Filomat 37:22 (2023), 7431–7444
https://doi.org/10.2298/FIL2322431K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Weyl correspondence in the linear canonical transform domain

Amit Kumar[a], Akhilesh Prasad[a], Pankaj Jain[b]

[a]*Department of Mathematics and Computing, Indian Institute of Technology (Indian School of Mines), Dhanbad, Jharkhand-826004, India*
[b]*Department of Mathematics, South Asian University, New Delhi-110023, India*

Abstract. The main objective of the paper is to generalize and enrich the Weyl transform by introducing the Weyl correspondence in the linear canonical transform (LCT) domain. In this paper, we propose the linear canonical-Wigner transform in harmonic analysis of phase space along with the admissible Wigner-Ville distribution (WVD) and Weyl transform in the LCT domain and discuss some useful results. Further we establish the relationship between the Wigner-Ville distribution and the Weyl transform in the LCT domain.

1. Introduction

The linear canonical transform (LCT) is a phase space transform with roots in optics and quantum mechanics [1–4]. It was introduced by Stuart A. Collins Jr. [1] in paraxial optics and independently by Marcos Moshinsky and Christiane Quesne [2] in quantum mechanics, to understand the conservation of information and of uncertainty under linear maps of phase space. The LCT is the generalization of classical Fourier transform, fractional Fourier transform, Fresnel transform, Lorentz transform, etc.

The Wigner-Ville distribution (WVD) was proposed by Wigner [5] in 1932 to study quantum corrections for classical statistical mechanics. Wigner constructed such a function for the quantum mechanical wave function with which its probabilistic interpretation led naturally to the concept of a distribution function similar to the concept in probability theory [5]. The WVD is shown to be an important method in the linear-frequency-modulated signal detection and parameter estimation, which is also essential in the signal processing community [6, 7]. Moreover, B. Z. Li et al. [6–8] obtained several important properties of the WVD.

The Weyl correspondence was first envisaged by Weyl [9] arising in quantum mechanics, and is called Weyl transform by Wong [10]. The Weyl transform has many applications to time-frequency analysis, the theory of differential equations, linear system theory, etc. In 1950, H. Weyl [9] proved that the Weyl transform is a Hilbert-Schmidt operator when the symbol is square integrable. Further, J. C. T. Pool [11]

2020 *Mathematics Subject Classification*. Primary 43A32, 34B20; Secondary 81S30
Keywords. Linear canonical transform; Weyl transform; Wigner-Ville distribution; Linear canonical-Wigner transform.
Received: 11 June 2022; Accepted: 30 March 2023
Communicated by Dragan S. Djordjević
Authors are very thankful to the reviewer for his/her valuable and constructive comments.
This work is supported by Department of Science & Technology, India, under grant no. DST/INSPIRE Fellowship/2017/IF170292. Second author is also supported by Science & Engineering Research Board, New Delhi, under grant number: MTR/2021/000193.
Email addresses: amitlvmaths@gmail.com (Amit Kumar), apr_bhu@yahoo.com (Akhilesh Prasad), pankaj.jain@sau.ac.in (Pankaj Jain)

pointed out that for the symbol belonging to L_p, $(1 \leq p \leq 2)$, the Weyl transform is bounded and compact. Apart from this, B. Simon [12] proved that the Weyl transform of L_p function for $p > 2$ is not bounded. Moreover, M. W. Wong [10] studied some other important properties of the Weyl transform.

The present paper is organized into six Sections, Section 1 is introductory, in which a brief introduction of the LCT as well as the WVD and Weyl transform are given. Section 2 is devoted to the introductory of the linear canonical transform, the classical Wigner–Ville distribution, classical weyl transform, Schwartz type space etc. Section 3 is dealt with the linear canonical-Wigner transform, in which we discuss some of its properties. In section 4, the Wigner-Ville distribution is discussed in LCT domain in brief. Section 5 is concerned with the Weyl transform in LCT domain and further establish the relationship between the Weyl transform and Wigner–Ville distribution function in the frame of LCT domain. Section 6 is the conclusion of the paper.

2. Priliminary

2.1. The linear canonical transform (LCT)

For a function $\varphi \in L^1(\mathbb{R})$, the LCT is denoted by $\mathcal{L}_A\varphi$ and defined as follows

$$(\mathcal{L}_A\varphi)(\xi) = \int_{\mathbb{R}} K_A(\xi, x)\varphi(x)dx \tag{1}$$

where $K_A(\xi, x)$ represents the kernel of the LCT and is given as

$$K_A(\xi, x) = \begin{cases} \frac{1}{\sqrt{2\pi i a_2}} \exp\left(i\frac{a_1}{2a_2}x^2 - i\frac{1}{a_2}x\xi + i\frac{a_4}{2a_2}\xi^2\right), & a_2 \neq 0, \\ \frac{1}{\sqrt{a_1}} \exp\left(i\frac{a_3}{2a_1}\xi^2\right)\delta\left(x - \frac{\xi}{a_1}\right), & a_2 = 0. \end{cases}$$

It is a three parameter class of linear integral transformation i.e. parameters are in terms of 2×2 unimodular matrix $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, which have a quadratic-phase kernel. For typographical convenience, we denote the matrix as $A = (a_1, a_2; a_3, a_4)$ in the text and we shall be dealing with the case when $a_2 \neq 0$.

The inversion of (1) is denoted by $\mathcal{L}_A^{-1} = \mathcal{L}_{A^{-1}}$ and defined as

$$\varphi(x) = (\mathcal{L}_{A^{-1}}(\mathcal{L}_A\varphi))(x) = \int_{\mathbb{R}} K_{A^{-1}}(x, \xi)(\mathcal{L}_A\varphi)(\xi)d\xi \tag{2}$$

where the kernel $K_{A^{-1}}(x, \xi)$ is given as

$$K_{A^{-1}}(x, \xi) = \sqrt{\frac{i}{2\pi a_2}} \exp\left(-i\frac{a_4}{2a_2}\xi^2 + i\frac{1}{a_2}\xi x - i\frac{a_1}{2a_2}x^2\right), \quad a_2 \neq 0.$$

Proposition 2.1. *[3] If $K_A(\xi, x)$ and $K_{A^{-1}}(x, \xi)$ are respectively the kernel of the LCT and inverse of the LCT, then*

$$\Delta_{A,x}K_A(\xi, x) = \left(-\frac{\xi}{a_2}\right)K_A(\xi, x) \quad and \quad \overline{\Delta}_{A,x}K_{A^{-1}}(x, \xi) = \left(-\frac{\xi}{a_2}\right)K_{A^{-1}}(x, \xi),$$

where the differential operators $\Delta_{A,x}$ and $\overline{\Delta}_{A,x}$ are defined as

$$\Delta_{A,x} = \frac{1}{i}\frac{d}{dx} - x\frac{a_1}{a_2} \quad and \quad \overline{\Delta}_{A,x} = -\frac{1}{i}\frac{d}{dx} - x\frac{a_1}{a_2}. \tag{3}$$

Here, $\Delta_{A,x}$ and $\overline{\Delta}_{A,x}$ are adjoint operators to each other.
Similarly as above, we have

$$\Delta_{A^{-1},x}K_A(\xi, x) = \left(-\frac{\xi}{a_2}\right)K_A(\xi, x) \quad and \quad \overline{\Delta}_{A^{-1},x}K_{A^{-1}}(x, \xi) = \left(-\frac{\xi}{a_2}\right)K_{A^{-1}}(x, \xi),$$

where the differential operators $\Delta_{A^{-1},x}$ and $\overline{\Delta}_{A^{-1},x}$ are respectively defined as

$$\Delta_{A^{-1},x} = \frac{1}{i}\frac{d}{dx} + x\frac{a_4}{a_2} \quad and \quad \overline{\Delta}_{A^{-1},x} = -\frac{1}{i}\frac{d}{dx} + x\frac{a_4}{a_2}. \tag{4}$$

Here, $\Delta_{A^{-1},x}$ and $\overline{\Delta}_{A^{-1},x}$ are also adjoint operators to each other.

The Schwartz space $S(\mathbb{R})$ is the space of rapidly decreasing complex-valued infinitely differentiable functions φ in $\mathbb{R}$ such that for every choice of β and N of non-negative integers

$$\left|D_x^{\beta}\varphi(x)\right| \leq C_{N,\beta}\,(1+|x|)^{-N}, \;\; D_x = -i\frac{d}{dx}. \tag{5}$$

Equivalent condition : for all non-negative α, β, $\exists$ $C_{\alpha,\beta} < \infty$ such that

$$\left|x^{\alpha}D_x^{\beta}\varphi(x)\right) \leq C_{\alpha,\beta}, \;\; D_x = -i\frac{d}{dx}. \tag{6}$$

To suit our discussion on the LCT, we define a variant of the Schwartz space S.

Definition 2.2. *[3] The test function space $S_A(\mathbb{R})$ is the set of all infinitely differentiable complex-valued functions φ on $\mathbb{R}$ such that for all non-negative integers β and N, there exist $C_{N,\beta}$, such that*

$$\left|\left(\overline{\Delta}_{A,x}^{\beta}\varphi\right)(x)\right| \leq C_{N,\beta}\left(1+\left|\frac{x}{a_2}\right|\right)^{-N}. \tag{7}$$

Equivalent condition : for all non-negative integers α, β, $\exists$ $C_{\alpha,\beta} < \infty$

$$\sup_{x\in\mathbb{R}}\left|\left(\frac{x}{a_2}\right)^{\alpha}\left(\overline{\Delta}_{A,x}^{\beta}\varphi\right)(x)\right| \leq C_{\alpha,\beta} < \infty, \tag{8}$$

or

$$\sup_{x\in\mathbb{R}}\left|x^{\alpha}\left(\overline{\Delta}_{A,x}^{\beta}\varphi\right)(x)\right| < \infty, \tag{9}$$

where $\overline{\Delta}_{A,x}$ is as in Proposition 2.1 and a_2 is as above.

Lemma 2.3. *[3] If $\varphi(x), \psi(x) \in S_A(\mathbb{R})$. Then for all $\beta \in \mathbb{N}_0$*

$$(i)\left(\overline{\Delta}_{A,x}^{\beta}\right)[\varphi(x)\psi(x)] = \sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}(-1)^{\gamma}\overline{\Delta}_{A,x}^{\beta-\gamma}\psi(x)D_x^{\gamma}\varphi(x),$$

Lemma 2.4. *For the differential operator $\Delta_{A,y}$ as in Proposition 2.1, we have*

$$\begin{aligned}&\left(1-\Delta_{A,y}^2\right)^n \exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right)\\&=\left[1+\left(\frac{\xi}{a_2}\right)^2\right]^n \exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right), \;\; \forall n\in\mathbb{N}_0.\end{aligned} \tag{10}$$

Proof. Using Proposition 2.1, we get

$$\begin{aligned}&\left(1-\Delta_{A,y}^2\right)^n \exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right)\\&=\sum_{r=0}^{n}\binom{n}{r}\left(-\Delta_{A,y}^2\right)^r \exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right)\end{aligned}$$

$$= \sum_{r=0}^{n} \binom{n}{r} (-1)^r \left(\Delta_{A,y}\right)^{2r} \exp\left(-i\frac{a_1}{2a_2}\left(x^2 - y^2\right) + i\frac{1}{a_2}(x-y)\xi\right)$$

$$= \sum_{r=0}^{n} \binom{n}{r} (-1)^r \left[-\left(\frac{\xi}{a_2}\right)^2\right]^r \exp\left(-i\frac{a_1}{2a_2}\left(x^2 - y^2\right) + i\frac{1}{a_2}(x-y)\xi\right)$$

$$= \left[1 + \left(\frac{\xi}{a_2}\right)^2\right]^n \exp\left(-i\frac{a_1}{2a_2}\left(x^2 - y^2\right) + i\frac{1}{a_2}(x-y)\xi\right). \tag{11}$$

□

2.2. The classical Wigner-Ville distribution (WVD)

The classical Wigner-Ville distribution $W(f,g)$ on $\mathbb{R}^2$ of two functions f and g be in $S(\mathbb{R})$ is defined by

$$W(f,g)(x,\xi) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-i\xi p} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)} dp, \ x, \xi \in \mathbb{R}, \tag{12}$$

can be used to interpret quantum mechanics as a form of non-deterministic statistical dynamics [10]. The Moyal identity is true for the classical WVD. For all f_1, g_1, f_2 and g_2 in $S(\mathbb{R})$, we have the Moyal identity

$$\langle W(f_1,g_1),\ W(f_2,g_2)\rangle = \langle f_1, f_2\rangle \overline{\langle g_1, g_2\rangle}. \tag{13}$$

2.3. The classical Weyl transform

Let $\sigma \in S^m, m \in \mathbb{R}$. Then, for any function φ in $S(\mathbb{R})$ a linear operator W_σ from $S(\mathbb{R})$ into $S(\mathbb{R})$ is defined on $\mathbb{R}$ by

$$(W_\sigma \varphi)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(x-y)\xi} \sigma\left(\frac{x+y}{2}, \xi\right) \varphi(y) dy d\xi, \ x \in \mathbb{R}, \tag{14}$$

where the integral in (14) is understood to be an iterated integral and call the classical Weyl transform associated with the symbol σ. The following result illuminates the relationship between the classical Weyl transform and the classical Wigner-Ville distribution and play a major role in the development of the theory of the classical Weyl transform.

$$\langle W_\sigma f, g\rangle = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma(x,\xi)\, W(f,g)(x,\xi) dx d\xi, \ x, \xi \in \mathbb{R}, \tag{15}$$

where $\sigma \in S^m,\ m \in \mathbb{R}$ and $f, g \in S(\mathbb{R})$.

3. The linear canonical-Wigner transform

In this section, we shall formally introduce the linear canonical-Wigner transform as analogy to the Fourier-Wigner transform [10] and study some of its properties in Lebesgue space. For this, we define the function $\rho_A(z,p)f$ on $\mathbb{R}$ by

$$(\rho_A(z,p)f)(x) = \exp\left(-i\frac{a_1}{2a_2}z^2 + i\frac{1}{a_2}zx - i\frac{a_4}{2a_2}x^2 + i\frac{1}{2a_2}zp - i\frac{a_4}{2a_2}p\left(x + \frac{p}{4}\right)\right) \times f(x+p), \ x \in \mathbb{R}. \tag{16}$$

Proposition 3.1. *$\rho_A(z,p) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a unitary operator for all z and $p \in \mathbb{R}$.*

Proof. We have to prove that

$$\left\|(\rho_A(z,p))\,f\right\|_{L^2(\mathbb{R})} \quad = \quad \left\|f\right\|_{L^2(\mathbb{R})}, f \in L^2(\mathbb{R}),$$

and $\rho_A(z,p)$ is onto for all z and $p \in \mathbb{R}$ only. By (16),

$$\begin{aligned}
&\left\|(\rho_A(z,p))\,f\right\|_{L^2(\mathbb{R})}\\
&= \int_{\mathbb{R}} \left|\exp\left(-i\frac{a_1}{2a_2}z^2 + i\frac{1}{a_2}zx - i\frac{a_4}{2a_2}x^2 + i\frac{1}{2a_2}zp - i\frac{a_4}{2a_2}p\left(x+\frac{p}{4}\right)\right)f(x+p)\right|^2 dx\\
&= \int_{\mathbb{R}} \left|f(x+p)\right|^2 dx\\
&= \int_{\mathbb{R}} \left|f(x)\right|^2 dx\\
&= \left\|f\right\|_{L^2(\mathbb{R})},\ f \in \mathbb{R},
\end{aligned}$$

for all z and $p \in \mathbb{R}$. Now, to prove that $\rho_A(z,p)$ is onto, we let $g \in L^2(\mathbb{R})$ and define the function f on $\mathbb{R}$ by

$$f(x) \quad = \quad \exp\left(i\frac{a_1}{2a_2}z^2 - i\frac{1}{a_2}zx + i\frac{a_4}{2a_2}x^2 + i\frac{1}{2a_2}zp - i\frac{a_4}{2a_2}p\left(x-\frac{p}{4}\right)\right)g(x-p),\quad x \in \mathbb{R}. \tag{17}$$

Here, f is obviously in $L^2(\mathbb{R})$ and by (16) and (17),

$$\begin{aligned}
&(\rho_A(z,p)f)\\
&= \exp\left(-i\frac{a_1}{2a_2}z^2 + i\frac{1}{a_2}zx - i\frac{a_4}{2a_2}x^2 + i\frac{1}{2a_2}zp - i\frac{a_4}{2a_2}p\left(x+\frac{p}{4}\right)\right)f(x+p)\\
&= \exp\left(-i\frac{a_1}{2a_2}z^2 + i\frac{1}{a_2}zx - i\frac{a_4}{2a_2}x^2 + i\frac{1}{2a_2}zp - i\frac{a_4}{2a_2}p\left(x+\frac{p}{4}\right)\right)\\
&\quad\times\exp\left(i\frac{a_1}{2a_2}z^2 - i\frac{1}{a_2}z(x+p) + i\frac{a_4}{2a_2}(x+p)^2 + i\frac{1}{2a_2}zp - i\frac{a_4}{2a_2}p\left(x+p-\frac{p}{4}\right)\right)g(x)\\
&= g(x),\quad x \in \mathbb{R}.
\end{aligned}$$

□

Remark 3.2. *It is clear from the proof of Proposition 3.1 that* $\rho_A(z,p)^{-1} = \rho_{-A^{-1}}(-z,-p),\quad z,p \in \mathbb{R}$. *In fact,* ρ_A *is a projective representation, i.e., a unitary representation upto phase factors, of the phase space* $\mathbb{R}^2$ *on* $L^2(\mathbb{R})$.

Definition 3.3. *Let* f *and* g *be in* $S_A(\mathbb{R})$. *Then, we define the function* $V_A(f,g)$ *on* $\mathbb{R}^2$ *by*

$$V_A(f,g)(z,p) \quad = \quad \sqrt{\frac{i}{2\pi a_2}}\,\langle\rho_A(z,p)f,g\rangle,\ \ z,p \in \mathbb{R} \tag{18}$$

where $\langle,\rangle$ *is the inner product in* $L^2(\mathbb{R})$. *We call* $V_A(f,g)$, *the linear canonical-Wigner transform of* f *and* g.

Proposition 3.4. *Let* f *and* g *be in* $S_A(\mathbb{R})$. *Then*

$$V_A(f,g)(z,p) \quad = \quad \sqrt{\frac{i}{2\pi a_2}}\int_{\mathbb{R}}\exp\left(-i\frac{a_1}{2a_2}z^2 + i\frac{1}{a_2}zy - i\frac{a_4}{2a_2}y^2\right)f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}dy \tag{19}$$

for all z *and* p *in* $\mathbb{R}$.

Proof. By (16) and (18), we get

$$\begin{aligned} & V_A(f,g)(z,p) \\ &= \sqrt{\frac{i}{2\pi a_2}}\,\langle \rho_A(z,p)f, g\rangle \\ &= \sqrt{\frac{i}{2\pi a_2}}\int_{\mathbb{R}} \exp\left(-i\frac{a_1}{2a_2}z^2 + i\frac{1}{a_2}zx - i\frac{a_4}{2a_2}x^2 + i\frac{1}{2a_2}zp - i\frac{a_4}{2a_2}p\left(x+\frac{p}{4}\right)\right) f(x+p)\overline{g(x)}dx. \end{aligned} \tag{20}$$

Now, let $x = y - \frac{p}{2}$ in (20), we get the desired result (19). □

Lemma 3.5. *Let $\varphi \in S_A(\mathbb{R}^2)$. Then, the function Ψ on $\mathbb{R}^2$ defined by*

$$\Psi(z,p) = \int_{\mathbb{R}} \exp\left(-i\frac{a_1}{2a_2}z^2 + i\frac{1}{a_2}zy - i\frac{a_4}{2a_2}y^2\right)\varphi(y,p)dy, \;\; z,p \in \mathbb{R}, \tag{21}$$

is also in $S_A(\mathbb{R}^2)$, provided $a_1^2 + a_2a_3 + 1 = 0$.

Proof. Let α, β, γ and λ be non-negative integers. Then, by (21) and Proposition 2.1, we have

$$\begin{aligned} & \left(\frac{z}{a_2}\right)^{\alpha}\left(\frac{p}{a_2}\right)^{\beta}\left(\overline{\Delta}_{A,q}^{\gamma}\overline{\Delta}_{A,p}^{\lambda}\Psi\right)(z,p) \\ &= \left(\frac{z}{a_2}\right)^{\alpha}\left(\frac{p}{a_2}\right)^{\beta}\int_{\mathbb{R}} \overline{\Delta}_{A,z}^{\gamma}\exp\left(-i\frac{a_1}{2a_2}z^2 + i\frac{1}{a_2}zy - i\frac{a_4}{2a_2}y^2\right)\overline{\Delta}_{A,p}^{\lambda}\varphi(y,p)dy \\ &= \left(\frac{z}{a_2}\right)^{\alpha}\left(\frac{p}{a_2}\right)^{\beta}\int_{\mathbb{R}} \left(-\frac{y}{a_2}\right)^{\gamma}\exp\left(-i\frac{a_1}{2a_2}z^2 + i\frac{1}{a_2}zy - i\frac{a_4}{2a_2}y^2\right)\overline{\Delta}_{A,p}^{\lambda}\varphi(y,p)dy \\ &= \left(\frac{p}{a_2}\right)^{\beta}\int_{\mathbb{R}} \Delta_{A^{-1},y}^{\alpha}\exp\left(-i\frac{a_1}{2a_2}z^2 + i\frac{1}{a_2}zy - i\frac{a_4}{2a_2}y^2\right)\left(-\frac{y}{a_2}\right)^{\gamma}\overline{\Delta}_{A,p}^{\lambda}\varphi(y,p)dy \\ &= \int_{\mathbb{R}} \exp\left(-i\frac{a_1}{2a_2}z^2 + i\frac{1}{a_2}zy - i\frac{a_4}{2a_2}y^2\right)\overline{\Delta}_{A^{-1},y}^{\alpha}\left(\left(-\frac{y}{a_2}\right)^{\gamma}\left(\frac{p}{a_2}\right)^{\beta}\overline{\Delta}_{A,p}^{\lambda}\varphi\right)(y,p)dy \\ &= \int_{\mathbb{R}} \exp\left(-i\frac{a_1}{2a_2}z^2 + i\frac{1}{a_2}zy - i\frac{a_4}{2a_2}y^2\right)\overline{\Delta}_{A,y}^{\alpha}\left(\left(-\frac{y}{a_2}\right)^{\gamma}\left(\frac{p}{a_2}\right)^{\beta}\overline{\Delta}_{A,p}^{\lambda}\varphi\right)(y,p)dy \end{aligned} \tag{22}$$

for all z and p in $\mathbb{R}$ and provided the last integral exist. Now, $\exists\; C_{\alpha,\beta,\gamma,\lambda}$ depending on α, β, γ and λ only, such that

$$\left|\overline{\Delta}_{A,y}^{\alpha}\left(\left(-\frac{y}{a_2}\right)^{\gamma}\left(\frac{p}{a_2}\right)^{\beta}\overline{\Delta}_{A,p}^{\lambda}\varphi\right)(y,p)\right| \;\le\; C_{\alpha,\beta,\gamma,\lambda}\left(1+\left|\frac{y}{a_2}\right|^2\right)^{-N}, \;\; y \in \mathbb{R}, \tag{23}$$

where N is some positive integer greater than $\frac{1}{2}$. Hence, by (22) and (23),

$$\sup_{z,p\in\mathbb{R}}\left|\left(\frac{z}{a_2}\right)^{\alpha}\left(\frac{p}{a_2}\right)^{\beta}\left(\overline{\Delta}_{A,q}^{\gamma}\overline{\Delta}_{A,p}^{\lambda}\Psi\right)(z,p)\right| \;\le\; C_{\alpha,\beta,\gamma,\lambda}\int_{\mathbb{R}}\left(1+\left|\frac{y}{a_2}\right|^2\right)^{-N}dy.$$

□

Proposition 3.6. *$V_A : S_A(\mathbb{R}) \times S_A(\mathbb{R}) \to S_A(\mathbb{R}^2)$ is a bilinear mapping.*

Proof. For all f and g in $S_A(\mathbb{R})$, the function φ on $S_A(\mathbb{R}^2)$ defined by

$$\varphi(y,p) = f(y)\overline{g(p)}, \;\; y,p \in \mathbb{R},$$

is obviously in $S_A(\mathbb{R}^2)$. Hence, the function Ψ on $\mathbb{R}^2$ defined by

$$\Psi(y,p) = f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}, \;\; y,p \in \mathbb{R}. \tag{24}$$

is also in $S_A(\mathbb{R}^2)$. Therefore, by (19), (24) and Lemma 3.5, $V_A(f,g) \in S_A(\mathbb{R}^2)$. □

4. The Wigner-Ville distribution

The Wigner-Ville distribution is a powerful time-frequency analysis tool. Wigner first introduced the classical Wigner distribution function associated with Fourier transform and then Wigner's investigation was substantially completed by Ville and named Wigner-Ville distribution. Moreover, for Wigner-Ville distribution involving certain integral transforms like fractional Fourier transform, linear canonical transform etc, we may refer [8, 13]. The theory of Weyl transform associated to the Wigner-Ville distribution is a large subject of remarkable interest both in mathematical analysis and physics, see, for instance,[10, 14–16]. In this section, we study the Wigner-Ville distribution in the LCT domain and investigate some of its properties [6, 7].

Definition 4.1. *[6] Let f and g be in $S_A(\mathbb{R})$. Then, the function $W_A(f,g)$ on $\mathbb{R}^2$, defined by*

$$W_A(f,g)(x,\xi) = \frac{1}{\sqrt{2\pi i a_2}}\int_{\mathbb{R}} \exp\left(i\frac{a_1}{a_2}xp - i\frac{1}{a_2}\xi p\right) f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}dp, \quad x,\xi\in\mathbb{R}, \tag{25}$$

is called the Wigner-Ville distribution of f and g in the LCT domain.

Theorem 4.2. *For all f_1, g_1, f_2 and g_2 in $S_A(\mathbb{R})$, we have*

$$\langle W_A(f_1,g_1), W_A(f_2,g_2)\rangle = \langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle}. \tag{26}$$

Proof. Define $\tilde{W}: S_A(\mathbb{R}^2)\to S_A(\mathbb{R}^2)$ by

$$\left(\tilde{W}_A F\right)(x,\xi) = \frac{1}{\sqrt{2\pi i a_2}}\int_{\mathbb{R}} \exp\left(i\frac{a_1}{a_2}xp - i\frac{1}{a_2}\xi p\right) F\left(x+\frac{p}{2}, x-\frac{p}{2}\right)dp, \quad x,\xi\in\mathbb{R}, \tag{27}$$

for all F in $S_A(\mathbb{R}^2)$.
Now, let F_1 and F_2 be functions on $\mathbb{R}^2$ defined by

$$F_1(u,v) = f_1(u)\overline{g_1(v)}, \quad u,v\in\mathbb{R} \tag{28}$$

and

$$F_2(u,v) = f_2(u)\overline{g_2(v)}, \quad u,v\in\mathbb{R}. \tag{29}$$

Then, by using (27), we have

$$\begin{aligned}
&\left\langle \tilde{W}_A F_1, \tilde{W}_A F_2\right\rangle \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\tilde{W}_A F_1\right)(x,\xi)\overline{\left(\tilde{W}_A F_2\right)(x,\xi)}dxd\xi \\
&= \int_{\mathbb{R}}\left\{\int_{\mathbb{R}} \left(\tilde{W}_A F_1\right)(x,\xi)\overline{\left(\tilde{W}_A F_2\right)(x,\xi)}d\xi\right\}dx \\
&= \frac{1}{2\pi a_2}\int_{\mathbb{R}}\int_{\mathbb{R}}\left\{\int_{\mathbb{R}} \exp\left(i\frac{a_1}{a_2}xp - i\frac{1}{a_2}\xi p\right)F_1\left(x+\frac{p}{2}, x-\frac{p}{2}\right)dp\right\} \\
&\quad\times\left\{\int_{\mathbb{R}} \exp\left(-i\frac{a_1}{a_2}xp' + i\frac{1}{a_2}\xi p'\right)F_2\left(x+\frac{p'}{2}, x-\frac{p'}{2}\right)dp'\right\}d\xi dx \\
&= \frac{1}{2\pi a_2}\int_{\mathbb{R}}\left\{\int_{\mathbb{R}} \exp\left(i\frac{a_1}{a_2}xp\right)F_1\left(x+\frac{p}{2}, x-\frac{p}{2}\right)dp\right\} \\
&\quad\times\left\{\int_{\mathbb{R}} \exp\left(-i\frac{a_1}{a_2}xp'\right)F_2\left(x+\frac{p'}{2}, x-\frac{p'}{2}\right)dp'\right\} \\
&\quad\times\left\{\int_{\mathbb{R}} \exp\left(i\frac{1}{a_2}\xi(p'-p)\right)d\xi\right\}dx
\end{aligned}$$

$$= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \exp\left(i\frac{a_1}{a_2}xp\right) F_1\left(x+\frac{p}{2}, x-\frac{p}{2}\right) dp \right\}$$
$$\times \left\{ \int_{\mathbb{R}} \exp\left(i\frac{a_1}{a_2}xp'\right) F_2\left(x+\frac{p'}{2}, x-\frac{p'}{2}\right) \delta(p'-p) dp' \right\} dx$$
$$= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} F_1\left(x+\frac{p}{2}, x-\frac{p}{2}\right) \overline{F_2\left(x+\frac{p}{2}, x-\frac{p}{2}\right)} dp \right\} dx$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} F_1\left(x+\frac{p}{2}, x-\frac{p}{2}\right) \overline{F_2\left(x+\frac{p}{2}, x-\frac{p}{2}\right)} dp dx \tag{30}$$

for all F_1 and F_2 in $S_A(\mathbb{R}^2)$. Let $u = x + \frac{p}{2}$ and $v = x - \frac{p}{2}$. Then, by (30), we get

$$\begin{aligned} \left\langle \tilde{W}_A F_1, \tilde{W}_A F_2 \right\rangle &= \int_{\mathbb{R}} \int_{\mathbb{R}} F_1(u,v) \overline{F_2(u,v)} du dv \\ &= \langle F_1, F_2 \rangle, \quad F_1, F_2 \in S_A(\mathbb{R}^2). \end{aligned} \tag{31}$$

Then, by (25), (27)-(29) and (31), we get

$$\begin{aligned} \langle W_A(f_1,g_1), W_A(f_2,g_2) \rangle &= \left\langle \tilde{W}_A F_1, \tilde{W}_A F_2 \right\rangle = \langle F_1, F_2 \rangle \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} F_1(u,v) \overline{F_2(u,v)} du dv \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f_1(u) \overline{g_1(v) f_2(u)} g_2(v) du dv \\ &= \left(\int_{\mathbb{R}} f_1(u) \overline{f_2(u)} du \right) \left(\int_{\mathbb{R}} \overline{g_1(v)} g_2(v) dv \right) \\ &= \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}. \end{aligned}$$

□

Remark 4.3. *The relation* (26) *is called the Moyal identity as [6] to the Wigner-Ville distribution in the LCT domain for* $f_1 = f_2 = g_1 = g_2 = f$.

Remark 4.4. *The function* $W_A(f,g)$ *is of greatest intrinsic interest in the case* $g = f$. *In this case we shall write*

$$W_A(f,f) = W_A(f)$$

and call $W_A(f)$*, the Wigner-Ville distribution of* f *in the LCT domain.*

Proposition 4.5. *We have frequency marginal property :*

$$\sqrt{\frac{i}{2\pi a_2}} \int_{\mathbb{R}} W_A(f)(x,\xi) dx = \left| (\mathcal{L}_A f)(\xi) \right|^2.$$

Proof. We have

$$\sqrt{\frac{i}{2\pi a_2}} \int_{\mathbb{R}} W_A(f)(x,\xi)$$
$$= \frac{1}{2\pi a_2} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\left(i\frac{a_1}{a_2}xp - i\frac{1}{a_2}\xi p\right) f\left(x+\frac{p}{2}\right) \overline{f\left(x-\frac{p}{2}\right)} dp dx.$$

Now, setting $u = x + \frac{p}{2}$ and $v = x - \frac{p}{2}$, we get

$$\sqrt{\frac{i}{2\pi a_2}} \int_{\mathbb{R}} W_A(f)(x,\xi)$$

$$= \frac{1}{2\pi a_2}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp\left(i\frac{a_1}{2a_2}\left(u^2 - v^2\right) - i\frac{1}{a_2}\left(u\xi - v\xi\right)\right) \times f(u)\,\overline{f(v)}dudv$$

$$= \frac{1}{2\pi a_2}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp\left(i\frac{a_1}{2a_2}u^2 - i\frac{1}{a_2}u\xi + i\frac{a_4}{2a_2}\xi^2\right)f(u) \overline{\times \exp\left(i\frac{a_1}{2a_2}v^2 - i\frac{1}{a_2}v\xi + i\frac{a_4}{2a_2}\xi^2\right)f(v)}dudv$$

$$= \left(\mathcal{L}_A f\right)(\xi)\overline{\left(\mathcal{L}_A f\right)(\xi)}.$$

This completes the proof of the Proposition 4.5. □

We have the following results similar to [6].

Proposition 4.6. *We have energy distribution property :*

$$\sqrt{\frac{i}{2\pi a_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} W_A(f)(x,\xi)dxd\xi = \int_{\mathbb{R}}\left|f(x)\right|^2 dx = \langle f(x), f(x)\rangle.$$

Proof. We have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} W_A(f)(x,\xi)d\xi dx$$
$$= \frac{1}{\sqrt{2\pi i a_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\left\{\int_{\mathbb{R}} \exp\left(i\frac{a_1}{2a_2}xp - i\frac{1}{a_2}\xi p\right)f\left(x+\frac{p}{2}\right)\overline{f\left(x-\frac{p}{2}\right)}dp\right\}d\xi dx$$

Setting $x + \frac{p}{2} = u$ and $x - \frac{p}{2} = v$, we get

$$\int_{\mathbb{R}}\int_{\mathbb{R}} W_A(f)(x,\xi)d\xi dx$$
$$= \frac{1}{\sqrt{2\pi i a_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}\left\{\int_{\mathbb{R}} \exp\left(-i\frac{a_1}{2a_2}\left(v^2-u^2\right) + i\frac{1}{a_2}\xi(v-u)\right)f(u)\,\overline{f(v)}d\xi\right\}dudv$$
$$= \sqrt{\frac{2\pi a_2}{i}}\int_{\mathbb{R}}\int_{\mathbb{R}} \delta(v-u)\exp\left(-i\frac{a_1}{2a_2}\left(v^2-u^2\right)\right)f(u)\overline{f(v)}dudv$$
$$= \sqrt{\frac{2\pi a_2}{i}}\int_{\mathbb{R}} f(u)\overline{f(u)}du$$

This completes the proof of the Proposition 4.6 □

Proposition 4.7. *We have time marginal property :*

$$\sqrt{\frac{i}{2\pi a_2}}\int_{\mathbb{R}} W_A(f)(x,\xi)d\xi = \left|f(x)\right|^2.$$

Proof. Proof is straight forward as Proposition 4.6 and immediately attained. □

5. The Weyl transform

The Weyl correspondence was first envisaged by Weyl arising in quantum mechanics, and is called Weyl transform by Wong [10]. The Weyl transform has found many applications to time-frequency analysis, the theory of differential equations, linear system theory, etc. In the theory of partial differential equation, the Weyl operator is considered as a particular case of pseudo differential operators, see, for instance [10, 14–19]. In this section, we introduce the Weyl transform in the LCT domain and explicate the relationship with the

Wigner-Ville distribution in the LCT domain.

At first, we give definition of a class of symbols S^s_A.

Definition 5.1. *[3] Let $s \in \mathbb{R}$. Then we define S^s_A to be the set of all functions $q(x,\xi) \in C^\infty(\mathbb{R}\times\mathbb{R})$ such that for any two non-negative integers α and β, there is a positive constant $C_{\alpha,\beta}$, depending on α and β only, such that*

$$\left|\left(D_x^\alpha \overline{\Delta}^\beta_{A^{-1},\xi} q(x,\xi)\right)\right| \leq C_{\alpha,\beta}\left(1+\left|\frac{\xi}{a_2}\right|\right)^{s-\beta}, \quad x,\xi \in \mathbb{R}, \tag{32}$$

where $\overline{\Delta}_{A^{-1},\xi}$ is as in (4), a_2 is as above and $D_x = \frac{1}{i}\frac{d}{dx}$.

Definition 5.2. *Let the symbol $q \in S^s_A(\mathbb{R}\times\mathbb{R})$. For all φ in $S_A(\mathbb{R})$, the Weyl transform in the framework of the LCT is a linear operator from $S_A(\mathbb{R})$ into $S_A(\mathbb{R})$ denoted by $W_q\varphi$ and defined by*

$$(W_q\varphi)(x) = (2\pi a_2)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right) q\left(\frac{x+y}{2},\xi\right)\varphi(y)\,dy\,d\xi, \ x \in \mathbb{R}, \tag{33}$$

where the integral in (33) is understood to be an iterated integral.

In order to obtain another representation of $W_q\varphi$, we let Θ be any function in C_0^∞ such that $\Theta(0) = 1$. Then we get following result :

Lemma 5.3. *The limit*

$$\lim_{\varepsilon\to 0^+}(2\pi a_2)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} \Theta(\varepsilon\xi)\exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right) q\left(\frac{x+y}{2},\xi\right)\varphi(y)\,dy\,d\xi$$

exists and is independent of the choice of the function Θ. Moreover, the convergence is uniform with respect to x on $\mathbb{R}$.

Proof. From Lemma 2.4, we have

$$\begin{aligned}&\left(1-\Delta^2_{A,y}\right)^N\left\{\exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right)\right\}\\ &= \left[1+\left(\frac{\xi}{a_2}\right)^2\right]^N \exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right), \quad x,y,\xi\in\mathbb{R}.\end{aligned} \tag{34}$$

So, by (34) and Proposition 2.1, we get

$$\begin{aligned}&(2\pi a_2)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} \Theta(\varepsilon\xi)\exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right) q\left(\frac{x+y}{2},\xi\right)\varphi(y)\,dy\,d\xi\\ &= (2\pi a_2)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} \Theta(\varepsilon\xi)\left[1+\left(\frac{\xi}{a_2}\right)^2\right]^{-N}\\ &\quad\times\left[\left(1-\Delta^2_{A,y}\right)^N\left\{\exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right)\right\}\right] q\left(\frac{x+y}{2},\xi\right)\varphi(y)\,dy\,d\xi\\ &= (2\pi a_2)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} \Theta(\varepsilon\xi)\left[1+\left(\frac{\xi}{a_2}\right)^2\right]^{-N}\exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right)\\ &\quad\times\left[\left(1-\overline{\Delta}^2_{A,y}\right)^N\left\{q\left(\frac{x+y}{2},\xi\right)\varphi(y)\right\}\right]dy\,d\xi\end{aligned} \tag{35}$$

for all $x \in \mathbb{R}$ and all positive numbers ε. Now, by (35) and Lemma 2.3, we get

$$(2\pi a_2)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} \Theta(\varepsilon\xi)\exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right) q\left(\frac{x+y}{2},\xi\right)\varphi(y)\,dy\,d\xi$$

$$= \sum_{r=0}^{N}\sum_{k=0}^{2r}\binom{N}{r}\binom{2r}{k}(-1)^{r+k}(2\pi a_2)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}}\Theta(\varepsilon\xi)\exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right)$$
$$\times\left(D_y^k q\right)\left(\frac{x+y}{2},\xi\right)\left(\overline{\Delta}_{A,y}^{2r-k}\varphi\right)(y)dyd\xi \tag{36}$$

for all $x \in \mathbb{R}$ and all positive numbers ε. Now, for each fixed x in $\mathbb{R}$,

$$\Theta(\varepsilon\xi)\left[1+\left(\frac{\xi}{a_2}\right)^2\right]^{-N}\exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right)\left(D_y^k q\right)\left(\frac{x+y}{2},\xi\right)\left(\overline{\Delta}_{A,y}^{2r-k}\varphi\right)(y)$$
$$\longrightarrow\left[1+\left(\frac{\xi}{a_2}\right)^2\right]^{-N}\exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right)\left(D_y^k q\right)\left(\frac{x+y}{2},\xi\right)\left(\overline{\Delta}_{A,y}^{2r-k}\varphi\right)(y) \tag{37}$$

for all y and ξ in $\mathbb{R}$ as $\varepsilon \to 0^+$. Furthermore, there exists a positive constant C, such that

$$\left|\Theta(\varepsilon\xi)\left[1+\left(\frac{\xi}{a_2}\right)^2\right]^{-N}\exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right)\left(D_y^k q\right)\left(\frac{x+y}{2},\xi\right)\left(\overline{\Delta}_{A,y}^{2r-k}\varphi\right)(y)\right|$$
$$\le C\left[1+\left(\frac{\xi}{a_2}\right)^2\right]^{-N}\left[1+\left|\frac{\xi}{a_2}\right|\right]^{s}\left|\left(\overline{\Delta}_{A,y}^{2r-k}\varphi\right)(y)\right|,\quad y,\xi\in\mathbb{R}. \tag{38}$$

Since

$$\left[1+\left(\frac{\xi}{a_2}\right)^2\right]^{-N}\left[1+\left|\frac{\xi}{a_2}\right|\right]^{s}\left|\left(\overline{\Delta}_{A,y}^{2r-k}\varphi\right)(y)\right|$$

is in $L^1(\mathbb{R}^2)$ as a function of (y,ξ) on $\mathbb{R}^2$ if $2N - s > 1$ i.e., $N > \frac{s+1}{2}$, it follows from (36), (37), (38) and the Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon\to 0^+}(2\pi a_2)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}}\Theta(\varepsilon\xi)\exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right)q\left(\frac{x+y}{2},\xi\right)\varphi(y)dyd\xi$$

exists and is independent of the choice of Θ. It can also be checked that the convergence is uniform with respect to x on $\mathbb{R}$. This completes the proof. □

Definition 5.4. *From the proof of Lemma 5.3, we can get another formula for $W_q\varphi$, when φ is in $S_A(\mathbb{R})$.*

$$(W_q\varphi)(x) = (2\pi a_2)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}}\left[1+\left(\frac{\xi}{a_2}\right)^2\right]^{-N}\exp\left(-i\frac{a_1}{2a_2}\left(x^2-y^2\right)+i\frac{1}{a_2}(x-y)\xi\right)$$
$$\times\left[\left(1-\overline{\Delta}_{A,y}^{2}\right)^N\left\{q\left(\frac{x+y}{2},\xi\right)\varphi(y)\right\}\right]dyd\xi,$$

for all x in $\mathbb{R}$, where N is any positive integer greater than $\frac{s+1}{2}$, $s \in \mathbb{R}$.

Proposition 5.5. *Let $q \in S_A^s$, $s \in \mathbb{R}$. Then, $W_q : S_A(\mathbb{R}) \to S_A(\mathbb{R})$ is continuous.*

Proof. As W_q is a particular case of pseudo-differential operator [3, P. 9]. □

Theorem 5.6. *Let $q \in S_A^s(\mathbb{R})$. Then,*

$$\left\langle W_q f, g\right\rangle = \sqrt{\frac{i}{2\pi a_2}}\int_{\mathbb{R}}\int_{\mathbb{R}}q(x,\xi)W_A(f,g)(x,\xi)dxd\xi,\quad f,g\in\mathbb{R}. \tag{39}$$

Proof. Let Θ be any function in $C_0^\infty(\mathbb{R})$ such that $\Theta(0) = 1$. Then, by (25), the Lebesgue dominated convergence theorem and Fubini's theorem, we get

$$\begin{aligned}
&\sqrt{\frac{i}{2\pi a_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} q(x,\xi)W_A(f,g)(x,\xi)dxd\xi \\
&= \lim_{\varepsilon\to 0^+}\sqrt{\frac{i}{2\pi a_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} \Theta(\varepsilon\xi)q(x,\xi)W_A(f,g)(x,\xi)dxd\xi \\
&= \lim_{\varepsilon\to 0^+}\frac{1}{2\pi a_2}\int_{\mathbb{R}}\int_{\mathbb{R}} \Theta(\varepsilon\xi)q(x,\xi) \\
&\quad\times\left\{\int_{\mathbb{R}}\exp\left(i\frac{a_1}{a_2}xp - i\frac{1}{a_2}\xi p\right)f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}dp\right\}dxd\xi \\
&= \lim_{\varepsilon\to 0^+}\frac{1}{2\pi a_2}\int_{\mathbb{R}}\Theta(\varepsilon\xi) \\
&\quad\times\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} q(x,\xi)\exp\left(i\frac{a_1}{a_2}xp - i\frac{1}{a_2}\xi p\right)f\left(x+\frac{p}{2}\right)\overline{g\left(x-\frac{p}{2}\right)}dpdx\right\}d\xi.
\end{aligned}$$

Setting $u = x + \frac{p}{2}$ and $v = x - \frac{p}{2}$, we get

$$\begin{aligned}
&\sqrt{\frac{i}{2\pi a_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} q(x,\xi)W_A(f,g)(x,\xi)dxd\xi \\
&= \lim_{\varepsilon\to 0^+}\frac{1}{2\pi a_2}\int_{\mathbb{R}}\Theta(\varepsilon\xi) \\
&\quad\times\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} q\left(\frac{u+v}{2},\xi\right)\exp\left(i\frac{a_1}{2a_2}\left(u^2-v^2\right) - i\frac{1}{a_2}(u-v)\xi\right)f(u)\,\overline{g(v)}dudv\right\}d\xi \\
&= \lim_{\varepsilon\to 0^+}\frac{1}{2\pi a_2}\int_{\mathbb{R}}\overline{g(v)} \\
&\quad\times\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} \Theta(\varepsilon\xi)q\left(\frac{u+v}{2},\xi\right)\exp\left(i\frac{a_1}{2a_2}\left(u^2-v^2\right) - i\frac{1}{a_2}(u-v)\xi\right)f(u)\,dud\xi\right\}dv \\
&= \int_{\mathbb{R}}\overline{g(v)}(W_q f)(v)dv \\
&= \left\langle W_q f, g\right\rangle.
\end{aligned}$$

This completes the proof. □

We denote the C^*-algebra of all bounded linear operators from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ by $B\left(L^2(\mathbb{R})\right)$.

Theorem 5.7. *There exists a unique bounded linear operator $\Psi : L^2(\mathbb{R}) \to B\left(L^2(\mathbb{R})\right)$ such that*

$$\langle(\Psi q)\,f, g\rangle = \sqrt{\frac{i}{2\pi a_2}}\int_{\mathbb{R}}\int_{\mathbb{R}} q(x,\xi)W_A(f,g)(x,\xi)dxd\xi \tag{40}$$

and

$$\left\|\Psi q\right\|_* \leq \frac{1}{\sqrt{2\pi a_2}}\left\|q\right\|_{L^2(\mathbb{R}^2)} \tag{41}$$

for all f and g in $L^2(\mathbb{R})$ and q in $L^2(\mathbb{R}^2)$, where $\|.\|_*$ denotes the norm in $B\left(L^2(\mathbb{R})\right)$.

Proof. Let $q \in S_A(\mathbb{R}^2)$. Then, for any f in $S_A(\mathbb{R})$, we define $(\Psi q)\, f$ by

$$(\Psi q)\, f = W_q f. \tag{42}$$

Then, by Theorem 5.6 and (42),

$$\begin{aligned}\langle (\Psi q)\, f, g\rangle &= \left\langle W_q f, g\right\rangle \\ &= \sqrt{\frac{i}{2\pi a_2}} \int_{\mathbb{R}}\int_{\mathbb{R}} q(x,\xi) W_A(f,g)(x,\xi)\,dx\,d\xi \end{aligned}\tag{43}$$

for all f and g in $S_A(\mathbb{R})$. Therefore, by Theorem 4.2 and (43),

$$\begin{aligned}\left|\langle (\Psi q)\, f, g\rangle\right| &\leq \frac{1}{\sqrt{2\pi a_2}} \left\|q\right\|_{L^2(\mathbb{R}^2)} \left\|W_A(f,g)\right\|_{L^2(\mathbb{R}^2)} \\ &= \frac{1}{\sqrt{2\pi a_2}} \left\|q\right\|_{L^2(\mathbb{R}^2)} \left\|f\right\|_{L^2(\mathbb{R})} \left\|g\right\|_{L^2(\mathbb{R})} \end{aligned}\tag{44}$$

for all f and g in $S_A(\mathbb{R})$. Hence, by (44),

$$\left\|(\Psi q)\, f\right\|_{L^2(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi a_2}} \left\|q\right\|_{L^2(\mathbb{R}^2)} \left\|f\right\|_{L^2(\mathbb{R})} \tag{45}$$

for all f in $S_A(\mathbb{R})$. Therefore, by (45),

$$\left\|\Psi q\right\|_* \leq \frac{1}{\sqrt{2\pi a_2}} \left\|q\right\|_{L^2(\mathbb{R}^2)}, \quad q \in S_A(\mathbb{R}^2). \tag{46}$$

Now, let $q \in L^2(\mathbb{R}^2)$. Let $\{q_m\}_{m=1}^{\infty}$ be a sequence of functions in $S_A(\mathbb{R}^2)$ such that $q_m \to q$ in $L^2(\mathbb{R}^2)$ as $m \to \infty$. Then, by (46),

$$\left\|\Psi q_m - \Psi q_k\right\|_* \leq \frac{1}{\sqrt{2\pi a_2}} \left\|q_m - q_k\right\|_{L^2(\mathbb{R}^2)} \to 0$$

as $m,k \to \infty$. Thus, $\{\Psi q_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $B(L^2(\mathbb{R}))$. We define Ψq to be limit in $B\left(L^2(\mathbb{R})\right)$ of the sequence $\{\Psi q_m\}_{m=0}^{\infty}$. This definition is independent of the choice of the sequence $\{q_m\}_{m=1}^{\infty}$. Indeed, let $\{\sigma_m\}_{m=1}^{\infty}$ be another sequence of functions in $S_A(\mathbb{R})$ such that $\sigma_m \to q$ in $L^2(\mathbb{R}^2)$ as $m \to \infty$. Then, again by (45), we get

$$\left\|\Psi q_m - \Psi \sigma_m\right\|_* \frac{1}{\sqrt{2\pi a_2}} \left\|q_m - \sigma_m\right\|_{L^2(\mathbb{R}^2)} \to 0$$

as $m \to \infty$. Thus, the limit in $B(L^2(\mathbb{R}))$ of $\{\Psi q_m\}_{m=1}^{\infty}$ and $\{\Psi \sigma_m\}_{m=1}^{\infty}$ are equal.

Next, let $q \in L^2(\mathbb{R}^2)$ and let $\{q_m\}_{m=1}^{\infty}$ be a sequence of functions in $S_A(\mathbb{R}^2)$ such that $q_m \to q$ in $L^2(\mathbb{R}^2)$ as $m \to \infty$. Then, by (46), we have

$$\left\|\Psi q\right\|_* = \lim_{m\to\infty} \left\|\Psi q_m\right\|_* \leq \frac{1}{\sqrt{2\pi a_2}} \lim_{m\to\infty} \left\|q_m\right\|_{L^2(\mathbb{R}^2)} = \frac{1}{\sqrt{2\pi a_2}} \left\|q\right\|_{L^2(\mathbb{R}^2)}$$

and (41) is proved.
Now, if f and g are in $S_A(\mathbb{R})$, then by (33),

$$\begin{aligned}\langle (\Psi q)\, f, g\rangle &= \lim_{m\to\infty} \langle (\Psi q_m)\, f, g\rangle \\ &= \lim_{m\to\infty} \sqrt{\frac{i}{2\pi a_2}} \int_{\mathbb{R}}\int_{\mathbb{R}} q_m(x,\xi) W_A(f,g)(x,\xi)\,dx\,d\xi \\ &= \sqrt{\frac{i}{2\pi a_2}} \int_{\mathbb{R}}\int_{\mathbb{R}} q(x,\xi) W_A(f,g)(x,\xi)\,dx\,d\xi. \end{aligned}$$

Finally, let f and g be in $L^2(\mathbb{R})$. Then, we pick sequences $\{f_m\}_{m=1}^{\infty}$ and $\{g_m\}_{m=1}^{\infty}$ in $S_A(\mathbb{R})$ such that $f_m \to f$ in $L^2(\mathbb{R})$ and $g_m \to g$ in $L^2(\mathbb{R})$ as $m \to \infty$. We have

$$\begin{aligned} \langle (\Psi q) f, g \rangle &= \lim_{m\to\infty} \langle (\Psi q) f_m, g_m \rangle \\ &= \lim_{m\to\infty} \sqrt{\frac{i}{2\pi a_2}} \int_{\mathbb{R}} \int_{\mathbb{R}} q(x,\xi) W_A(f_m, g_m)(x,\xi) dx d\xi \\ &= \sqrt{\frac{i}{2\pi a_2}} \int_{\mathbb{R}} \int_{\mathbb{R}} q(x,\xi) W_A(f, g)(x,\xi) dx d\xi. \end{aligned}$$

It is obvious that $\Psi : L^2(\mathbb{R}^2) \to B(L^2(\mathbb{R}))$ is the only bounded linear operator satisfying (40) for all f and g in $L^2(\mathbb{R})$ and q in $L^2(\mathbb{R}^2)$. Therefore, this completes the proof. □

6. Conclusion

Based on the LCT and classical Fourier-Wigner transform, this paper proposes the definition of the linear canonical-Wigner transform in harmonic analysis of phase space alongwith some of its results. Moreover the Wigner-Ville distribution in the LCT domain as [6] is discussed briefly. Further the Weyl transform in the LCT domain is defined and established a relation between WVD along with Weyl transform in the LCT domain. The future works will be applicable to study the boundedness and compactness of defined Weyl transform in LCT domain on the L^p space.

References

[1] S. A. Collins, *Lens-system diffraction integral written in terms of matrix optics*, J. Opt. Soc. Amer., **60**(9)(1970), 1168-1177.

[2] M. Moshinsky, C. Quesne, *Linear canonical transformations and their unitary representation*, J. Maths. Phys., **12**(1971), 1772-1783.

[3] A. Prasad, A. Kumar, *The canonical potential and L^p-Sobolev space associated with linear canonical transform*, Integral Transforms Spec. Funct., https://doi.org/10.1080/10652469.2022.2118737

[4] K. B. Wolf, *Construction and properties of canonical transforms*, in Integral Transform in Science and Engineering (Plenum, New York, (1979)). Chap. 9

[5] E. Wigner, *On the quantum correction for thermodynamic equilibrium*, Phys. Rev., **40**(1932), 749-759.

[6] D. Urynbassarova, B. Z. Li, R. Tao, *The Wigner-Ville distribution in the linear canonical transform domain*, IAENG Int. J. Appl. Math., **46**(4)(2016), 559-563.

[7] R. F. Bai, B. Z. Li, Q. Y. Cheng, *Wigner-Ville distribution associated with the linear canonical transform domain*, J. Appl. Math., **46**(4)(2012), 1-4.

[8] D. Urynbassarova, B. Z. Li, R. Tao, *Convolution and correlation theorems for Wigner-Ville distribution associated with the offset linear canonical transform*, Optik, **157**(2018), 455-466.

[9] H. Weyl, *The theory of groups und quantum mechanics*, Dover, New York, (1950).

[10] M. W. Wong, *Weyl transform*, Singapore, New York, (1998).

[11] J. C. T. Pool, *Mathematical aspects of the Weyl correspondence*, J. Math. Phys., **7**(1)(1966), 66-76.

[12] B. Simon, *The Weyl transforms and L_p functions on phase space*, Proc. Amer. Math. Soc., **116**(4)(1992), 10451047.

[13] D. Mustard, *The fractional Fourier transform and the Wigner distribution*, J. Austral. Math. Soc. Ser. B, **38**(1996), 209-219.

[14] L. T. Rachdi, K. Trimeche, *Weyl transforms associated with the spherical mean operator*, Anal. Appl., **1**(2)(2003), 141-164.

[15] A. Dachraoui, *Weyl-Bessel transform*, J. Comp. Appl. Math., **133**(2001), 263-276.

[16] G. B. Folland, *Harmonic analysis in phase space*, Princeton University Press, New Jersey, (1989).

[17] J. v. Neumann, *Mathematical foundations of quantum mechanics*, Princeton University Press, Princeton, (1996).

[18] G. B. Folland, *Real analysis(Modern techniques and their applications)*, John Wiley and Sons, (1984).

[19] A. Grossman, G. Loupias, E. M. Stein, *An algebra of pseudo-differential operators and quantum mechanics in phase space*, Ann. Inst. Fourier(Grenoble), **18**(1968), 343-368.