed;
(3) for each $t \in T$, both $N(t)$ and $M(t)$ are connected subsets;
(4) for each $t \in T$, the family $\mathcal{F}(t)$ defined by

$$\{N(t),\ M(t),\ \ N(M(t)),\ M(N(t)),\ N(M(N(t))),\ M(N(M(t))),\ \cdots\}$$

contains only finite many different elements. Morover, each element of $\mathcal{F}(t)$ is connected.
(5) if Y is a component and $t \in Y$, then each element of $\mathcal{F}(t)$ is contained in Y.

Proposition 3.4. *Let (T, τ) be an FTS. Then for each $t \in T$, the maximal element in $\mathcal{F}(t)$ is a component. In other words, there exists a component $A \in \mathcal{F}(t)$ such that $A = N(A) = M(A)$.*

Theorem 3.5. *Let (T, τ) be an FTS. If $T_1, T_2, \cdots, T_k$ are all disjoint non-empty components, then for each $Y \subset T$, we have the modularized computations as follows*

$$Y^\circ = (Y \cap T_1)^\circ \cup (Y \cap T_2)^\circ \cup \cdots (Y \cap T_k)^\circ, \qquad \overline{Y} = \overline{Y \cap T_1} \cup \overline{Y \cap T_2} \cup \cdots \overline{Y \cap T_k}.$$

Proof. Because T_i is both open and closed, all $Y^\circ \cap T_i, i = 1, 2, \cdots, k$ are open, and thus $Y^\circ \cap T_i$ is the biggest open subset contained in $Y \cap T_i$, so it is equal to $(Y \cap T_i)^\circ$. Hence,

$$Y^\circ = \cup_{i=1}^{k}(Y^\circ \cap T_i) = \cup_{i=1}^{k}(Y \cap T_i)^\circ.$$

Similarly, all $\overline{Y} \cap T_i, i = 1, 2, \cdots, k$ are closed, and thus $\overline{Y} \cap T_i$ is the smallest closed subset containing $Y \cap T_i$, so it is equal to $\overline{Y \cap T_i}$. Hence,

$$\overline{Y} = \cup_{i=1}^{k}(\overline{Y} \cap T_i) = \cup_{i=1}^{k}\overline{Y \cap T_i}.$$

The proof is completed. □

Remark 3.6. In fact, if some T_j in Theorem 3.5 is replaced by several components whose union is T_j, then Theorem 3.5 is also correct. But we hope each T_j is small enough, so in this paper, we use the case of that all T_j $(1 \le j \le k)$ are components which constitute a partition of T. Thus T is modularized.

If X is a subspace of T and X is not a component of T, then for any subset $Y \subset X$, the interior of Y in the subspace X generally does not equal the interior of Y in T. Dually, the closure of Y in X generally does not equal the closure of Y in T. Please see the following example.

Example 3.7. Let (T, τ) be an FTS with a subbase ϑ for the topology τ, where $T = \{t_1, t_2, \cdots, t_{26}\}$ and ϑ is given by

$$\left\{\begin{array}{lll} \{t_1,t_8\},\{t_8\},\{t_{10}\}, & \{t_3,t_{10}\},\{t_{16}\},\{t_{10}\}, & \{t_5,t_{20}\},\{t_2,t_4\}, \\ \{t_{12},t_{15}\},\{t_{13},t_{15}\}, & \{t_7,t_{11}\},\{t_{11}\}\{t_2\}, & \{t_{13}\},\{t_6,t_{21}\},\{t_{21}\}, \\ \{t_{22},t_{23}\},\{t_{25},t_{26}\}, & \{t_{26}\},\{t_9,t_{14},t_{16}\}, & \{t_2,t_4,t_{12},t_{13},t_{15},t_{18}\}, \\ \{t_7,t_9,t_{11},t_{14},t_{16},t_{19}\}, & \{t_6,t_{21},t_{22},t_{23},t_{24},t_{25},t_{26}\}, & \{t_1,t_3,t_5,t_8,t_{10},t_{17},t_{20}\}. \end{array}\right\}$$

The smallest open sets are enumerated as follows:

$$\begin{array}{llll} N(t_1) = \{t_1,t_8\}, & N(t_2) = \{t_2\}, & N(t_3) = \{t_3,t_{10}\}, & N(t_4) = \{t_2,t_4\}, \\ N(t_5) = \{t_5,t_{20}\}, & N(t_6) = \{t_6,t_{21}\}, & N(t_7) = \{t_7,t_{11}\}, & N(t_8) = \{t_8\}, \\ N(t_9) = \{t_9,t_{14},t_{16}\}, & N(t_{10}) = \{t_{10}\}, & N(t_{11}) = \{t_{11}\}, & N(t_{12}) = \{t_{12},t_{15}\}, \\ N(t_{13}) = \{t_{13}\}, & N(t_{14}) = \{t_9,t_{14},t_{16}\}, & N(t_{15}) = \{t_{15}\}, & N(t_{16})) = \{t_{16}\}, \end{array}$$

$$\begin{array}{ll} N(t_{17}) = \{t_1,t_3,t_5,t_8,t_{10},t_{17},t_{20}\}, & N(t_{18}) = \{t_2,t_4,t_{12},t_{13},t_{15},t_{18}\}, \\ N(t_{19}) = \{t_7,t_9,t_{11},t_{14},t_{16},t_{19}\}, & N(t_{20}) = \{t_5,t_{20}\}, \\ N(t_{21}) = \{t_{21}\}, & N(t_{22}) = \{t_{22},t_{23}\}, \\ N(t_{23}) = \{t_{22},t_{23}\}, & N(t_{24}) = \{t_6,t_{21},t_{22},t_{23},t_{24},t_{25},t_{26}\}, \\ N(t_{25}) = \{t_{25},t_{26}\}, & N(t_{26}) = \{t_{26}\}. \end{array}$$

For the subspace $T_1 = \{t_1, t_3, t_5, t_8, t_{10}, t_{17}, t_{20}\}$ and $X = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9\}$ of T, we consider a subset $Y = \{t_3, t_5, t_8\}$. The interiors and closures of Y in subspaces T_1 and X are respectively enumerated as follows:

$$\begin{array}{ll} (Y|_{T_1})^\circ = \{t_8\} = Y^\circ, & \overline{Y|_{T_1}} = \{t_1, t_3, t_5, t_8, t_{17}, t_{20}\} = \overline{Y}, \\ (Y|_X)^\circ = \{t_3, t_5, t_8\} \neq Y^\circ, & \overline{Y|_X} = \{t_1, t_3, t_5, t_8\} \neq \overline{Y}. \end{array}$$

Here we can obtain the modular computations of Kuratowski 14 sets.

Theorem 3.8. *Let (T, τ) be an FTS and $\{T_i : i = 1, 2, \cdots, k\}$ be all the components of T. Then the Kuratowski 14 sets of Y can be computed by the modularization technique:*

$$\begin{array}{lll} Y = \cup_{i=1}^k (Y \cap T_i), & Y^\circ = \cup_{i=1}^k (Y \cap T_i)^\circ, & \overline{Y^\circ} = \cup_{i=1}^k \overline{(Y \cap T_i)^\circ}, \\ \overline{Y^\circ}^\circ = \cup_{i=1}^k \overline{(Y \cap T_i)^\circ}^\circ, & \overline{Y} = \cup_{i=1}^k \overline{Y \cap T_i}, & \overline{Y}^\circ = \cup_{i=1}^k \overline{Y \cap T_i}^\circ, \\ \overline{\overline{Y}^\circ} = \cup_{i=1}^k \overline{\overline{Y \cap T_i}^\circ}, & Y^c = \cup_{i=1}^k (T_i - Y \cap T_i), & Y^{\circ c} = \cup_{i=1}^k (T_i - (Y \cap T_i)^\circ), \\ \overline{Y^\circ}^c = \cup_{i=1}^k (T_i - \overline{(Y \cap T_i)^\circ}), & \overline{Y^\circ}^{\circ c} = \cup_{i=1}^k (T_i - \overline{(Y \cap T_i)^\circ}^\circ), & \overline{Y}^c = \cup_{i=1}^k (T_i - \overline{Y \cap T_i}), \\ \overline{Y}^{\circ c} = \cup_{i=1}^k (T_i - \overline{Y \cap T_i}^\circ), & \overline{\overline{Y}^\circ}^c = \cup_{i=1}^k (T_i - \overline{\overline{Y \cap T_i}^\circ}). & \end{array}$$

If we compute the Kuratowski 14 sets of a subset Y of an FTS (T, τ), we only need to consider components which intersect with Y instead of the whole space T. On this basis, Y is partitioned into disjoint parts within related components to compute separately. When the subset Y is very small relative to whole T, this method greatly reduces the range involved. So the modular method is relatively simpler, more efficient and accurate.

Definition 3.9. In an FTS (T, τ), if the order of all elements is fixed as $T = \{t_1, t_2, \cdots, t_n\}$, and if Y is a subset of T, then the first element of Y is denoted by $f_s(Y)$, i.e.,

$$f_s(Y) = t_k, \quad k = min\{i : t_i \in Y\}.$$

In the following, an algorithm for modularizing a topological space is proposed.

Algorithm 1: Modularity of the finite topological space

Input: $T = \{t_1, t_2, \cdots, t_n\}$
1 Let $Y = T$, $i = 1$;
2 **while** $Y \neq \emptyset$ **do**
3 | $A = N(f_s(Y))$, $B = M(A)$;
4 | **while** $B \neq A$ **do**
5 | | $A = N(B)$, $B = M(A)$ and return to step 4;
6 | **end**
| **Output:** $T_i = B$;
7 | Update: $Y = Y - T_i$, $i = i + 1$ and return to step 2;
8 **end**
Output: all modules $\{T_i\}$ of T.

The following example demonstrates the process of modularizing a topological space using this algorithm.

Example 3.10. Let (T, τ) be the topological space defined in Example 3.7. In the following, we show how to modularize the topological space T in accordance with Algorithm 1.

- Input: $T = \{t_i : 1 \leq i \leq 26\}$ and let $Y = T$, $i = 1$.

Since $Y \neq \emptyset$ and $f_s(Y) = t_1$, we set

$$A = N(f_s(Y)) = N(t_1) = \{t_1, t_8\},\ B = M(A) = \{t_1, t_8, t_{17}\}.$$

Since $B \neq A$, update:

$$A = N(B) = \{t_1, t_3, t_5, t_8, t_{10}, t_{17}, t_{20}\},\ B = M(A) = \{t_1, t_3, t_5, t_8, t_{10}, t_{17}, t_{20}\}.$$

* There is $A = B$, then output: $T_1 = \{t_1, t_3, t_5, t_8, t_{10}, t_{17}, t_{20}\}$.

- Then update: $i = 2$,

$$Y = Y - T_1 = \{t_2, t_4, t_6, t_7, t_9, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}, t_{16}, t_{18}, t_{19}, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}, t_{26}\}.$$

Since $Y \neq \emptyset$ and $f_s(Y) = t_2$, let $A = N(\{t_2\}) = \{t_2\},\ B = M(A) = \{t_2, t_4, t_{18}\}$.
Since $B \neq A$, update:

$$A = N(B) = \{t_2, t_4, t_{12}, t_{13}, t_{15}, t_{18}\},\ B = M(A) = \{t_2, t_4, t_{12}, t_{13}, t_{15}, t_{18}\}.$$

* Since $A = B$, output: $T_2 = \{t_2, t_4, t_{12}, t_{13}, t_{15}, t_{18}\}$.

- Update: $i = 3$, $Y = Y - T_2 = \{t_6, t_7, t_9, t_{11}, t_{14}, t_{16}, t_{19}, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}, t_{26}\}$.
Because $Y \neq \emptyset$ and $f_s(Y) = t_6$, we set

$$A = N(t_6) = \{t_6, t_{21}\},\ B = M(A) = \{t_6, t_{21}, t_{24}\}.$$

Since $B \neq A$, update:

$$A = N(B) = \{t_6, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}, t_{26}\},\ B = M(A) = \{t_6, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}, t_{26}\}.$$

* Since $A = B$, output: $T_3 = \{t_6, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}, t_{26}\}$.

- Update: $i = 4$, $Y = Y - T_3 = \{t_7, t_9, t_{11}, t_{14}, t_{16}, t_{19}\}$.
Since $Y \neq \emptyset$ and $f_s(Y) = t_7$, let $A = N(\{t_7\}) = \{t_7, t_{11}\},\ B = M(A) = \{t_7, t_{11}, t_{19}\}$.
Since $B \neq A$, update:

$$A = N(B) = \{t_7, t_9, t_{11}, t_{14}, t_{16}, t_{19}\},\ B = M(A) = \{t_7, t_9, t_{11}, t_{14}, t_{16}, t_{19}\}.$$

* Since $A = B$, output: $T_4 = \{t_7, t_9, t_{11}, t_{14}, t_{16}, t_{19}\}$.

** Update: $Y = Y - T_4$, and since $Y = \emptyset$, stop and output: all modules $\{T_1, T_2, T_3, T_4\}$.

The following example demonstrates the modular method of computing Kuratowski 14 sets.

Example 3.11. Let (T, τ) be the topological space defined in Example 3.10. Then all modules of T are

$$T_1 = \{t_1, t_3, t_5, t_8, t_{10}, t_{17}, t_{20}\}, \qquad T_2 = \{t_2, t_4, t_{12}, t_{13}, t_{15}, t_{18}\},$$
$$T_3 = \{t_6, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}, t_{26}\}, \quad T_4 = \{t_7, t_9, t_{11}, t_{14}, t_{16}, t_{19}\}.$$

For a subset $Y = \{t_2, t_3, t_5, t_8, t_{12}\}$, we enumerate all the Kuratowski 14 sets of Y as follows. Since $Y \cap T_3 = Y \cap T_4 = \emptyset$, we need only to consider the modules T_1 and T_2. As $Y \cap T_1 = \{t_3, t_5, t_8\},\ Y \cap T_2 = \{t_2, t_{12}\}$, we have that
$Y = (Y \cap T_1) \cup (Y \cap T_2) = \{t_3, t_5, t_8\} \cap \{t_2, t_{12}\} = \{t_2, t_3, t_5, t_8, t_{12}\}$,
$Y^\circ = (Y \cap T_1)^\circ \cup (Y \cap T_2)^\circ = \{t_8\} \cup \{t_2\} = \{t_2, t_8\}$,
$\overline{Y^\circ} = \overline{(Y \cap T_1)^\circ} \cup \overline{(Y \cap T_2)^\circ} = \{t_1, t_8, t_{17}\} \cup \{t_2, t_4, t_{18}\} = \{t_1, t_2, t_4, t_8, t_{17}, t_{18}\}$,
$\overline{Y^\circ}^\circ = \overline{(Y \cap T_1)^\circ}^\circ \cup \overline{(Y \cap T_2)^\circ}^\circ = \{t_1, t_8\} \cup \{t_2, t_4\} = \{t_1, t_2, t_4, t_8\}$,
$\overline{Y} = \overline{Y \cap T_1} \cup \overline{Y \cap T_2} = \{t_1, t_3, t_5, t_8, t_{17}, t_{20}\} \cup \{t_2, t_4, t_{12}, t_{18}\} = \{t_1, t_2, t_3, t_4, t_5, t_8, t_{12}, t_{17}, t_{18}, t_{20}\}$,

$\overline{Y}^{\circ} = \overline{Y \cap T_1}^{\circ} \cup \overline{Y \cap T_2}^{\circ} = \{t_1, t_5, t_8, t_{20}\} \cup \{t_2, t_4\} = \{t_1, t_2, t_4, t_5, t_8, t_{20}\}$,
$\overline{\overline{Y}^{\circ}} = \overline{\overline{Y \cap T_1}^{\circ}} \cup \overline{\overline{Y \cap T_2}^{\circ}} = \{t_1, t_5, t_8, t_{17}, t_{20}\} \cup \{t_2, t_4, t_{18}\} = \{t_1, t_2, t_4, t_5, t_8, t_{17}, t_{18}, t_{20}\}$,
$Y^c = \cup_{i=1}^{4}(T_i - Y \cap T_i) = \{t_1, t_4, t_6, t_7, t_9, t_{10}, t_{11}, t_{13}, t_{14}, t_{15}, t_{16}, t_{17}, t_{18}, t_{19}, t_{20}, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}, t_{26}\}$,
$Y^{\circ c} = \cup_{i=1}^{4}(T_i - (Y \cap T_i)^{\circ})$
$= \{t_1, t_3, t_4, t_5, t_6, t_7, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}, t_{16}, t_{17}, t_{18}, t_{19}, t_{20}, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}, t_{26}\}$,
$\overline{Y^{\circ}}^{c} = \cup_{i=1}^{4}(T_i - \overline{(Y \cap T_i)^{\circ}}) = \{t_3, t_5, t_6, t_7, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}, t_{16}, t_{19}, t_{20}, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}, t_{26}\}$,
$\overline{Y^{\circ}}^{\circ c} = \cup_{i=1}^{4}(T_i - \overline{(Y \cap T_i)^{\circ}}^{\circ})$
$= \{t_3, t_5, t_6, t_7, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}, t_{16}, t_{17}, t_{18}, t_{19}, t_{20}, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}, t_{26}\}$,
$\overline{Y}^{c} = \cup_{i=1}^{4}(T_i - \overline{Y \cap T_i}) = \{t_6, t_7, t_9, t_{10}, t_{11}, t_{13}, t_{14}, t_{15}, t_{16}, t_{19}, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}, t_{26}\}$,
$\overline{Y}^{\circ c} = \cup_{i=1}^{k}(T_i - \overline{Y \cap T_i}^{\circ}) = \{t_3, t_6, t_7, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}, t_{16}, t_{17}, t_{18}, t_{19}, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}, t_{26}\}$,
$\overline{\overline{Y}^{\circ}}^{c} = \cup_{i=1}^{k}(T_i - \overline{\overline{Y \cap T_i}^{\circ}}) = \{t_3, t_6, t_7, t_9, t_{10}, t_{11}, t_{12}, t_{13}, t_{14}, t_{15}, t_{16}, t_{19}, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}, t_{26}\}$.

4. The modular matrix computation method

The matrix method for computing the Kuratowski 14 sets in Theorem 3.8 was proved in paper [5], which is simpler and more efficient than the traditional way because it does not need logical analysis. However, when the space is very large, the orders of corresponding matrices will also be very large, and a great number of calculations will lead to the waste of time and space. If the matrix can be divided into smaller matrices for computations, then time and space will be saved to a greater extent, and accuracy will be improved. Therefore, we will explore the modular matrix computation method.

Definition 4.1. Let (T, τ) be an FTS in which the order of elements is given as $T = \{t_1, t_2, \cdots, t_n\}$, and T_i $(1 \leq i \leq k)$ be all components of T where $T_i = \{t_{i_1}, t_{i_2}, \cdots, t_{i_{k_i}}\}$ $(k_i = |T_i|)$ and the order of elements in T_i is the same as that in T. Then the characteristic vector of a subset $Y \subset T_i$ is defined as

$$\phi^i(Y) = (a_1, a_2, \cdots, a_{k_i})^T,$$

where

$$a_j = \begin{cases} 1, & t_{i_j} \in Y \\ 0, & t_{i_j} \notin Y \end{cases}, \quad j = 1, 2, \cdots, k_i.$$

It is evident that the set Y can be uniquely determined by its characteristic vector $\phi^i(Y)$. In fact, $Y = \{t_{i_j} : a_j = 1\}$. Thus, the inverse map of ϕ^i is defined as $(\phi^i)^{-1}[(a_1, a_2, \cdots, a_{k_i})^T] = Y$ for $i = 1, 2, \cdots, k$.

For each $i = 1, 2, \cdots, k$, the modular base matrix B_i are defined as follows:
the jth row α_j of B_i is $\alpha_j = (\phi^i(N(t_{i_j})))^T$, $j = 1, 2, \cdots, k_i$.

The next proposition follows directly from this definition and Proposition 3.3.

Proposition 4.2. *Suppose that* (T, τ) *and* T_i $(1 \leq i \leq k)$ *are the same as those of Definition 4.1, and* $B_i = (\beta_1, \beta_2, \cdots, \beta_{k_i})$ *for* $i = 1, 2, \cdots, k$. *Then*

$$(\phi^i)^{-1}[\beta_j] = \overline{\{t_{i_j}\}}, \ j = 1, 2, \cdots, k_i, i = 1, 2, \cdots, k.$$

We then give an Algorithm 2 based on the modular matrices to compute the 14 sets of a subset Y in a topological space (T, τ).

Then we use Algorithm 2 to compute the 14 sets of a subset.

Example 4.3. Let (T, τ) be the same as that of Example 3.11. For a subset $Y = \{t_2, t_3, t_5, t_8, t_{12}\}$, we use Algorithm 2 to compute the Kuratowski 14 sets of Y.

Input Y and $k = 4$ and all modules $\{T_1, T_2, T_3, T_4\}$ of T, where

$$T_1 = \{t_1, t_3, t_5, t_8, t_{10}, t_{17}, t_{20}\}, \quad T_2 = \{t_2, t_4, t_{12}, t_{13}, t_{15}, t_{18}\},$$
$$T_3 = \{t_6, t_{21}, t_{22}, t_{23}, t_{24}, t_{25}, t_{26}\}, \quad T_4 = \{t_7, t_9, t_{11}, t_{14}, t_{16}, t_{19}\}.$$

Algorithm 2: Modular matrix computations of Kuratowski 14 sets

Input: Y, T
1 Use Algorithm 1 **Output:** $T_1, T_2, \cdots, T_k$
2 Use Definition 8 **Output:** $B_1, B_2, \cdots, B_k$
3 Let $i = 1$;
4 **while** $i \neq (k+1)$ **do**
5   $Y_i = Y \cap T_i$;
6   **if** $Y_i \neq \emptyset$ **then**
    Output: $\alpha_1^i = \phi^i(Y_i)$, $\alpha_2^i = B_i * \phi^i(Y_i)$, $\alpha_3^i = B_i \cdot (B_i * \phi^i(Y_i))$, $\alpha_4^i = B_i * [B_i \cdot (B_i * \phi^i(Y_i))]$, $\alpha_5^i = B_i \cdot \phi^i(Y_i)$, $\alpha_6^i = B_i * (B_i \cdot \phi^i(Y_i))$, $\alpha_7^i = B_i \cdot [B_i * (B_i \cdot \phi^i(Y_i))]$.
7   Update: $i = i + 1$, and return to step 4.
8   **else**
9     $i = i + 1$, and return to step 4.
10   **end**
11 **end**
12 Let $j = 1, i = 1, X = \emptyset$;
13 **while** $j \neq 8,\ i \neq (k+1)$ **do**
14   $Y_i(j) = (\phi^i)^{-1}[\alpha_j^i]$;
15   Update: $Y(j) = X \cup Y_i(j),\ X = Y(j),\ i = i + 1$ and return to step 13;
   Output: $Y(j),\ T - Y(j)$
16   Update: $j = j + 1$, and return to step 13.
17 **end**
Output: Kuratowski 14 sets $Y(j),\ Y(j+7) = T - Y(j)\ (j = 1, 2, \cdots, 7)$ of Y.

Obviously, the modular matrices of modules are

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

$i = 1$. Since $i \neq 5$, let $Y_1 = Y \cap T_1 = \{t_3, t_5, x_8\}$. We have $Y_1 \neq \emptyset$, then output:

$$\begin{aligned} &\alpha_1^1 = \phi^1(Y_1) = (0,1,1,1,0,0,0,0)^T, && \alpha_3^1 = B_1 \cdot (B_1 * \phi^1(Y_1)) = (1,0,0,1,0,1,0)^T, \\ &\alpha_2^1 = B_1 * \phi^1(Y_1) = (0,0,0,1,0,0,0)^T, && \alpha_4^1 = B_1 * [B_1 \cdot (B_1 * \phi^1(Y_1))] = (1,0,0,1,0,0,0)^T, \\ &\alpha_5^1 = B_1 \cdot \phi^1(Y_1) = (1,1,1,1,0,1,1)^T, && \alpha_6^1 = B_1 * (B_1 \cdot \phi^1(Y_1)) = (1,0,1,1,0,0,1)^T, \\ & && \alpha_7^1 = B_1 \cdot [B_1 * (B_1 \cdot \phi^1(Y_1))] = (1,0,1,1,0,1,1)^T. \end{aligned}$$

$i = 2$. Since $i \neq 5$, let $Y_2 = Y \cap T_2 = \{t_2, t_{12}\}$. We have $Y_2 \neq \emptyset$, then output:

$$\alpha_1^2 = \phi^2(Y_2) = (1,0,1,0,0,0)^T, \quad \alpha_3^2 = B_2 \cdot (B_2 * \phi^2(Y_2)) = (1,1,0,0,0,1)^T,$$
$$\alpha_2^2 = B_2 * \phi^2(Y_2) = (0,1,0,0,0,0)^T, \quad \alpha_4^2 = B_2 * [B_2 \cdot (B_2 * \phi^2(Y_2))] = (1,1,0,0,0,0)^T,$$
$$\alpha_5^2 = B_2 \cdot \phi^2(Y_2) = (1,1,1,0,1,1)^T, \quad \alpha_6^2 = B_2 * (B_2 \cdot \phi^2(Y_2)) = (1,1,1,0,0,1)^T$$
$$\alpha_7^2 = B_2 \cdot [B_2 * (B_2 \cdot \phi^2(Y_2))] = (1,1,1,0,1,1)^T.$$

$i = 3$. Since $i \neq 5$, let $Y_3 = Y \cap T_3 = \emptyset$.
$i = 4$. Since $i \neq 5$, let $Y_4 = Y \cap T_4 = \emptyset$.
$i = 5$, end.

Let	$\mathbf{j = 1}, i = 1, X = \emptyset$.
Since	$j \neq 8,\ i \neq 5, \quad Y_1(1) = (\phi^1)^{-1}[\alpha_1^1] = \{t_3, t_5, t_8\}$,
Update:	$Y(1) = X \cup Y_1(1) = \{t_3, t_5, t_8\},\ X = \{t_3, t_5, t_8\}$.
$i = 2$.	Since $\quad i \neq 5, \quad Y_2(1) = (\phi^2)^{-1}[\alpha_1^2] = \{t_2, t_{12}\}$,
Update:	$Y(1) = X \cup Y_2(1) = \{t_2, t_3, t_5, t_8, t_{12}\}, X = \{t_2, t_3, t_5, t_8, t_{12}\}$.
$i = 3$.	Since $\quad i \neq 5, \quad Y_3(1) = (\phi^3)^{-1}[\alpha_1^3] = \emptyset$,
Update:	$Y(1) = X \cup Y_3(1) = \{t_2, t_3, t_5, t_8, t_{12}\}, X = \{t_2, t_3, t_5, t_8, t_{12}\}$.
$i = 4$.	Since $\quad i \neq 5, \quad Y_4(1) = (\phi^4)^{-1}[\alpha_1^4] = \emptyset$,
Update:	$Y(1) = X \cup Y_4(1) = \{t_2, t_3, t_5, t_8, t_{12}\}, X = \{t_2, t_3, t_5, t_8, t_{12}\}$.
$i = 5$,	end.

$\mathbf{j = 2}$,	$i = 1$. Since $\quad j \neq 8,\ i \neq 5, \quad Y_1(2) = (\phi^2)^{-1}[\alpha_2^1] = \{t_8\}$,
Update:	$Y(2) = X \cup Y_1(2) = \{t_8\},\ X = \{t_8\}$.
$i = 2$.	Since $\quad i \neq 5, \quad Y_2(2) = (\phi^2)^{-1}[\alpha_2^2] = \{t_2\}$,
Update:	$Y(2) = X \cup Y_2(2) = \{t_2, t_8\},\ X = \{t_2, t_8\}$.
$i = 3$.	Since $\quad i \neq 5, \quad Y_3(2) = (\phi^3)^{-1}[\alpha_2^3] = \emptyset$,
Update:	$Y(2) = X \cup Y_3(2) = \{t_2, t_8\},\ X = \{t_2, t_8\}$.
$i = 4$.	Since $\quad i \neq 5, \quad Y_4(2) = (\phi^4)^{-1}[\alpha_2^4] = \emptyset$,
Update:	$Y(2) = X \cup Y_4(2) = \{t_2, t_8\},\ X = \{t_2, t_8\}$.
$i = 5$,	end.

$\mathbf{j = 3}$,	$i = 1. \quad$ Since $\quad j \neq 8,\ i \neq 5, \quad Y_1(3) = (\phi^3)^{-1}[\alpha_3^1] = \{t_1, t_8, t_{17}\}$,
Update:	$Y(3) = X \cup Y_1(3) = \{t_1, t_8, t_{17}\},\ X = \{t_1, t_8, t_{17}\}$.
$i = 2$.	Since $\quad i \neq 5, \quad Y_2(3) = (\phi^2)^{-1}[\alpha_3^2] = \{t_2, t_4, t_{18}\}$,
Update:	$Y(3) = X \cup Y_2(3) = \{t_1, t_2, t_4, t_8, t_{17}, t_{18}\}, \quad X = \{t_1, t_2, t_4, t_8, t_{17}, t_{18}\}$.
$i = 3$.	Since $\quad i \neq 5, \quad Y_3(3) = (\phi^3)^{-1}[\alpha_3^3] = \emptyset$,
Update:	$Y(3) = X \cup Y_3(3) = \{t_1, t_2, t_4, t_8, t_{17}, t_{18}\}, \quad X = \{t_1, t_2, t_4, t_8, t_{17}, t_{18}\}$.
$i = 4$.	Since $\quad i \neq 5, \quad Y_4(3) = (\phi^4)^{-1}[\alpha_3^4] = \emptyset$,
Update:	$Y(3) = X \cup Y_4(3) = \{t_1, t_2, t_4, t_8, t_{17}, t_{18}\}, \quad X = \{t_1, t_2, t_4, t_8, t_{17}, t_{18}\}$.

$i=5$, end.

j = 4, $i=1$. Since $j \neq 8,\ i \neq 5$, $Y_1(4)=(\phi^4)^{-1}[\alpha_4^1]=\{t_1,t_8\}$,

Update : $Y(4)=X\cup Y_1(4)=\{t_1,t_8\}$, $X=\{t_1,t_8\}$.

$i=2$. Since $i\neq 5$, $Y_2(4)=(\phi^2)^{-1}[\alpha_4^2]=\{t_2,t_4\}$,

Update: $Y(4)=X\cup Y_2(4)=\{t_1,t_2,t_4,t_8\}$, $X=\{t_1,t_2,t_4,t_8\}$.

$i=3$. Since$i\neq 5$, $Y_3(4)=(\phi^3)^{-1}[\alpha_4^3]=\emptyset$,

Update: $Y(4)=X\cup Y_3(4)=\{t_1,t_2,t_4,t_8\}$, $X=\{t_1,t_2,t_4,t_8\}$.

$i=4$. Since $i\neq 5$, $Y_4(4)=(\phi^4)^{-1}[\alpha_4^4]=\emptyset$,

Update: $Y(4)=X\cup Y_4(4)=\{t_1,t_2,t_4,t_8\}$, $X=\{t_1,t_2,t_4,t_8\}$.

$i=5$, end.

j = 5, $i=1$. Since $j\neq 8,\ i\neq 5$, $Y_1(5)=(\phi^5)^{-1}[\alpha_5^1]=\{t_1,t_3,t_5,t_8,t_{17},t_{20}\}$,

Update : $Y(5)=X\cup Y_1(5)=\{t_1,t_3,t_5,t_8,t_{17},t_{20}\}$, $X=\{t_1,t_3,t_5,t_8,t_{17},t_{20}\}$.

$i=2$. Since $i\neq 5$, $Y_2(5)=(\phi^2)^{-1}[\alpha_5^2]=\{t_2,t_4,t_{12},t_{18}\}$,

Update: $Y(5)=X\cup Y_2(5)=\{t_1,t_2,t_3,t_4,t_5,t_8,t_{12},t_{17},t_{18},t_{20}\}$,
$X=\{t_1,t_2,t_3,t_4,t_5,t_8,t_{12},t_{17},t_{18},t_{20}\}$.

$i=3$. Since $i\neq 5$, $Y_3(5)=(\phi^3)^{-1}[\alpha_5^3]=\emptyset$,

Update: $Y(5)=X\cup Y_3(5)=\{t_1,t_2,t_3,t_4,t_5,t_8,t_{12},t_{17},t_{18},t_{20}\}$,
$X=\{t_1,t_2,t_3,t_4,t_5,t_8,t_{12},t_{17},t_{18},t_{20}\}$.

$i=4$. Since $i\neq 5$, $Y_4(5)=(\phi^5)^{-1}[\alpha_5^4]=\emptyset$,

Update: $Y(5)=X\cup Y_4(5)=\{t_1,t_2,t_3,t_4,t_5,t_8,t_{12},t_{17},t_{18},t_{20}\}$,
$X=\{t_1,t_2,t_3,t_4,t_5,t_8,t_{12},t_{17},t_{18},t_{20}\}$.

$i=5$, end.

j = 6, $i=1$.Since $j\neq 8,\ i\neq 5$, $Y_1(6)=(\phi^6)^{-1}[\alpha_6^1]=\{t_1,t_5,t_8,t_{20}\}$,

Update: $Y(6)=X\cup Y_1(6)=\{t_1,t_5,t_8,t_{20}\}$, $X=\{t_1,t_5,t_8,t_{20}\}$.

$i=2$. Since $i\neq 5$, $Y_2(6)=(\phi^2)^{-1}[\alpha_6^2]=\{t_2,t_4\}$,

Update: $Y(6)=X\cup Y_2(6)=\{t_1,t_2,t_4,t_5,t_8,t_{20}\}$,
$X=\{t_1,t_2,t_4,t_5,t_8,t_{20}\}$.

$i=3$. Since $i\neq 5$, $Y_3(6)=(\phi^3)^{-1}[\alpha_6^3]=\emptyset$,

Update: $Y(6)=X\cup Y_3(6)=\{t_1,t_2,t_4,t_5,t_8,t_{20}\}$,
$X=\{t_1,t_2,t_4,t_5,t_8,t_{20}\}$.

$i=4$. Since $i\neq 5$, $Y_4(6)=(\phi^6)^{-1}[\alpha_6^4]=\emptyset$,

Update : $Y(6)=X\cup Y_4(6)=\{t_1,t_2,t_4,t_5,t_8,t_{20}\}$,
$X=\{t_1,t_2,t_4,t_5,t_8,t_{20}\}$.

$i=5$, end.

j = 7, $i=1$.Since$j\neq 8,\ i\neq 5$, $Y_1(7)=(\phi^7)^{-1}[\alpha_7^1]=\{t_1,t_5,t_8,t_{17},t_{20}\}$,

Update:	$Y(7) = X \cup Y_1(7) = \{t_1, t_5, t_8, t_{17}, t_{20}\},\ X = \{t_1, t_5, t_8, t_{17}, t_{20}\}.$
$i = 2.$	Since $i \neq 5, \quad Y_2(7) = (\phi^2)^{-1}[\alpha_7^2] = \{t_2, t_4, t_{18}\},$
Update:	$Y(7) = X \cup Y_2(7) = \{t_1, t_2, t_4, t_5, t_8, t_{17}, t_{18}, t_{20}\},$
	$X = \{t_1, t_2, t_4, t_5, t_8, t_{17}, t_{18}, t_{20}\}.$
$i = 3.$	Since $i \neq 5, \quad Y_3(7) = (\phi^3)^{-1}[\alpha_7^3] = \emptyset,$
Update:	$Y(7) = X \cup Y_3(7) = \{t_1, t_2, t_4, t_5, t_8, t_{17}, t_{18}, t_{20}\},$
	$X = \{t_1, t_2, t_4, t_5, t_8, t_{17}, t_{18}, t_{20}\}.$
$i = 4.$	Since $i \neq 5, \quad Y_4(7) = (\phi^7)^{-1}[\alpha_7^4] = \emptyset,$
Update:	$Y(7) = X \cup Y_4(7) = \{t_1, t_2, t_4, t_5, t_8, t_{17}, t_{18}, t_{20}\},$
	$X = \{t_1, t_2, t_4, t_5, t_8, t_{17}, t_{18}, t_{20}\}.$
$i = 5,$	end.
j = **8**,	end.

Output: all 14 sets

$Y(1) = \{t_2, t_3, t_5, t_8, t_{12}\},$	$Y(8) = T - Y(1),$
$Y(2) = \{t_2, t_8\},$	$Y(9) = T - Y(2),$
$Y(3) = \{t_1, t_2, t_4, t_8, t_{17}, t_{18}\},$	$Y(10) = T - Y(3),$
$Y(4) = \{t_1, t_2, t_4, t_8\},$	$Y(11) = T - Y(4),$
$Y(5) = \{t_1, t_2, t_3, t_4, t_5, t_8, t_{12}, t_{17}, t_{18}, t_{20}\},$	$Y(12) = T - Y(5),$
$Y(6) = \{t_1, t_2, t_4, t_5, t_8, t_{20}\},$	$Y(13) = T - Y(6),$
$Y(7) = \{t_1, t_2, t_4, t_5, t_8, t_{17}, t_{18}, t_{20}\},$	$Y(14) = T - Y(7).$

5. Conclusions

In this paper, the modular computation method is formulated in the finite topological spaces. After the modularization of a given space, computation process will be smoother and more efficient. We have also given the algorithms that how to modularize a topological space and how to use the modular matrix method to compute the Kuratowski 14 sets in a subspace of a finite topological space.

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